



Existence of maximal solutions for some very singular nonlinear fractional diffusion equations in 1D

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Abstract. We consider nonlinear parabolic equations involving fractional diffusion of the form $\partial_t u + (-\Delta)^s \Phi(u) = 0$, with $0 < s < 1$, and solve an open problem concerning the existence of solutions for very singular nonlinearities Φ in power form, precisely $\Phi(u) = c u^{-(n+1)}$ for some $0 < n < 1$. We also include the logarithmic diffusion equation $\partial_t u + (-\Delta)^s \log(u) = 0$, which appears as the case $n = 0$. We consider the Cauchy problem with nonnegative and integrable data $u_0(x)$ in one space dimension, since the same problem in higher dimensions admits no nontrivial solutions according to recent results of the author and collaborators. The *limit solutions* we construct are unique, conserve mass, and are in fact maximal solutions of the problem. We also construct self-similar solutions of Barenblatt type, which are used as a cornerstone in the existence theory, and we prove that they are asymptotic attractors (as $t \rightarrow \infty$) of the solutions with general integrable data. A new comparison principle is introduced.

1. Introduction

In this paper, we consider a class of nonlinear parabolic equations involving fractional diffusion of the form

$$\partial_t u + (-\Delta)^s \Phi(u) = 0. \quad (1.1)$$

The symbol $(-\Delta)^s$ denotes the fractional Laplacian operator with $0 < s < 1$, i.e., the nonlocal operator defined by

$$(-\Delta)^s v(x) = c(N, s) \text{ p.v. } \int_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x - y|^{N+2s}} dy, \quad \forall x \in \mathbb{R}^N, \quad (1.2)$$

acting on the whole Euclidean space at least for functions in the Schwartz class \mathcal{S} . The formula is valid for all dimensions $N \geq 1$. The constant $c(N, s)$ is given in the literature, but it is not needed in what follows and p.v. means principal value of the integral.

The existence and properties of solutions for this type of equations with fractional diffusion have been studied by the author and collaborators for nonlinearities Φ that are positive and increasing for $u > 0$, in particular when $\Phi(u) = u^m$ with $m > 0$, cf.

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[8, 9, 18–20, 42, 43]. This includes singular cases for $0 < m < 1$ since then $\Phi'(u) = mu^{m-1} \rightarrow \infty$ as $u \rightarrow 0$.

Here, we are interested in *very singular nonlinearities*, more precisely, when $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a monotone increasing function of u with a singularity in $u = 0$ such that $\Phi(0+) = -\infty$. Consequently, nonnegative data and solutions are considered. The standard cases we have in mind are $\Phi_n(u) = -1/u^n$ for some $n > 0$, or $\Phi_0(u) = \log(u)$. They correspond to $\Phi'(u) \sim u^{-(n+1)}$ with $n + 1 \geq 1$; thus, the denomination *very singular* is introduced in the literature for this type of equations with standard Laplacian, cf. [39]. We will keep this tradition for equations with a fractional Laplacian. These very singular diffusion equations are also described in the literature as *very fast diffusion*, *superfast diffusion*, or *ultra-fast diffusion*, cf. e.g. [25, 31, 39].

For such equations, the existence of solutions is not at all obvious. Thus, we have proved in a recent paper with Bonforte and Segatti [7] that when the space dimension is $N \geq 2$ and we try to solve the Cauchy problem in the whole space \mathbb{R}^N with integrable initial data, then there exist no nontrivial solutions, even if we accept local-in-time solutions defined for a short time interval, $0 < t < T$.¹ The same happens for the problem posed in a bounded domain with zero Dirichlet data.

The purpose of this paper is to prove that there is a range of existence of solutions for very singular equations of the form (1.1) with $\Phi = \Phi_n$ if the space dimension is 1. We will also prove that the solutions have the good properties of the nonsingular range of parameters $\Phi(u) = u^m$ with $m > 0$. A very crude explanation of the existence result is as follows: By becoming strictly positive for $t > 0$, the solutions avoid the singularity in a way that suffices to grant first nontrivial existence and then the rest of the properties.

Before stating the results, let us point out that the standard notation for the nonsingular equation is $\Phi(u) = cu^m$ with $c, m > 0$. In this paper, the exponent of the nonlinearity Φ_n is written in terms of $n = -m$ with $n \geq 0$, and the detailed calculations are done for $n > 0$. The reason for this notation is to avoid the use of negative exponents that might confuse the reader in interpreting the results.² The space dimension is N , mostly $N = 1$ here.

THEOREM 1.1. *Let $N = 1$ and let $\Phi(u) = \Phi_n(u)$ with $n > 0$ or $\Phi(u) = \log(u)$ (case $n = 0$). Equation (1.1) posed in $Q = \mathbb{R} \times (0, \infty)$ with initial data*

$$u(x, 0) = u_0(x) \in L^1(\mathbb{R}), \quad u_0 \geq 0, \quad (1.3)$$

admits a positive very weak solution if $s > 1/2$ and $0 \leq n < 2s - 1$. There could be nonuniqueness of the solutions, but we construct a unique limit solution for every initial data, and we prove that it is maximal among all solutions. This solution exists globally in time, $u \in C([0, \infty) : L^1(\mathbb{R}))$, and is positive everywhere.

¹By trivial solution, we mean $u(x, t) \equiv 0$ in the whole domain of definition.

²The reader may also wonder, why the minus sign in the coefficient of Φ_n ? It is needed to make Φ_n an increasing function, so that the equation will be parabolic in some sense.

The range of exponents $1/2 < s < 1$ and $0 \leq n < 2s - 1$ is almost sharp. There is indeed another isolated case of existence of integrable solutions in 1D, namely $s = 1/2$ and $n = 0$ (logarithmic diffusion). The very peculiar properties of this case deserve a separate study, but we give a preliminary idea in Sect. 12 that supports the assertion of existence of solutions at least for short times. For the other exponents $s \in (0, 1)$, $n \geq 0$, in dimension one, nontrivial solutions do not exist by the mentioned results of [7].

The construction of solutions proceeds by approximation, taking approximate initial data that are uniformly positive, so that the problem is no more singular. Passing then to the limit in the approximations, we obtain a solution that is shown to be nontrivial after some effort. It is called the *limit solution* (*upper limit solution*, to be precise). It is subsequently proved to be a very weak solution. Here we define very weak solution of Eq. (1.1) as a nonnegative function $u \in C((0, \infty) : L^1(\mathbb{R}^N))$ such that

$$\int_0^\infty \int_{\mathbb{R}} u \frac{\partial \zeta}{\partial t} \, dx dt = \int_0^\infty \int_{\mathbb{R}} \Phi(u) (-\Delta)^s \zeta \, dx dt, \tag{1.4}$$

and the last integral is absolutely convergent for all ζ smooth and compactly supported. In the theorem, $\Phi(u) = \Phi_n(u)$.

In the course of the present paper, we will also establish the main properties of the constructed solutions. We select here the main results for easy reference.

THEOREM 1.2. *The limit solution preserves mass, $\int u(x, t) \, dx = \int u_0(x) \, dx$. Moreover, it is a weak solution for $t > 0$ and satisfies the bounds*

$$C_1(t)(1 + |x|^2)^{s/(1+n)} \leq u(x, t) \leq C_2 \|u_0\|_1^\delta t^{-\alpha} \tag{1.5}$$

with $\alpha = 1/(2s - n - 1)$, $\delta = 2s\alpha$, a continuous function $C_1(t) > 0$ that may depend on the solution, and a constant $C_2(n, s) > 0$. The collection of limit solutions generates an ordered, L^1 -contraction semigroup in $L^1_+(\mathbb{R})$.

A specially important feature of the paper is the construction and properties of the fundamental solutions.

THEOREM 1.3. *There exists a special function of the form*

$$U(x, t) = t^{-\alpha} F(x t^{-\alpha}) \tag{1.6}$$

that is a very weak positive solution of the problem for $t \geq \tau > 0$ and takes on a Dirac mass as initial data, $u(x, t) \rightarrow \delta(x)$ as $x \rightarrow 0$ in the sense of positive Radon measures. The profile F is positive everywhere, integrable, symmetric, $F(x) = F(-x)$, and F monotone decreasing for $x > 0$. Moreover;

$$\lim_{|x| \rightarrow \infty} |x|^{2s/(1+n)} F(|x|) = C(s, n) > 0. \tag{1.7}$$

The constant $C(s, n)$ can be calculated as the constant appearing in the very singular solution, a special solution with formula $\tilde{U}(x, t) = C(s, n) t^{1/(1+n)} |x|^{-2s/(1+n)}$, that

has a nonintegrable singularity at $x = 0$. Finally, the solution with initial data $M \delta(x)$, $M > 0$, is just

$$U_M(x, t) = M U(x, M^{-(1+n)}t), \text{ so that } F_M(\xi) = M^{2s\alpha} F(M^{(1+n)\alpha}\xi). \quad (1.8)$$

Using the terminology of [40], we call a *fundamental* any solution with $u(x, 0)$ equal to a Dirac delta, and Barenblatt solution a fundamental solution that is also self-similar. On the one hand, the Barenblatt solutions of the theorem play an important role in completing the existence theory described in Theorem 1.2. In the theory, different comparison theorems are also used, in particular a new shifting comparison result, that we prove as Theorem 4.2. On the other hand, the Barenblatt solutions explain the asymptotic behaviour of general solutions, according to the following general theorem.

THEOREM 1.4. *Let $u_0 \in L^1(\mathbb{R})$, let $M = \int u_0(x) dx$, and let U_M be the self-similar Barenblatt solution with mass M . Then as $t \rightarrow \infty$ the solutions $u(x, t)$ and $U_M(x, t)$ are increasingly close and we have*

$$\lim_{t \rightarrow \infty} \|u(\cdot, t) - U_M(\cdot, t)\|_1 = 0, \quad (1.9)$$

Indeed, convergence happens in all L^p norms, $1 \leq p < \infty$, in the form

$$\lim_{t \rightarrow \infty} t^{\alpha p} \|u(\cdot, t) - U_M(\cdot, t)\|_{L^p(\mathbb{R})} = 0, \quad \alpha_p = \frac{p-1}{p(2s-1+n)}. \quad (1.10)$$

There are some other results worth recalling. Thus, as a side result of our analysis, we construct in Sect. 8 the very singular solution (VSS), which is explicit (see Theorem 8.1) and has very special properties. Very singular solutions have played a special role in the theory of fast diffusion equations, as attested, e.g. in [12]. Our VSS will give us a first clue to the lower bound $O(|x|^{-2s/(1+n)})$ for the spatial decay of all positive solutions, that we have stated in (1.5), and plays a role in different passages of the existence theory that we will develop below.

After all these theorems are proved and shifting comparison is established, we devote a short section to a preliminary presentation of the special case $s = 1/2, n = 0$. The paper concludes with a section on comments, open problems.

Precedents and commentary

(1) Many results are known about Problems (1.1)–(1.3), mainly for standard diffusion $s = 1$, where the Laplacian is used instead of the fractional Laplacian. The nonsingular case $\Phi(u) = u^m$ with $m > 0$ is known as the porous medium equation when $m > 1$, the heat equation for $m = 1$ and the fast diffusion equation, $0 < m < 1$; their theory has been studied in great detail and is described in monographs like [2, 17, 38, 39]. As a basic existence result, each of these equations generates a mild solution for every initial data $u_0 \in L^1(\mathbb{R}^N)$, and the collection of such solutions forms an ordered L^1 contraction semigroup for every fixed m .

(2) For equations with fractional Laplacians of the form (1.1)–(1.2) with $0 < s < 1$ and the same of power-like nonlinearity $\Phi(u) = u^m$, $m > 0$, the study of the Cauchy problem with nonnegative data in $L^1(\mathbb{R}^N)$, $N \geq 1$, has been done in the papers [18, 19], and most of the basic results are still true though the techniques may be quite different. More precisely, existence and uniqueness of solutions in the class of very weak or strong solutions have been proved, and the main qualitative and quantitative properties are established. Thus, when $N(m - 1) + 2s > 0$, the solutions are positive everywhere in $Q = \mathbb{R}^N \times (0, \infty)$; the so-called smoothing effect asserts that L^1 initial data produce bounded solutions for positive times and indeed

$$u(x, t) \leq C(N, m, s) \|u_0\|_1^\gamma t^{-\alpha}$$

where $\alpha = N/(N(m - 1) + 2s)$ and $\gamma = 2s/(N(m - 1) + 2s)$, both positive in this range. A main feature of the theory is the existence of fundamental solutions and their use in establishing the asymptotic behaviour of general solutions. This was proved in [40], also under the necessary restriction $N(m - 1) + 2s > 0$.

(3) Since this condition on the exponents to obtain good behaviour becomes formally $m > 1 - 2s$ in 1D, a similar theory could be expected to hold for very singular exponents $m < 0$, if $2s - 1 > n = -m > 0$ when $N = 1$. However, the difficulties of dealing the singular nonlinearities prevented the inclusion of this extension in the works [18, 19, 40]. We supply in this paper the approach and tools to fill such a gap in the range of exponents of Theorem 1.1. In particular, we construct the fundamental solutions and show that they are responsible for the asymptotic behaviour of general solutions. The text below shows that such extended theory is far from immediate and needs some involved tools.

(4) The case of singular powers $\Phi(u) = -u^{-n}$ with $n > 0$, or $\Phi_0(u) = \log(u)$, was considered by the author many years ago in [35] for the standard Laplacian, $s = 1$. A remarkable result of non-existence of integrable solutions was proved for all $n \geq 0$ if $N \geq 3$, for $n > 0$ if $N = 2$, and for $n \geq 1$ if $N = 1$. That paper is a remote precedent for the present work (previously, non-existence for the particular limit case $n = 1$ in $N = 1$ had been proved in [22] using a special transformation). More precisely, when we perform the natural approximation by regular problems, as explained in the next section, the sequence of approximations collapses to zero for all $x \in \mathbb{R}$ and all $t > 0$ for all initial data in the integrable class. This radical phenomenon is called instantaneous extinction. On the other hand, existence occurs in the remaining cases, $N = 1$, $0 \leq n < 1$, and $N = 2$, $n = 0$. This agrees with the results we are going to prove in the fractional case. Interesting properties arise in the existence cases, see a detailed account in [39, Section 9]. The non-existence results of [35] were extended to optimal classes of (nonintegrable) initial data in [14–16].

(5) In the case of singular nonlinearities and fractional Laplacians, which is our framework here, the non-existence of solutions has been recently established in collaboration with Bonforte and Segatti [7], in the range that perfectly complements the positive result of Theorem 1.1 plus the announced existence result for $s = 1/2$,

$n = 0$ in 1D. Non-existence happens for all singular power cases ($n \geq 0$) in dimension $N \geq 2$. In all those non-existence cases, we got *instantaneous extinction* for the initial value problem with any data $u_0 \in L^1_+(\mathbb{R})$. Putting these results together, we obtain a complete picture of the solubility problem with integrable data for all singular parameters.

More on notations

We use the sign $f \sim g$ to denote that both functions are proportional in a certain limit or range of values I (which may be explicit or understood from the context). If the proportionality ratio goes to 1 in some limit, then we write $f \approx g$. In the proofs, we will often use *rearranged* functions defined on the line. This means that they are nonnegative, symmetric and monotone nonincreasing for $x > 0$. Other notations will be explained as they appear.

2. Problem, approximation, and limit solutions

Let us discuss the way to prove existence for the Cauchy Problem (1.1)–(1.3), i.e. to find solutions of the equation posed in $Q = \mathbb{R} \times (0, \infty)$ with $T > 0$, taking on initial data $u_0(x)$, assumed to be nonnegative and integrable. In this section, we also assume that u_0 is bounded, a restriction that is made for convenience and will be removed later on. The nonlinear function Φ is defined, increasing and smooth for $u > 0$, with $\Phi(s) \rightarrow -\infty$ as $s \rightarrow 0$. More precisely, we will construct a complete theory for the case where Φ is chosen from the list Φ_n , $n \geq 0$, as mentioned in Introduction. From this moment on, we assume n to be fixed. Following [35], a strategy of proof of existence or non-existence of solutions is based on approximating problem (1.1)–(1.3) by the family problems

$$\begin{cases} \partial_t u_\varepsilon + (-\Delta)^s(\Phi(u_\varepsilon)) = 0, & n > 0, \\ u_\varepsilon(x, 0) = u_0(x) + \varepsilon & \text{for } x \in \mathbb{R}^N, \end{cases} \quad (2.1)$$

for any $\varepsilon > 0$, so that we avoid data with values on the singular level $u = 0$. The standard theory applies to these problems and a classical solution $u_\varepsilon(t, x)$ exists for all $\varepsilon > 0$, and it is strictly positive: $u_\varepsilon \geq \varepsilon$ (see details in the next subsection). Moreover, the maximum principle holds for these classical solutions, and we have $u_\varepsilon \geq u_{\varepsilon'}$ for $\varepsilon \geq \varepsilon' > 0$. Therefore, we can take the monotone limit

$$\bar{u}(x, t) = \lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t). \quad (2.2)$$

This function is a kind of generalized solution of the problem, which belongs to the class of *limit solutions*. It is now an important step of the theory to decide in which sense this limit solution is a solution of the equation in a more traditional functional sense (like very weak, weak, strong or viscosity solution), and also in which sense it takes the initial data. In cases of nonuniqueness of such solutions, the unique limit

obtained by the above method has been called by various names: maximal solution, SOLA, proper solution, However, this kind of considerations is not the main issue of this paper, which is concerned with describing the existence and behaviour of the class of limit solutions. In order to recall the way the limit is taken (via approximations from above) and to avoid possible confusions, we propose the more precise term *upper limit solutions* for the limits (2.2), but we will allow the simpler name limit solutions when there is no fear of confusion.

This approximation method has been used in [35] to prove non-existence in the case of standard Laplacian as mentioned above, and in [7] to prove the non-existence results for fractional diffusion and those singular Φ that do not fall into the cases treated in this paper.

2.1. Existence and properties of the approximate solution

It is convenient to write $u_\varepsilon(x, t) = v_\varepsilon(x, t) + \varepsilon$ and then try to solve the Cauchy problem

$$\begin{cases} \partial_t v + (-\Delta)^s(\Phi_\varepsilon(v)) = 0 & \text{with } \Phi_\varepsilon(v) := \Phi(v + \varepsilon) - \Phi(\varepsilon) \\ v(0) = u_0 & \text{for } x \in \mathbb{R}. \end{cases} \quad (2.3)$$

for all $\varepsilon > 0$. This is a modified problem prepared to avoid the singular level $u = 0$ of the equation by displacement of the axes. Note that for $\varepsilon > 0$ and for nonnegative arguments Φ_ε is a smooth, positive, monotone increasing function with $\Phi_\varepsilon(0) = 0$ and $\Phi'_\varepsilon(v)$ positive, bounded and decreasing for all $v \geq 0$; $\Phi'_\varepsilon(v)$ is uniformly positive if v is bounded. The theory of existence and uniqueness of weak solutions v_ε to the nonsingular Problem (2.3) is given in [42, Theorem 8.2]. After obtaining these solutions, we restore for any $\varepsilon > 0$ the original u -level by defining $u_\varepsilon := v_\varepsilon + \varepsilon$, as stated at the beginning. Clearly, we have that $u_\varepsilon \geq \varepsilon$ (actually, $u_\varepsilon > \varepsilon$) and hence $v_\varepsilon \geq 0$ in Q .

Let us list some further properties of v_ε and u_ε . For the proof, we may again refer to [7, 19, 42].

- *Boundedness and regularity.* These solutions are shown to be bounded for strictly positive times. More precisely, for every $t > 0$ and every $p \in [1, \infty]$ there holds

$$\|v_\varepsilon(\cdot, t)\|_p \leq \|u_0\|_p \quad (2.4)$$

As a consequence, if u_0 belongs to L^∞ , then v_ε is regular enough to satisfy the equation in the classical sense at least when $t > 0$, by the results of [42]. Therefore, $u_\varepsilon = v_\varepsilon + \varepsilon$ is smooth and satisfies the original equation in the classical sense in Q . Under these circumstances, the initial data are also taken, at least in the sense of convergence in $L^1(\mathbb{R}^N)$. For unbounded data, this result about initial data follows from density and contraction in L^1 , see next paragraph.

• *L¹-contraction and comparison.* The evolution (2.3) is an L^1 contraction, namely for two solutions $u_{1,\varepsilon}, u_{2,\varepsilon}$ we have

$$\int_{\mathbb{R}^N} (u_{1\varepsilon}(x, t) - u_{2\varepsilon}(x, t))_+ dx \leq \int_{\mathbb{R}^N} (u_{01} - u_{02})_+ dx \quad \text{for } t > 0, \tag{2.5}$$

Here, $(\cdot)_+$ denotes the positive part function. In particular, standard comparison follows: If $u_{01} \leq u_{02}$ a.e., then for every $t > 0$ we get $u_{1\varepsilon}(\cdot, t) \leq u_{2\varepsilon}(\cdot, t)$ a.e.

• *Mass conservation.* Nonnegative solutions to the evolution Eq. (2.3) conserve the mass, cf. [19,42]. More precisely, we have for all $t \geq 0$

$$\int_{\mathbb{R}^N} v_\varepsilon(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx \quad \text{i.e.,} \quad \int_{\mathbb{R}^N} (u_\varepsilon(x, t) - \varepsilon) dx = \int_{\mathbb{R}^N} u_0(x) dx. \tag{2.6}$$

• *Monotonicity with respect to ε .* An easy version of the above comparison argument shows also that for $0 < \varepsilon < \varepsilon'$ we have $0 < \varepsilon \leq u_\varepsilon \leq u_{\varepsilon'}$.

• *Time monotonicity.* There is an important monotonicity property valid of all nonnegative solutions, known as the Bénilan–Crandall inequality

$$\partial_t u_\varepsilon \leq \frac{u_\varepsilon}{(n + 1)t} \quad \forall (x, t) \in Q. \tag{2.7}$$

The argument only uses the scaling invariance of the equation and the maximum principle, so [5]’s argument applies.

• *The smoothing effect.* It says that all solutions with integrable data are in fact bounded for positive times. This follows from Theorem 8.2 of [42] that we adapt to our dimension and notations as follows.

PROPOSITION 2.1. *Let $\Phi \in C^1(\mathbb{R})$ be such that $\Phi'(u) \geq C|u|^{-n-1}$ for some $n \in \mathbb{R}$ and $|u| \geq C$. If $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, where $p \geq 1$ satisfies $2sp > 1 + n$, then a weak L^1 -energy solution to the Cauchy problem (1.1)–(1.3) is bounded in $\mathbb{R} \times (\tau, \infty)$ for all $\tau > 0$. More precisely, it satisfies*

$$\sup_{x \in \mathbb{R}^N} |u(x, t)| \leq \max \left\{ C, C_1 t^{-\alpha_p} \|u_0\|_p^{\delta_p} \right\} \tag{2.8}$$

with $\alpha_p = 1/(2sp - n - 1)$ and $\delta_p = 2sp\gamma_p$, the constant C_1 depending on n, p, σ, C .

See also [19] in that respect. The statement of [42] does not assume that $m = -n > 0$, only that $2sp > 1 - m$. The assumptions on Φ are satisfied by the nonlinearities Φ_ε of the approximate problems (2.3); hence, the result applies to the approximations v_ε . Recall that we write $m = -n$ and note that the constants in the formula may also depend on ε . We will solve the latter difficulty later on.

2.2. Passing to the limit

We may now pass to the limit $\varepsilon \rightarrow 0$ using the monotonicity in ε of the family u_ε . By the monotone convergence theorem, the limit \bar{u} is taken in the local L^1 sense, i.e. in $L^1(B)$ for every compact subset B of $\mathbb{R}^N \times [0, \infty]$. We have

PROPOSITION 2.2. *If u_0 is a nonnegative function in $L^1(\mathbb{R}^N)$, there exists the monotone limit $\bar{u} = \lim_{\varepsilon \rightarrow 0} u_\varepsilon$ with local convergence in $L^1(Q)$.*

3. Existence of nontrivial limit solutions

The main problem with this procedure concerns the possibility that the limit may become identically zero in Q_T for some relevant class of initial data. This is what happens for $N \geq 2$, as established in [7]. In the cases we study here, this failure of existence will not happen as we will show below. In fact, the limit will be nontrivial for all nontrivial initial data. The proof of this general result is long and proceeds in steps. The first of such steps consists in exhibiting at least one nontrivial solution.

A reminder: In the next sections, we concentrate on the case of exponents $s > 1/2$ and $0 \leq n < 2s - 1$. Note that even if the diffusion is singular, the range is formally the same as the good fast diffusion range $1 > m > (N - 2s)/N$, considered in the general theory of [19]; hence, it is supercritical in the notation of that paper. But there only exponents $m > 0$ were considered.

3.1. The very singular solution, I

In paper [40], a formal solution of equation $u_t + (-\Delta)^s u^m = 0$ is constructed with the form

$$U(x, t) = H(t)F(x) = C(N, s, m) t^{1/(1-m)} |x|^{-2s/(1-m)} \tag{3.1}$$

when $m > (N - 2s)/N$, so that the spatial profile has a nonintegrable singularity at $x = 0$. This type is called *very singular solution* in the literature (VSS for short). It is also proved that such formal solution is a limit of a monotone increasing sequence of standard solutions.

Following that paper we try the same formula here for $m = -n \leq 0$ with factors

$$H_1(t) = C t^{1/(1+n)}, \quad F_1(x) = |x|^{-2s/(1+n)}.$$

• Let us check that it works in the parameter range, $s > 1/2$, $n + 1 < 2s$ with $n > 0$. We have $\phi(F(x)) = -|x|^{2sn/(1+n)}$, so that a simple calculation gives

$$\mathcal{L}^s \phi(F(x)) = -K(s, n) |x|^{2sn/(1+n)-2s} = -K(s, n)F(x). \tag{3.2}$$

We prove in the note below that $K(s, n)$ is positive in this range of values of s, n . It is now easy to see that if we put

$$U_1(x, t) = H_1(t)F_1(x) \quad \text{with} \quad C^{1+n} = K(n + 1), \tag{3.3}$$

then U_1 is a solution of the 1D equation in the range $0 < n < 2s - 1$, unless at the singular point $x = 0$.

The singularity prevents the VSS from being acceptable as an example of nontrivial solution in the theory we are considering for Problem (1.1)–(1.3), at this stage. But we will return to this topic in style in Sect. 8.

Note on fractional Laplacians of power functions. The s -Laplacian of a power, $(-\Delta)^s |x|^\alpha$, $\alpha > 0$, $\alpha \neq 2s$, is the power $k(\alpha, s) |x|^{\alpha-2s}$, with a constant factor that for $N = 1$ equals

$$k(\alpha, s) = 2^{2s} \frac{\Gamma((1 + \alpha)/2) \Gamma((-\alpha + 2s)/2)}{\Gamma((1 + \alpha - 2s)/2) \Gamma(-\alpha/2)} \tag{3.4}$$

In our case, we take $\alpha = 2sn/(1 + n) > 0$. Hence, $\Gamma((1 + \alpha)/2) > 0$; besides, $(-\alpha + 2s)/2 = s/(1 + n) > 0$ so that $\Gamma((-\alpha + 2s)/2) > 0$; moreover, $\alpha/2 = sn/(1 + n) < 1$ (since $s < 2$ and $n/(1 + n) < 1/2$), so that $\Gamma(-\alpha/2) < 0$. Finally, for our choice of α

$$1 + \alpha - 2s = \frac{2ns}{1 + n} + 1 - 2s = \frac{2sn - (1 + n)(2s - 1)}{1 + n} = \frac{1 + n - 2s}{1 + n} \in (0, 1).$$

It means that $\Gamma((-\alpha + 2s)/2) < 0$. Therefore, $k(\alpha, s) > 0$ for these particular values of α and s . This is what we have called $K(s, n)$ some lines above.

3.2. A bounded subsolution

Though the singularity prevents the VSS from being directly useful as an example of nontrivial solution, the idea is not completely lost. We will start from it to construct a useful smooth variant, but it will be only a subsolution.

LEMMA 3.1. *There exists a bounded and smooth function in separate variable form*

$$\tilde{U}(x, t) = H(t)F_2(x) \tag{3.5}$$

which is a positive subsolution of the equation for some $0 < t < T$ and all $x \in \mathbb{R}$. Moreover, F_2 is symmetric, radially decreasing and $F_2(x) \approx c|x|^{2s/(1+n)}$ as $x \rightarrow \infty$. Changing the form of H , we can get a supersolution. In both cases, H is continuous and the initial value $H(0) > 0$ can be chosen.

Proof. (i) In order to avoid the problem with the singularity of the VSS, we round the function F_1 in a small ball near $x = 0$ to get a smooth positive F_2 , so that instead of the exact formula $(-\Delta)^s \phi(F_1) = -K(s)F_1(x)$ for $x \neq 0$ we get an approximate equation for all x that we can still use. Indeed, in view of the perturbation we get that $(-\Delta)^s \phi(F_2)$ is bounded on bounded sets; on the other hand, the difference

$$(-\Delta)^s \phi(F_1) - (-\Delta)^s \phi(F_2) = O(|x|^{-(1+2s)}) \text{ as } |x| \rightarrow \infty.$$

The last formula comes directly from the representation formula for $(-\Delta)^s$ plus the fact that $\phi(F_1) - \phi(F_2)$ has compact support. This correction $O(|x|^{-(1+2s)})$ is lower order with respect to $(-\Delta)^s \phi(F_1)$ at infinity since $(-\Delta)^s \phi(F_1) \sim F_1 = F_2 \sim |x|^{-2s/(1+n)}$ for all large $|x|$. Summing up, there exist constants $K_1, K_2 > 0$ such that

$$-K_1 F_2(x) \leq -(-\Delta)^s \phi(F_2) \leq K_2 F_2(x). \tag{3.6}$$

If we now take $H(0) = a > 0$ and $H' \leq -K_1 H^{-n}$, we get a subsolution,

$$\partial_t \tilde{U} + (-\Delta)^s \Phi_n(\tilde{U}) \leq 0,$$

in some time interval. In the last case, maybe H decays and vanishes in finite time (even if this is not realistic for actual solutions, as we will see, it is just the form of the modified subsolution). Take $F = F_2$ and this choice of H to end the construction of the desired subsolution \tilde{U} . Finally, note that since $H(0)$ can be taken at will we find a family of bounded subsolutions and the L^∞ norm can be taken as small as wanted.

(ii) In the same way, if we take F_1 as before and $H(0) = a > 0$ such that $H' \geq K_2 H^{-n}$, then we get a formal supersolution. In this case, it will exist for all times. \square

Case $n = 0$ This case is settled by replacing the power $-u^{-n}$ by $\log(u)$ and repeating the above procedure. We will need the calculation the s -Laplacian of the logarithm. We have

$$(-\Delta)^s (\log |x|) = c(s) |x|^{-2s}. \tag{3.7}$$

A short proof is as follows: Using the previous formula for $\alpha > 0$, α very small we get

$$k(\alpha, s) \sim \alpha 2^{2s-2} \frac{(2s-1)\Gamma(1/2)\Gamma(s)}{\Gamma((3-2s)/2)} \tag{3.8}$$

where we have used $\Gamma(-\alpha/2) \sim \Gamma(1)(-2/\alpha)$, and $\Gamma((1+\alpha-2s)/2) \sim 2/(1-2s)\Gamma((3-2s)/2)$. We now use the expression $\log(x) = \lim_{\alpha \rightarrow 0} (x^\alpha - 1)/\alpha$, $x > 0$, to conclude that the formula is true with $c(s) = \lim_{\alpha \rightarrow 0} k(\alpha, s)/\alpha > 0$ if $2s > 1$.

Then the rest of the steps is quite similar, and the conclusions of Lemma 3.1 hold.

PROPOSITION 3.2. *Let $N = 1$ and $0 < n < 2s - 1$. Let the initial data u_0 be positive and integrable and satisfy $u_0(x) \geq CF_2(x)$ for all x . Then the limit solution constructed as in Sect. 2 is nontrivial. In fact, it sits on top of one of the constructed subsolutions; hence, it is a positive very weak solution of the equation, at least in a certain time interval $0 < t < T$. Nontrivial solutions are also obtained for $n = 0$, $1/2 < s < 1$.*

Proof. The subsolution can be compared by classical results with all approximate problems and remains below for all the existence time. This allows us to prove that the limit solution does not vanish identically when we take initial data that decay equal or slower than F . By the L^1 contraction, the same nontrivial limit happens for any continuous and bounded initial data. \square

REMARK. This is the first successful step in a long road that leads to the proof that all upper limits corresponding to nontrivial data are in fact positive weak solutions.

3.3. Some properties of nontrivial limit solutions

• **Scaling property.** If $u(x, t)$ is a nontrivial limit solution, then so is

$$u_{AL}(x, t) = Au(Lx, L^{2s} A^{-(1+n)}t) \tag{3.9}$$

for all parameters $A, B > 0$. We leave the easy proof to the reader (see similar arguments in [39]).

• **The smoothing effect.** It says that all limit solutions with integrable data are in fact bounded so that we use bounded solutions in the proofs. We have for all limit solutions

$$\sup_{x \in \mathbb{R}} |u(x, t)| \leq C_2 t^{-\alpha} \|u_0\|_1^\delta \tag{3.10}$$

with $\alpha = 1/(2s - n - 1)$ and $\delta = 2s/(2s - n - 1)$, the constant C_2 depending on n, s .

Proof. Use the smoothing effect in the rough form already proved for the approximate solutions and let $t = 1$ to conclude that the result holds for $M = \|u_0\|_1 = 1$. When $M \neq 1$ and $t \neq 1$ use the scaling rule (3.9) in the usual way to reduce the proof to the particular case. This is a well-known scaling trick. □

• **Time monotonicity.** The limit solutions satisfy

$$\partial_t u \leq \frac{u}{(1+n)t}. \tag{3.11}$$

It follows from the same property for the approximations. As we said there, the argument has a proof using scaling arguments originally due to Bénilan and Crandall [5].

Also the properties of L^p boundedness, pointwise comparison, and L^1 contraction of the approximate solutions pass to the limit without change.

• **Space monotonicity. Aleksandrov’s principle.** The Aleksandrov–Serrin reflection method is a well-established tool to prove monotonicity of solutions of wide classes of (possibly nonlinear) elliptic and parabolic equations, cf. [1,32]. It has been quite useful in particular in the case of the PME, as documented in [11,38]. This is the version proved in [40, Theorem 15.2] for nonlinear parabolic equations with fractional diffusion of the type (1.1), and adapted to our situation

PROPOSITION 3.3. *Let v_ε the unique solution of (2.3) with initial data $u_0 \in L^1(\mathbb{R})$, $u_0 \geq 0$. Under the assumption that*

$$u_0(x) \leq u_0(2a - x) \text{ for } x > a \tag{3.12}$$

for some $a > 0$, we have for all $t > 0$

$$v_\varepsilon(x, t) \leq v_\varepsilon(2a - x, t) \text{ for } x > a. \tag{3.13}$$

In plain words, the result deals with comparison of a solution with its space reflection with respect to the point $x = a$. If it is true for $t = 0$, then it is true forever. After passage to the limit, the same comparison is true for the obtained upper solutions. An immediate consequence of this that has been used in the literature and we will use below is

COROLLARY 3.4. *An upper limit solution with initial data supported in the half-fine $\{x < a\}$ is monotone nonincreasing in x in the region $\{x > a, t > 0\}$. If the initial data are supported in the half-fine $\{x > -a\}$, the solution is monotone nondecreasing in x in the region $\{x < -a, t > 0\}$.*

4. Comparison results

The standard comparison theorem (Maximum Principle) applies the approximate problems; hence, it will be valid in the limit $\varepsilon \rightarrow 0$ for the class of upper limit solutions. The Aleksandrov principle is another comparison theorem. In the sequel, we will use two other comparison results, which we discuss next.

4.1. Symmetrization and concentration comparison

Symmetrization techniques are a very popular tool of obtaining a priori estimates for the solutions of different partial differential equations, notably those of elliptic and parabolic type. The application of Schwarz symmetrization allows to obtain sharp a priori estimates for elliptic problem by comparison with a model symmetric problem. For parabolic problems, the usual pointwise comparison of the solutions of the two problems fails and is replaced by comparison of integrals [3]. In the case of the porous medium equation $u_t = \Delta u^m$ that result was established in [33,37] and holds for all $m > 0$. In order to state the result we will use, the following definition is needed:

DEFINITION. Let $f, g \in L^1_{\text{loc}}(\mathbb{R}^N)$ be two radially symmetric functions on \mathbb{R}^N . We say that f is less concentrated than g , and we write $f < g$, if for all $R > 0$ we get

$$\int_{B_R(0)} f(x) \, dx \leq \int_{B_R(0)} g(x) \, dx. \quad (4.1)$$

The partial order relationship $<$ is called *comparison of mass concentrations*. In the applications, we are going to assume that f and g are *rearranged functions*.

The following result is proved in [43].

THEOREM 4.1. *Let u_1, u_2 be two nonnegative, weak solutions of the equation $u_t + (-\Delta)^s \Phi(u) = 0$, posed in $Q = \mathbb{R}^N \times (0, \infty)$, with nonnegative initial data $u_{01}, u_{02} \in L^1(\mathbb{R}^N)$. Assume that both u_{02} and u_{01} are rearranged and $u_{02} < u_{01}$. Assume moreover that the nonlinear function $\Phi(u)$ is positive, smooth and concave for $u > 0$. Then, for all $t > 0$ the functions $u_1(\cdot, t)$ and $u_2(\cdot, t)$ are rearranged and we have*

$$u_2(\cdot, t) < u_1(\cdot, t). \quad (4.2)$$

In particular, we have $\|u_2(\cdot, t)\|_p \leq \|u_1(\cdot, t)\|_p$ for every $t > 0$ and every $p \in [1, \infty)$.

This result applies to the solutions v_ε of the regularized problems (2.1). In the limit, it will apply to all nontrivial solutions of the singular fractional FDE (1.1) with $\Phi = \Phi_n$ and $n \geq 0$, for all $0 < s < 1$.

Note that in 1D the integrals in formula (4.1) have a more classical interpretation: The integral $\int_0^x f(x) dx$ is just the distribution function of the mass density f , that we are assuming to be nonnegative, integrable and monotone decreasing for $x > 0$. Therefore, formula (4.2) is just a comparison of distribution functions.

4.2. Shifting comparison

A new comparison result that is related to symmetrization in spirit and techniques is based on lateral displacement of the solution, viewed as a mass distribution.

THEOREM 4.2. *Let two functions $u_{01}, u_{02} \in L^1(\mathbb{R})$ with the following properties:*

- (i) *They are nonnegative and rearranged around their points of maximum x_1 and x_2 resp., with $x_1 < x_2$. The total mass is the same.*
- (ii) *We assume moreover that $\int_{-\infty}^x u_{02} dx \leq \int_{-\infty}^x u_{01} dx$ for every $x \in \mathbb{R}$.*

Besides, we assume that Φ is monotone, concave and defined on \mathbb{R}_+ with $\Phi(0) = 0$ and $0 < \Phi'(0) < \infty$. Then, the following comparison inequalities hold for the corresponding (limit) solutions

$$\int_{-\infty}^x u_2(x, t) dx \leq \int_{-\infty}^x u_1(x, t) dx, \tag{4.3}$$

for every $x \in \mathbb{R}$ and every $t > 0$.

This result is called the *Shifting comparison lemma*. It is essentially one-dimensional, and it was established in the PME case by Vázquez [34] and it proved useful in studies of free boundary location or asymptotic behaviour. It is related to mass transport and Wasserstein distances [36,45]. It will be crucial in some proofs below, like the proof of the existence for general initial data and the asymptotic behaviour. Since it has a rather technical and long proof, we will delay to Sect. 11 at the end of the paper.

5. Mass conservation and global solutions

The property of mass conservation plays an important role in passing from local-in-time existence to global solutions, since it prevents the phenomenon of finite-time extinction. A first result is as follows.

PROPOSITION 5.1. *Let us assume that $N = 1, s > 1/2$ and $0 \leq n < 2s - 1$ and let us assume that u_0 is integrable and $u_0(x) \geq cF_2(x)$, i.e.*

$$u_0(x) \geq c/(1 + |x|^2)^{s/(1+n)} \tag{5.1}$$

for some $c > 0$. Then the limit solution not only is positive in a certain time interval $0 < t < T$, but it also conserves mass in that interval:

$$\int_{\mathbb{R}} u(x, t) dx = \int_{\mathbb{R}} u_0(x, t) dx. \tag{5.2}$$

Proof. (i) Let us first assume that $n > 0$. We will prove conservation of mass for small times. We take a nonnegative nonincreasing cutoff function $\zeta(s)$ such that $\zeta(s) = 1$ for $0 \leq |s| \leq 1$, $\zeta(s) = 0$ for $|s| \geq 2$, and define $\zeta_R(x) = \zeta(|x|/R)$. We have $(-\Delta)^s \zeta_1 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$; moreover, $|(-\Delta)^s \zeta_1| \sim |x|^{-(1+2s)}$ as $|x| \rightarrow \infty$. The radial cutoff function ζ_R has the scaling property

$$(-\Delta)^s \zeta_R(x) = R^{-2s} (-\Delta)^s \zeta_1(x/R). \tag{5.3}$$

We take the approximate solutions u_ε of Eq. (1.1) with nonlinearity Φ_ε as in Sect. 2, multiply by ζ_R and integrate by parts. We have

$$\int_{\mathbb{R}} (u_\varepsilon(t_1) - u_0 - \varepsilon) \zeta_R \, dx dt = \int_0^{t_1} \int_{\mathbb{R}} u_\varepsilon^{-n} (-\Delta)^s \zeta_R \, dx.$$

Note that the last integral is absolutely integrable. Passing to the limit $\varepsilon \rightarrow 0$ we get for every $t > 0$,

$$\int_{\mathbb{R}} u(t_1) \zeta_R \, dx - \int_{\mathbb{R}} u_0 \zeta_R \, dx = \int_0^{t_1} \int_{\mathbb{R}} u^{-n} (-\Delta)^s \zeta_R \, dx dt. \tag{5.4}$$

Let us split the right-hand side of (5.4) into the integrals for $|x| \leq R$ and $|x| \geq R$. We only estimate the integrals in x , forgetting for the moment the time integration. We get

$$|I_1(R)| \leq \int_{|x| \geq R} u^{-n}(t) |(-\Delta)^s \zeta_R| \, dx \leq \frac{C}{R^{2s}} \int_{|x| \geq R} |x|^{2ns/(1+n)} (|x|/R)^{-(1+2s)} \, dx.$$

Putting $x = Ry$, we get

$$I_1(R) \leq \frac{CR^{2ns/(1+n)}}{R^{2s-1}} \int_{|y| \geq 1} |y|^{\frac{2ns}{1+n} - (1+2s)} \, dy = CR^{-\gamma} \int_{|y| \geq 1} \frac{dy}{|y|^{2+\gamma}}$$

Since $\gamma = 2s - 1 - 2ns/(1+n) = 2s/(n+1) - 1 > 0$, we get $I_1(R) \rightarrow 0$ as $R \rightarrow \infty$. On the other hand, for the analogous integral in the set $|x| \leq R$ we get

$$|I_2(R)| \leq \int_{|x| \leq R} u^{-n}(t) |(-\Delta)^s \zeta_R| \, dx \leq \frac{C}{R^{2s}} \int_{|x| \leq R} |x|^{2ns/(1+n)} \, dx \leq CR^{-\gamma}$$

that goes also to zero as $R \rightarrow \infty$. Since these estimates do not depend on t (for small t) we may go back to Eq. (5.4) and let $R \rightarrow \infty$ to get

$$\int_{\mathbb{R}} u(x, t_1) \, dx = \int_{\mathbb{R}} u_0(x) \, dx,$$

which is the mass conservation law. This law holds for the small times for which we have the lower estimate for the limit solution u (for instance, the estimate coming from comparison with the subsolution constructed in Sect. 3, as used in Proposition 3.2).

This method of proof is inspired in a common technique that was used to prove the property in the case $s = 1$ for $m > (N - 2)/N$, $N \geq 2$. The beginning is the same, but the further details are quite different.

(ii) In the case $n = 0$, the nonlinearity is logarithmic and not power-like. We arrive at formulas $I_1(R)$ and $I_2(R)$ with the bound $c \log(1 + |x|^2)$ instead of $c|x|^{2ns/(1+n)}$, and we put $n = 0$ in the rest of the places. The remaining steps follow easily. \square

PROPOSITION 5.2. *If the initial data u_0 is a rearranged function with mass $M > 0$, the limit solution $u(x, t)$ exists globally in time, is positive everywhere and the mass is conserved for all times.*

Proof. (i) We first prove that for rearranged initial data in the same class $u_0 \geq c F_2(x)$ the solution exists for all times and conservation of mass holds also for all times. We know that u is positive and conserves mass for $0 < t < T_1 = T_1(u)$. Assume that this maximal time is finite. We use the scaling property of the equation to define of new solutions

$$u_L(x, t) = Lu \left(Lx, L^{2s-(1+n)}t \right) \tag{5.5}$$

with $L > 1$. Let us introduce the notation $u_L = \mathcal{T}_L u$ for future reference. This scaling, a particular case of (3.9), keeps the mass of the solutions u and u_L identical. Now, since the time of existence of the family u_L shrinks to the time $T_L = T_1/L^{2s-(1+n)}$, we seem have a problem in using rescaling. But the problem can be fixed by using symmetrization (concentration comparison), and we end up with an expected gain.

Indeed, the new solution u_L for $L > 1$ is clearly more concentrated than $u_1(x, t) = u(x, t)$ at at time $t = 0$; hence, it will be more concentrated at all times by the result of Sect. 4.1. Then, the concentration relation (4.2) immediately implies that the mass of u_L must be conserved as long as the mass of u is, say until $T_1(u)$; the justification for $T_L \leq t \leq T_1$ is done by doing symmetrization comparison on the approximate problems and passing to the limit, which cannot be trivial because of this argument. This means that the nontrivial existence time with conservation of mass for u_L is $T(u_L) \geq T_1$. Undoing the scaling we get the same property for u in a time $T(u) \geq L^{2s-(1+n)}T_1$, hence T_1 must be infinite.

(ii) Next, we prove that the solution is positive everywhere. Let us do an analysis of what happens if a rearranged solution touches zero, and show that this cannot happen. First, we use the monotonicity properties in space and time to show that, if the solution vanishes at $t = t_1$ and $x = R$, then we must have $u(x, t) = 0$ for all $|x| \geq R$ and $t \geq t_1$. Using the proof of the mass formula of Proposition 5.1, we have for the approximations u_ε

$$\frac{d}{dt} \int_{\mathbb{R}} u_\varepsilon(x, t) \zeta(x) dx = - \int_{\mathbb{R}} \Phi_n(u_\varepsilon) (-\Delta)^s \zeta dx$$

Taking a cutoff function supported in $[-R_1, R_1]$, with $R_1 < R$, we know that $-(-\Delta)^s \zeta > 0$ for $|x| \geq R_1$, and since $\Phi_n(u_\varepsilon)$ tends to minus infinity in the set we have described, we have

$$- \int_{|x| \geq R_1} \Phi_n(u_\varepsilon) (-\Delta)^s \zeta dx \rightarrow -\infty$$

as $\varepsilon \rightarrow 0$. The integral for $|x| \leq R_1$ has a bounded limit since the limit u is bounded below; hence, $|\Phi(u)|$ is bounded. Applying this between times t_1 and $t_1 + \tau$, we can show that the weighted mass must be zero for all times larger than t_1 . Since the solution is rearranged, this implies that the mass will be zero, which is excluded by the previous step.

(iii) The next step is to consider rearranged initial data that do not satisfy a bound from below like (5.1), for instance u_0 may be compactly supported. Let $M > 0$ be the initial mass. In that case we make a small perturbation by adding to u_0 a tail of the form $\eta^\delta(x) = \delta(1 + |x|^2)^{-s/(1+n)}$ and besides we truncated the initial data on top to make it bounded. We may do all this and conserve the mass M . In this way, we obtain a solution u^δ to the problem where the previous analysis applies, and mass is conserved. We now use the L^1 contraction property and the conservation of mass for u^δ , and we conclude that the limit solution u corresponding to initial data u_0 must have mass

$$\int u(x, t) \, dx \geq M - \|u_0^\delta - u_0\|_1$$

which is positive for δ small; hence, u is a global solution. The argument is justified as limit solution, i.e., in the limit of approximate problems. Finally, by letting $\delta \rightarrow 0$ we derive the mass conservation property for u . □

Positivity can now be obtained for all solutions with continuous initial data, not necessarily rearranged, by standard comparison with a solution with compactly supported rearranged data (after possibly a space displacement to fit it under u_0). Such a subsolution is positive for all positive times; hence, u is too. We will return to this question later on, after we get some quantitative estimates on the behaviour.

5.1. Concept of solution: the mass function

As a preliminary for the next developments, we need to clarify the type of solution that we get at this stage when we pass to the “limit solutions.” In the end, we would like to prove that our positive solutions are very weak in stated sense that

$$\int \int u \, \zeta_t \, dxdt = \int \int \Phi_n(u) (-\Delta)^s \zeta \, dxdt \tag{5.6}$$

for all smooth test functions with compact support. However, the last term offers a difficulty as long as we do not know the decay of u at infinity, i.e., as long as we do not control the growth of u^{-n} , since typically $(-\Delta)^s \zeta$ behaves like $O(|x|^{1+2s})$ as $|x| \rightarrow \infty$.

The conditions of Proposition 5.1 do allow for a correct passage to the limit $\varepsilon \rightarrow 0$ in the definition, but the conditions of Proposition 5.2 do not. Keeping the assumption of rearranged data, we find a remedy by weakening the definition by integration in space to get a new function $V(r, t) = \int_0^r u(x, t) \, dx$ so we have $V_r = u$ for all $r > 0$ and we can write the equation formally as

$$V_t = \int_0^r u_t \, dx = - \int_0^r (-\Delta)^s \Phi_n(u) \, dx \tag{5.7}$$

Putting $(-\Delta)^s = -\partial_{xx}^2 (-\Delta)^{s'}$ with $s' = 1 - s > 0$, and integrating we get

$$V_t = \partial_r (-\Delta)^{-s'} (\Phi_n(u)), \quad V_r = u. \tag{5.8}$$

This is an integrated version of the weak solution that makes perfect sense for the approximate problems and for their limits in the very weak sense. We will call $V(x, t)$ the *mass function*; it is the distribution function in Probability, but that name might lead to confusion here.

6. The Barenblatt solutions in 1D

After establishing mass conservation for all times, we can construct the Barenblatt solutions and derive the main properties.

6.1. Existence

Take one of the rearranged solutions $u_1(x, t)$ of the previous section (Proposition 5.2) with initial data $u_{01}(x)$ that we may assume continuous. It exists globally in time, is positive everywhere and conserves mass. Let us fix the L^1 norm of u_1 to 1. Take then the rescaled family $\{u_L(x, t) = \mathcal{T}_L u : L \geq 1\}$ defined by formula (5.5) of the previous section. Finally, pass to the limit

$$U(x, t) = \lim_{L \rightarrow \infty} u_L(x, t). \tag{6.1}$$

We have to show that this limit exists and has the desired properties.

- (i) In principle, the family $u_L(x, t)$ converges weakly in L^1 for $t \geq \tau > 0$, and there may also be a nonunique limit. It is best to use an argument based on concentrations, which makes it natural to argue with the family of mass functions $V_L(x, t) = \int_0^x u_L(x, t) \, dx$. By the concentration comparison result, we see that the family $\{V_L\}$ is monotone increasing in the parameter L for $x > 0$ (resp. negative and decreasing for $x < 0$). Hence, the limit $V_\infty(x, t)$ exists and does not depend on subsequences $L_k \rightarrow \infty$. The convergence $V_L(x, t) \rightarrow V_\infty(x, t)$ is uniform convergence in x for every fixed positive time.
- (ii) Differentiation in x gives the unique weak limit $U = \lim_{L \rightarrow \infty} U_L$. Stronger convergence will be proved later on.
- (iii) For $t = 0$, we have $V_\infty(x, 0) = 1/2$ for all $x > 0$, $V_\infty(x, 0) = -1/2$ for all $x < 0$, in other words the limit of the initial data is a delta function. We have to show that for all positive times $U(\cdot, t)$ is not trivial away from $x = 0$ [i.e. it is not equal to $\delta(x)$]. This follows from the smoothing effect (proved in Sect. 3.3) that applies uniformly to all functions u_L . We conclude that $U(\cdot, t)$ is a bounded function for all $t > 0$; therefore, $U(\cdot, t)$ is not a Dirac delta, and $V_\infty(\cdot, t)$ is Lipschitz continuous in x uniformly if $t \geq \tau > 0$.

- (iv) Let us prove next that the limit must be a self-similar function. The argument is based on passing to the limit in the scaling family using the group of transformations. In fact, for all $L, k > 0$ we have $\mathcal{T}_L \mathcal{T}_k u_1 = \mathcal{T}_{kL} u_1$, Passing to the limit $k \rightarrow \infty$ and using the uniqueness of the limit because of the comparison of concentrations, we get $U(x, t) = \mathcal{T}_L U(x, t) = LU(Lx, L^{2s-1-n}t)$, hence $LU(Lx, L^{2s-1-n}t) = U(x, 1)$. In the usual way, it follows that

$$U(x, t) = t^{-\alpha} F(xt^{-\alpha}), \quad \alpha = 1/(2s - (1 + n)), \tag{6.2}$$

with $F(x) = U(x, 1)$.

- (v) It is immediate that the profile F is positive everywhere, bounded, integrable, and rearranged (i. e., $F(x) = F(-x)$ and F monotone decreasing for $x > 0$).
- (vi) To construct the Barenblatt solutions with initial data $M \delta(x)$ with any mass $M > 0$, we use another scaling \mathcal{T}' defined by $(\mathcal{T}'u)(x, t) = M u(x, M^{-(1+n)}t)$, which transforms solutions of mass 1 into solutions of mass M . We get a self-similar solution with the same formula as before, but now the profile is

$$F_M(x) = M^{2s\alpha} F(xM^{(1+n)\alpha}). \tag{6.3}$$

6.2. Alternative approach and better convergence

The idea is to apply the same type of approach to the approximate problems (2.1) to obtain a fundamental solution of each of those problems. After careful inspection, we see that we can find a solution $U_{\infty\epsilon}(x, t)$ with initial data $U_{0\epsilon}(x, 0) = \delta_0(x) + \epsilon$. This is better done by using the formulation $v_\epsilon(x, t) = u_\epsilon(x, t) - \epsilon$ that solves Problem (2.3), to which we can apply the usual L^1 theory and comparison of concentrations. Therefore, we obtain a fundamental solution with this argument, and the difference is that this time we cannot conclude that it is self-similar. On the other hand, the whole collection of rescaled solutions $U_{L,\epsilon}(x, t)$ are uniformly bounded for $t \geq \tau > 0$; hence, by the regularity results of [42], they are uniformly C^α continuous for a certain $\alpha > 0$. The family is thus locally compact in L^1_{loc} (in both space and time), which means that the convergence $U_{L,\epsilon}(x, t) \rightarrow U_{\infty,\epsilon}(x, t)$ takes place in the strong sense of $L^1(\mathbb{R})$ for every $t > 0$. This is a key improvement in the situation.

Now we take the monotone limit of these fundamental solutions to get

$$\lim_{\epsilon \rightarrow 0} U_{\infty,\epsilon}(x, t) = U_\infty(x, t)$$

By a simple comparison, $U_\infty(x, t) \geq U(x, t)$, where U is the previously constructed Barenblatt solution, formula (6.2). By the equality of masses, we conclude that both are the same function, $U_\infty = U$.

It is now easy to see that $U_L(x, t) \rightarrow U(x, t)$ in $L^1(\mathbb{R})$ for every $t \geq \tau > 0$. In fact,

$$\begin{aligned} \|(U_L(x, t) - U(x, t))_+\|_{L^1(-R, R)} &\leq \|(U_{L,\epsilon}(x, t) - U(x, t))_+\|_{L^1(-R, R)} \\ &\leq \|(U_{L,\epsilon}(x, t) - U_{\infty,\epsilon}(x, t))_+\|_{L^1(-R, R)} + o(\epsilon), \end{aligned}$$

where the last term is the contribution of $\|U_{\infty,\varepsilon}(x, t) - U_{\infty}(x, t)\|_{L^1(-R, R)}$. Fixing a small $\varepsilon > 0$ and using the convergence $U_{L,\varepsilon}(x, t) \rightarrow U_{\infty,\varepsilon}(x, t)$ in $L^1(\mathbb{R})$, there is an $L_\varepsilon > 0$ such that have $\|U_L(x, t) - U(x, t)\| \leq \varepsilon$ for $L \geq L_\varepsilon$. All together, this means that $(U_L(x, t) - U(x, t))_+ \rightarrow 0$ in $L^1(-R, R)$ as $L \rightarrow \infty$. Since the mass in the far field is uniformly small, and there is total mass equality, we get $U_L(x, t) - U(x, t) \rightarrow 0$ in $L^1(\mathbb{R})$. This is the improved convergence that we needed.

6.3. Uniqueness

PROPOSITION 6.1. *The fundamental solution is independent of the rearranged function u_0 that starts the construction.*

Proof. For bounded functions of compact support, this is done by symmetrization comparison and iterated limits. Indeed, let us start from functions u_1 and \tilde{u}_1 and let U and \tilde{U} the corresponding Barenblatt solutions obtained in the limit of the scaling process. Then, it is not difficult to see that the scaled function u_{0k} is more concentrated than \tilde{u}_0 if k is large enough. Therefore, u_{kL} is more concentrated than \tilde{u}_{0L} , hence in the limit $U(\cdot, t)$ is more concentrated than $\tilde{U}(\cdot, t)$. The reverse relation also holds; hence, $U = \tilde{U}$.

For general data, we use L^1 contraction. □

7. Study of the profile

We want to know more about the profile F of the Barenblatt solution, in particular its equation and its asymptotics for large $|x|$.

7.1. Profile Equation

We can apply to U the concept of solution of last section and perform the computation when U is self-similar. Then, with $\xi = x t^{-\alpha}$,

$$\begin{aligned} V_t &= \frac{d}{dt} \int_0^r t^{-\alpha} F(x t^{-\alpha}) dx = -\alpha t^{-(\alpha+1)} \int_0^r (F(\xi) + \xi F'(\xi)) dx \\ &= -\alpha t^{-1} \xi F(\xi). \end{aligned}$$

Therefore, Eq. (5.8) becomes at $t = 1$

$$\partial_x (-\Delta)^{-s'} (F^{-n}(x)) = \alpha x F(x), \quad x > 0. \tag{7.1}$$

This is the integro-differential equation satisfied by the profile F . Note that $1 - 2s' = 2s - 1 > 0$, so that the operator in the LHS has a positive degree of differentiation, $\partial_x \mathcal{L}^{-s'} = H \mathcal{L}^{s-1/2}$. We can also write the equation as

$$\int_0^r (-\Delta)^s F^{-n}(r) dr = \alpha x F(x), \quad \int_0^r F^{-n}(r) dr = \alpha (-\Delta)^{-s} (x F(x)). \tag{7.2}$$

7.2. Asymptotic behaviour of F

According to the analogy with the Barenblatt solutions constructed for the case $m > 0$ in [40], since we are in the case $m < m_1 = N/(N + 2s)$ (a value that was important in that respect), we expect the following behaviour.

PROPOSITION 7.1. *The profile F decays as $|x| \rightarrow \infty$ like $r^{-\gamma}$, with $\gamma = 2s/(1 + n) > 1$. Indeed, there exists the finite positive limit*

$$\lim_{r \rightarrow +\infty} F(r) r^\gamma = c_\infty, \tag{7.3}$$

and moreover $F(r) \leq c_\infty r^{-\gamma}$ for all $r > 0$.

REMARK. Later on we will calculate the constant c_∞ explicitly as a corollary of our work on very singular solutions, Sect. 8.

Proof. (i) We need a first estimate of the decay of F . Since it is an integrable and rearranged function, we immediately get $F(x) \leq C/|x|$, which is too rough. A better estimate is obtained as follows: We start the scaling procedure to construct U by taking as initial data u_{01} the function $F_2(x)$ of Lemma 3.1. By comparison with the subsolution of the Lemma, we have a decay rate $u_1(x) \geq c(t)|x|^{-2s/(1+n)}$ for all $|x| \geq 1$ uniformly in some time interval $0 < t < T_1$.

We can show a similar decay rate for the limit solution $U(x, t)$ by applying concentration comparison. Indeed, the mass of u_1 in region $|x| \geq R > 1$ is estimated as $C(t)R^{1-2s/(1+n)}$, and the mass of U has to be less than that by the comparison. By using the monotonicity of U w.e.t. $|x|$, we get the conclusion that U is less than $c_1(t)|x|^{-2s/(1+n)}$. Recall that $F(x)$ is just $U(x, 1)$. Therefore,

$$F(x) \leq C|x|^{-2s/(1+n)}.$$

The reader can find similar arguments in [40, Section 12].

(ii) The power-like bound from below can be obtained following the ideas of [40]: We start from the homogeneity estimate (3.11) that says that $(1 + n)t u_t \leq u$. In terms of the self-similar profile, this just means that $-(1 + n)\alpha(F + rF'(r)) \leq F$, $r = |x| > 0$, hence

$$\frac{-rF'(r)}{F(r)} \leq 1 + \frac{1}{(1 + n)\alpha} = \frac{2s}{1 + n}.$$

Integration of this inequality gives the following lower bound, valid for all $r \geq 1$, all $s \in (0, 1)$:

$$F(r) \geq C r^{-2s/(1+n)}. \tag{7.4}$$

Moreover, the function $J(r) := F(r) r^{-2s/(1+n)}$ is monotone nondecreasing with r , so that it has a limit as $r \rightarrow \infty$. Let us put

$$\lim_{|x| \rightarrow \infty} F(x) |x|^{-2s/(1+n)} = c_\infty(s, n). \tag{7.5}$$

In principle, the limit may be finite and positive or infinite. By part (i), it is finite. \square

8. The very singular solution

The a priori bounds on the profile of the Barenblatt solutions make it easy to pass to the limit $M \rightarrow \infty$ and obtain a special function, called the very singular solution (VSS), much as we have done in [40] for $m > 0$. This is the result

THEOREM 8.1. *The Very Singular Solution calculated in Sect. 3,*

$$\tilde{U}(x, t) = C(n, s) t^{1/(1+n)} |x|^{-2s/(1+n)}, \tag{8.1}$$

is the limit of the Barenblatt solutions $U_M(x, t)$ as the mass M goes to infinity.

Proof. We recall that the value of the constant $C(n, s)$ can be explicitly computed from the calculations there as $C(n, s) = K(n, s)(1 + n)$.

By the established comparison properties, it is clear that the sequence of Barenblatt solutions $U_M(x, t)$ is monotone increasing with $M > 0$. Next, we check that they are all bounded above by the VSS. A direct comparison of U_M with \tilde{U} is difficult to justify directly; hence, we argue in another way to get our conclusion.

We first fix the mass equal to one and use the upper profile bound, $F(r) \leq c_\infty r^{-2s/(1+n)}$ for all $r > 0$, to conclude that

$$U_1(x, t) \leq c_\infty t^{1/(1+n)} |x|^{-2s/(1+n)}.$$

We know that the whole sequence U_M can be obtained from $M = 1$ by the rescaling $U_M(x, t) = (\mathcal{I}_M u)(x, t) := M u(x, M^{-(1+n)}t)$. We immediately see that $U_M(x, t)$ satisfies the same upper bound, even with the same constant, $c_\infty(M) = c_\infty(1)$. Once we have the same upper bound for the whole sequence, we may pass to the monotone limit and get

$$U_\infty(x, t) \leq c_\infty t^{1/(1+n)} |x|^{-2s/(1+n)}.$$

But since the functions U_M are invariant under the mass conserving scaling, so is the unique limit, hence $U_\infty(x, t)$ is self-similar, $U_\infty(x, t) = t^{-\alpha} F_\infty(x t^{-\alpha})$. Also $F_\infty(y) \geq F_M(y)$ for all $y > 0$ and all $M > 0$, and $F_\infty(y) \leq c_\infty |y|^{-2s/(1+n)}$. It easily follows from a tail analysis that $F_\infty(y) = c_\infty |y|^{-2s/(1+n)}$, hence

$$U_\infty(x, t) = c_\infty t^{1/(1+n)} |x|^{-2s/(1+n)}.$$

We conclude that U_∞ is another possible very singular solution of the equation obtained as limit of upper limit solutions. In order to see that U_∞ and \tilde{U} must be the same we only have to check that both are weak solutions for the equation for $x \neq 0$; in other words, the profiles must be weak solutions for F Eq. (7.1). This is what selects the constant in a unique way. We conclude that $c_\infty = C(n, s)$ and $U_\infty = \tilde{U}$. \square

As a corollary of the above result, we can refine the information on the Barenblatt solutions obtained in Proposition 7.1 as follows

PROPOSITION 8.2. *The profiles F_M , $M > 0$, of the Barenblatt solution satisfy the uniform bound $F(r) \leq C(n, s)r^{-2s/(1+n)}$ for all $r > 0$, and this estimate is sharp at infinity*

$$\lim_{r \rightarrow +\infty} F(r) r^{2s/(1+n)} = C(n, s), \tag{8.2}$$

where $C(n, s)$ is the constant of the VSS.

9. The theory for general initial data

Here we want to complete the proof of Theorems 1.1 and 1.2 on the existence and properties of solutions of the Cauchy Problem. We establish existence, positivity and behaviour as $|x| \rightarrow \infty$. The basic analysis is done for compactly supported and rearranged data. Then, we perform a series of extensions of the results to greater generality.

• **Compactly supported and rearranged data.** (i) *Existence and conservation of mass.* Assume to fix ideas that u_0 is supported in the interval $[-R, R]$, and has total mass M . We may use shifting comparison with the Barenblatt solutions with the initial masses located on either end of the support to prove bounds from above and below for the mass function of u , defined by $v(x, t) = \int_{-\infty}^x u(x, t) dx$, in terms of displaced versions of the mass function of the Barenblatt solution with the same mass, V_M corresponding to U_M . We get

$$V_M(x - R, t) \leq v(x, t) \leq V_M(x + R, t).$$

Recall that $V_M(\infty, t) = M$ for every $t > 0$ by mass conservation. This not only proves nontrivial existence for all times, but also conservation of the total mass and existence of a nontrivial mass (i.e., L^1 integral) on every interval of length larger than $2R$.

We have to justify that shifting comparison applies to Barenblatt solutions, and this is done by starting the construction with compactly supported $\tilde{u}_1(x)$ and its scalings $u_{0,L}(x) = Lu_1(Lx)$.

(ii) *Positivity.* By the Aleksandrov principle, see Corollary 3.4, the function $u(x, t)$ is monotone decreasing in x for $x > R$ and $t > 0$. Together with the previous mass analysis, this implies that $u(x, t) > 0$ for $x > R$. A similar argument happens for $x < -R$ and ensures for positivity for all $|x| > R$ and all $t > 0$.

In order to establish the positivity for $|x| \leq R$ at times $t > 0$, we take time $t_1 > 0$ and move the the origin or coordinates to a point $x_1 > 2R$. Setting $y = x - x_1$, we see that $u(y + x_1, t)$ is positive near the new origin and bounded below by a rearranged function $\tilde{u}_0(y)$ with small support $[-R_1, R_1]$ to which the preceding result applies so that $\tilde{u}(y, t)$ is positive for $|y| > R_1$. By comparison, $u(x, t)$ is positive with a uniform lower bound for $|x| \leq R$ and $t = 2t_1$.

(iii) *Asymptotic behaviour as $|x| \rightarrow \infty$.*

PROPOSITION 9.1. *For every solution with rearranged and compactly supported data with mass 1, we have*

$$C_1|x|^{2s/(1+n)} \leq u(x, t) t^{-1/(1+n)} \leq C_2|x|^{2s/(1+n)} \tag{9.1}$$

for all $|x| \geq 2R$, and the constants do not depend on the particular u , nor on t .

Proof. The upper bound comes from the mass analysis of (i) and the monotonicity in x for large $|x|$. We get

$$x u(x, t) \leq 2 \int_{x/2}^x u(x, t) dx \leq 2 \int_{x/2}^\infty u(x, t) dx$$

and by the shifting comparison result, this mass is less than the mass of the displaced Barenblatt $U_\infty(x - R, t)$ which is proportional to

$$(c_\infty/\gamma) t^{1/(1+n)} |x - R|^{-\gamma}, \quad \gamma = (2s - 1 - n)/(1 + n).$$

We conclude that there is a constant c_1 such that for all $x \geq 2R$.

$$u(x, t) \leq c_1 t^{1/(1+n)} |x|^{2s/(1+n)}.$$

- The lower bound follows from similar arguments, but now we compare with the mass of $U_\infty(x + R, t)$. We get

$$(k - 1)xu(x, t) \geq \int_x^{kx} u(x, t) dx = \int_x^\infty u(x, t) dx - \int_{kx}^\infty u(x, t) dx = I_1 - I_2.$$

The first integral I_1 is estimated from below by the mass of $U_\infty(x + R, t)$ in the same interval which is accurately given by $C_2 t^{1/(1+n)} |x|^{-\gamma}$, while the second is estimated from above by the mass $U_\infty(x - R, t)$ and gives $C_1 t^{1/(1+n)} |kx|^{-\gamma}$ in first approximation. Hence, for $k > C_2/C_1$ and x large enough we get $u(x, t) \geq c_3 t^{1/(1+n)} |x|^{2s/(1+n)}$, which implies the stated lower bound. □

REMARK. The translation of these results for data with mass $M \neq 1$ is easy by using the mass-changing transformation \mathcal{T} .

- **Rearranged data.** If the initial data are rearranged but not compactly supported, we use approximation of the data from below with compactly supported data, so that by comparison the property of positivity follows. Conservation of mass comes from the L^1 contraction property. The asymptotic lower bound in (9.1) still holds, but the upper bound need not hold (it depends on the behaviour of the initial data for large $|x|$).

- **Continuous data.** In this case, there is a possible problem with the positivity of the solution, which is still not guaranteed by us. This is proved for continuous data u_0 by putting below some rearranged data with compact support, \tilde{u}_0 , to which the previous theory applies. Then we can apply comparison to see that u has large tails, above the minimum decay estimate of (9.1). Then, conservation of mass is true for $t \geq \tau > 0$.

In order to get conservation of mass since the beginning, we add a small perturbation with suitable tail to u_0 . After an easy argument, this implies general conservation of mass and general minimum decay estimate.

- **General data with compact support.** We attack the general case for data with compact support in the interval $I_R = [-R, R]$. We propose to use convolution with a smooth kernel to obtain a smooth approximation $u_{0\delta}$, that produces a solution to which the previous paragraph applies. By L^1 contraction, we get conservation of mass, so that the solution must be global in time.

Due to mass conservation and the smoothing effect, the mass of the solution u cannot be contained in the original interval for large times. Then there is a time T_1 such that half the mass is outside I_R . By the space monotonicity away from I_R , we can put a displaced rearranged subsolution below, and then, there are tails with at least the minimal decay rate for $t > T_1$. Now we use the monotonicity in time to derive the same conclusion also for all $t \geq \tau > 0$.

- Finally, when u_0 is not compactly supported, rearranged or continuous, we just approximate from below with compactly supported data and pass to the monotone limit. The rest of the argument follows.

9.1. L^1 continuity and initial data

We want to show that the limit solution is continuous as an orbit $t \mapsto u(\cdot, t)$, as stated in Theorem 1.1. At $t = 0$, this means that it takes the initial data in $L^1(\mathbb{R})$. We may use the fact that this holds for the approximate problems and then pass to the limit, using a Fatou argument plus conservation of mass.

The continuity at $t > 0$ can be made into a stronger result, and in fact we obtain Lipschitz continuity of the orbit for all positive times. This is a consequence of the time monotonicity and the conservation of mass. Indeed, the first implies that for $h > 0$

$$u(x, t + h) - u(x, t) \leq \left((1 + (h/t))^{1/(1+n)} - 1 \right) u(x, t) \leq Ch u(x, t)$$

which is uniform if $t \geq \tau > 0$ and $h \leq c\tau$. Since u is bounded, this implies a pointwise Lipschitz bound from above. The L^1 bound from below comes from conservation of mass.

9.2. Upper limit solutions are very weak solutions

This subsection extends to the singular case the result of papers [18–20]. Since the approximate solutions u_ε of Problem (2.1) are smooth, they satisfy the very weak formulation. The main difficulty in passing to the limit when $\varepsilon \rightarrow 0$ is the control of the possible growth of $\Phi_n(u)$ as $|x| \rightarrow \infty$. But this depends on having a good lower bound for u , and such a bound is contained in the left-hand side of (9.1). It follows that the integrals involved in the passage to the limit are uniformly absolutely integrable.

Now we can pass to the limit in the ε approximate equations written in very weak form to show that the limit solution is indeed a very weak solution.

9.3. Upper limit solutions form a semigroup

We consider the maps $S_t : L^1_+(\mathbb{R}) \rightarrow L^1_+(\mathbb{R})$ that map any initial data u_0 to the solution of the equation at time $t > 0$, $S_t u_0 = u(\cdot, t)$. Since the approximate problems (2.1) produce unique classical solutions, the semigroup property is true for them, $S_{t+t_1}^\varepsilon u_0 = S_t^\varepsilon(S_{t_1}^\varepsilon u_0)$. Passing to the limit, we obtain the same property for our upper limit solutions arguing as follows:

Fix u_0 and $t_1 > 0$ and recall the monotone convergence $u_\varepsilon(x, t_1) \rightarrow u(x, t_1)$ that takes place in $L^1_{loc}(\mathbb{R})$. Consider now $v_0 = u(\cdot, t_1)$ as initial data for a new lap of the evolution. In order to get the upper limit solution $S_t(v_0)$, we take $\varepsilon' > 0$ and solve the approximate problem with $v_{0,\varepsilon'} = v_0 + \varepsilon'$. If we now take ε' smaller than $\varepsilon/2$, we get an estimate of $\|(v_{0,\varepsilon'} - u_\varepsilon(x, t_1))_+\|_1$ in the following way. We first note that set of points K where $u(x, t_1) > \varepsilon/2$ has measure less than $2\|u(x, t_1)\|_1/\varepsilon$. Therefore

$$\int_K (v_{0,\varepsilon'} - u_\varepsilon(x, t_1))_+ dx = \int_K (v_0(x) + \varepsilon' - u_\varepsilon(x, t_1))_+ dx \leq |K|\varepsilon'.$$

On the other hand, since $u_\varepsilon(x, t_1) \geq \varepsilon$ everywhere, and $v_{0,\varepsilon'}(x) \leq \varepsilon/2 + \varepsilon'$ on $\mathbb{R} \setminus K$, on that set we have $v_{0,\varepsilon'} - u_\varepsilon(x, t_1) \leq 0$. We conclude that

$$\|(v_{0,\varepsilon'} - u_\varepsilon(x, t_1))_+\|_1 \leq \frac{\varepsilon'}{\varepsilon} \|u_0\|_1.$$

By the ordered contraction property, this estimate remains true during the evolution, hence

$$\left\| \left(S_t^{\varepsilon'} v_{0,\varepsilon'} - u_\varepsilon(x, t + t_1) \right)_+ \right\|_1 \leq \frac{\varepsilon'}{\varepsilon} \|u_0\|_1$$

holds for all $t > 0$. Let now $\varepsilon' \rightarrow 0$ to get by the very definition of the upper limit solution that

$$\|(S_t v_0(x) - u_\varepsilon(x, t + t_1))_+\|_1 \leq 0.$$

This means that $S_t v_0(x) \leq u_\varepsilon(x, t + t_1)$ for a.e. $x \in \mathbb{R}$. In other words, $S_t(S_{t_1} u_0) \leq S_{t+t_1} u_0$. By the conservation of mass, both functions are the same, which proves the semigroup property $S_t(S_{t_1} u_0) = S_{t+t_1} u_0$.

9.4. Upper limit solutions are maximal solutions

In the standard Laplacian case $s = 1$ with singular diffusion, it is known that in the range of exponents $0 < n < 1$ where there is existence of nontrivial solutions, the limit solution is not the only possible solution defined in the whole line $x \in \mathbb{R}$. On the contrary, there are infinitely many other solutions with the same initial data determined by some “flux at infinity,” as described in [21,28]. But the upper limit solution is the maximal element in that class, in fact it is the only one for which the total mass does not decrease in time.

In order to repeat the proof of maximality in a short way, we only consider a class of solutions that admits comparison with classical supersolutions (which will be the solutions u_ε of the approximate problems). Let us call good solutions the elements of such a class.

THEOREM 9.2. *Let $u_g(x, t)$ denote a good solution of (1.1) for our choice of Φ defined in an interval $0 < t < T$ and having nonnegative initial data $u_0 \in L^1(\mathbb{R})$, and let $\bar{u}(x, t)$ the upper limit solution with same initial data. Then, $u_g \leq \bar{u}$ in $\mathbb{R} \times (0, T)$.*

The further exploration of the existence of such solutions falls out of the scope of this paper for reasons of space.

10. Asymptotic behaviour of general solutions

We want to prove Theorem 1.4, which means that we want to show that the Barenblatt solutions are asymptotic attractors of the solutions with general data restricted only by the running conditions on the initial data: $u_0(x) \geq 0, u_0 \in L^1(\mathbb{R})$. We address first the question of convergence in the L^1 norm, which is split into a number of cases.

- We consider first rearranged initial data. The result is just a reformulation of the proof of construction of the Barenblatt solution in Sect. 6. The argument is well known in the literature, see [38]. We just recall that the rescaled family $u_L(x, t)$ converges to U_M in $L^1(\mathbb{R})$ at any time $t > 0$, and fix $t = 1$ to get

$$\lim_{L \rightarrow \infty} \|u_L(x, 1) - U_M(x, 1)\|_1 \rightarrow 0$$

We then undo the scaling, see formula (5.5), to find that

$$\lim_{L \rightarrow \infty} \|u(x, L^{2s-1-n}) - U_M(x, L^{2s-1-n})\|_1 \rightarrow 0$$

Putting $t = L^{2s-1-n} \rightarrow \infty$ we obtain the result.

- The main novelty lies in establishing the result for general data that are not rearranged. We have to use the trick introduced in paper [27] with Portilheiro for general data with compact support. We argue as follows: we fix $t_1 \gg 1$ and assume the support of u_0 be included in $[-R, R]$. Define

$$\tilde{u}_1(r) := \inf_{|x|=r} u(x, t_1), \quad \tilde{u}_2(r) := \max_{|x|=r} u(x, t_1).$$

We easily verify that $\tilde{u}_1(r), \tilde{u}_2(r)$ are nonnegative and radially symmetric functions, they are nonincreasing as functions of r for $r \geq R$, we have the immediate comparison

$$\tilde{u}_1(r) \leq u(x, t_1) \leq \tilde{u}_2(r),$$

where $|x| = r$. We also have by the Aleksandrov reflection comparison

$$\tilde{u}_2(r) \geq \tilde{u}_1(r) \geq \tilde{u}_2(r + 2R)$$

for all $r \geq R$. It is then easy to verify that the $1-d$ mass of $\tilde{u}_2(r) - \tilde{u}_1(r)$ is less than $CRt_1^{-1/(n+1)}$, which can be made very small.

We restart the evolution at time t_1 and get radially symmetric solutions $\tilde{u}_1(x, t), \tilde{u}_2(x, t)$ with initial data $\tilde{u}_1(r), \tilde{u}_2(r)$ resp., and we also have the original $u(x, t + t_1)$ (now displaced in time) that stays between them. The asymptotic behaviour says that $\tilde{u}_1(x, t)$ converges to the Barenblatt U_{M_1} and $\tilde{u}_2(x, t)$ converges to the Barenblatt U_{M_2} . Moreover, the masses satisfy $M_1 \leq M \leq M_2$ and $M_2 - M_1 \leq \varepsilon$. The asymptotic formula for convergence of $u(\cdot, t)$ to $U_M(\cdot, t)$ follows easily.

- If u_0 does not have compact support, we use approximation and L^1 contraction.
- The estimate in the L^p norms is just an interpolation between the convergence result for $p = 1$ just proved, and the L^∞ bound of the form $u(x, t) \leq C t^{-\alpha}$ with $\alpha = 1/(2s - 1 - n)$.

This result allows to extend the uniqueness of the fundamental solutions as follows

THEOREM 10.1. *Every upper limit solution for positive times that is self-similar and integrable is a Barenblatt solution $U_M(x, t)$ for some $M > 0$. It can be obtained by rescaling from any integral and nonnegative initial data with mass M .*

Proof. Let u_1 be that solution and M its mass. Since it is self-similar, we have

$$\|u_1(\cdot, t_1) - U_M(\cdot, t_1)\|_1 = \|u_1(\cdot, t_2) - U_M(\cdot, t_2)\|_1$$

Fix now $t_1 > 0$ and let $t_2 \rightarrow \infty$. Applying Theorem 1.4, the right-hand side goes to zero. Hence, $u_1 \equiv U_M$. The second assertion is easier. □

11. Proof of the shifting comparison lemma

In this section, we will give a complete proof of Theorem 4.2. Note that this result applies to the approximate problems that have nonsingular functions Φ . The passage to the limit allows to apply it to our upper limit solutions.

11.1. Elliptic problem: extended problem

The implicit time discretization scheme [4, 13] directly connects the analysis of the parabolic Eq. (1.1) to solving a sequence of elliptic equations of the form

$$(-\Delta)^{\sigma/2} v + B(v) = f(x) \quad x \in \mathbb{R}, \tag{11.1}$$

where $\sigma \in (0, 2)$ and f is an integrable function defined in \mathbb{R} . We assume that the nonlinearity is given by a function $B : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is smooth and monotone increasing with $B(0) = 0$ and $B'(v) > 0$. It is not essential to consider negative values for our main results, but the general theory can be done in that greater generality. We are using here nonnegative data and solutions. In the parabolic application $B = \Phi^{-1}$, see [38, 43]. Then, we need to prove the following result

THEOREM 11.1. *Let us consider two functions $f_1, f_2 \in L^1(\mathbb{R})$ with the following properties:*

- (i) *They are nonnegative, and rearranged around their points of maximum x_1 and x_2 resp., with $x_1 < x_2$. BY this mean that $f_i(x - x_i)$ is rearranged.*
- (ii) *We assume moreover that $\int_{-\infty}^x f_2 \, dx \leq \int_{-\infty}^x f_1 \, dx$.*
- (iii) *the mass is the same, $\int f_1(x) \, dx = \int f_2(x) \, dx = M > 0$.*

Besides, we assume that B is convex and defined on \mathbb{R}_+ with $B(0) = 0$ and $B'(0) > 0$. Then, the following comparison inequalities hold

$$\int_{-\infty}^x v_2(x) \, dx \leq \int_{-\infty}^x v_1(x) \, dx, \quad \int_{-\infty}^x B(v_2(x)) \, dx \leq \int_{-\infty}^x B(v_1(x)) \, dx \quad (11.2)$$

for every $x \in \mathbb{R}$.

Proof. (a) It will be convenient to formulate the elliptic problem by using a proper extension problem, which is defined as the trace of a properly defined Dirichlet-Neumann problem as follows. If w is a weak solution to the local problem

$$\begin{cases} \operatorname{div}_{x,y} (y^{1-\sigma} \nabla w) = 0 & \text{in } Q_+, \\ -\frac{1}{\kappa_\sigma} \lim_{y \rightarrow 0^+} y^{1-\sigma} \frac{\partial w}{\partial y}(x, y) + B(w(x, 0)) = f(x) \text{ for } x \in \mathbb{R} \end{cases} \quad (11.3)$$

where $Q_+ := \mathbb{R} \times (0, +\infty)$ is the upper half-plane and κ_σ is the constant that is not important in what follows. See [10] which is the main reference in the issue of this extension. We can define again a suitable meaning of weak solution in terms of this extended problem. The functional setting is perfectly explained in [43], Sect. 3.1. There is a solution to problem (11.3). Then, the trace of w over $\mathbb{R} \times \{0\}$, $\operatorname{Tr}_{\mathbb{R}}(w) = w(\cdot, 0) := v$ is said a solution to problem (11.1). Using the change of variables $z = Cy^\sigma$ for a convenient constant $c > 0$, the problem can also be written as

$$\begin{cases} cz^\nu \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} = 0 & \text{in } Q_+ \\ -\frac{\partial w}{\partial z}(x, 0) = f(x) - B(w(x, 0)) \text{ for } x \in \mathbb{R}. \end{cases} \quad (11.4)$$

We use the extension formulation and write $w_i(x, 0) = v_i(x)$ and $u_i = B(v_i), i = 1, 2$.

(b) Note that the properties (i) of the f_i 's hold also for the solutions v_1 and v_2 . Conservation of mass applies so that $B(v_1)$ and $B(v_2)$ have the same mass M . Due to the properties of B , the functions v_1 and v_2 are also integrable, though their masses are not controlled. By the properties of the extension to the upper half-plane, the functions $w_2(z, z)$ and $w_1(x, z)$ are also integrable in x for every fixed $z > 0$.

(c) Using a similar strategy to the symmetrization proof in paper [43], we introduce the function

$$Z(s, z) = \int_{-\infty}^s (w_2(\tau, z) - w_1(\tau, z)) d\tau. \quad (11.5)$$

Then, it is clear that

$$c z^{\nu} Z_{zz} + Z_{xx} = 0 \tag{11.6}$$

and

$$Z(-\infty, z) = 0, \quad Z(\infty, z) = 0. \tag{11.7}$$

A crucial point in our arguments below is played by the derivative of Z with respect to z . Due to the boundary conditions contained in (11.4), we have

$$Z_z(x, 0) \geq \int_{-\infty}^x (B(w_2(\tau, 0)) - B(w_1(\tau, 0))) d\tau \tag{11.8}$$

Recall that $w_i(\tau, 0) = v_i(\tau)$. Observe also that the function

$$Y(x, 0) = \int_{-\infty}^x B(w_2(\tau, 0)) - B(w_1(\tau, 0)) d\tau$$

has the same points of maximum or minimum and the same regions of monotonicity than $Z(x, 0)$.

(d) Then we argue as follows. Due to the maximum principle and the boundary conditions (11.7), a positive maximum of Z can be achieved only on the line $\{z = 0\}$. On the other hand, in the interval $I = \{x : x_1 < x < x_2\}$ we know that $\partial_x v_1 \leq 0$ and $\partial_x v_2 \geq 0$ hence in this interval I

$$Y_{xx} = \partial_x B(v_2) - \partial_x B(v_1) \geq 0$$

and the maximum of Y must lie outside of I ; hence, the same happens for Z . Suppose the maximum of Z happens at $(x_0, 0)$ with $x_0 \geq x_2$. We must also have $Z_z(x_0, 0) < 0$ by Hopf's maximum principle, and by (11.8), this leads to $Y(x_0, 0) < 0$. But for $x > x_0$

$$\begin{aligned} Y(x, 0) - Y(x_0, 0) &= \int_{x_0}^x [B(v_2(\tau)) - B(v_1(\tau))] d\tau \\ &\leq \int_{s_0}^s B'(v_2(\tau))(v_2(\tau) - v_1(\tau)) d\tau. \end{aligned}$$

Here, we have used the convexity of B so that B' is an increasing real function and

$$B(v_2(\tau)) - B(v_1(\tau)) \leq B'(v_2(\tau))(v_2(\tau) - v_1(\tau)).$$

After integration by parts in the expression for the increment of Y , we get

$$\begin{aligned} Y(x, 0) - Y(x_0, 0) &\leq [B'(v_2(\tau))(Z(\tau, 0) - Z(x_0, 0))]_{x_0}^x \\ &\quad - \int_{x_0}^x B''(v_2(\tau))v_{2,x}(\tau)(Z(\tau, 0) - Z(x_0, 0))d\tau. \end{aligned}$$

Since Z has a maximum at x_0 and B' is positive, the first term in the RHS is nonpositive. As for the second, we have: $B'' > 0$, $v_{2,x} < 0$, and $Z(x, 0) - Z(x_0, 0) \leq 0$; hence, the

last term is also nonpositive. We conclude that $Y(x, 0) \leq Y(x_0, 0) < 0$ for all $x > x_0$. This is a contradiction, because by the conservation of mass property at plus infinity we have $Y(\infty, 0) = 0$. Therefore, there is no positive maximum for Z on this side.

(ii) Similar argument on the other side, $x < x_1$ reversing the roles of v_1 and v_2 and the direction of integration, that starts now at $+\infty$. We conclude that $Z(x, 0) \leq 0$ everywhere.

(iii) Once we have $Z(x, 0) \leq 0$, we also want to prove that $Y(x, 0) \leq 0$. We may use Lemma 11.2 below (a well-known result), taking advantage of the convexity of B and choosing any convex, increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$. This ends the proof of the comparison theorem in this case. \square

LEMMA 11.2. *Let $f, g \in L^1(\Omega)$ be two rearranged functions on a ball $\Omega = B_R(0)$. Then, $f < g$ if and only if for every convex nondecreasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ we have*

$$\int_{\Omega} \Phi(f(x)) \, dx \leq \int_{\Omega} \Phi(g(x)) \, dx. \tag{11.9}$$

This result still holds if $R = \infty$ and $f, g \in L^1_{\text{loc}}(\mathbb{R}^N)$ with $g \rightarrow 0$ as $|x| \rightarrow \infty$.

REMARK. We have imposed severe conditions on the shape of u_{01}, u_{02} , and we have required Φ to be concave. Are these conditions necessary? They are not for the same result with standard Laplacian instead of fractional Laplacian.

12. A preview of logarithmic diffusion with $s = 1/2$

This is a kind of exceptional case in the parameter diagram. The study is a bit different from the previous analysis, and much of our intuition comes from a similar special case that happens for standard diffusion $s = 1$, in dimension $N = 2$ with logarithmic diffusion $n = 0$. An explicit solution exists then, and it was used in [35]. The formula is

$$U(x, t) = \frac{8(T - t)}{(1 + |x|^2)^2}, \quad x \in \mathbb{R}^2, \quad 0 < t < T. \tag{12.1}$$

Note that U is positive only in the time interval $0 < t < T$. As a very weak solution, it can be defined for all $t > 0$, and it vanishes identically for $t \geq T$. Another important property is the total mass decay with a constant rate

$$\frac{d}{dt} \int_{\mathbb{R}^2} U(x, t) \, dx = -8\pi. \tag{12.2}$$

But, as shown by a number of studies, this rate does not correspond the limit solutions, which are characterized by the rule $dM(t)/dt = -4\pi$. A gap of nonuniqueness opens up, and this is carefully described in [39], where related literature can be found.

12.1. Existence of a positive solution

Luckily, in our one-dimensional case there also exists an explicit solution of the evolution equation

$$\partial_t u + (-\Delta)^{1/2}(\log u) = 0 \quad \text{in } \mathbb{R} \times (0, T) \tag{12.3}$$

with a smooth initial condition $u(x, 0) \in L^1(\mathbb{R})$. It is given by the formula

$$U(x, t) = \frac{2\lambda(T-t)}{\lambda^2 + |x|^2} \quad \text{in } \mathbb{R} \times (0, +\infty), \tag{12.4}$$

with any $\lambda > 0$, so it is indeed a whole family of solutions related by scaling. We see that U has the separate variable type as in the previous example, and $U(\cdot, t)$ is in $L^1(\mathbb{R})$ for any $0 \leq t < T$. It is very peculiar that the solution becomes identically zero in finite time. This is the so-called *finite-time extinction* phenomenon, which is typical of some ranges of fast diffusion, see [39] for standard diffusion and [19,26] for fractional diffusion.

In order to prove that (12.4) is a classical solution of the equation we write it in the form $U(x, t) = (T - t)F(x)$ and we need to find an integrable profile $F > 0$ such that

$$(-\Delta)^{1/2}F(x) = F(x). \tag{12.5}$$

It only remains to check that $F(x) = 2/(1 + |x|^2)$ is a solution of this nonlinear elliptic equation. This is only a technical calculus result.

Observe that the initial mass is in all cases (12.4)

$$M_0 = \|U_0\|_1 = 2T \int \frac{dx}{1 + x^2} = 2\pi T,$$

so that in the extinction time is given by $T = \|U_0\|_1/2\pi$. Accordingly, the mass decay rule is $M'(t) = -2\pi$, which is related to what happens for $N = 2$ and standard diffusion (see above).

In the same way as in the previous analysis of the case $s > 1/2$, we can use this example and standard comparison to prove that for initial data $u_0(x) \geq c/(1 + |x|^2)$ with $c > 0$, the limit solution is nontrivial, more precisely positive for $0 < t < T$ with $T = c/2$. We do not expect upper limit solutions to conserve mass since this does happen for the special case of standard diffusion $s = 1, N = 2$, as described in [30,39,41].

13. Comments, extensions, and open problems

- We have given preference in the paper to the treatment of the more singular case of exponents $n > 0$. However, the logarithmic equation is also covered, and the main results are proved to be true, as special case $n = 0$. This is shown not only at the formal level, but also technical details are given as needed. There is

another variant of the limit case $n = 0$, the sign diffusion $u_t = \Delta \text{sign}(u)$, that is related to the total variation flow and was treated in [6]. We are not covering the fractional version of this variant.

- The actual behaviour of the solutions of logarithmic diffusion on the line with exponent $s = 1/2$ is a question to be investigated, and this will be done in a separate work. For the nonsingular equation $\partial_t u + (-\Delta)^s(1 + \log u) = 0$, the study was done in [20], see also [42].
- The existence of nonmaximal solutions, which are not upper limit solutions and do not conserve mass, is an interesting open problem. For the standard Laplacian, it was solved in [21, 28] where it is proved that there are infinitely many solutions for every integrable initial data, and they are determined by some flux conditions at infinity. A large related literature has developed, see, e.g. [23, 24, 28, 29].
- Equation (5.8) is a kind of fractional p -Laplacian equation for the mass function. It would be interesting to perform a study of its properties and applicability.
- Elliptic problems and the Crandall–Liggett approach. A natural way to solve the evolution equation is by implicit discretization in time, a method that became basic in the early studies of the Porous Medium Equation, see the original paper CL71 or [38, Chapter 10]. In the present situation, it means that we have to solve elliptic problems of the form $(-\Delta)^s \Phi(u) + u = f$. Note that when $\Phi(u) = \log(u)$, this equation takes the suggestive form $(-\Delta)^s v + e^v = f$. Non-existence results are carefully described in [7] for the range of parameters $2s < n + 1$. The existence theory in our parameter range is not difficult. We refrain from further details for reasons of space, but see [7, Section 9].
- Dirichlet problem in bounded domains. In the case of the standard Laplacian, $s = 1$, no nontrivial solutions exist for the Dirichlet problem with zero boundary data. The question is open for our equations.
- Problems with more general nonlinearities Φ . Symmetrization can be applied in some cases using the results of [44] to compare a general Φ with the power cases we deal with. It can give the starting results on nontrivial existence. Non-existence results in that direction are explained in [7]. Of course, in the simple case where $\Phi(u) = -c u^{-n}$ (or $\Phi(u) = c \log(u)$) with some $c > 0$, the constant may be absorbed into the time variable and we need not make any changes to the theory.
- Problems with other classes of initial data are worth studying. A simple example is the constant solutions do not belong to our class of integrable upper limit solutions. It is not difficult to construct solutions for initial data in $L^p(\mathbb{R})$ with $p > 1$ using the smoothing effect and approximation by integrable data. We can consider in this way initial data that are not integrable with different decay rates at infinity. It is even possible to consider data that grow at infinity.
- A different direction is establishing existence of upper limit solutions with non-negative Radon measures as initial data. An example for that extension is the Barenblatt solutions that we have just constructed. Since the basic process for

general measures is easy following the indications of [40, Section 4], we leave it to the interested reader.

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