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On the Cauchy problem for the compressible Hall-magneto-hydrodynamics equations

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Abstract. In this paper, we consider the large time behavior of solutions to the three-dimensional compressible Hall-magneto-hydrodynamics equations. We first establish the uniform estimates of the global smooth solution with respect to the Hall coefficient ϵ. Then we obtain the optimal decay estimates with the aid of a negative Sobolev space. We next show that the unique smooth solution of the compressible Hall-magneto-hydrodynamics system converges globally in time to the smooth solution of the compressible magneto-hydrodynamics system as ϵ tends to zero. We also give the convergence rate estimates for any given positive time.

1. Introduction

Spacecraft observations of magnetic and velocity fluctuations in the solar wind show a distinct steepening of the $f^{-\frac{5}{3}}$ power law inertial range spectrum at frequencies above the Doppler-shifted ion cyclotron frequency, where f is the spacecraft rest frame frequency. This is commonly attributed to dissipation due to wave–particle interactions. To investigate the extent to which this steepening can be described, the following three-dimensional magneto-hydrodynamic (MHD) formulation that includes the so-called Hall term has been presented by Ghosh et al. [13]:

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \\ \partial_t (n\mathbf{u}) + \nabla \cdot (n\mathbf{u} \otimes \mathbf{u}) + \nabla p(n) = \mu \Delta \mathbf{u} + (\mu + \nu)\nabla(\nabla \cdot \mathbf{u}) + (\nabla \times \mathbf{b}) \times \mathbf{b}, \\ \partial_t \mathbf{b} + \nabla \times (\mathbf{b} \times \mathbf{u}) + \epsilon \nabla \times \left(\dfrac{(\nabla \times \mathbf{b}) \times \mathbf{b}}{n} \right) = -\nabla \times (\nabla \times \mathbf{b}). \end{cases} \tag{1.1}$$

Here the unknowns n denotes the density of the fluid, $\mathbf{u}$ the velocity of the fluid, and $\mathbf{b}$ the magnetic field, respectively. This system is closed by using a polytropic relation between pressure and density $p(n) = an^\gamma$ with positive constants a and $\gamma > 1$. The constant viscosity coefficients μ and ν satisfy the usual physical conditions

$$\mu > 0 \quad \text{and} \quad 2\mu + 3\nu > 0.$$

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The small parameter $\epsilon := \frac{\omega_A}{\Omega_i}$ is called Hall coefficient, where ω_A is the Alfven frequency of the lowest wave number and Ω_i is the ion cyclotron frequency. Hence the reciprocal of ϵ is the resonant wave number at which the nondispersive Alfven frequency resonates with the ion cyclotron frequency. In particular, when $\epsilon = 0$, Eq. (1.1) become the classical compressible MHD equations. Compared with the classical MHD equations, the Hall-MHD equations (1.1) have the Hall term $\nabla \times \frac{(\nabla\times\mathbf{b})\times\mathbf{b}}{n}$ in $(1.1)_2$, which is believed to be the key for understanding the problem of magnetic reconnection and cannot be described in the framework of ideal MHD, due to the frozen-field effect. Thus the Hall-MHD system (1.1) is very important in describing many phenomena such as magnetic reconnection in space plasmas, star formation, neutron stars and geo-dynamo (see [15,18,19,21] and references therein).

From a mathematical point of view, while the classical compressible and incompressible MHD equations are well understood for quite long time (see e.g., [17]), the Hall-MHD equations have received little attention from mathematicians. Until recently, Acheritogaray et al. [1] derived the compressible and incompressible Hall-MHD equations from either two fluids model or kinetic models in a mathematically rigorous way. Then for the incompressible cases, the global existence of weak solutions, local well-posedness of classical solution and global existence of smooth solutions with small initial data are established by Chae et al. [3]. Very recently, temporal decay for the weak solution and smooth solution is established by Chae and Schonbek [5], and the blow-up criterion and small data global existence are obtained by Chae and Lee [4]. For more related works on incompressible Hall-MHD, we refer to [6–8,10,11,20] and references therein. On the other hand, for the compressible cases (1.1), we only find the paper [9,12], where the authors proved the existence of global small solutions with small initial data and established some decay estimates.

In the present paper, we study the Cauchy problem of the MHD equations (1.1), which is supplemented with the following initial data

$$(n, \mathbf{u}, \mathbf{b})(x, 0) = (n_0(x), \mathbf{u}_0(x), \mathbf{b}_0(x)) \to (\bar{n}, 0, 0) \quad \text{as} \quad |x| \to +\infty,$$

where $\bar{n}$ is a positive constant. We will also assume that $\nabla \cdot \mathbf{b}_0 = 0$. Notice that by taking the operation div on both sides of $(1.1)_3$ we have $\partial_t(\nabla \cdot \mathbf{b}) = 0$. This means that the divergence-free condition of $\mathbf{b}_0$ can be propagated. Thus we can formulate our problem as follows:

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{u}) = 0, \\ \partial_t(n\mathbf{u}) + \nabla \cdot (n\mathbf{u} \otimes \mathbf{u}) + \nabla p(n) = \mu\Delta\mathbf{u} + (\mu + \nu)\nabla(\nabla \cdot \mathbf{u}) + (\nabla \times \mathbf{b}) \times \mathbf{b}, \\ \partial_t\mathbf{b} + \nabla \times (\mathbf{b} \times \mathbf{u}) + \epsilon\nabla \times \left(\dfrac{(\nabla \times \mathbf{b}) \times \mathbf{b}}{n}\right) = \Delta\mathbf{b}, \\ \nabla \cdot \mathbf{b} = 0, \\ (n, \mathbf{u}, \mathbf{b})(x, 0) = \big(n_0, \mathbf{u}_0, \mathbf{b}_0\big)(x) \end{cases} \tag{1.2}$$

in $\mathbb{R}^3 \times (0, +\infty)$.

We shall establish the global existence and optimal decay estimates of the classical solution to system (1.2) when the initial data are small perturbations around the given constant state $(\bar{n}, 0, 0)$. The novelty is twofold: first, to allow the higher-order derivatives of initial data to be of large oscillations with constant state at far field; second, to establish the uniform estimates of the global smooth solutions with respect to the Hall coefficient ϵ. With the help of the latter, we will also investigate the vanishing Hall limit $\epsilon \to 0$ of the global-in-time solutions to equations (1.2).

We introduce the set:

$$\mathcal{M}_{k,M_0} := \left\{\varphi \in H^k(\mathbb{R}^3) \,\middle|\, \|\nabla^2\varphi\|_{H^1} \le M_0\right\}$$

for any $k \ge 3$ and $M_0 \ge 1$, and state our global existence result as follows.

THEOREM 1.1. (Global existence) *Assume that* $(n_0 - \bar{n}, \mathbf{u}_0, \mathbf{b}_0) \in \mathcal{M}_{k,M_0}$ *and*

$$\|n_0 - \bar{n}\|_{H^1}^2 + \|\mathbf{u}_0\|_{H^1}^2 + \|\mathbf{b}_0\|_{H^1}^2 \le \varepsilon^2$$

for some positive constant ε. *Then if* ε *is suitable small, the compressible Hall-MHD equations* (1.2) *admit a unique global solution* $(n, \mathbf{u}, \mathbf{b})$ *satisfying the uniform estimates:*

$$\begin{aligned}
&\left(\|n(t) - \bar{n}\|_{H^k}^2 + \|\mathbf{u}(t)\|_{H^k}^2 + \|\mathbf{b}(t)\|_{H^k}^2\right)\\
&\quad + \int_0^t \left(\|\nabla n(\tau)\|_{H^{k-1}}^2 + \|\nabla \mathbf{u}(\tau)\|_{H^k}^2 + \|\nabla \mathbf{b}(\tau)\|_{H^k}^2\right) d\tau\\
&\le C\left(\|n_0 - \bar{n}\|_{H^k}^2 + \|\mathbf{u}_0\|_{H^k}^2 + \|\mathbf{b}_0\|_{H^k}^2\right)
\end{aligned} \tag{1.3}$$

for all $t \ge 0$, *where the positive constant* C *is independent of* ϵ *and* t.

REMARK 1.1. From a physical viewpoint, the Hall term restores the influence of the electric current in the Lorentz force occurring in Ohm's law, which was neglected in conventional MHD models. This term is quadratic in the magnetic field and involves second-order derivatives, and thus its influence becomes dominant in the cases where the magnetic shear is large. This intuitively explains why our results are more subtle than the standard MHD equations even in the classical framework (e.g., for initial data with small H^3 norm).

We next investigate the decay rate of solutions of the Cauchy problem (1.1) around the steady state $(1, 0)$. Precisely, we will apply the energy method together with the negative index Sobolev spaces to prove the optimal decay rate of the solution to the problem (1.2).

THEOREM 1.2. (Decay estimates) *Under the assumptions Theorem 1.1, if we further assume that* $(n_0 - \bar{n}, \mathbf{u}_0, \mathbf{b}_0) \in \dot{H}^{-s}$ *for some* $s \in \left[0, \frac{3}{2}\right)$, *then the global solution* $(n, \mathbf{u}, \mathbf{b})$ *satisfies*

$$\|n(t) - \bar{n}\|_{\dot{H}^{-s}}^2 + \|\mathbf{u}(t)\|_{\dot{H}^{-s}}^2 + \|\mathbf{b}(t)\|_{\dot{H}^{-s}}^2 \le C \tag{1.4}$$

and

$$\|\nabla^\ell(n-\bar{n})(t)\|_{L^2}^2+\|\nabla^\ell\mathbf{u}(t)\|_{L^2}^2+\|\nabla^\ell\mathbf{b}(t)\|_{L^2}^2 \le C(1+t)^{-(\ell+s)} \quad (\ell=0,1,\ldots,k-1) \tag{1.5}$$

for all $t \ge 0$, *where the positive constant* C *is independent of* ϵ *and* t.

REMARK 1.2. For $\epsilon = 1$, the corresponding decay estimate has been established by [9,12] under the assumption that the H^3 norm of initial data is small and that the L^1 norm is finite. Notice that if $f \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, we have $f \in H^{-s}(\mathbb{R}^3)$ for $s \in \left[0, \frac{3}{2}\right)$ by the Littlewood-Payley decomposition. Thus we relaxed the regularity and smallness conditions on the initial data required in [9,12].

REMARK 1.3. The key point of our result is that all of our estimates are uniform in the Hall coefficient ϵ in Theorems 1.1 and 1.2.

We now turn to show that the unique smooth solution of the three-dimensional compressible Hall-magneto-hydrodynamics system converges globally in time to the smooth solution of the three-dimensional compressible magneto-hydrodynamics system as the Hall coefficient ϵ tends to zero. We also give the convergence rate estimates for any given positive time.

THEOREM 1.3. (Vanishing Hall limit) *Assume that* $(n^\epsilon, \mathbf{u}^\epsilon, \mathbf{b}^\epsilon)$ *and* $(n^0, \mathbf{u}^0, \mathbf{b}^0)$ *are two solutions to equations* (1.2) *obtained in Theorem* 1.1 *corresponding to* $\epsilon > 0$ *and* $\epsilon = 0$, *respectively. Then it holds that*

$$n^\epsilon \to n^0, \quad \mathbf{u}^\epsilon \to \mathbf{u}^0 \quad \text{and} \quad \mathbf{b}^\epsilon \to \mathbf{b}^0 \qquad \text{strongly in } C\big(0, t;\ H^{k-\sigma}_{loc}\big)$$

with $\sigma \in (0, \frac{1}{2})$, *as* $\epsilon \to 0$. *Moreover, we have the following the convergence rate estimates:*

$$\begin{aligned}&\|n^\epsilon(t) - n^0(t)\|_{H^{k-2}}^2 + \|\mathbf{u}^\epsilon(t) - \mathbf{u}(t)\|_{H^{k-2}}^2 + \|\mathbf{b}^\epsilon(t) - \mathbf{b}(t)\|_{H^{k-2}}^2 \\ &\quad \le \epsilon^2 e^{Ct} \qquad \text{for any } \ t \in [0,\infty).\end{aligned}$$

The rest of this paper is organized as follows. In Sect. 2, we establish the uniform estimates and global existence of smooth solution with respect to the Hall coefficient ϵ. Then we obtain the optimal decay estimates with the aid of a negative Sobolev space in Sect. 3. We next give the convergence rate estimates for any given positive time in Sect. 4. In Appendix, finally, we state several basic inequalities used in this paper.

Notations: Throughout this paper, ∇^ℓ with a nonnegative integer ℓ stands for the usual spatial derivatives of order ℓ. The letters c and C denote generic positive constants which may vary in the context.

2. Global existence

In this section, we will investigate the uniform estimate and global existence of classical solutions to (1.1) around the state $(\bar{n}, 0, 0)$. Without loss of generality, we can take $\bar{n} = 1$ for simplicity. Then by setting $\rho = n - 1$ and

$$g(\rho)=\frac{\rho}{\rho+1},\qquad \varphi(\rho)=\frac{1}{\rho+1},\qquad \text{and}\qquad h(\rho)=a\gamma\Big((\rho+1)^{\gamma-2}-1\Big),$$

we can rewrite (1.2) as the following form:

$$\begin{cases}\partial_t\rho+\nabla\cdot\mathbf{u}=-\rho\nabla\cdot\mathbf{u}-\mathbf{u}\cdot\nabla\rho,\\ \partial_t\mathbf{u}-\mu\Delta\mathbf{u}-(\mu+\nu)\nabla(\nabla\cdot\mathbf{u})+a\gamma\nabla\rho\\ \qquad=-\mathbf{u}\cdot\nabla\mathbf{u}-\mu g(\rho)\Delta\mathbf{u}-(\mu+\nu)g(\rho)\nabla(\nabla\cdot\mathbf{u})\\ \qquad\quad+\varphi(\rho)\Big(\mathbf{b}\cdot\nabla\mathbf{b}-\frac{1}{2}\nabla|\mathbf{b}|^2\Big)-h(\rho)\nabla\rho,\\ \partial_t\mathbf{b}-\Delta\mathbf{b}=-\epsilon\nabla\times\Big((\nabla\times\mathbf{b})\times\big(\varphi(\rho)\mathbf{b}\big)\Big)-\mathbf{u}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{u}-\mathbf{b}\nabla\cdot\mathbf{u},\\ \nabla\cdot\mathbf{b}=0,\\ (\rho,\mathbf{u},\mathbf{b})(x,0)=\big(n_0-1,\mathbf{u}_0,\mathbf{b}_0\big)(x)\end{cases}\tag{2.1}$$

in $\mathbb{R}^3\times(0,+\infty)$. We first state the local well-posedness of system (2.1) as follows.

THEOREM 2.1. (Local well-posedness) *Assume that* $(\rho_0,u_0,b_0)\in H^k$ *with* $k\geq 3$. *Then there exist* $T>0$ *and a unique* $(\rho,\mathbf{u},\mathbf{b})\in C([0,T];H^k)\cap L^2([0,T];H^{k+1})$ *solving equations* (2.1).

Proof. The proof of Theorem 2.1 is standard, and we refer to [9] for its details. □

For simplicity, throughout this paper, we will set

$$\begin{aligned}\mathcal{E}_\ell(t)&=a\gamma\Big(\|\nabla^\ell\rho\|_{L^2}^2+\|\nabla^{\ell+1}\rho\|_{L^2}^2\Big)+\|\nabla^\ell\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2+\|\nabla^\ell\mathbf{b}\|_{L^2}^2\\&\quad+\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2+2\delta\int\nabla^{\ell+1}\rho\cdot\nabla^\ell\mathbf{u}\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}_\ell(t)&=a\gamma\delta\|\nabla^{\ell+1}\rho\|_{L^2}^2+c_0\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big)\\&\quad-\delta\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2-\delta C_0\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}.\end{aligned}$$

Clearly, we can fix a small constant $\delta>0$ such that

$$\mathcal{E}_\ell(t)\simeq\Big(\|\nabla^\ell\rho\|_{L^2}^2+\|\nabla^{\ell+1}\rho\|_{L^2}^2+\|\nabla^\ell\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2+\|\nabla^\ell\mathbf{b}\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2\Big)\tag{2.2}$$

and

$$\mathcal{F}_\ell(t)\simeq\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big),\tag{2.3}$$

where $A\simeq B$ means that $c_1B\leq A\leq\frac{1}{c_1}B$ for some fixed positive constant c_1.

Then we turn to derive the uniform nonlinear energy estimates for the system (2.1). Some of our ideas are motivated by [2,14]. Precisely, we first give an a priori assumption on solutions and then derive the higher-order uniform estimates of solutions. We shall complete our argument by closing the above *a priori* assumption. For this purpose, we assume that

$$\|\nabla^2\rho(t)\|_{H^1}^2 + \|\nabla^2\mathbf{u}(t)\|_{H^1}^2 + \|\nabla^2\mathbf{b}(t)\|_{H^1}^2 \le \frac{2M_0^2}{c_1^2} \tag{2.4}$$

and

$$\|\rho(t)\|_{H^1}^2 + \|\mathbf{u}(t)\|_{H^1}^2 + \|\mathbf{b}(t)\|_{H^1}^2 \le \varepsilon_0^2 \tag{2.5}$$

for any $t \in [0, T]$. Without loss of generality, we assume that $M_0 > 1$, and $\varepsilon_0 \in (0, 1)$ suitably small. Then we immediately obtain

$$|g(\rho)| \le C|\rho|, \quad |\varphi(\rho)| \le C, \quad |h(\rho)| \le C|\rho|$$

and

$$|g^{(k)}(\rho)| \le C, \quad |\varphi^{(k)}(\rho)| \le C, \quad |h^{(k)}(\rho)| \le C$$

for any $k \ge 1$. We now begin to derive a series of *a priori* estimates.

LEMMA 2.1. *Let* $0 \le \ell \le k-1$ *and* (2.4)–(2.5) *hold. Then there exist two positive constants* c_0 *and* C *such that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(a\gamma\|\nabla^\ell\rho(t)\|_{L^2}^2 + \|\nabla^\ell\mathbf{u}(t)\|_{L^2}^2 + \|\nabla^\ell\mathbf{b}(t)\|_{L^2}^2\Big) \\
&\quad + c_0\Big(\|\nabla^{\ell+1}\mathbf{u}(t)\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}(t)\|_{L^2}^2\Big) \\
&\le CM_0^{\frac{5}{2}}\varepsilon_0^{\frac{1}{2}}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 \\
&\quad + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big).
\end{aligned}$$

Proof. We first consider the case $\ell = 0$. Multiplying $(2.1)_1$, $(2.1)_2$ and $(2.1)_3$ by $a\gamma\rho$, $\mathbf{u}$ and $\mathbf{b}$, respectively, summing up and integrating the resulting equations over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\rho(t)\|_{L^2}^2 + \|\mathbf{u}(t)\|_{L^2}^2 + \|\mathbf{b}(t)\|_{L^2}^2\Big) \\
&\quad + \Big(\mu\|\nabla\mathbf{u}(t)\|_{L^2}^2 + (\mu+\nu)\|\nabla\cdot\mathbf{u}(t)\|_{L^2}^2 + \|\nabla\mathbf{b}(t)\|_{L^2}^2\Big) \\
&= \int\bigg(-\epsilon\mathbf{b}\cdot\nabla\times\Big((\nabla\times\mathbf{b})\times\big(\varphi(\rho)\mathbf{b}\big)\Big) + a\gamma\rho\mathbf{u}\cdot\nabla\rho - \mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) \\
&\quad - g(\rho)\mathbf{u}\cdot\Big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla\big(\nabla\cdot\mathbf{u}\big)\Big) + \varphi(\rho)\mathbf{u}\cdot\Big(\mathbf{b}\cdot\nabla\mathbf{b} - \frac{1}{2}\nabla|\mathbf{b}|^2\Big) \\
&\quad - h(\rho)\mathbf{u}\cdot\nabla\rho - \mathbf{b}\cdot(\mathbf{u}\cdot\nabla\mathbf{b}) + \mathbf{b}\cdot\Big(\mathbf{b}\cdot\nabla\mathbf{u} - \mathbf{b}\nabla\cdot\mathbf{u}\Big)\bigg)dx.
\end{aligned}$$

For the Hall term, we have

$$\int \mathbf{b} \cdot \Big(\nabla \times \Big((\nabla \times \mathbf{b}) \times \big(\varphi(\rho)\mathbf{b}\big)\Big)\Big)\,\mathrm{d}x = \int (\nabla \times \mathbf{b}) \cdot (\nabla \times \mathbf{b}) \times \big(\varphi(\rho)\mathbf{b}\big)\mathrm{d}x = 0,$$

which implies that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\rho(t)\|_{L^2}^2 + \|\mathbf{u}(t)\|_{L^2}^2 + \|\mathbf{b}(t)\|_{L^2}^2\Big)\\
&\quad + \Big(\mu\|\nabla\mathbf{u}(t)\|_{L^2}^2 + (\mu+\nu)\|\nabla\cdot\mathbf{u}(t)\|_{L^2}^2 + \|\nabla\mathbf{b}(t)\|_{L^2}^2\Big)\\
&= \int \Big(a\gamma\rho\mathbf{u}\cdot\nabla\rho - \mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) - g(\rho)\mathbf{u}\cdot\Big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla\big(\nabla\cdot\mathbf{u}\big)\Big) - h(\rho)\mathbf{u}\cdot\nabla\rho\\
&\quad + \varphi(\rho)\mathbf{u}\cdot\Big(\mathbf{b}\cdot\nabla\mathbf{b} - \frac{1}{2}\nabla|\mathbf{b}|^2\Big) - \mathbf{b}\cdot(\mathbf{u}\cdot\nabla\mathbf{b})\\
&\quad + \mathbf{b}\cdot\Big(\mathbf{b}\cdot\nabla\mathbf{u} - \mathbf{b}\nabla\cdot\mathbf{u}\Big)\Big)dx.
\end{aligned}$$

It then follows from Hölder's inequality and Sobolev embedding that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\rho(t)\|_{L^2}^2 + \|\mathbf{u}(t)\|_{L^2}^2 + \|\mathbf{b}(t)\|_{L^2}^2\Big) + c_0\Big(\|\nabla\mathbf{u}(t)\|_{L^2}^2 + \|\nabla\mathbf{b}(t)\|_{L^2}^2\Big)\\
&\le C\Big(\|\rho\|_{L^3}\|\mathbf{u}\|_{L^6}\|\nabla\rho\|_{L^2} + \|\mathbf{u}\|_{L^3}\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^2} + \|g'(\rho)\|_{L^\infty}\|\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\\
&\quad + \|g(\rho)\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}^2 + \|h(\rho)\|_{L^6}\|\mathbf{u}\|_{L^3}\|\nabla\rho\|_{L^2}\\
&\quad + \|\mathbf{b}\|_{L^6}\Big(\|\mathbf{u}\|_{L^3}\|\nabla\mathbf{b}\|_{L^2} + \|\mathbf{b}\|_{L^3}\|\nabla\mathbf{u}\|_{L^2}\Big)\Big)\\
&\le C\Big(\|\rho\|_{L^3} + \|\mathbf{u}\|_{L^3} + \|\mathbf{b}\|_{L^3} + \|\rho\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty}\Big)\Big(\|\nabla\rho\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2\Big)\\
&\le C\Big(\|\rho\|_{H^1} + \|\mathbf{u}\|_{H^1} + \|\mathbf{b}\|_{H^1}\\
&\quad + \|\nabla\rho\|_{L^2}^{\frac{1}{2}}\|\nabla^2\rho\|_{L^2}^{\frac{1}{2}} + \|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{1}{2}}\Big)\Big(\|\nabla\rho\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{b}\|_{L^2}^2\Big)\\
&\le C\Big(\varepsilon_0 + M_0^{\frac{1}{2}}\varepsilon_0^{\frac{1}{2}}\Big)\Big(\|\nabla\rho(t)\|_{L^2}^2 + \|\nabla\mathbf{u}(t)\|_{L^2}^2 + \|\nabla\mathbf{b}(t)\|_{L^2}^2\Big).
\end{aligned}$$

Here we also used the facts $|g'(\rho)| \le C$, $|\nabla(h(\rho))| \le |\rho|$, $|g(\rho)| \le |\rho|$ and $|\varphi(\rho)| \le C$.

We now turn to the case $1 \le \ell \le k-1$. Applying ∇^ℓ to $(2.1)_1$, $(2.1)_2$ and $(2.1)_3$, multiplying the resulting equations by $a\gamma\nabla^\ell\rho$, $\nabla^\ell\mathbf{u}$ and $\nabla^\ell\mathbf{b}$, respectively, summing up and then integrating the resulting equations over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\nabla^\ell\rho(t)\|_{L^2}^2 + \|\nabla^\ell\mathbf{u}(t)\|_{L^2}^2 + \|\nabla^\ell\mathbf{b}(t)\|_{L^2}^2\Big)\\
&\quad + \Big(\mu\|\nabla^{\ell+1}\mathbf{u}(t)\|_{L^2}^2 + (\mu+\nu)\|\nabla^\ell\nabla\cdot\mathbf{u}(t)\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}(t)\|_{L^2}^2\Big)\\
&= -\epsilon\int \nabla^\ell\mathbf{b}\cdot\nabla^\ell\Big(\nabla\times\Big((\nabla\times\mathbf{b})\times(\varphi(\rho)\mathbf{b})\Big)\Big) - a\gamma\int \nabla^\ell\rho\nabla^\ell\nabla\cdot\big(\rho\mathbf{u}\big)
\end{aligned}$$

$$
\begin{aligned}
&-\int\Big(\nabla^\ell\mathbf{u}\cdot\nabla^\ell(\mathbf{u}\cdot\nabla\mathbf{u})\\
&\quad+\nabla^\ell\mathbf{b}\cdot\nabla^\ell\big(\mathbf{u}\cdot\nabla\mathbf{b}-\mathbf{b}\cdot\nabla\mathbf{u}+\mathbf{b}\nabla\cdot\mathbf{u}\big)\Big)-\int\nabla^\ell\mathbf{u}\cdot\nabla^\ell\Big(h(\rho)\nabla\rho\Big)\\
&-\int\nabla^\ell\mathbf{u}\cdot\nabla^\ell\Big(g(\rho)\big(\mu\Delta\mathbf{u}+(\mu+\nu)\nabla(\nabla\cdot\mathbf{u})\big)\Big)\\
&+\int\nabla^\ell\mathbf{u}\cdot\nabla^\ell\Big(\varphi(\rho)\big(\mathbf{b}\cdot\nabla\mathbf{b}-\frac{1}{2}\nabla|\mathbf{b}|^2\big)\Big)\\
:=\;&\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6.
\end{aligned}\tag{2.6}
$$

We now estimate the terms $\mathcal{I}_1,\mathcal{I}_2,\ldots,\mathcal{I}_6$ one by one. Firstly, for the Hall term $\mathcal{I}_1$, a direct calculation yields that

$$
\begin{aligned}
\mathcal{I}_1&=-\epsilon\int\nabla^\ell(\nabla\times\mathbf{b})\cdot\nabla^\ell\Big((\nabla\times\mathbf{b})\times\big(\varphi(\rho)\mathbf{b}\big)\Big)\\
&=-\epsilon\int\nabla^\ell(\nabla\times\mathbf{b})\cdot\bigg(\nabla^\ell\Big((\nabla\times\mathbf{b})\times\big(\varphi(\rho)\mathbf{b}\big)\Big)-\Big(\nabla^\ell(\nabla\times\mathbf{b})\Big)\times\big(\varphi(\rho)\mathbf{b}\big)\bigg)\\
&\le C\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}\Big\|\nabla^\ell\Big((\nabla\mathbf{b})\big(\varphi(\rho)\mathbf{b}\big)\Big)-\big(\nabla^\ell\nabla\mathbf{b}\big)\big(\varphi(\rho)\mathbf{b}\big)\Big\|_{L^2}.
\end{aligned}\tag{2.7}
$$

To estimate the second factor on the right-hand side of (2.7), we can use the standard commutator estimates and Lemma 5.1 to obtain

$$
\begin{aligned}
&\Big\|\nabla^\ell\Big((\nabla\mathbf{b})\big(\varphi(\rho)\mathbf{b}\big)\Big)-\big(\nabla^\ell\nabla\mathbf{b}\big)\big(\varphi(\rho)\mathbf{b}\big)\Big\|_{L^2}\\
&\le C\big\|\nabla\big(\varphi(\rho)\mathbf{b}\big)\big\|_{L^3}\|\nabla^{\ell-1}(\nabla\mathbf{b})\|_{L^6}+C\|\nabla\mathbf{b}\|_{L^3}\big\|\nabla^\ell\big(\varphi(\rho)\mathbf{b}\big)\big\|_{L^6}\\
&\le C\Big(\|\nabla\varphi(\rho)\|_{L^3}\|\mathbf{b}\|_{L^\infty}+\|\varphi(\rho)\|_{L^\infty}\|\nabla\mathbf{b}\|_{L^3}\Big)\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}+C\|\nabla\mathbf{b}\|_{L^3}\big\|\nabla^{\ell+1}\big(\varphi(\rho)\mathbf{b}\big)\big\|_{L^2}\\
&\le C\Big(\|\nabla\rho\|_{L^3}\|\mathbf{b}\|_{L^\infty}+\|\nabla\mathbf{b}\|_{L^3}\Big)\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}+C\|\nabla\mathbf{b}\|_{L^2}^{\frac12}\|\nabla^2\mathbf{b}\|_{L^2}^{\frac12}\big\|\nabla^{\ell+1}\big(\varphi(\rho)\mathbf{b}\big)\big\|_{L^2}\\
&\le C(\|\rho\|_{H^2}+1)\|\nabla\mathbf{b}\|_{L^2}^{\frac12}\|\nabla^2\mathbf{b}\|_{L^2}^{\frac12}\Big(\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}+\big\|\nabla^{\ell+1}\big(\varphi(\rho)\mathbf{b}\big)\big\|_{L^2}\Big)
\end{aligned}\tag{2.8}
$$

It remains to estimate the last term $\|\nabla^{\ell+1}\big(\varphi(\rho)b\big)\|_{L^2}$. For this purpose, we can use the product estimates, the boundedness of φ, Lemma 5.1 and the interpolation to get

$$
\begin{aligned}
\big\|\nabla^{\ell+1}\big(\varphi(\rho)\mathbf{b}\big)\big\|_{L^2}&\le C\big\|\varphi(\rho)\big\|_{L^\infty}\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}+C\|\mathbf{b}\|_{L^\infty}\big\|\nabla^{\ell+1}\big(\varphi(\rho)\big)\big\|_{L^2}\\
&\le C\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}+C\|\mathbf{b}\|_{H^2}\|\nabla^{\ell+1}\rho\|_{L^2}.
\end{aligned}\tag{2.9}
$$

Then substituting (2.8) and (2.9) into (2.7), we have

$$
\begin{aligned}
\mathcal{I}_1&\le C(\|\rho\|_{H^2}+1)(\|\mathbf{b}\|_{H^2}+1)\|\nabla\mathbf{b}\|_{L^2}^{\frac12}\|\nabla^2\mathbf{b}\|_{L^2}^{\frac12}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{b}\|_{H^2}^2\Big)\\
&\le CM_0^{\frac52}\varepsilon_0^{\frac12}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2\Big).
\end{aligned}\tag{2.10}
$$

For the term $\mathcal{I}_2$, we first use the integration by parts and Hölder's inequality to obtain

$$\mathcal{I}_2 = a\gamma \int \nabla^{\ell+1}\rho \cdot \nabla^{\ell}(\rho\mathbf{u}) \leq C\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla^{\ell}(\rho\mathbf{u})\|_{L^2}. \tag{2.11}$$

Notice that by the product estimates and Sobolev embedding, we have

$$\begin{aligned}\|\nabla^{\ell}(\rho\mathbf{u})\|_{L^2} &\leq C\Big(\|\nabla^{\ell}\rho\|_{L^6}\|\mathbf{u}\|_{L^3} + \|\nabla^{\ell}\mathbf{u}\|_{L^6}\|\rho\|_{L^3}\Big)\\ &\leq C\Big(\|\nabla^{\ell+1}\rho\|_{L^2}\|\mathbf{u}\|_{H^1} + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\|\rho\|_{H^1}\Big),\end{aligned}$$

which together with (2.11) gives that

$$\begin{aligned}\mathcal{I}_2 &\leq C\Big(\|\rho\|_{H^1} + \|\mathbf{u}\|_{H^1}\Big)\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2\Big)\\ &\leq C\varepsilon_0\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2\Big). \end{aligned}\tag{2.12}$$

Similarly, for the term $\mathcal{I}_3$, we have

$$\begin{aligned}\mathcal{I}_3 &= \int \Big(\nabla^{\ell+1}\mathbf{u}\cdot\nabla^{\ell-1}(\mathbf{u}\cdot\nabla\mathbf{u}) + \nabla^{\ell+1}\mathbf{b}\cdot\nabla^{\ell-1}\big(\mathbf{u}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{u} + \mathbf{b}\nabla\cdot\mathbf{u}\big)\Big)\\ &\leq C\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\|\nabla^{\ell-1}(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2}\\ &\quad +C\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}\Big(\|\nabla^{\ell-1}(\mathbf{u}\cdot\nabla\mathbf{b})\|_{L^2}\\ &\quad +\|\nabla^{\ell-1}(\mathbf{b}\cdot\nabla\mathbf{u})\|_{L^2} + \|\nabla^{\ell-1}(\mathbf{b}\nabla\cdot\mathbf{u})\|_{L^2}\Big). \end{aligned}\tag{2.13}$$

It follows from the product estimates and the interpolation that

$$\begin{aligned}\|\nabla^{\ell-1}(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2} &\leq C\|\nabla^{\ell-1}\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3} + C\|\mathbf{u}\|_{L^3}\|\nabla^{\ell}\mathbf{u}\|_{L^6}\\ &\leq C\|\nabla^{\ell}\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^3} + C\|\mathbf{u}\|_{H^1}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\\ &\leq C\|\mathbf{u}\|_{L^2}^{\frac{1}{\ell+1}}\|\nabla^{\frac{\ell+1}{2\ell}}\mathbf{u}\|_{L^2}^{\frac{\ell}{\ell+1}}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2} + C\|\mathbf{u}\|_{H^1}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\\ &\leq C\|\mathbf{u}\|_{H^1}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}. \end{aligned}\tag{2.14}$$

Similarly, we can deduce that

$$\begin{aligned}&\|\nabla^{\ell-1}(\mathbf{u}\cdot\nabla\mathbf{b})\|_{L^2} + \|\nabla^{\ell-1}(\mathbf{b}\cdot\nabla\mathbf{u})\|_{L^2} + \|\nabla^{\ell-1}(\mathbf{b}\nabla\cdot\mathbf{u})\|_{L^2}\\ &\leq C\Big(\|\nabla^{\ell}\mathbf{u}\|_{L^2}\|\nabla\mathbf{b}\|_{L^3}+\|\mathbf{u}\|_{H^1}\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}+\|\nabla^{\ell}\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^3}+\|\mathbf{b}\|_{H^1}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\Big)\\ &\leq C\Big(\|\mathbf{u}\|_{H^1} + \|\mathbf{b}\|_{H^1}\Big)\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2} + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}\Big). \end{aligned}\tag{2.15}$$

Substituting (2.14) and (2.15) into (2.13), we obtain

$$\begin{aligned}\mathcal{I}_3 &\leq C\Big(\|\mathbf{u}\|_{H^1} + \|\mathbf{b}\|_{H^1}\Big)\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2\Big)\\ &\leq C\varepsilon_0\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2\Big). \end{aligned}\tag{2.16}$$

For the term $\mathcal{I}_4$, it follows from the integration by parts and Hölder's inequality that

$$\mathcal{I}_4 = \int \nabla^{\ell+1}\mathbf{u}\cdot\nabla^{\ell-1}\big(h(\rho)\nabla\rho\big) \le \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\|\nabla^{\ell-1}\big(h(\rho)\nabla\rho\big)\|_{L^2}.$$

Noticing that

$$\begin{aligned}\|\nabla^{\ell-1}\big(h(\rho)\nabla\rho\big)\|_{L^2} &\le C\|\nabla^{\ell-1}\big(h(\rho)\big)\|_{L^6}\|\nabla\rho\|_{L^3} + C\|h(\rho)\|_{L^3}\|\nabla^{\ell}\rho\|_{L^6}\\ &\le C\|\nabla^{\ell}\big(h(\rho)\big)\|_{L^2}\|\nabla\rho\|_{L^3} + C\|h(\rho)\|_{L^3}\|\nabla^{\ell+1}\rho\|_{L^2}\\ &\le C\|\nabla^{\ell}\rho\|_{L^2}\|\nabla\rho\|_{L^3} + C\|\rho\|_{L^3}\|\nabla^{\ell+1}\rho\|_{L^2}\\ &\le C\|\rho\|_{L^2}^{1-\frac{\ell}{\ell+1}}\|\nabla^{\frac{\ell+1}{2\ell}}\rho\|_{L^2}^{\frac{\ell}{\ell+1}}\|\nabla^{\ell+1}\rho\|_{L^2} + C\|\rho\|_{L^3}\|\nabla^{\ell+1}\rho\|_{L^2}\\ &\le C\|\rho\|_{H^1}\|\nabla^{\ell+1}\rho\|_{L^2}.\end{aligned}$$

we have

$$\mathcal{I}_4 \le C\|\rho\|_{H^1}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2\Big) \le C\varepsilon_0\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2\Big). \tag{2.17}$$

For $\mathcal{I}_5$, it follows from the integration by parts and Hölder's inequality that

$$\begin{aligned}\mathcal{I}_5 &= \int \nabla^{\ell+1}\mathbf{u}\cdot\nabla^{\ell-1}\Big(g(\rho)\big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla(\nabla\cdot\mathbf{u})\big)\Big)\\ &\le C\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\|\nabla^{\ell-1}\big(g(\rho)\nabla^2\mathbf{u}\big)\|_{L^2}.\end{aligned} \tag{2.18}$$

By the product estimates, Lemma 5.1 and the interpolation, we have

$$\begin{aligned}\|\nabla^{\ell-1}\big(g(\rho)\nabla^2\mathbf{u}\big)\|_{L^2} &\le C\|\nabla^{\ell-1}g(\rho)\|_{L^6}\|\nabla^2\mathbf{u}\|_{L^3} + C\|g(\rho)\|_{L^\infty}\|\nabla^{\ell-1}\nabla^2\mathbf{u}\|_{L^2}\\ &\le C\|\nabla^{\ell}g(\rho)\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^3} + C\|\rho\|_{L^\infty}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\\ &\le C\|\nabla^{\ell}\rho\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^3} + C\|\nabla\rho\|_{L^2}^{\frac12}\|\nabla^2\rho\|_{L^2}^{\frac12}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\\ &\le C\|\rho\|_{L^2}^{1-\frac{\ell}{\ell+1}}\|\nabla^{\ell+1}\rho\|_{L^2}^{\frac{\ell}{\ell+1}}\|\mathbf{u}\|_{L^2}^{\frac{\ell}{\ell+1}}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^{1-\frac{\ell}{\ell+1}}\\ &\quad +C\|\nabla\rho\|_{L^2}^{\frac12}\|\nabla^2\rho\|_{L^2}^{\frac12}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\\ &\le C\Big(\|\rho\|_{L^2} + \|\mathbf{u}\|_{L^2} + \|\nabla\rho\|_{L^2}^{\frac12}\|\nabla^2\rho\|_{L^2}^{\frac12}\Big)\Big(\|\nabla^{\ell+1}\rho\|_{L^2}\\ &\quad +\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\Big).\end{aligned} \tag{2.19}$$

Substituting (2.19) into (2.18), we obtain

$$\mathcal{I}_5 \le CM_0^{\frac12}\varepsilon_0^{\frac12}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2\Big). \tag{2.20}$$

Finally, we investigate the term $\mathcal{I}_6$. We first notice $\varphi(\rho) = 1 - g(\rho)$ and thus have

$$\begin{aligned}\mathcal{I}_6 &= \int \nabla^\ell \mathbf{u} \cdot \nabla^\ell \Big((\mathbf{b}\cdot\nabla\mathbf{b}) - \frac{1}{2}\nabla|\mathbf{b}|^2 \Big) \\ &\quad - \int \nabla^\ell \mathbf{u} \cdot \nabla^\ell \Big(g(\rho)(\mathbf{b}\cdot\nabla\mathbf{b}) - \frac{1}{2} g(\rho)\nabla|\mathbf{b}|^2 \Big) \\ &:= \mathcal{I}_{61} + \mathcal{I}_{62}. \end{aligned} \tag{2.21}$$

For $\mathcal{I}_{61}$, we can use similar procedure as $\mathcal{I}_3$ to obtain

$$\mathcal{I}_{61} \le C\|\mathbf{b}\|_{H^1}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\|\nabla^{\ell+1}\mathbf{b}\|_{L^2} \le C\varepsilon_0\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2\Big). \tag{2.22}$$

For $\mathcal{I}_{62}$, by the divergence free of $\mathbf{b}$ and the product estimates, we have

$$\begin{aligned}\mathcal{I}_{62} &= \int \nabla^{\ell+1}\mathbf{u} \cdot \nabla^{\ell-1}\Big(g(\rho)(\nabla(\mathbf{b}\otimes\mathbf{b}) - \frac{1}{2}\nabla|\mathbf{b}|^2) \Big) \\ &\le C\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\Big(\|\nabla^{\ell-1}(g(\rho))\|_{L^6}\|\nabla(\mathbf{bb})\|_{L^3} + \|g(\rho)\|_{L^\infty}\|\nabla^\ell(\mathbf{bb})\|_{L^2} \Big).\end{aligned}$$

The by Lemma 5.1 and the product estimates again, we obtain

$$\begin{aligned}\mathcal{I}_{62} &\le C\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\Big(\|\nabla^\ell\rho\|_{L^2}\|\mathbf{b}\nabla\mathbf{b}\|_{L^3} + \|\rho\|_{L^\infty}\|\nabla^\ell(\mathbf{bb})\|_{L^2} \Big) \\ &\le C\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\Big(\|\nabla^\ell\rho\|_{L^2}\|\mathbf{b}\|_{L^6}\|\nabla\mathbf{b}\|_{L^6} + \|\rho\|_{L^\infty}\|\nabla^\ell\mathbf{b}\|_{L^6}\|\mathbf{b}\|_{L^3} \Big) \\ &\le C\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\Big(\|\rho\|_{L^2}^{1-\frac{\ell}{\ell+1}}\|\nabla^{\ell+1}\rho\|_{L^2}^{\frac{\ell}{\ell+1}}\|\mathbf{b}\|_{L^2}^{1-\frac{1}{\ell+1}}\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^{\frac{1}{\ell+1}}\|\nabla^2\mathbf{b}\|_{L^2} \\ &\quad + \|\rho\|_{H^2}\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}\|\mathbf{b}\|_{H^1} \Big) \\ &\le C\Big(\|\rho\|_{H^1} + \|\mathbf{b}\|_{H^1}\Big)\Big(\|\rho\|_{H^2} + \|\mathbf{b}\|_{H^2}\Big)\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 \\ &\quad + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2 \Big).\end{aligned} \tag{2.23}$$

Substituting (2.22) and (2.23) into (2.21), we obtain

$$\mathcal{I}_6 \le CM_0\varepsilon_0\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2 \Big). \tag{2.24}$$

Summarily, substituting (2.10), (2.12), (2.16), (2.17), (2.20) and (2.24) into (2.6), we conclude that there exists a positive constant c_0 such that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\nabla^\ell\rho(t)\|_{L^2}^2 + \|\nabla^\ell\mathbf{u}(t)\|_{L^2}^2 + \|\nabla^\ell\mathbf{b}(t)\|_{L^2}^2 \Big) \\ &\quad + c_0\Big(\|\nabla^{\ell+1}\mathbf{u}(t)\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}(t)\|_{L^2}^2 \Big) \\ &\le CM_0^{\frac{5}{2}}\varepsilon_0^{\frac{1}{2}}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 \\ &\quad + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2 \Big).\end{aligned}$$

This completes the proof of Lemma 2.1. □

LEMMA 2.2. *Let* $0 \le \ell \le k-1$ *and* (2.4)–(2.5) *hold. Then there exist two positive constants* c_0 *and* C *such that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(a\gamma\|\nabla^{\ell+1}\rho(t)\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}(t)\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}(t)\|_{L^2}^2\Big)\\
&\quad + c_0\Big(\|\nabla^{\ell+2}\mathbf{u}(t)\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{b}(t)\|_{L^2}^2\Big)\\
&\le CM_0^{\frac{7}{4}}\varepsilon_0^{\frac{1}{4}}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big).
\end{aligned}$$

Proof. Similar to (2.6), we can deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\nabla^{\ell+1}\rho(t)\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}(t)\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}(t)\|_{L^2}^2\Big)\\
&\qquad+\Big(\mu\|\nabla^{\ell+2}\mathbf{u}(t)\|_{L^2}^2 + (\mu+\nu)\|\nabla^{\ell+1}\nabla\cdot\mathbf{u}(t)\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{b}(t)\|_{L^2}^2\Big)\\
&\quad= -\epsilon\int \nabla^{\ell+1}(\nabla\times\mathbf{b})\cdot\nabla^{\ell+1}\Big((\nabla\times\mathbf{b})\times\big(\varphi(\rho)\mathbf{b}\big)\Big)\\
&\qquad-a\gamma\int\nabla^{\ell+1}\rho\nabla^{\ell+1}\big(\rho\nabla\cdot\mathbf{u}\big) - a\gamma\int\nabla^{\ell+1}\rho\nabla^{\ell+1}\big(\mathbf{u}\cdot\nabla\rho\big)\\
&\qquad-\int\Big(\nabla^{\ell+1}\mathbf{u}\cdot\nabla^{\ell+1}(\mathbf{u}\cdot\nabla\mathbf{u}) + \nabla^{\ell+1}\mathbf{b}\cdot\nabla^{\ell+1}(\mathbf{u}\cdot\nabla\mathbf{b}-\mathbf{b}\cdot\nabla\mathbf{u}+\mathbf{b}\nabla\cdot\mathbf{u})\Big)\\
&\qquad-\int\nabla^{\ell+1}\mathbf{u}\cdot\nabla^{\ell+1}\Big(g(\rho)\big(\mu\Delta\mathbf{u}+(\mu+\nu)\nabla(\nabla\cdot\mathbf{u})\big)\Big)\\
&\qquad+\int\nabla^{\ell+1}\mathbf{u}\cdot\nabla^{\ell+1}\Big(\varphi(\rho)\big(\mathbf{b}\cdot\nabla\mathbf{b}-\frac{1}{2}\nabla|\mathbf{b}|^2\big)\Big) - \int\nabla^{\ell+1}\mathbf{u}\cdot\nabla^{\ell+1}\Big(h(\rho)\nabla\rho\Big)\\
&\quad:= \mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4+\mathcal{J}_5+\mathcal{J}_6+\mathcal{J}_7.
\end{aligned}\tag{2.25}$$

The estimates of $\mathcal{J}_1, \ldots, \mathcal{J}_7$ are similar to those of $\mathcal{I}_1, \ldots, \mathcal{I}_6$, but we need to be more careful since the higher-order estimates are involved. Firstly, for the Hall term $\mathcal{J}_1$, a direct calculation yields that

$$\begin{aligned}
\mathcal{J}_1 &= -\epsilon\int\nabla^{\ell+1}(\nabla\times\mathbf{b})\cdot\Big(\nabla^{\ell+1}\Big((\nabla\times\mathbf{b})\times\big(\varphi(\rho)\mathbf{b}\big)\Big)\\
&\qquad-\Big(\nabla^{\ell+1}(\nabla\times\mathbf{b})\Big)\times\big(\varphi(\rho)\mathbf{b}\big)\Big)\\
&\le C\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}\Big\|\nabla^{\ell+1}\Big((\nabla\mathbf{b})\big(\varphi(\rho)\mathbf{b}\big)\Big) - \big(\nabla^{\ell+1}\nabla\mathbf{b}\big)\big(\varphi(\rho)\mathbf{b}\big)\Big\|_{L^2}.
\end{aligned}\tag{2.26}$$

By the standard commutator estimates, Sobolev embedding and Lemma 5.1, we have

$$\begin{aligned}
&\Big\|\nabla^{\ell+1}\Big((\nabla\mathbf{b})\big(\varphi(\rho)\mathbf{b}\big)\Big) - \big(\nabla^{\ell+1}\nabla\mathbf{b}\big)\big(\varphi(\rho)\mathbf{b}\big)\Big\|_{L^2}\\
&\le C\big\|\nabla\big(\varphi(\rho)\mathbf{b}\big)\big\|_{L^3}\|\nabla^{\ell}(\nabla\mathbf{b})\|_{L^6} + C\|\nabla\mathbf{b}\|_{L^\infty}\big\|\nabla^{\ell+1}\big(\varphi(\rho)\mathbf{b}\big)\big\|_{L^2}\\
&\le C\Big(\|\nabla\varphi(\rho)\|_{L^3}\|\mathbf{b}\|_{L^\infty} + \|\varphi(\rho)\|_{L^\infty}\|\nabla\mathbf{b}\|_{L^3}\Big)\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}\\
&\quad+C\|\nabla\mathbf{b}\|_{L^\infty}\big\|\nabla^{\ell+1}\big(\varphi(\rho)\mathbf{b}\big)\big\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\le C\Big(\|\nabla\rho\|_{L^3}\|\mathbf{b}\|_{L^\infty}+\|\nabla\mathbf{b}\|_{L^3}\Big)\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}\\
&\quad +C\|\nabla\mathbf{b}\|_{L^\infty}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}\|\mathbf{b}\|_{L^\infty}+\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}\Big)\\
&\le C(\|\rho\|_{H^2}+1)\|\nabla\mathbf{b}\|_{L^2}^{\frac12}\|\nabla^2\mathbf{b}\|_{L^2}^{\frac12}\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}\\
&\quad +C\|\nabla\mathbf{b}\|_{L^2}^{\frac14}\|\nabla^3\mathbf{b}\|_{L^2}^{\frac34}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}\|\mathbf{b}\|_{H^2}+\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}\Big),
\end{aligned}
$$

which together with (2.26) yields that

$$
\mathcal{J}_1 \le CM_0^{\frac74}\varepsilon_0^{\frac14}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big). \tag{2.27}
$$

For the term $\mathcal{J}_2$, it follows from Hölder's inequality, the product estimates and Sobolev embedding that

$$
\begin{aligned}
\mathcal{J}_2 &\le C\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla^{\ell+1}(\rho\nabla\mathbf{u})\|_{L^2}\\
&\le C\|\nabla^{\ell+1}\rho\|_{L^2}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla\mathbf{u})\|_{L^\infty}+\|\rho\|_{L^\infty}\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\Big)\\
&\le C\|\nabla^{\ell+1}\rho\|_{L^2}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{\frac14}\|\nabla^3\mathbf{u}\|_{L^2}^{\frac34}+\|\nabla\rho\|_{L^2}^{\frac12}\|\nabla^2\rho\|_{L^2}^{\frac12}\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\Big)\\
&\le CM_0^{\frac34}\varepsilon_0^{\frac14}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2\Big).
\end{aligned} \tag{2.28}
$$

To estimate the term $\mathcal{J}_3$, we rewrite it as

$$
\mathcal{J}_3 = -a\gamma\int \nabla^{\ell+1}\rho\Big(\nabla^{\ell+1}\big(\mathbf{u}\cdot\nabla\rho\big)-\mathbf{u}\cdot\nabla\nabla^{\ell+1}\rho\big)\Big)+\frac{a\gamma}{2}\int|\nabla^{\ell+1}\rho|^2\nabla\cdot\mathbf{u}.
$$

Then by Hölder's inequality and the commutator estimates, we have

$$
\begin{aligned}
\mathcal{J}_3 &\le C\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla^{\ell+1}\big(\mathbf{u}\cdot\nabla\rho\big)-\mathbf{u}\cdot\nabla\nabla^{\ell+1}\rho\|_{L^2}+C\|\nabla^{\ell+1}\rho\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^\infty}\\
&\le C\|\nabla^{\ell+1}\rho\|_{L^2}\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\|\nabla\rho\|_{L^\infty}+\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^{\ell+1}\rho\|_{L^2}\Big)\\
&\quad +C\|\nabla^{\ell+1}\rho\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^\infty}\\
&\le C\Big(\|\nabla\rho\|_{L^\infty}+\|\nabla\mathbf{u}\|_{L^\infty}\Big)\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+1}\rho\|_{L^2}^2\Big).
\end{aligned}
$$

Thus by Sobolev embedding, we obtain

$$
\begin{aligned}
\mathcal{J}_3 &\le C\Big(\|\nabla\rho\|_{L^2}^{\frac14}\|\nabla^3\rho\|_{L^2}^{\frac34}+\|\nabla\mathbf{u}\|_{L^2}^{\frac14}\|\nabla^3\mathbf{u}\|_{L^2}^{\frac34}\Big)\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+1}\rho\|_{L^2}^2\Big)\\
&\le CM_0^{\frac34}\varepsilon_0^{\frac14}\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+1}\rho\|_{L^2}^2\Big).
\end{aligned} \tag{2.29}
$$

We now turn to the term $\mathcal{J}_4$. we have

$$
\begin{aligned}
\mathcal{J}_4 &= \int\Big(\nabla^{\ell+2}\mathbf{u}\cdot\nabla^\ell(\mathbf{u}\cdot\nabla\mathbf{u})+\nabla^{\ell+2}\mathbf{b}\cdot\nabla^\ell\big(\mathbf{u}\cdot\nabla\mathbf{b}-\mathbf{b}\cdot\nabla\mathbf{u}+\mathbf{b}\nabla\cdot\mathbf{u}\big)\Big)\\
&\le C\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\|\nabla^\ell(\mathbf{u}\nabla\mathbf{u})\|_{L^2}+C\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}\Big(\|\nabla^\ell(\mathbf{u}\nabla\mathbf{b})\|_{L^2}+\|\nabla^\ell(\mathbf{b}\nabla\mathbf{u})\|_{L^2}\Big).
\end{aligned} \tag{2.30}
$$

It follows from the product estimates and the interpolation that

$$\begin{aligned}\|\nabla^\ell(\mathbf{u}\nabla\mathbf{u})\|_{L^2} &\le C\|\nabla^\ell\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3} + C\|\mathbf{u}\|_{L^3}\|\nabla^{\ell+1}\mathbf{u}\|_{L^6}\\ &\le C\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{\frac12}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac12} + C\|\mathbf{u}\|_{H^1}\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\\ &\le CM_0^{\frac12}\varepsilon_0^{\frac12}\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2} + \|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\Big).\end{aligned} \tag{2.31}$$

Similarly, we can deduce that

$$\begin{aligned}&\|\nabla^\ell(\mathbf{u}\nabla\mathbf{b})\|_{L^2} + \|\nabla^\ell(\mathbf{b}\nabla\mathbf{u})\|_{L^2}\\ &\quad\le C\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\|\nabla\mathbf{b}\|_{L^3} + \|\mathbf{u}\|_{H^1}\|\nabla^{\ell+2}\mathbf{b}\|_{L^2} + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}\|\nabla\mathbf{u}\|_{L^3}\\ &\qquad + \|\mathbf{b}\|_{H^1}\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\Big)\\ &\quad\le CM_0^{\frac12}\varepsilon_0^{\frac12}\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2} + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2} + \|\nabla^{\ell+2}\mathbf{u}\|_{L^2} + \|\nabla^{\ell+2}\mathbf{b}\|_{L^2}\Big).\end{aligned} \tag{2.32}$$

Substituting (2.31) and (2.32) into (2.30), we obtain

$$\mathcal{J}_4 \le CM_0^{\frac12}\varepsilon_0^{\frac12}\Big(\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big). \tag{2.33}$$

For the term $\mathcal{I}_5$, it follows from the integration by parts and Hölder's inequality that

$$\mathcal{J}_5 = \int \nabla^{\ell+2}\mathbf{u}\cdot\nabla^\ell\Big(g(\rho)\big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla(\nabla\cdot\mathbf{u})\big)\Big) \le C\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\|\nabla^\ell\big(g(\rho)\nabla^2\mathbf{u}\big)\|_{L^2}. \tag{2.34}$$

By the product estimates, Lemma 5.1 and the interpolation, we have

$$\begin{aligned}\|\nabla^\ell\big(g(\rho)\nabla^2\mathbf{u}\big)\|_{L^2} &\le C\|\nabla^\ell g(\rho)\|_{L^6}\|\nabla^2\mathbf{u}\|_{L^3} + C\|g(\rho)\|_{L^\infty}\|\nabla^\ell\nabla^2\mathbf{u}\|_{L^2}\\ &\le C\|\nabla^{\ell+1}g(\rho)\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{\frac14}\|\nabla^3\mathbf{u}\|_{L^2}^{\frac34} + C\|\rho\|_{L^\infty}\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\\ &\le C\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{\frac14}\|\nabla^3\mathbf{u}\|_{L^2}^{\frac34} + C\|\nabla\rho\|_{L^2}^{\frac12}\|\nabla^2\rho\|_{L^2}^{\frac12}\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}.\end{aligned}$$

Thus by (2.34), we obtain

$$\mathcal{J}_5 \le CM_0^{\frac34}\varepsilon_0^{\frac14}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2\Big). \tag{2.35}$$

For the term $\mathcal{J}_6$, we use $\varphi(\rho) = 1 - g(\rho)$ to rewrite it as

$$\begin{aligned}\mathcal{J}_6 &= \int \nabla^{\ell+1}\mathbf{u}\cdot\nabla^{\ell+1}\Big((\mathbf{b}\cdot\nabla\mathbf{b}) - \frac12\nabla|\mathbf{b}|^2\Big)\\ &\quad - \int \nabla^{\ell+1}\mathbf{u}\cdot\nabla^{\ell+1}\Big(g(\rho)(\mathbf{b}\cdot\nabla\mathbf{b}) - \frac12 g(\rho)\nabla|\mathbf{b}|^2\Big)\\ &:= \mathcal{J}_{61} + \mathcal{J}_{62}.\end{aligned} \tag{2.36}$$

Similar to $\mathcal{J}_4$, we have

$$\mathcal{J}_{61} \le CM_0^{\frac12}\varepsilon_0^{\frac12}\Big(\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big). \tag{2.37}$$

For the term $\mathcal{J}_{62}$, it follows from the divergence free of $\mathbf{b}$ and the product estimates that

$$\begin{aligned}\mathcal{J}_{62} &= \int \nabla^{\ell+2}\mathbf{u}\cdot\nabla^{\ell}\Big(g(\rho)(\nabla(\mathbf{b}\otimes\mathbf{b})-\frac{1}{2}\nabla|\mathbf{b}|^2)\Big)\\ &\le C\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\Big(\|\nabla^{\ell}(g(\rho))\|_{L^6}\|\nabla(\mathbf{bb})\|_{L^3}+\|g(\rho)\|_{L^\infty}\|\nabla^{\ell+1}(\mathbf{bb})\|_{L^2}\Big),\end{aligned}$$

which together with Lemma 5.1 and the product estimates again gives that

$$\begin{aligned}\mathcal{J}_{62} &\le C\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}\|\mathbf{b}\nabla\mathbf{b}\|_{L^3}+\|\rho\|_{L^\infty}\|\nabla^{\ell+1}(\mathbf{bb})\|_{L^2}\Big)\\ &\le C\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}\|\mathbf{b}\|_{L^6}\|\nabla\mathbf{b}\|_{L^6}+\|\rho\|_{L^\infty}\|\nabla^{\ell+1}\mathbf{b}\|_{L^6}\|\mathbf{b}\|_{L^3}\Big)\\ &\le C\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla\mathbf{b}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}+\|\rho\|_{H^2}\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}\|\mathbf{b}\|_{H^1}\Big)\\ &\le CM_0\varepsilon_0\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big).\end{aligned}\tag{2.38}$$

Substituting (2.37) and (2.38) into (2.36), we obtain

$$\mathcal{J}_6 \le CM_0\varepsilon_0\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big).\tag{2.39}$$

Finally, for the term $\mathcal{J}_7$, we first use the integration by parts and Hölder's inequality to obtain

$$\mathcal{J}_7 = \int \nabla^{\ell+2}\mathbf{u}\cdot\nabla^{\ell}\big(h(\rho)\nabla\rho\big) \le \|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\|\nabla^{\ell}\big(h(\rho)\nabla\rho\big)\|_{L^2}.\tag{2.40}$$

Similar to $\mathcal{J}_{62}$, we have

$$\begin{aligned}\|\nabla^{\ell}\big(h(\rho)\nabla\rho\big)\|_{L^2} &\le C\|\nabla^{\ell}\big(h(\rho)\big)\|_{L^6}\|\nabla\rho\|_{L^3}+C\|h(\rho)\|_{L^\infty}\|\nabla^{\ell+1}\rho\|_{L^2}\\ &\le C\|\nabla^{\ell+1}\big(h(\rho)\big)\|_{L^2}\|\nabla\rho\|_{L^2}^{\frac{1}{2}}\|\nabla^2\rho\|_{L^2}^{\frac{1}{2}}+C\|\rho\|_{L^\infty}\|\nabla^{\ell+1}\rho\|_{L^2}\\ &\le C\|\nabla\rho\|_{L^2}^{\frac{1}{2}}\|\nabla^2\rho\|_{L^2}^{\frac{1}{2}}\|\nabla^{\ell+1}\rho\|_{L^2}.\end{aligned}\tag{2.41}$$

Substituting (2.41) into (2.40), we obtain

$$\mathcal{J}_7 \le CM_0^{\frac{1}{2}}\varepsilon_0^{\frac{1}{2}}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2\Big).\tag{2.42}$$

Summarily, substituting (2.27), (2.28), (2.29), (2.33), (2.35), (2.39) and (2.42) into (2.25), we conclude that there exists a constant c_0 such that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\nabla^{\ell+1}\rho(t)\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{u}(t)\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{b}(t)\|_{L^2}^2\Big)\\ &\quad+c_0\Big(\|\nabla^{\ell+2}\mathbf{u}(t)\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{b}(t)\|_{L^2}^2\Big)\\ &\le CM_0^{\frac{7}{4}}\varepsilon_0^{\frac{1}{4}}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2+\|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big).\end{aligned}$$

This completes the proof of Lemma 2.2. □

LEMMA 2.3. *Let* $0 \le \ell \le k-1$ *and* (2.4)–(2.5) *hold. Then there exist two positive constants* C_0 *and* C *such that*

$$\frac{d}{dt}\int \nabla^\ell \nabla\rho \cdot \nabla^\ell \mathbf{u} + a\gamma\|\nabla^{\ell+1}\rho\|_{L^2}^2 - \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 - C_0\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}$$
$$\le CM_0\varepsilon_0^{\frac{1}{4}}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big). \tag{2.43}$$

Proof. Applying ∇^ℓ to Eq. $(2.1)_2$ and taking the L^2 inner product with $\nabla^\ell\nabla\rho$, we have

$$\int \nabla^\ell\nabla\rho\cdot\nabla^\ell\partial_t\mathbf{u} + a\gamma\|\nabla^\ell\nabla\rho\|_{L^2}^2 - \int \nabla^\ell\nabla\rho\cdot\nabla^\ell\Big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla(\nabla\cdot\mathbf{u})\Big)$$
$$= -\int \nabla^\ell\nabla\rho\cdot\nabla^\ell(\mathbf{u}\cdot\nabla\mathbf{u}) - \int \nabla^\ell\nabla\rho\cdot\nabla^\ell\Big(g(\rho)\big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla(\nabla\cdot\mathbf{u})\big)\Big)$$
$$+ \int \nabla^\ell\nabla\rho\cdot\nabla^\ell\Big(\varphi(\rho)(\mathbf{b}\cdot\nabla\mathbf{b} - \frac{1}{2}\nabla|\mathbf{b}|^2)\Big) - \int \nabla^\ell\nabla\rho\cdot\nabla^\ell\Big(h(\rho)\nabla\rho\Big).$$

Similarly, applying $\nabla^{\ell+1}$ to Eq. $(2.1)_1$ and taking the L^2 inner product with $\nabla^\ell\mathbf{u}$, we have

$$\int \nabla^\ell\nabla\partial_t\rho\cdot\nabla^\ell\mathbf{u} + \int \nabla^\ell\mathbf{u}\cdot\nabla^{\ell+1}\nabla\cdot\mathbf{u} = -\int \nabla^\ell\mathbf{u}\cdot\nabla^{\ell+1}\nabla\cdot(\rho\mathbf{u}).$$

By the above two equalities and the integration by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \nabla^\ell\nabla\rho\cdot\nabla^\ell\mathbf{u} + a\gamma\|\nabla^{\ell+1}\rho\|_{L^2}^2 - \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2$$
$$- \int \nabla^{\ell+1}\rho\cdot\nabla^\ell\Big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla(\nabla\cdot\mathbf{u})\Big)$$
$$= \int \nabla^{\ell+1}\rho\cdot\nabla^\ell(\mathbf{u}\cdot\nabla\mathbf{u}) - \int \nabla^{\ell+1}\rho\cdot\nabla^\ell\Big(g(\rho)\big(\mu\Delta\mathbf{u} + (\mu+\nu)\nabla(\nabla\cdot\mathbf{u})\big)\Big)$$
$$+ \int \nabla^{\ell+1}\rho\cdot\nabla^\ell\Big(\varphi(\rho)(\mathbf{b}\cdot\nabla\mathbf{b}$$
$$- \frac{1}{2}\nabla|\mathbf{b}|^2)\Big) - \int \nabla^{\ell+1}\rho\cdot\nabla^\ell\Big(h(\rho)\nabla\rho\Big) + \int \nabla^{\ell+1}\mathbf{u}\cdot\nabla^{\ell+1}(\rho\mathbf{u}).$$

It then follows from Hölder's inequality that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \nabla^\ell\nabla\rho\cdot\nabla^\ell\mathbf{u} + a\gamma\|\nabla^{\ell+1}\rho\|_{L^2}^2 - \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 - C_0\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}$$
$$\le C\|\nabla^{\ell+1}\rho\|_{L^2}\Big(\|\nabla^\ell(\mathbf{u}\nabla\mathbf{u})\|_{L^2} + \|\nabla^\ell(g(\rho)\nabla^2\mathbf{u})\|_{L^2} + \|\nabla^\ell(\varphi(\rho)(\mathbf{b}\nabla\mathbf{b}))\|_{L^2}$$
$$+\|\nabla^\ell(h(\rho)\nabla\rho)\|_{L^2}\Big)$$
$$+\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\|\nabla^{\ell+1}(\rho\mathbf{u})\|_{L^2}, \tag{2.44}$$

where $C_0 = 2\mu + \nu$. Notice that from the estimates of $\mathcal{J}_4$, $\mathcal{J}_5$, $\mathcal{J}_6$ and $\mathcal{J}_7$, we have

$$\begin{aligned}&\|\nabla^\ell(\mathbf{u}\nabla\mathbf{u})\|_{L^2} + \|\nabla^\ell(g(\rho)\nabla^2\mathbf{u})\|_{L^2} + \|\nabla^\ell(\varphi(\rho)(\mathbf{b}\nabla\mathbf{b}))\|_{L^2} + \|\nabla^\ell(h(\rho)\nabla\rho)\|_{L^2}\\&\le CM_0\varepsilon_0^{\frac{1}{4}}\Big(\|\nabla^{\ell+1}\rho\|_{L^2} + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2} + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2} + \|\nabla^{\ell+2}\mathbf{u}\|_{L^2} + \|\nabla^{\ell+2}\mathbf{b}\|_{L^2}\Big).\end{aligned} \tag{2.45}$$

On the other hand, by the product estimates, we have

$$\begin{aligned}\|\nabla^{\ell+1}(\rho\mathbf{u})\|_{L^2} &\le C\|\nabla^{\ell+1}\rho\|_{L^2}\|\mathbf{u}\|_{L^\infty} + \|\rho\|_{L^\infty}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\\&\le C\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{1}{2}} + \|\nabla\rho\|_{L^2}^{\frac{1}{2}}\|\nabla^2\rho\|_{L^2}^{\frac{1}{2}}\|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\\&\le CM_0^{\frac{1}{2}}\varepsilon_0^{\frac{1}{2}}\Big(\|\nabla^{\ell+1}\rho\|_{L^2} + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}\Big).\end{aligned} \tag{2.46}$$

Substituting (2.45) and (2.46) into (2.44), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int \nabla^\ell\nabla\rho\cdot\nabla^\ell\mathbf{u} + a\gamma\|\nabla^{\ell+1}\rho\|_{L^2}^2 - \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 - C_0\|\nabla^{\ell+1}\rho\|_{L^2}\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}\\&\le CM_0\varepsilon_0^{\frac{1}{4}}\Big(\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big).\end{aligned}$$

This completes the proof of Lemma 2.3. □

PROPOSITION 2.1. *Let* $0 \le \ell \le k-1$ *and* (2.4)–(2.5) *hold. For any given* M_0*, if* $\varepsilon_0 > 0$ *is suitable small, then we have*

$$\frac{d}{dt}\mathcal{E}_\ell(t) + \mathcal{F}_\ell(t) \le 0 \qquad \text{for any} \quad t \in [0, T]. \tag{2.47}$$

Proof. By Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\ell(t) + \mathcal{F}_\ell(t) \le CM_0^{\frac{5}{2}}\varepsilon_0^{\frac{1}{4}}\Big(&\|\nabla^{\ell+1}\rho\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+1}\mathbf{b}\|_{L^2}^2\\&+\|\nabla^{\ell+2}\mathbf{u}\|_{L^2}^2 + \|\nabla^{\ell+2}\mathbf{b}\|_{L^2}^2\Big).\end{aligned}$$

Thus if we choose ε_0 such that $CM_0^{\frac{5}{2}}\varepsilon_0^{\frac{1}{4}} \le \frac{1}{2}c_1$, we can use (2.3) to conclude that (2.47) holds. □

Proof of Theorem 1.1. Integrating the inequality (2.47) of Proposition 2.1 from 0 to t, we obtain

$$\mathcal{E}_\ell(t) + \int_0^t \mathcal{F}_\ell(\tau)d\tau \le \mathcal{E}_\ell(0), \qquad t \in [0, T] \tag{2.48}$$

for any $0 \le \ell \le k-1$. Then by taking $\ell = 0$ in (2.48) and using (2.2)–(2.3), we conclude that

$$\begin{aligned}&\Big(\|\rho(t)\|_{H^1}^2+\|\mathbf{u}(t)\|_{H^1}^2+\|\mathbf{b}(t)\|_{H^1}^2\Big)\\&\quad+\int_0^t\Big(\|\nabla\rho(\tau)\|_{L^2}^2+\|\nabla\mathbf{u}(\tau)\|_{H^1}^2+\|\nabla\mathbf{b}(\tau)\|_{H^1}^2\Big)d\tau\\&\le\frac{1}{c_1^2}\Big(\|\rho_0\|_{H^1}^2+\|\mathbf{u}_0\|_{H^1}^2+\|\mathbf{b}_0\|_{H^1}^2\Big)\le\frac{\varepsilon^2}{c_1^2}\quad\text{for any}\quad t\in[0,T],\end{aligned}\tag{2.49}$$

which closes the a priori assumption (2.5) provided that $\varepsilon^2\le c_1^2\varepsilon_0$. On the other hand, by summing up (2.48) from $\ell=1$ to 2 and using (2.2)–(2.3) again, we conclude that

$$\begin{aligned}&\Big(\|\nabla\rho(t)\|_{H^2}^2+\|\nabla\mathbf{u}(t)\|_{H^2}^2+\|\nabla\mathbf{b}(t)\|_{H^2}^2\Big)\\&\quad+\int_0^t\Big(\|\nabla^2\rho(\tau)\|_{H^1}^2+\|\nabla^2\mathbf{u}(\tau)\|_{H^2}^2+\|\nabla^2\mathbf{b}(\tau)\|_{H^2}^2\Big)d\tau\\&\le\frac{2}{c_1^2}\Big(\|\nabla\rho_0\|_{H^2}^2+\|\nabla\mathbf{u}_0\|_{H^2}^2+\|\nabla\mathbf{b}_0\|_{H^2}^2\Big)\le\frac{2M_0^2}{c_1^2}\quad\text{for any}\quad t\in[0,T],\end{aligned}\tag{2.50}$$

which closes the a priori assumption (2.4).

Once we have closed the a priori assumptions (2.4) and (2.5), we can sum the inequality (2.48) from $\ell=0$ to $k-1$ and then complete the proof of Theorem 1.1. □

3. Decay estimates

In this section, we establish the time decay rates for solutions to Eq. (2.1) with the help of the global energy estimates (1.3) and negative index Sobolev space. We first give the following energy-type estimates.

LEMMA 3.1. *There exist two positive constants c_0 and C such that for $s\in\big(0,\frac{1}{2}\big]$, we have*

$$\begin{aligned}&\frac{d}{dt}\Big(\|\Lambda^{-s}\rho\|_{L^2}^2+\|\Lambda^{-s}\mathbf{u}\|_{L^2}^2+\|\Lambda^{-s}\mathbf{b}\|_{L^2}^2\Big)+c_0\Big(\|\Lambda^{-s}\nabla\mathbf{u}\|_{L^2}^2+\|\Lambda^{-s}\nabla\mathbf{b}\|_{L^2}^2\Big)\\&\quad\le C\big\|\big(\nabla\rho,\nabla\mathbf{u},\nabla\mathbf{b}\big)\big\|_{L^2\times H^1\times H^1}^{1+2s}\big\|\big(\nabla\rho,\nabla\mathbf{u},\nabla\mathbf{b}\big)\big\|_{H^1\times H^2\times H^2}^{1-2s}\Big(\|\Lambda^{-s}\rho\|_{L^2}\\&\qquad+\|\Lambda^{-s}\mathbf{u}\|_{L^2}+\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big),\end{aligned}\tag{3.1}$$

while for $s\in\big(\frac{1}{2},\frac{3}{2}\big)$, we have

$$\begin{aligned}&\frac{d}{dt}\Big(\|\Lambda^{-s}\rho\|_{L^2}^2+\|\Lambda^{-s}\mathbf{u}\|_{L^2}^2+\|\Lambda^{-s}\mathbf{b}\|_{L^2}^2\Big)+c_0\Big(\|\Lambda^{-s}\nabla\mathbf{u}\|_{L^2}^2+\|\Lambda^{-s}\nabla\mathbf{b}\|_{L^2}^2\Big)\\&\le C\|(\rho,\mathbf{u},\mathbf{b})\|_{L^2}^{s-\frac{1}{2}}\|\nabla(\rho,\mathbf{u},\mathbf{b})\|_{H^1}^{\frac{5}{2}-s}\Big(\|\Lambda^{-s}\rho\|_{L^2}+\|\Lambda^{-s}\mathbf{u}\|_{L^2}+\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big).\end{aligned}\tag{3.2}$$

Proof. Applying Λ^{-s} to $(2.1)_1$, $(2.1)_2$ and $(2.1)_3$, multiplying the resulting equations by $\gamma\Lambda^{-s}\rho$, $\Lambda^{-s}\mathbf{u}$ and $\Lambda^{-s}\mathbf{b}$, respectively, summing up and then integrating the resulting equations over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\gamma\|\Lambda^{-s}\rho(t)\|_{L^2}^2+\|\Lambda^{-s}\mathbf{u}(t)\|_{L^2}^2+\|\Lambda^{-s}\mathbf{b}(t)\|_{L^2}^2\Big)\\
&\quad+\mu\|\Lambda^{-s}\nabla\mathbf{u}(t)\|_{L^2}^2+(\mu+\nu)\|\Lambda^{-s}\nabla\cdot\mathbf{u}(t)\|_{L^2}^2+\|\Lambda^{-s}\nabla\mathbf{b}(t)\|_{L^2}^2\\
&=-\epsilon\int\Lambda^{-s}\mathbf{b}\cdot\Lambda^{-s}\nabla\times\Big((\nabla\times\mathbf{b})\times(\varphi(\rho)\mathbf{b})\Big)-\int\gamma\Lambda^{-s}\rho\Lambda^{-s}\nabla\cdot\big(\rho\mathbf{u}\big)\\
&\quad-\int\Lambda^{-s}\mathbf{u}\cdot\Lambda^{-s}(\mathbf{u}\cdot\nabla\mathbf{u})\\
&\quad-\int\Lambda^{-s}\mathbf{u}\cdot\Lambda^{-s}\Big(g(\rho)\big(\mu\Delta\mathbf{u}+(\mu+\nu)\nabla(\nabla\cdot\mathbf{u})\big)\Big)\\
&\quad+\int\Lambda^{-s}\mathbf{u}\cdot\Lambda^{-s}\Big(\varphi(\rho)(\mathbf{b}\cdot\nabla\mathbf{b}-\frac{1}{2}\nabla|\mathbf{b}|^2)\Big)\\
&\quad-\int\Lambda^{-s}\mathbf{u}\cdot\Lambda^{-s}\Big(h(\rho)\nabla\rho\Big)-\int\Lambda^{-s}\mathbf{b}\cdot\Lambda^{-s}\Big(\mathbf{u}\cdot\nabla\mathbf{b}-\mathbf{b}\cdot\nabla\mathbf{u}+\mathbf{b}\nabla\cdot\mathbf{u}\Big)\\
&:=\mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3+\mathcal{K}_4+\mathcal{K}_5+\mathcal{K}_6+\mathcal{K}_7.
\end{aligned}\tag{3.3}$$

We will estimate the terms $\mathcal{K}_1$, ..., $\mathcal{K}_7$ one by one.

For $s\in\big(0,\frac{1}{2}\big]$, we have $\frac{1}{2}+\frac{s}{3}<1$ and $\frac{3}{s}\geq 6$. To estimate the Hall term $\mathcal{K}_1$, we can use the estimate for Riesz potential to obtain

$$\begin{aligned}
\mathcal{K}_1&\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\|\Lambda^{-s}\nabla\big(\varphi(\rho)\mathbf{b}\nabla\mathbf{b}\big)\|_{L^2}\\
&\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big(\|\varphi'(\rho)\nabla\rho\mathbf{b}\nabla\mathbf{b}\|_{L^{\frac{6}{2s+3}}}+\big\|\varphi(\rho)|\nabla\mathbf{b}|^2\big\|_{L^{\frac{6}{2s+3}}}\\
&\quad+\|\varphi(\rho)\mathbf{b}\Delta\mathbf{b}\|_{L^{\frac{6}{2s+3}}}\Big).
\end{aligned}$$

By the boundedness of φ, φ', the interpolation and Sobolev embedding, we deduce that

$$\begin{aligned}
\mathcal{K}_1&\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big(\|\nabla\rho\mathbf{b}\nabla\mathbf{b}\|_{L^{\frac{6}{2s+3}}}+\|\nabla\mathbf{b}\|^2_{L^{\frac{12}{2s+3}}}+\|\mathbf{b}\Delta\mathbf{b}\|_{L^{\frac{6}{2s+3}}}\Big)\\
&\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big(\|\mathbf{b}\|_{L^\infty}\|\nabla\rho\|_{L^2}\|\nabla\mathbf{b}\|_{L^{\frac{3}{s}}}+\|\nabla\mathbf{b}\|_{L^2}^{\frac{1+2s}{2}}\|\nabla^2\mathbf{b}\|_{L^2}^{\frac{3-2s}{2}}+\|\mathbf{b}\|_{L^{\frac{3}{s}}}\|\Delta\mathbf{b}\|_{L^2}\Big)\\
&\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big(\|\nabla\mathbf{b}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\mathbf{b}\|_{L^2}^{\frac{1}{2}}\|\nabla\rho\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}^{\frac{1}{2}+s}\|\nabla^3\mathbf{b}\|_{L^2}^{\frac{1}{2}-s}\\
&\quad+\|\nabla\mathbf{b}\|_{H^1}^2+\|\nabla\mathbf{b}\|_{H^1}\|\nabla^2\mathbf{b}\|_{L^2}\Big)\\
&\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big(\|\nabla\rho\|_{L^2}^2+\|\nabla\mathbf{b}\|_{H^1}^2\Big).
\end{aligned}$$

Here we used (1.3) and Young's inequality in the last inequality. For $\mathcal{K}_2$, we have

$$\begin{aligned}\mathcal{K}_2 &\le C\|\Lambda^{-s}\rho\|_{L^2}\|\Lambda^{-s}\nabla\cdot\big(\rho\mathbf{u}\big)\|_{L^2} \le C\|\Lambda^{-s}\rho\|_{L^2}\|\nabla\cdot\big(\rho\mathbf{u}\big)\|_{L^{\frac{6}{2s+3}}}\\ &\le C\|\Lambda^{-s}\rho\|_{L^2}\Big(\|\nabla\rho\|_{L^2}\|\mathbf{u}\|_{L^{\frac{3}{s}}}+\|\rho\|_{L^6}\|\nabla\mathbf{u}\|_{L^{\frac{3}{s+1}}}\Big)\\ &\le C\|\Lambda^{-s}\rho\|_{L^2}\Big(\|\nabla\rho\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}+s}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}-s}+\|\nabla\rho\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}+s}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{1}{2}-s}\Big)\\ &\le C\|\Lambda^{-s}\rho\|_{L^2}\Big(\|\nabla\rho\|_{H^1}^2+\|\nabla\mathbf{u}\|_{H^1}^2\Big).\end{aligned}$$

Similarly, we can deduce that

$$\begin{aligned}\mathcal{K}_3+\mathcal{K}_7 &\le C\Big(\|\Lambda^{-s}\mathbf{u}\|_{L^2}+\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big)\Big(\|\mathbf{u}\nabla\mathbf{u}\|_{L^{\frac{6}{2s+3}}}\\ &\quad+\|\mathbf{u}\nabla\mathbf{b}\|_{L^{\frac{6}{2s+3}}}+\|\mathbf{b}\nabla\mathbf{u}\|_{L^{\frac{6}{2s+3}}}\Big)\\ &\le C\Big(\|\Lambda^{-s}\mathbf{u}\|_{L^2}+\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big)\Big(\|\mathbf{u}\|_{L^{\frac{3}{s}}}\|\nabla\mathbf{u}\|_{L^2}\\ &\quad+\|\mathbf{u}\|_{L^{\frac{3}{s}}}\|\nabla\mathbf{b}\|_{L^2}+\|\mathbf{b}\|_{L^{\frac{3}{s}}}\|\nabla\mathbf{u}\|_{L^2}\Big)\\ &\le C\Big(\|\Lambda^{-s}\mathbf{u}\|_{L^2}+\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big)\Big(\|\nabla\mathbf{u}\|_{H^1}^2\\ &\quad+\|\nabla\mathbf{b}\|_{H^1}^2\Big).\end{aligned}$$

The term $\mathcal{K}_5$ can be similarly dealt with by noticing that

$$\begin{aligned}\mathcal{K}_5 &\le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|\Lambda^{-s}\big(\varphi(\rho)(\mathbf{b}\nabla\mathbf{b})\big)\|_{L^2} \le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|\varphi(\rho)(\mathbf{b}\nabla\mathbf{b})\|_{L^{\frac{6}{2s+3}}}\\ &\le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|\mathbf{b}\nabla\mathbf{b}\|_{L^{\frac{6}{2s+3}}}.\end{aligned}$$

For the term $\mathcal{K}_4$, since

$$\begin{aligned}\mathcal{K}_4 &\le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|\Lambda^{-s}\big(g(\rho)\nabla^2\mathbf{u}\big)\|_{L^2}\\ &\le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|g(\rho)\nabla^2\mathbf{u}\|_{L^{\frac{6}{2s+3}}} \le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|\rho\nabla^2\mathbf{u}\|_{L^{\frac{6}{2s+3}}},\end{aligned}$$

we have

$$\begin{aligned}\mathcal{K}_4 &\le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|\rho\nabla^2\mathbf{u}\|_{L^{\frac{6}{2s+3}}} \le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|\rho\|_{L^6}\|\nabla^2\mathbf{u}\|_{L^{\frac{3}{s+1}}}\\ &\le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|\nabla\rho\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{1}{2}+s}\|\nabla^3\mathbf{u}\|_{L^2}^{\frac{1}{2}-s}.\end{aligned}$$

Similarly, for the term $\mathcal{K}_6$, we have

$$\begin{aligned}\mathcal{K}_6 &\le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|h(\rho)\nabla\rho\|_{L^{\frac{6}{2s+3}}} \le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|\rho\nabla\rho\|_{L^{\frac{6}{2s+3}}}\\ &\le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|\rho\|_{L^6}\|\nabla\rho\|_{L^{\frac{3}{s+1}}} \le C\|\Lambda^{-s}\mathbf{u}\|_{L^2}\|\nabla\rho\|_{L^2}^{\frac{3}{2}+s}\|\nabla^2\rho\|_{L^2}^{\frac{1}{2}-s}.\end{aligned}$$

Substituting the above estimates into (3.3), we obtain (3.1).

For $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, we have $\frac{1}{2} + \frac{s}{3} < 1$ and $2 < \frac{3}{s} < 6$. In this case, the estimates are more subtle. For the Hall term$\mathcal{K}_1$, we first use the estimates for Riesz potential and the boundedness of φ and φ' to obtain

$$\begin{aligned}\mathcal{K}_1 &\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\|\Lambda^{-s}\nabla\big(\varphi(\rho)\mathbf{b}\nabla\mathbf{b}\big)\|_{L^2}\\ &\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big(\|\varphi'(\rho)\nabla\rho\mathbf{b}\nabla\mathbf{b}\|_{L^{\frac{6}{2s+3}}} + \big\|\varphi(\rho)|\nabla\mathbf{b}|^2\big\|_{L^{\frac{6}{2s+3}}} + \|\varphi(\rho)\mathbf{b}\Delta\mathbf{b}\|_{L^{\frac{6}{2s+3}}}\Big)\\ &\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big(\|\nabla\rho\mathbf{b}\nabla\mathbf{b}\|_{L^{\frac{6}{2s+3}}} + \|\nabla\mathbf{b}\|^2_{L^{\frac{12}{2s+3}}} + \|\mathbf{b}\Delta\mathbf{b}\|_{L^{\frac{6}{2s+3}}}\Big).\end{aligned}$$

Then by the interpolation, Sobolev embedding and (1.3), we can deduce that

$$\begin{aligned}\mathcal{K}_1 &\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big(\|\nabla\rho\|_{L^\infty}\|\mathbf{b}\|_{L^{\frac{3}{s}}}\|\nabla\mathbf{b}\|_{L^2} + \|\nabla\mathbf{b}\|_{L^2}^{s+\frac{1}{2}}\|\nabla^2\mathbf{b}\|_{L^2}^{\frac{3}{2}-s} + \|\mathbf{b}\|_{L^{\frac{3}{s}}}\|\Delta\mathbf{b}\|_{L^2}\Big)\\ &\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big(\|\rho\|_{H^3}\|\mathbf{b}\|_{L^2}^{s-\frac{1}{2}}\|\nabla\mathbf{b}\|_{L^2}^{\frac{5}{2}-s} + \|\nabla\mathbf{b}\|_{L^2}^{s-\frac{1}{2}}\|\nabla\mathbf{b}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}^{\frac{3}{2}-s}\\ &\quad + \|\mathbf{b}\|_{L^2}^{s-\frac{1}{2}}\|\nabla\mathbf{b}\|_{L^2}^{\frac{3}{2}-s}\|\nabla^2\mathbf{b}\|_{L^2}\Big)\\ &\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\Big(\|\nabla\mathbf{b}\|_{L^2}^{\frac{5}{2}-s} + \|\nabla\mathbf{b}\|_{L^2}\|\nabla^2\mathbf{b}\|_{L^2}^{\frac{3}{2}-s} + \|\nabla\mathbf{b}\|_{L^2}^{\frac{3}{2}-s}\|\nabla^2\mathbf{b}\|_{L^2}\Big)\\ &\leq C\|\Lambda^{-s}\mathbf{b}\|_{L^2}\|\nabla\mathbf{b}\|_{H^1}^{\frac{5}{2}-s}.\end{aligned}$$

For the term $\mathcal{K}_2$, we have

$$\begin{aligned}\mathcal{K}_2 &\leq C\|\Lambda^{-s}\rho\|_{L^2}\|\Lambda^{-s}\nabla\cdot\big(\rho\mathbf{u}\big)\|_{L^2} \leq C\|\Lambda^{-s}\rho\|_{L^2}\|\nabla\cdot\big(\rho\mathbf{u}\big)\|_{L^{\frac{6}{2s+3}}}\\ &\leq C\|\Lambda^{-s}\rho\|_{L^2}\Big(\|\nabla\rho\|_{L^2}\|\mathbf{u}\|_{L^{\frac{3}{s}}} + \|\rho\|_{L^{\frac{3}{s}}}\|\nabla\mathbf{u}\|_{L^2}\Big)\\ &\leq C\|\Lambda^{-s}\rho\|_{L^2}\Big(\|\nabla\rho\|_{L^2}\|\mathbf{u}\|_{L^2}^{s-\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{3}{2}-s} + \|\rho\|_{L^2}^{s-\frac{1}{2}}\|\nabla\rho\|_{L^2}^{\frac{3}{2}-s}\|\nabla\mathbf{u}\|_{L^2}\Big)\\ &\leq C\|\Lambda^{-s}\rho\|_{L^2}\Big(\|\nabla\rho\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^{\frac{3}{2}-s} + \|\nabla\rho\|_{L^2}^{\frac{3}{2}-s}\|\nabla\mathbf{u}\|_{L^2}\Big)\\ &\leq C\|\Lambda^{-s}\rho\|_{L^2}\Big(\|\nabla\rho\|_{L^2}^{\frac{5}{2}-s} + \|\nabla\mathbf{u}\|_{L^2}^{\frac{5}{2}-s}\Big).\end{aligned}$$

The terms $\mathcal{K}_3, \ldots, \mathcal{K}_7$ can be similarly dealt with. Summarily, we can conclude the desired estimate (3.2). This completes the proof of Lemma 3.1. □

Proof of Theorem 1.2. For simplicity, we set

$$\mathcal{E}_{-s}(t) = \|\Lambda^{-s}\rho(t)\|_{L^2}^2 + \|\Lambda^{-s}\mathbf{u}(t)\|_{L^2}^2 + \|\Lambda^{-s}\mathbf{b}(t)\|_{L^2}^2.$$

For $s \in \big(0, \frac{1}{2}\big]$, we can integrate inequality (3.1) from 0 to t and use Hölder's inequality to obtain

$$\begin{aligned}\mathcal{E}_{-s}(t) &\leq \mathcal{E}_{-s}(0) + C\int_0^t \big\|\big(\nabla\rho, \nabla\mathbf{u}, \nabla\mathbf{b}\big)\big\|_{L^2\times H^1\times H^1}^{1+2s}\big\|\big(\nabla\rho, \nabla\mathbf{u}, \nabla\mathbf{b}\big)\big\|_{H^1\times H^2\times H^2}^{1-2s}\sqrt{\mathcal{E}_{-s}(\tau)}d\tau\\ &\leq C + C\Big(\int_0^t \big\|\big(\nabla\rho, \nabla\mathbf{u}, \nabla\mathbf{b}\big)\big\|_{L^2\times H^1\times H^1}^2 d\tau\Big)^{\frac{1+2s}{2}}\\ &\Big(\int_0^t \big\|\big(\nabla\rho, \nabla\mathbf{u}, \nabla\mathbf{b}\big)\big\|_{H^1\times H^2\times H^2}^2 d\tau\Big)^{\frac{1-2s}{2}} \sup_{0\leq\tau\leq t}\sqrt{\mathcal{E}_{-s}(\tau)}.\end{aligned}$$

It then follows from (2.49) and (2.50) that

$$\mathcal{E}_{-s}(t) \leq C + CM_0^{1-2s}\varepsilon^{1+2s} \sup_{0\leq\tau\leq t} \sqrt{\mathcal{E}_{-s}(\tau)},$$

which implies the desired estimates (1.4). We now deduce the precise decay estimates for $s \in \left(0, \frac{1}{2}\right]$. Notice that for $\ell = 0, 1, 2, \ldots, k-1$, we have

$$\|\nabla^{\ell+1} f\|_{L^2} \geq c\|\Lambda^{-s} f\|_{L^2}^{-\frac{1}{\ell+s}} \|\nabla^\ell f\|_{L^2}^{1+\frac{1}{\ell+s}}$$

for some $c > 0$, which together with (1.3) and (1.4) yields that

$$\mathcal{F}_\ell(t) \geq c\mathcal{E}_\ell(t)^{1+\frac{1}{\ell+s}}. \tag{3.4}$$

Here we also used $\|\nabla^{\ell+1}\rho\|_{L^2} \geq c\|\nabla^{\ell+1}\rho\|_{L^2}^{1+\frac{1}{\ell+s}}$, which follows from (1.3). Substituting (3.4) into (2.47), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\ell(t) + c\mathcal{E}_\ell(t)^{1+\frac{1}{\ell+s}} \leq 0. \tag{3.5}$$

A direct calculation gives that

$$\mathcal{E}_\ell(t) \leq C(1+t)^{-(\ell+s)} \qquad (\ell = 0, 1, 2, \ldots, k-1), \tag{3.6}$$

which implies the desired estimates (1.5).

For $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, we have $(\rho_0, \mathbf{u}_0, \mathbf{b}_0) \in \dot{H}^{-\frac{1}{2}}$ by $\dot{H}^{-s} \cap L^2 \subset \dot{H}^{-s'}$ for any $s' \in [0, s)$. Thus by using the decay estimates (3.6) with $\ell = 0, 1, 2$, we conclude that

$$\|\rho(t)\|_{L^2}^2 + \|\mathbf{u}(t)\|_{L^2}^2 + \|\mathbf{b}(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{1}{2}},$$

and

$$\|\nabla\rho(t)\|_{H^1}^2 + \|\nabla\mathbf{u}(t)\|_{H^1}^2 + \|\nabla\mathbf{b}(t)\|_{H^1}^2 \leq C(1+t)^{-\frac{3}{2}},$$

which together with (3.1) yields that

$$\begin{aligned}
\mathcal{E}_{-s}(t) &\leq \mathcal{E}_{-s}(0) + C\int_0^t \|(\rho, \mathbf{u}, \mathbf{b})(\tau)\|_{L^2}^{s-\frac{1}{2}} \|\nabla(\rho, \mathbf{u}, \mathbf{b})(\tau)\|_{H^1}^{\frac{5}{2}-s} \sqrt{\mathcal{E}_{-s}(\tau)} d\tau \\
&\leq C + C\int_0^t (1+\tau)^{-\frac{1}{4}(s-\frac{1}{2})}(1+\tau)^{-\frac{3}{4}(\frac{5}{2}-s)} \sqrt{\mathcal{E}_{-s}(\tau)} d\tau \\
&= C + C\int_0^t (1+\tau)^{-\left(\frac{7}{4}-\frac{s}{2}\right)} d\tau \sup_{0\leq\tau\leq t} \sqrt{\mathcal{E}_{-s}(\tau)} \\
&\leq C + C \sup_{0\leq\tau\leq t} \sqrt{\mathcal{E}_{-s}(\tau)}
\end{aligned}$$

by $s < \frac{3}{2}$, which implies the desired estimates (1.4). Then we may repeat the arguments to obtain (3.5) and thus (3.6) with $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, which leads to the decay estimates (1.5) for $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$. This completes the proof of Theorem 1.2. □

4. Vanishing Hall limit

In this section, we shall use the basic energy methods to derive the convergence rate estimates from Hall-MHD equations (2.1) with $\epsilon > 0$ to the classical MHD equations (2.1) with $\epsilon = 0$ for any given positive time.

Proof of Theorem 1.3. Let $(\rho^\epsilon, \mathbf{u}^\epsilon, \mathbf{b}^\epsilon)$ and $(\rho^0, \mathbf{u}^0, \mathbf{b}^0)$ be two solutions to equations (2.1) obtained in Theorem 1.1 corresponding to $\epsilon > 0$ and $\epsilon = 0$, respectively. Then by the proof of Theorem 1.1, we have the uniform estimates with respect to ϵ:

$$\|\rho^\epsilon(t)\|_{H^k}^2 + \|\mathbf{u}^\epsilon(t)\|_{H^k}^2 + \|\mathbf{b}^\epsilon(t)\|_{H^k}^2 \le C \tag{4.1}$$

and

$$\int_0^t \Big(\|\nabla\rho^\epsilon(\tau)\|_{H^{k-1}}^2 + \|\nabla\mathbf{u}^\epsilon(\tau)\|_{H^k}^2 + \|\nabla\mathbf{b}^\epsilon(\tau)\|_{H^k}^2\Big)d\tau \le C. \tag{4.2}$$

Moreover, by equations (2.1), we also have

$$\|\partial_t\rho^\epsilon(\cdot,t)\|_{H^{k-1}} \le C, \quad \|\partial_t\mathbf{u}^\epsilon(\cdot,t)\|_{H^{k-2}} \le C \quad \text{and} \quad \|\partial_t\mathbf{b}^\epsilon(\cdot,t)\|_{H^{k-2}} \le C.$$

Since the constants C on the right-hand side of the above inequalities are independent of ϵ, these uniform estimates together with Aubin-Lions lemma yield the existence of a subsequence (denoted still by $(\rho^\epsilon, \mathbf{u}^\epsilon, \mathbf{b}^\epsilon)$) and $(\tilde{\rho}^0, \tilde{\mathbf{u}}^0, \tilde{\mathbf{b}}^0)$ such that

$$\begin{aligned} \rho^\epsilon &\to \tilde{\rho}^0 \qquad \text{strongly in } C\big(0,t;\ H_{loc}^{k-\sigma}\big), \\ \mathbf{u}^\epsilon &\to \tilde{\mathbf{u}}^0 \qquad \text{strongly in } C\big(0,t;\ H_{loc}^{k-\sigma}\big), \end{aligned}$$

and

$$\mathbf{b}^\epsilon \to \tilde{\mathbf{b}}^0 \qquad \text{strongly in } C\big(0,t;\ H_{loc}^{k-\sigma}\big)$$

with $\sigma \in (0, \frac{1}{2})$, as $\epsilon \to 0$. Lemmas 5.1 and 5.3 together with the uniform estimates (4.1) and (4.2) yield that

$$\int_0^t \Big\|\frac{\mathbf{b}^\epsilon\nabla\mathbf{b}^\epsilon}{n^\epsilon}(\cdot,\tau)\Big\|_{H^k}^2 d\tau = \int_0^t \big\|\varphi(\rho^\epsilon)\mathbf{b}^\epsilon\nabla\mathbf{b}^\epsilon(\cdot,\tau)\big\|_{H^k}^2 d\tau \le C,$$

which implies that

$$\epsilon\nabla\times\Big(\frac{(\nabla\times\mathbf{b}^\epsilon)\times\mathbf{b}^\epsilon}{n^\epsilon}\Big) \to 0 \qquad \text{strongly in } L^2\big(0,t;H^{k-1}\big)$$

as $\epsilon \to 0$. This allows ϵ pass to the zero, and the limit $(\tilde{\rho}^0, \tilde{\mathbf{u}}^0, \tilde{\mathbf{b}}^0)$ satisfies (2.1) with zero Hall coefficient $\epsilon = 0$. By the uniqueness of solution to the classical MHD equations (2.1) with $\epsilon = 0$, we have $(\tilde{\rho}^0, \tilde{\mathbf{u}}^0, \tilde{\mathbf{b}}^0) = (\rho^0, \mathbf{u}^0, \mathbf{b}^0)$.

We now turn to deriving the convergence rate. For simplicity, we first define

$$\rho = \rho^\epsilon - \rho^0, \quad u = \mathbf{u}^\epsilon - \mathbf{u}^0, \quad \text{and} \quad b = \mathbf{b}^\epsilon - \mathbf{b}^0$$

and then have

$$
\begin{cases}
\partial_t \rho + \nabla \cdot u = -\Big(\rho \nabla \cdot \mathbf{u}^\epsilon + \rho^0 \nabla \cdot u + u \cdot \nabla \rho^\epsilon + \mathbf{u}^0 \cdot \nabla \rho\Big), \\
\partial_t u - \mu \varphi(\rho^0) \Delta u - (\mu + \nu) \varphi(\rho^0) \nabla(\nabla \cdot u) + a\gamma \nabla \rho \\
\quad = -\Big(u \cdot \nabla \mathbf{u}^\epsilon + \mathbf{u}^0 \cdot \nabla u\Big) - \mu \Big(g(\rho^\epsilon) - g(\rho^0)\Big) \Delta \mathbf{u}^\epsilon \\
\qquad - (\mu + \nu)\Big(g(\rho^\epsilon) - g(\rho^0)\Big) \nabla \nabla \cdot \mathbf{u}^\epsilon \\
\qquad + \Big(\varphi(\rho^\epsilon) - \varphi(\rho^0)\Big)\Big(\mathbf{b}^\epsilon \cdot \nabla \mathbf{b}^\epsilon - \frac{1}{2} \nabla |\mathbf{b}^\epsilon|^2\Big) \\
\qquad + \varphi(\rho^0)\Big(b \cdot \nabla \mathbf{b}^\epsilon + \mathbf{b}^0 \cdot \nabla b - \frac{1}{2} \nabla\big(|\mathbf{b}^\epsilon|^2 - |\mathbf{b}^0|^2\big)\Big) \\
\qquad - \Big(\big(h(\rho^\epsilon) - h(\rho^0)\big) \nabla \rho^\epsilon + h(\rho^0) \nabla \rho\Big), \\
\partial_t b - \Delta b = -\epsilon \nabla \times \Big((\nabla \times \mathbf{b}^\epsilon) \times \big(\varphi(\rho^\epsilon) \mathbf{b}^\epsilon\big)\Big) \\
\qquad - \Big(u \cdot \nabla \mathbf{b}^\epsilon + \mathbf{u}^0 \cdot \nabla b - b \cdot \nabla \mathbf{u}^\epsilon - \mathbf{b}^0 \cdot \nabla u + b \nabla \cdot \mathbf{u}^\epsilon + \mathbf{b}^0 \nabla \cdot u\Big)
\end{cases}
\tag{4.3}
$$

with $\nabla \cdot b = 0$ and zero initial data.

Multiplying equations $(4.3)_1$, $(4.3)_2$ and $(4.3)_3$ by $a\gamma\rho$, u and b, respectively, summing up and integrating the resulting equations over $\mathbb{R}^3$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma \|\rho\|_{L^2}^2 + \|u\|_{L^2}^2 + \|b\|_{L^2}^2\Big) \\
&\quad + \mu \int \varphi(\rho^0) |\nabla u|^2 + (\mu + \nu) \int \varphi(\rho^0) |\nabla \cdot u|^2 + \|\nabla b\|_{L^2}^2 \\
&\quad = -\int \rho \Big(\rho \nabla \cdot \mathbf{u}^\epsilon + \rho^0 \nabla \cdot u + u \cdot \nabla \rho^\epsilon\Big) \\
&\qquad + \frac{1}{2} \int |\rho|^2 \nabla \cdot \mathbf{u}^0 - \int u \cdot \Big(u \cdot \nabla \mathbf{u}^\epsilon + \mathbf{u}^0 \cdot \nabla u\Big) \\
&\qquad - \mu \int u \cdot \Big(\big(g(\rho^\epsilon) - g(\rho^0)\big) \Delta \mathbf{u}^\epsilon + \nabla \varphi(\rho^0) \cdot \nabla u\Big) \\
&\qquad - (\mu + \nu) \int u \cdot \Big(\big(g(\rho^\epsilon) - g(\rho^0)\big) \nabla \nabla \cdot \mathbf{u}^\epsilon + \nabla \varphi(\rho^0) \nabla \cdot u\Big) \\
&\qquad + \int u \cdot \Big(\varphi(\rho^\epsilon) - \varphi(\rho^0)\Big)\Big(\mathbf{b}^\epsilon \cdot \nabla \mathbf{b}^\epsilon \\
&\qquad - \frac{1}{2} \nabla |\mathbf{b}^\epsilon|^2\Big) + \int u \cdot \varphi(\rho^0)\Big(b \cdot \nabla \mathbf{b}^\epsilon + b^0 \cdot \nabla b - \frac{1}{2} \nabla\big(|\mathbf{b}^\epsilon|^2 - |b^0|^2\big)\Big) \\
&\qquad - \int u \cdot \Big(\big(h(\rho^\epsilon) - h(\rho^0)\big) \nabla \rho^\epsilon + h(\rho^0) \nabla \rho\Big) - \epsilon \int b \cdot \nabla \times \Big((\nabla \times \mathbf{b}^\epsilon) \times \big(\varphi(\rho^\epsilon) \mathbf{b}^\epsilon\big)\Big) \\
&\qquad - \int b \cdot \Big(u \cdot \nabla \mathbf{b}^\epsilon + \mathbf{u}^0 \cdot \nabla b - b \cdot \nabla \mathbf{u}^\epsilon - \mathbf{b}^0 \cdot \nabla u + b \nabla \cdot \mathbf{u}^\epsilon + \mathbf{b}^0 \nabla \cdot u\Big).
\end{aligned}
$$

Then we can use the integration by parts and Hölder's inequality to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\rho\|_{L^2}^2+\|u\|_{L^2}^2+\|b\|_{L^2}^2\Big)+\mu\int\varphi(\rho^0)|\nabla u|^2\\
&\quad+(\mu+\nu)\int\varphi(\rho^0)|\nabla\cdot u|^2+\|\nabla b\|_{L^2}^2\\
&\le C\|\rho\|_{L^2}\Big(\|\rho\|_{L^2}\|\nabla\mathbf{u}^\epsilon\|_{L^\infty}\\
&\quad+\|\rho^0\|_{L^\infty}\|\nabla u\|_{L^2}+\|u\|_{L^2}\|\nabla\rho^\epsilon\|_{L^\infty}+\|\rho\|_{L^2}\|\nabla\mathbf{u}^0\|_{L^\infty}\Big)\\
&\quad+C\|u\|_{L^2}\Big(\|u\|_{L^6}\|\nabla\mathbf{u}^\epsilon\|_{L^3}+\|\mathbf{u}^0\|_{L^\infty}\|\nabla u\|_{L^2}\Big)\\
&+C\|u\|_{L^6}\big\|g(\rho^\epsilon)-g(\rho^0)\big\|_{L^2}\|\nabla^2\mathbf{u}^\epsilon\|_{L^3}\\
&\quad+C\|\nabla\varphi(\rho^0)\|_{L^\infty}\|u\|_{L^2}\|\nabla u\|_{L^2}\\
&\quad+C\|u\|_{L^6}\big\|\varphi(\rho^\epsilon)-\varphi(\rho^0)\big\|_{L^2}\|\mathbf{b}^\epsilon\|_{L^6}\|\nabla\mathbf{b}^\epsilon\|_{L^6}\\
&\quad+C\|u\|_{L^2}\|\varphi(\rho^0)\|_{L^\infty}\Big(\|b\|_{L^6}\|\nabla\mathbf{b}^\epsilon\|_{L^3}+\|\mathbf{b}^0\|_{L^\infty}\|\nabla b\|_{L^2}\Big)\\
&\quad+C\|u\|_{L^6}\big\|h(\rho^\epsilon)-h(\rho^0)\big\|_{L^2}\|\nabla\rho^\epsilon\|_{L^3}\\
&\quad+C\|\nabla u\|_{L^2}\|h(\rho^0)\|_{L^\infty}\|\rho\|_{L^2}+C\|u\|_{L^6}\|\nabla h(\rho^0)\|_{L^3}\|\rho\|_{L^2}\\
&+\epsilon\|\nabla b\|_{L^2}\|\nabla\mathbf{b}^\epsilon\|_{L^2}\big\|\varphi(\rho^\epsilon)\big\|_{L^\infty}\|\mathbf{b}^\epsilon\|_{L^\infty}\\
&\quad+C\|b\|_{L^2}\Big(\|u\|_{L^6}\|\nabla\mathbf{b}^\epsilon\|_{L^3}+\|\mathbf{u}^0\|_{L^\infty}\|\nabla b\|_{L^2}\\
&\quad+\|b\|_{L^6}\|\nabla\mathbf{u}^\epsilon\|_{L^3}+\|\mathbf{b}^0\|_{\infty}\|\nabla u\|_{L^2}\Big).
\end{aligned}
$$

Thus by Sobolev embedding and the uniform estimates obtained in Theorem 1.1, we can find two positive constants c_3 and C such that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\rho\|_{L^2}^2+\|u\|_{L^2}^2+\|b\|_{L^2}^2\Big)+c_3\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2\\
&\le C\Big(\|\rho\|_{L^2}+\|u\|_{L^2}+\|b\|_{L^2}\Big)\Big(\|\rho\|_{L^2}+\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2}\Big)+C\epsilon\|\nabla b\|_{L^2}.
\end{aligned}
\tag{4.4}
$$

Now we turn to the higher derivative estimates. To this end, for any $1\le\ell\le k-2$, applying ∇^ℓ to equations $(4.3)_1$, $(4.3)_2$ and $(4.3)_3$, and taking the inner product with $\nabla^\ell\rho$, $\nabla^\ell u$ and $\nabla^\ell b$, respectively, we can deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^\ell\rho\|_{L^2}^2+\int\nabla^\ell\rho\nabla^\ell\nabla\cdot u dx\\
&=-\int\nabla^\ell\rho\cdot\nabla^\ell\Big(\rho\cdot\nabla\mathbf{u}^\epsilon+\rho^0\nabla\cdot u+u\cdot\nabla\rho^\epsilon\Big)\\
&\quad-\int\nabla^\ell\rho\cdot\Big(\nabla^\ell(\mathbf{u}^0\cdot\nabla\rho)-\mathbf{u}^0\cdot\nabla\nabla^\ell\rho\Big)+\frac{1}{2}\int|\nabla^\ell\rho|^2\nabla\cdot\mathbf{u}^0\\
&:=\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3
\end{aligned}
$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^\ell u\|_{L^2}^2+\mu\int\varphi(\rho^0)|\nabla^{\ell+1}u|^2+(\mu+\nu)\int\varphi(\rho^0)|\nabla^\ell\nabla\cdot u|^2+a\gamma\int\nabla^\ell u\cdot\nabla^\ell\nabla\rho\\
&=\int\nabla^{\ell+1}u\cdot\nabla^{\ell-1}\Big(u\cdot\nabla\mathbf{u}^\epsilon+\mathbf{u}^0\cdot\nabla u\Big)\\
&\quad+\mu\int\nabla^{\ell+1}u\cdot\Big(\nabla^{\ell-1}\big(\big(g(\rho^\epsilon)-g(\rho^0)\big)\Delta\mathbf{u}^\epsilon\big)-\nabla^{\ell-1}\big(\varphi(\rho^0)\nabla^2u\big)\\
&\quad+\varphi(\rho^0)\nabla^{\ell+1}u\Big)+(\mu+\nu)\int\nabla^{\ell+1}u\cdot\nabla^{\ell-1}\Big(\big(g(\rho^\epsilon)\\
&\quad-g(\rho^0)\big)\nabla\nabla\cdot\mathbf{u}^\epsilon-\nabla^{\ell-1}\big(\varphi(\rho^0)\nabla^2u\big)\Big)\\
&\quad+\varphi(\rho^0)\nabla^\ell\nabla\cdot u\Big)-\int\nabla^{\ell+1}u\cdot\Big(\nabla^{\ell-1}\big((\varphi(\rho^\epsilon)-\varphi(\rho^0))(\mathbf{b}^\epsilon\cdot\nabla\mathbf{b}^\epsilon-\frac{1}{2}\nabla|\mathbf{b}^\epsilon|^2)\big)\Big)\\
&\quad-\int\nabla^{\ell+1}u\cdot\nabla^{\ell-1}\Big(\varphi(\rho^0)\big(b\cdot\nabla\mathbf{b}^\epsilon+\mathbf{b}^0\cdot\nabla b-\frac{1}{2}\nabla\big(|\mathbf{b}^\epsilon|^2-|\mathbf{b}^0|^2\big)\big)\Big)\\
&\quad+\int\nabla^{\ell+1}u\cdot\nabla^{\ell-1}\Big(\big(h(\rho^\epsilon)-h(\rho^0)\big)\nabla\rho^\epsilon+h(\rho^0)\nabla\rho\Big)\\
&:=\mathcal{N}_4+\mathcal{N}_5+\mathcal{N}_6+\mathcal{N}_7+\mathcal{N}_8+\mathcal{N}_9
\end{aligned}$$

as well as

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^\ell b\|_{L^2}^2+\|\nabla^{\ell+1}b\|_{L^2}^2&=-\epsilon\int\nabla^\ell b\cdot\nabla^\ell\nabla\times\Big((\nabla\times\mathbf{b}^\epsilon)\times\big(\varphi(\rho^\epsilon)\mathbf{b}^\epsilon\big)\Big)\\
&\quad+\int\nabla^{\ell+1}b\cdot\nabla^{\ell-1}\Big(u\cdot\nabla\mathbf{b}^\epsilon+\mathbf{u}^0\cdot\nabla b\\
&\quad-b\cdot\nabla\mathbf{u}^\epsilon-\mathbf{b}^0\cdot\nabla u+b\nabla\cdot\mathbf{u}^\epsilon+\mathbf{b}^0\nabla\cdot u\Big)\\
&:=\mathcal{N}_{10}+\mathcal{N}_{11}.
\end{aligned}$$

Thus we can find a positive constant c_3 such that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\nabla^\ell\rho\|_{L^2}^2+\|\nabla^\ell u\|_{L^2}^2+\|\nabla^\ell b\|_{L^2}^2\Big)+c_3\|\nabla^{\ell+1}u\|_{L^2}^2+\|\nabla^{\ell+1}b\|_{L^2}^2\\
&\le a\gamma(\mathcal{N}_1+\mathcal{N}_2+\mathcal{N}_3)+\mathcal{N}_4+\cdots+\mathcal{N}_{11}.
\end{aligned}\tag{4.5}$$

We will estimate the terms $\mathcal{N}_1,\ \ldots,\mathcal{N}_{11}$ one by one for $1\le\ell\le k-2$. Firstly, for $\mathcal{N}_1$ and $\mathcal{N}_2$, we can use Hölder's inequality and the product estimates to obtain

$$\begin{aligned}
\mathcal{N}_1&\le C\|\nabla^\ell\rho\|_{L^2}\Big(\|\nabla^\ell(\rho\cdot\nabla\mathbf{u}^\epsilon)\|_{L^2}+\|\nabla^\ell(\rho^0\nabla\cdot u)\|_{L^2}+\|\nabla^\ell(u\cdot\nabla\rho^\epsilon)\|_{L^2}\Big)\\
&\le C\|\nabla^\ell\rho\|_{L^2}\Big(\|\nabla^\ell\rho\|_{L^2}\|\nabla\mathbf{u}^\epsilon\|_{L^\infty}\\
&\quad+\|\rho\|_{L^6}\|\nabla^{\ell+1}\mathbf{u}^\epsilon\|_{L^3}+\|\nabla^\ell\rho^0\|_{L^3}\|\nabla u\|_{L^6}+\|\rho^0\|_{L^\infty}\|\nabla^{\ell+1}u\|_{L^2}\\
&\quad+\|\nabla^\ell u\|_{L^2}\|\nabla\rho^\epsilon\|_{L^\infty}+\|u\|_{L^6}\|\nabla^{\ell+1}\rho^\epsilon\|_{L^3}\Big)\\
&\le C\|\nabla^\ell\rho\|_{L^2}\Big(\|\rho\|_{H^{k-2}}\|\nabla\mathbf{u}^\epsilon\|_{H^{k-1}}+\|\nabla u\|_{H^{k-2}}\|\rho^0\|_{H^{k-1}}+\|u\|_{H^{k-2}}\|\rho^\epsilon\|_{H^k}\Big)\\
&\le C\Big(\|\mathbf{u}^\epsilon\|_{H^k}+\|\rho^0\|_{H^k}+\|\rho^\epsilon\|_{H^k}\Big)\|\rho\|_{H^{k-2}}\Big(\|\rho\|_{H^{k-2}}+\|u\|_{H^{k-2}}+\|\nabla u\|_{H^{k-2}}\Big)
\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}\mathcal{N}_2 &\le C\|\nabla^\ell\rho\|_{L^2}\left\|\nabla^\ell\big(\mathbf{u}^0\cdot\nabla\rho\big)-\mathbf{u}^0\cdot\nabla\nabla^\ell\rho\right\|_{L^2}\\ &\le C\|\nabla^\ell\rho\|_{L^2}\|\mathbf{u}^0\|_{H^k}\|\rho\|_{H^{k-2}}\le C\|\mathbf{u}^0\|_{H^k}\|\rho\|^2_{H^{k-2}}.\end{aligned}\tag{4.7}$$

For $\mathcal{N}_3$, it is clear that

$$\mathcal{N}_3\le C\|\nabla^\ell\rho\|^2_{L^2}\|\nabla\mathbf{u}^0\|_{L^\infty}\le C\|\mathbf{u}^0\|_{H^k}\|\rho\|^2_{H^{k-2}}.\tag{4.8}$$

Similarly, for $\mathcal{N}_4$, we have

$$\begin{aligned}\mathcal{N}_4 &\le \|\nabla^{\ell+1}u\|_{L^2}\|\nabla^{\ell-1}(u\cdot\nabla\mathbf{u}^\epsilon)\|_{L^2}+\|\nabla^{\ell-1}(\mathbf{u}^0\cdot\nabla u)\|_{L^2}\\ &\le C\|\nabla^{\ell+1}u\|_{L^2}\Big(\|\nabla^{\ell-1}u\|_{L^6}\|\nabla\mathbf{u}^\epsilon\|_{L^3}+\|u\|_{L^6}\|\nabla^\ell\mathbf{u}^\epsilon\|_{L^3}+\|\nabla^{\ell-1}\mathbf{u}^0\|_{L^3}\|\nabla u\|_{L^6}\\ &\quad+\|\mathbf{u}^0\|_{L^\infty}\|\nabla^\ell u\|_{L^2}\Big)\\ &\le C\Big(\|\mathbf{u}^\epsilon\|_{H^k}+\|\mathbf{u}^0\|_{H^k}\Big)\|\nabla u\|_{H^{k-2}}\|u\|_{H^{k-2}}.\end{aligned}\tag{4.9}$$

For $\mathcal{N}_5$ and $\mathcal{N}_6$, we can deduce that

$$\begin{aligned}\mathcal{N}_5+\mathcal{N}_6 &\le C\|\nabla^{\ell+1}u\|_{L^2}\Big(\|\nabla^{\ell-1}\big(g(\rho^\epsilon)-g(\rho^0))\nabla^2\mathbf{u}^\epsilon\big)\|_{L^2}\\ &\quad+\|\nabla^{\ell-1}(\varphi(\rho^0)\nabla^2 u)-\varphi(\rho^0)\nabla^{\ell-1}\nabla^2 u\|_{L^2}\Big)\\ &\le C\|\nabla^{\ell+1}u\|_{L^2}\Big(\|\nabla^{\ell-1}\big(g(\rho^\epsilon)-g(\rho^0))\|_{L^6}\|\nabla^2\mathbf{u}^\epsilon\|_{L^3}\\ &\quad+\|g(\rho^\epsilon)-g(\rho^0)\|_{L^6}\|\nabla^{\ell+1}\mathbf{u}^\epsilon\big\|_{L^3}\\ &\quad+\|\varphi(\rho^0)\|_{H^k}\|u\|_{H^{k-2}}\Big)\\ &\le C\|\nabla^{\ell+1}u\|_{L^2}\Big(\|\nabla^\ell\big(\rho^\epsilon-\rho^0\big)\|_{L^2}\|\nabla^2\mathbf{u}^\epsilon\|_{L^3}\\ &\quad+\|\nabla\big(\rho^\epsilon-\rho^0\big)\|_{L^2}\|\nabla^{\ell+1}\mathbf{u}^\epsilon\|_{L^3}+\|\rho^0\|_{H^k}\|u\|_{H^{k-2}}\Big)\\ &\le C\|\nabla^{\ell+1}u\|_{L^2}\Big(\|\rho\|_{H^{k-2}}\|\mathbf{u}^\epsilon\|_{H^k}+\|\rho^0\|_{H^k}\|u\|_{H^{k-2}}\Big)\\ &\le C\Big(\|\mathbf{u}^\epsilon\|_{H^k}+\|\rho^0\|_{H^k}\Big)\|\nabla u\|_{H^{k-2}}\Big(\|\rho\|_{H^{k-2}}+\|u\|_{H^{k-2}}\Big).\end{aligned}\tag{4.10}$$

Similarly, for $\mathcal{N}_7$, we have

$$\begin{aligned}\mathcal{N}_7 &\le C\|\nabla^{\ell+1}u\|_{L^2}\Big(\|\nabla^{\ell-1}\big((\varphi(\rho^\epsilon)-\varphi(\rho^0))\mathbf{b}^\epsilon\nabla\mathbf{b}^\epsilon\big)\|_{L^2}\\ &\le C\|\nabla^{\ell+1}u\|_{L^2}\Big(\|\nabla^{\ell-1}\big(\varphi(\rho^\epsilon)-\varphi(\rho^0))\|_{L^6}\|\mathbf{b}^\epsilon\nabla\mathbf{b}^\epsilon\|_{L^3}\\ &\quad+\|\varphi(\rho^\epsilon)-\varphi(\rho^0)\|_{L^6}\|\nabla^{\ell-1}(\mathbf{b}^\epsilon\nabla\mathbf{b}^\epsilon)\big\|_{L^3}\Big)\\ &\le C\|\nabla^{\ell+1}u\|_{L^2}\Big(\|\nabla^\ell\big(\rho^\epsilon-\rho^0\big)\|_{L^2}\|\mathbf{b}^\epsilon\|_{L^\infty}\|\nabla\mathbf{b}^\epsilon\|_{L^3}\end{aligned}$$

$$
\begin{aligned}
&+\|\nabla(\rho^\epsilon-\rho^0)\|_{L^2}\|\mathbf{b}^\epsilon\|_{H^2}\|\mathbf{b}^\epsilon\|_{H^k}\Big)\\
&\leq C\|\mathbf{b}^\epsilon\|_{H^k}^2\|\nabla u\|_{H^{k-2}}\|\rho\|_{H^{k-2}}.
\end{aligned}
\tag{4.11}
$$

Similarly, for the terms $\mathcal{N}_8$ and $\mathcal{N}_9$, we can obtain

$$
\begin{aligned}
\mathcal{N}_8+\mathcal{N}_9\leq C\|\nabla u\|_{H^{k-2}}\Big(\|\rho\|_{H^{k-2}}+\|b\|_{H^{k-2}}\Big)\Big(\|\rho^\epsilon\|_{H^k}+\big(1+\|\rho^0\|_{H^k}\big)\\
\big(1+\|b^0\|_{H^k}+\|\mathbf{b}^\epsilon\|_{H^k}\big)\Big).
\end{aligned}
\tag{4.12}
$$

For the Hall term $\mathcal{N}_{10}$, we have

$$
\begin{aligned}
\mathcal{N}_{10}&=\epsilon\int\nabla^\ell\nabla\times b\cdot\nabla^\ell\Big((\nabla\times\mathbf{b}^\epsilon)\times\big(\varphi(\rho^\epsilon)\mathbf{b}^\epsilon\big)\Big)\\
&\leq\epsilon\|\nabla^{\ell+1}b\|_{L^2}\|\nabla^\ell\big(\varphi(\rho^\epsilon)\mathbf{b}^\epsilon\nabla\mathbf{b}^\epsilon\big)\|_{L^2}\\
&\leq\epsilon\|\nabla b\|_{H^{k-2}}\big(1+\|\rho^\epsilon\|_{H^k}\big)\|\mathbf{b}^\epsilon\|_{H^k}^2.
\end{aligned}
\tag{4.13}
$$

Finally, we take a similar procedure as (4.9) to bound the term $\mathcal{N}_{11}$ as follows:

$$
\mathcal{N}_{11}\leq C\Big(\|\mathbf{u}^\epsilon\|_{H^k}+\|\mathbf{b}^\epsilon\|_{H^k}+\|\mathbf{u}^0\|_{H^k}+\|\mathbf{b}^0\|_{H^k}\Big)\|\nabla b\|_{H^{k-2}}\Big(\|u\|_{H^{k-2}}+\|b\|_{H^{k-2}}\Big).
\tag{4.14}
$$

Substituting (4.6)–(4.14) into (4.5) and using the uniform estimates obtained in Theorem 1.1, we can deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\nabla^\ell\rho\|_{L^2}^2+\|\nabla^\ell u\|_{L^2}^2+\|\nabla^\ell b\|_{L^2}^2\Big)+c_3\|\nabla^{\ell+1}u\|_{L^2}^2+\|\nabla^{\ell+1}b\|_{L^2}^2\\
&\leq C\Big(\|\rho\|_{H^{k-2}}+\|u\|_{H^{k-2}}+\|b\|_{H^{k-2}}\Big)\Big(\|\rho\|_{H^{k-2}}+\|\nabla u\|_{H^{k-2}}\\
&\quad+\|\nabla b\|_{H^{k-2}}\Big)+C\epsilon\|\nabla b\|_{H^{k-2}}
\end{aligned}
\tag{4.15}
$$

for $\ell=1,2,\ldots,k-2$. Then summing up (4.15) from $\ell=1$ to $k-2$ and using the energy estimate (4.4), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(a\gamma\|\rho\|_{H^{k-2}}^2+\|u\|_{H^{k-2}}^2+\|b\|_{H^{k-2}}^2\Big)+c_3\|\nabla u\|_{H^{k-2}}^2+\|\nabla b\|_{H^{k-2}}^2\\
&\leq C\Big(\|\rho\|_{H^{k-2}}+\|u\|_{H^{k-2}}+\|b\|_{H^{k-2}}\Big)\Big(\|\rho\|_{H^{k-2}}+\|\nabla u\|_{H^{k-2}}+\|\nabla b\|_{H^{k-2}}\Big)\\
&\quad+C\epsilon\|\nabla b\|_{H^{k-2}},
\end{aligned}
$$

which together with Young's inequality gives that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\rho\|_{H^{k-2}}^2+\|u\|_{H^{k-2}}^2+\|b\|_{H^{k-2}}^2\Big)+\|\nabla u\|_{H^{k-2}}^2+\|\nabla b\|_{H^{k-2}}^2\\
&\leq C\Big(\|\rho\|_{H^{k-2}}^2+\|u\|_{H^{k-2}}^2+\|b\|_{H^{k-2}}^2\Big)+C\epsilon^2.
\end{aligned}
$$

Hence, by Gronwall's inequality, we can conclude that

$$
\|\rho(t)\|_{H^{k-2}}^2+\|u(t)\|_{H^{k-2}}^2+\|b(t)\|_{H^{k-2}}^2\leq\epsilon^2e^{Ct}\quad\text{for any }\ t\in[0,\infty).
$$

That is,

$$\|\rho^\epsilon(t)-\rho^0(t)\|_{H^{k-2}}^2+\|\mathbf{u}^\epsilon(t)-\mathbf{u}(t)\|_{H^{k-2}}^2+\|\mathbf{b}^\epsilon(t)-\mathbf{b}(t)\|_{H^{k-2}}^2 \\ \le \epsilon^2 e^{Ct} \quad \text{for any} \quad t\in[0,\infty).$$

This completes the proof of Theorem 1.3. □

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Appendix: Basic inequalities

In this appendix, we state some basic inequalities used in this paper.

LEMMA 5.1. *Let* $1\le p\le+\infty$. *Assume that* $\|\nabla^{\frac{d}{p}}\xi\|_{L^p(\mathbb{R}^d)}\le 1$ *and that* $\psi(s)$ *is a smooth function of* s *with bound derivatives of any order. Then for any integer* $m\ge\frac{d}{p}$,*there exists a constant* $C>0$ *such that*

$$\|\nabla^m\big(\psi(\xi)\big)\|_{L^p(\mathbb{R}^d)}\le C\|\nabla^m\xi\|_{L^p(\mathbb{R}^d)}.$$

Proof. Notice that for $m\ge 1$, we have

$$\nabla^m\big(\psi(\xi)\big)=\text{a sum of product } \psi^{\gamma_1,\dots,\gamma_n}(\xi)\nabla^{\gamma_1}\xi\cdots\nabla^{\gamma_n}\xi,$$

where the functions $\psi^{\gamma_1,\dots,\gamma_n}(\xi)$ are some derivatives of $g(\xi)$ and $1\le\gamma_i\le m$ $(i=1,\dots,n)$ with $\gamma_1+\cdots+\gamma_n=m$. It then follows from Holder's inequality and the interpolation that

$$\begin{aligned}\|\nabla^m\big(\psi(\xi)\big)\|_{L^p}&\le C\|\nabla^{\gamma_1}\xi\cdots\nabla^{\gamma_n}\xi\|_{L^p}\\ &\le C\|\nabla^{\gamma_1}\xi\|_{L^{\frac{mp}{\gamma_1}}}\cdots\|\nabla^{\gamma_n}\xi\|_{L^{\frac{mp}{\gamma_n}}}\\ &\le C\Big(\|\nabla^{\frac{d}{p}}\xi\|_{L^p}^{1-\frac{\gamma_1}{m}}\|\nabla^m\xi\|_{L^p}^{\frac{\gamma_1}{m}}\Big)\cdots\Big(\|\nabla^{\frac{d}{p}}\xi\|_{L^p}^{1-\frac{\gamma_n}{m}}\|\nabla^m\xi\|_{L^p}^{\frac{\gamma_n}{m}}\Big)\\ &=C\|\nabla^{\frac{d}{p}}\xi\|_{L^p}^{n-1}\|\nabla^m\xi\|_{L^p}.\end{aligned}$$

Since $\|\nabla^{\frac{d}{p}}\xi\|_{L^p}\le 1$, this completes the proof of Lemma 5.1. □

LEMMA 5.2. (Interpolation Inequality) *For any* $f\in\mathcal{S}(\mathbb{R}^3)$*, the Schwartz class, there exists a constant* $C>0$ *such that*

$$\|f\|_{L^\infty(\mathbb{R}^3)}\le C\|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac12}\|\nabla^2 f\|_{L^2(\mathbb{R}^3)}^{\frac12}.$$

Proof. By using Bernstein's inequality, we have

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} \|\dot{\Delta}_j f\|_{L^\infty(\mathbb{R}^3)} \\
&\leq C \sum_{j=-\infty}^{\infty} 2^{\frac{3}{2}j} \|\dot{\Delta}_j f\|_2 \\
&= C\Big(\sum_{j=-\infty}^{N} 2^{\frac{1}{2}j} 2^j \|\dot{\Delta}_j f\|_{L^2(\mathbb{R}^3)} + \sum_{j=N+1}^{\infty} 2^{-\frac{1}{2}j} 2^{2j} \|\dot{\Delta}_j f\|_{L^2(\mathbb{R}^3)} \Big) \\
&\leq C\Big(2^{\frac{N}{2}} \Big(\sum_{j=-\infty}^{N} 2^{2j} \|\dot{\Delta}_j f\|^2_{L^2(\mathbb{R}^3)} \Big)^{\frac{1}{2}} + 2^{-\frac{N}{2}} \Big(\sum_{j=N+1}^{\infty} 2^{4j} \|\dot{\Delta}_j f\|^2_{L^2(\mathbb{R}^3)} \Big)^{\frac{1}{2}} \Big) \\
&\leq C\Big(2^{\frac{N}{2}} \|\nabla f\|_{L^2(\mathbb{R}^3)} + 2^{-\frac{N}{2}} \|\nabla^2 f\|_{L^2(\mathbb{R}^3)} \Big).
\end{aligned}$$

We optimize in N by taking N of the order $\log_2 \frac{\|\nabla^2 f\|_{L^2(\mathbb{R}^3)}}{\|\nabla f\|_{L^2(\mathbb{R}^3)}}$ and obtain

$$\sum_{j=-\infty}^{\infty} \|\dot{\Delta}_j f\|_{L^\infty(\mathbb{R}^3)} \leq C \|\nabla f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \|\nabla^2 f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}.$$

Plugging this estimate into the Littlewood-Payley decomposition $f = \sum_{j=-\infty}^{\infty} \dot{\Delta}_j f$, we obtain the desired estimates. □

LEMMA 5.3. (Commutator and Product Estimates) (*see Lemma 3.1 in* [16]). *Suppose that* $s > 0$ *and* $p \in (1, +\infty)$. *For any* $f, g \in \mathcal{S}$, *the Schwartz class, there exists a constant* $C > 0$ *such that*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C\Big(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}} \Big)$$

and

$$\|\Lambda^s(fg)\|_{L^p} \leq C\Big(\|f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}} \Big)$$

with $p_2, p_3 \in (1, +\infty)$ *such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

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