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Well-posedness of the Ostrovsky–Hunter Equation under the combined effects of dissipation and short-wave dispersion

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Abstract. The Ostrovsky–Hunter equation provides a model for small-amplitude long waves in a rotating fluid of finite depth. It is a nonlinear evolution equation. In this paper we study the well-posedness for the Cauchy problem associated with this equation in presence of some weak dissipation effects.

1. Introduction

Many physical problems (such as nonlinear shallow-water waves and wave motion in plasmas) are described by the following nonlinear evolution equation

$$\partial_t u + \partial_x f(u) - \alpha\partial_{xx}^2 u - \beta\partial_{xxx}^3 u = 0, \quad \alpha, \beta \in \mathbb{R}, \quad f(u) = \frac{u^2}{2}, \tag{1.1}$$

which was derived by Korteweg–deVries (see [12]). (1.1) is also known as the Korteweg–de Vries–Burgers equation (see [2,9,26]), where $\alpha\partial_{xx}^2 u$ is a viscous dissipation term. If (1.1) describes the evolution of nonlinear shallow-water waves, then the function $u(t, x)$ is the amplitude of an appropriate linear long wave mode, with linear long wave speed C_0. However, when the effects of background rotation through the Coriolis parameter κ need to be taken into account, an extra term is needed, and (1.1) is replaced by

$$\partial_x(\partial_t u + \partial_x f(u) - \alpha\partial_{xx}^2 u - \beta\partial_{xxx}^3 u) = \gamma u, \tag{1.2}$$

where $\gamma = \frac{\kappa^2}{2C_0}$ (see [7,11]). If $\alpha = \beta = 0$, then (1.2) reads

$$\partial_x(\partial_t u + \partial_x f(u)) = \gamma u. \tag{1.3}$$

(1.3) is known under different names such as the reduced Ostrovsky equation [6, 23,25], the Ostrovsky–Hunter equation [1], the short-wave equation [10], and the

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Vakhnenko equation [20,24]. The well-posedness of (1.3) in class of discontinuous solutions has been proved in [3,4].

If $\alpha = 0$, (1.2) reads

$$\partial_x(\partial_t u + \partial_x f(u) - \beta\partial^3_{xxx} u) = \gamma u, \tag{1.4}$$

which is known as the Ostrovsky equation (see [22]). Mathematical properties of (1.4) were studied recently in many details, including the local and global well-posedness in energy space [8,15,18,28], stability of solitary waves [13,16,19], wave breaking [17], and convergence of solutions in the limit of the Korteweg–deVries equation [14,19].

Let us assume, in (1.2), that $\alpha = 1,\ \beta = 0$. Therefore, we have

$$\partial_x(\partial_t u + \partial_x f(u) - \partial^2_{xx} u) = \gamma u. \tag{1.5}$$

(1.5) describes the combined effects of dissipation and short-wave dispersion, and is analogous to the (1.1) for dissipative long waves. It can be deduced considering two asymptotic expansions of the shallow-water equations, first with respect to the rotation frequency and then with respect to the amplitude of the waves (see [7,11]).

We are interested in the initial value problem for (1.5), so we augment (1.5) with the initial condition

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}. \tag{1.6}$$

Integrating (1.5) on $(-\infty, x)$ we gain the integro-differential formulation of problem (1.5), and (1.6) (see [18])

$$\begin{cases} \partial_t u + \partial_x f(u) = \gamma \int_{-\infty}^{x} u(t, y)\mathrm{d}y + \partial^2_{xx} u, & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.7}$$

that is equivalent to

$$\begin{cases} \partial_t u + \partial_x f(u) = \gamma P + \partial^2_{xx} u, & t > 0,\ x \in \mathbb{R}, \\ \partial_x P = u, & t > 0,\ x \in \mathbb{R}, \\ P(t, -\infty) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.8}$$

On the initial datum we assume that

$$u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x)\mathrm{d}x = 0. \tag{1.9}$$

On the function

$$P_0(x) = \int_{-\infty}^{x} u_0(y)\mathrm{d}y, \quad x \in \mathbb{R}, \tag{1.10}$$

we assume that

$$\begin{aligned} \|P_0\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left(\int_{-\infty}^{x} u_0(y)\mathrm{d}y \right)^2 \mathrm{d}x < \infty, \\ \int_{\mathbb{R}} P_0(x)\mathrm{d}x &= \int_{\mathbb{R}} \left(\int_{-\infty}^{x} u_0(y)\mathrm{d}y \right) \mathrm{d}x = 0. \end{aligned} \tag{1.11}$$

The flux f is assumed to be smooth, genuinely nonlinear, and subquadratic, namely:

$$f \in C^2(\mathbb{R}), \quad |\{f'' = 0\}| = 0, \quad |f'(u)| \le C_0|u|, \quad u \in \mathbb{R}, \tag{1.12}$$

for some a positive constant C_0.

The main result of this paper is the following theorem.

THEOREM 1.1. *Let $T > 0$. Assume (1.9), (1.10), (1.11) and (1.12). Then there exists a unique classical solution for the Cauchy problem of (1.7), or (1.8), u such that*

$$\begin{aligned} &u \in L^\infty((0,T)\times\mathbb{R}) \cap C((0,T); H^\ell(\mathbb{R})), \quad \forall \ell \in \mathbb{N}, \\ &P \in L^\infty((0,T)\times\mathbb{R}) \cap L^2((0,T)\times\mathbb{R}), \\ &\int_{\mathbb{R}} u(t,x)\mathrm{d}x = 0, \quad t \ge 0. \end{aligned} \tag{1.13}$$

Moreover, if u and v are two solutions of (1.7), or (1.8), the following inequality holds

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^2(\mathbb{R})} \le e^{C(T)t} \|u_0 - v_0\|_{L^2(\mathbb{R})}, \tag{1.14}$$

for some suitable $C(T) > 0$, and every $0 \le t \le T$.

The existence argument is based on passing to limit using a compensated compactness argument [27] in the parabolic-elliptic approximation of (1.8):

$$\partial_t u_\delta + \partial_x f(u_\delta) = \gamma P_\delta + \partial^2_{xx} u_\delta, \quad -\delta\partial^2_{xx} P_\delta + \partial_x P_\delta = u_\delta. \tag{1.15}$$

In (1.8) P is not a real unknown of the problem, indeed we can rewrite (1.3) as the integro-differential problem (1.7). The same applies to (1.15). Indeed P_δ has the integral form

$$P_\delta(t,x) = \frac{1}{2\sqrt{\delta}} \int_{\mathbb{R}} e^{-\frac{|x-y|}{2\sqrt{\delta}}} u_\delta(t,y)\mathrm{d}y$$

and we can rewrite (1.15) in the integro-differential form

$$\partial_t u_\delta + \partial_x f(u_\delta) = \frac{\gamma}{2\sqrt{\delta}} \int_{\mathbb{R}} e^{-\frac{|x-y|}{2\sqrt{\delta}}} u_\delta(t,y)\mathrm{d}y + \partial^2_{xx} u_\delta.$$

The paper is organized as follows. In Sect. 2 we prove several a priori estimates on the parabolic-elliptic. Those play a key role in the proof of our main result, that is given in Sect. 3.

2. Parabolic-elliptic approximation

Our existence argument is based on passing to the limit in a parabolic-elliptic approximation. Fix $0 < \delta < 1$, and let $u_\delta = u_\delta(t, x)$ be the unique classical solution of the following mixed problem [5]:

$$\begin{cases} \partial_t u_\delta + \partial_x f(u_\delta) = \gamma P_\delta + \partial_{xx}^2 u_\delta, & t > 0,\ x \in \mathbb{R}, \\ -\delta \partial_{xx}^2 P_\delta + \partial_x P_\delta = u_\delta, & t > 0,\ x \in \mathbb{R}, \\ u_\delta(0, x) = u_{\delta,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $u_{\delta,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} &\left\| u_{\delta,0} \right\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})}, \quad \left\| u_{\delta,0} \right\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})}, \\ &\left\| \partial_x u_{\delta,0} \right\|_{L^2(\mathbb{R})} \le C_0, \quad \left\| \partial_{xx}^2 u_{\delta,0} \right\|_{L^2(\mathbb{R})} \le C_0 \\ &\left\| P_{\delta,0} \right\|_{L^2(\mathbb{R})} \le \|P_0\|_{L^2(\mathbb{R})}, \quad \delta \left\| \partial_x P_{\delta,0} \right\|_{L^2(\mathbb{R})} \le C_0, \end{aligned} \tag{2.2}$$

and C_0 is a constant independent on δ.

Let us prove some a priori estimates on u_δ and P_δ, denoting with C_0 the constants which depend on the initial data, and $C(T)$ the constants which depend also on T.

LEMMA 2.1. *For each $t \in (0, \infty)$,*

$$P_\delta(t, \infty) = \partial_x P_\delta(t, -\infty) = \partial_x P_\delta(t, \infty) = 0. \tag{2.3}$$

Moreover,

$$\delta^2 \left\| \partial_{xx}^2 P_\delta(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \|\partial_x P_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \|u_\delta(t, \cdot)\|_{L^2(\mathbb{R})}^2. \tag{2.4}$$

Proof. We begin by proving that (2.3) holds.

Differentiating the first equation of (2.1) with respect to x, we have

$$\partial_x(\partial_t u_\delta + \partial_x f(u_\delta) - \partial_{xx}^2 u_\delta) = \gamma \partial_x P_\delta. \tag{2.5}$$

From the smoothness of u_δ, it follows from (2.1) and (2.5) that

$$\begin{aligned} &\lim_{x\to\infty} (\partial_t u_\delta + \partial_x f(u_\delta) - \partial_{xx}^2 u_\delta) = \gamma P_\delta(t, \infty) = 0, \\ &\lim_{x\to-\infty} \partial_x(\partial_t u_\delta + \partial_x f(u_\delta) - \partial_{xx}^2 u_\delta) = \gamma \partial_x P_\delta(t, -\infty) = 0, \\ &\lim_{x\to\infty} \partial_x(\partial_t u_\delta + \partial_x f(u_\delta) - \partial_{xx}^2 u_\delta) = \gamma \partial_x P_\delta(t, \infty) = 0, \end{aligned}$$

which gives (2.3).

Let us show that (2.4) holds. Squaring the equation for P_δ in (2.1), we get

$$\delta^2 (\partial_{xx}^2 P_\delta)^2 + (\partial_x P_\delta)^2 - \delta \partial_x((\partial_x P_\delta)^2) = u_\delta^2.$$

Therefore, (2.4) follows from (2.3) and an integration on $\mathbb{R}$. □

LEMMA 2.2. *For each $t \in (0, \infty)$,*

$$\sqrt{\delta}\,\|\partial_x P_\delta(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \|u_\delta(t,\cdot)\|_{L^2(\mathbb{R})}\,, \tag{2.6}$$

$$\int_{\mathbb{R}} u_\delta(t,x) P_\delta(t,x)\mathrm{d}x \le \|u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})}\,. \tag{2.7}$$

Proof. We begin by proving that (2.6) holds.

Observe that

$$0 \le (-\delta\partial^2_{xx} P_\delta + \partial_x P_\delta)^2 = \delta^2(\partial^2_{xx} P_\delta)^2 + (\partial_x P_\delta)^2 - \delta\partial_x((\partial_x P_\delta)^2),$$

that is,

$$\delta\partial_x((\partial_x P_\delta)^2) \le \delta^2(\partial^2_{xx} P_\delta)^2 + (\partial_x P_\delta)^2. \tag{2.8}$$

Integrating (2.8) on $(-\infty, x)$, we have

$$\begin{aligned}\delta(\partial_x P_\delta)^2 &\le \delta^2 \int_{-\infty}^{x} (\partial^2_{xx} P_\delta)^2\mathrm{d}x + \int_{-\infty}^{x} (\partial_x P_\delta)^2\mathrm{d}x\\ &\le \delta^2 \int_{\mathbb{R}} (\partial^2_{xx} P_\delta)^2\mathrm{d}x + \int_{\mathbb{R}} (\partial_x P_\delta)^2\mathrm{d}x.\end{aligned} \tag{2.9}$$

It follows from (2.4) and (2.9) that

$$\delta(\partial_x P_\delta)^2 \le \delta^2 \int_{\mathbb{R}} (\partial^2_{xx} P_\delta)^2\mathrm{d}x + \int_{\mathbb{R}} (\partial_x P_\delta)^2\mathrm{d}x = \|u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})}\,.$$

Therefore,

$$\sqrt{\delta}|\partial_x P_\delta(t,x)| \le \|u_\delta(t,\cdot)\|_{L^2(\mathbb{R})}\,,$$

which gives (2.6).

Finally, we prove (2.7). Multiplying by P_δ the equation for P_δ in (2.1), we get

$$-\delta P_\delta\partial^2_{xx} P_\delta + P_\delta\partial_x P_\delta = u_\delta P_\delta.$$

An integration on $\mathbb{R}$ and (2.3) give

$$\begin{aligned}\int_{\mathbb{R}} u_\delta P_\delta\mathrm{d}x &= \frac{1}{2}\int_{\mathbb{R}} \partial_x(P_\varepsilon)^2\mathrm{d}x - \delta\int_{\mathbb{R}} P_\delta\partial^2_{xx} P_\delta\mathrm{d}x\\ &= -\delta\int_{\mathbb{R}} P_\delta\partial^2_{xx} P_\delta\mathrm{d}x = \delta\int_{\mathbb{R}} (\partial_x P_\delta)^2\mathrm{d}x,\end{aligned}$$

that is

$$\int_{\mathbb{R}} u_\delta P_\delta\mathrm{d}x = \delta\int_{\mathbb{R}} (\partial_x P_\delta)^2\mathrm{d}x.$$

Since $0 < \delta < 1$, from (2.4), we have (2.7). □

LEMMA 2.3. *For each $t \in (0, \infty)$, the following inequality holds*

$$\|u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2e^{2\gamma t}\int_0^t e^{-2\gamma s}\|\partial_x u_\delta(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le e^{2\gamma t}\|u_0\|^2_{L^2(\mathbb{R})}. \tag{2.10}$$

In particular, we have

$$\|\partial_x P_\delta(t,\cdot)\|_{L^2(\mathbb{R})},\ \delta\left\|\partial^2_{xx} P_\delta(t,\cdot)\right\|_{L^2(\mathbb{R})},\ \sqrt{\delta}\,\|\partial_x P_\delta(t,\cdot)\|_{L^\infty(\mathbb{R})} \le e^{\gamma t}\|u_0\|_{L^2(\mathbb{R})}. \tag{2.11}$$

Proof. Due to (2.1) and (2.7),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u_\delta^2\mathrm{d}x &= 2\int_{\mathbb{R}} u_\delta\partial_t u_\delta\mathrm{d}x\\ &= 2\int_{\mathbb{R}} u_\delta\partial^2_{xx}u_\delta\mathrm{d}x - 2\int_{\mathbb{R}} u_\delta f'(u_\delta)\partial_x u_\delta\mathrm{d}x + 2\gamma\int_{\mathbb{R}} u_\delta P_\delta\mathrm{d}x\\ &\le -2\int_{\mathbb{R}}(\partial_x u_\delta)^2\mathrm{d}x + 2\gamma\|u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

The Gronwall Lemma and (2.2) give (2.10).

Finally, (2.11) follows from (2.4), (2.6) and (2.10). □

LEMMA 2.4. *For each $t \ge 0$, we have that*

$$\int_0^{-\infty} P_\delta(t,x)\mathrm{d}x = a_\delta(t), \tag{2.12}$$

$$\int_0^{\infty} P_\delta(t,x)\mathrm{d}x = a_\delta(t), \tag{2.13}$$

where

$$a_\delta(t) = \frac{\delta}{\gamma}\partial^2_{tx}P_\delta(t,0) - \frac{1}{\gamma}\partial_t P_\delta(t,0) + \frac{1}{\gamma}f(0) - \frac{1}{\gamma}f(u_\delta(t,0)) + \frac{1}{\gamma}\partial_x u_\delta(t,0). \tag{2.14}$$

In particular,

$$\int_{\mathbb{R}} P_\delta(t,x)\mathrm{d}x = 0, \quad t \ge 0. \tag{2.15}$$

Proof. We begin by observing that, integrating the second equation of (2.1) on $(0, x)$, we have that

$$\int_0^x u_\delta(t,y)\mathrm{d}y = P_\delta(t,x) - P_\delta(t,0) - \delta\partial_x P_\delta(t,x) + \delta\partial_x P_\delta(t,0). \tag{2.16}$$

It follows from (2.3) that

$$\lim_{x\to-\infty}\int_0^x u_\delta(t,y)\mathrm{d}y = \int_0^{-\infty} u_\delta(t,x)\mathrm{d}x = \delta\partial_x P_\delta(t,0) - P_\delta(t,0). \tag{2.17}$$

Differentiating (2.17) with respect to t, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{-\infty} u_\delta(t,x)\mathrm{d}x = \int_0^{-\infty} \partial_t u_\delta(t,x)\mathrm{d}x = \delta\partial_{tx}^2 P_\delta(t,0) - \partial_t P_\delta(t,0). \tag{2.18}$$

Integrating the first equation of (2.1) on $(0, x)$, we obtain that

$$\begin{aligned}\int_0^x \partial_t u_\delta(t,y)\mathrm{d}y + f(u_\delta(t,x)) - f(u_\delta(t,0))&\\ - \partial_x u_\delta(t,x) + \partial_x u_\delta(t,0) = \gamma\int_0^x P_\delta(t,y)\mathrm{d}y.&\end{aligned} \tag{2.19}$$

Being u_δ a smooth solution of (2.1), we get

$$\lim_{x\to-\infty}\Big(f(u_\delta(t,x)) - \partial_x u_\delta(t,x)\Big) = f(0). \tag{2.20}$$

Sending $x \to -\infty$ in (2.19), from (2.18) and (2.20), we have

$$\begin{aligned}\gamma\int_0^{-\infty} P_\delta(t,x)\mathrm{d}x = \delta\partial_{tx}^2 P_\delta(t,0) - \partial_t P_\delta(t,0)&\\ + f(0) - f(u_\delta(t,0)) + \partial_x u_\delta(t,0),&\end{aligned}$$

which gives (2.12).

Let us show that (2.13) holds. We begin by observing that, for (2.3) and (2.16),

$$\int_0^{\infty} u_\delta(t,x)\mathrm{d}x = \delta\partial_x P_\delta(t,0) - P_\delta(t,0).$$

Therefore,

$$\lim_{x\to\infty}\int_0^x \partial_t u_\delta(t,y)\mathrm{d}y = \int_0^\infty \partial_t u_\delta(t,x)\mathrm{d}x = \delta\partial_{tx}^2 P_\delta(t,0) - \partial_t P_\delta(t,0). \tag{2.21}$$

Again by the regularity of u_δ,

$$\lim_{x\to\infty}\Big(f(u_\delta(t,x)) - \partial_x u_\delta(t,x)\Big) = f(0). \tag{2.22}$$

It follows from (2.19), (2.21) and (2.22) that

$$\begin{aligned}\gamma\int_0^{\infty} P_\delta(t,x)\mathrm{d}x = \delta\partial_{tx}^2 P_\delta(t,0) - \partial_t P_\delta(t,0)&\\ + f(0) - f(u_\delta(t,0)) + \partial_x u_\delta(t,0),&\end{aligned}$$

which gives (2.13).

Finally, we prove (2.15). It follows from (2.12) that

$$\int_{-\infty}^0 P_\delta(t,x)\mathrm{d}x = -a_\delta(t).$$

Therefore, for (2.13),

$$\int_{-\infty}^0 P_\delta(t,x)\mathrm{d}x + \int_0^\infty P_\delta(t,x) = \int_{\mathbb{R}} P_\delta(t,x)\mathrm{d}x = -a_\delta(t) + a_\delta(t) = 0,$$

that is (2.15). □

Lemma 2.4 says that $P_\delta(t,x)$ is integrable at $\pm\infty$. Therefore, for each $t \geq 0$, we can consider the following function

$$F_\delta(t,x) = \int_{-\infty}^{x} P_\delta(t,y)\mathrm{d}y. \tag{2.23}$$

LEMMA 2.5. *Let $T > 0$. There exists $C(T) > 0$, independent on δ, such that*

$$\|P_\delta\|_{L^\infty(I_{T,1})} \leq C(T), \tag{2.24}$$

$$\|P_\delta(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(T), \tag{2.25}$$

$$\delta\,\|\partial_x P_\delta(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(T), \tag{2.26}$$

where

$$I_{T,1} = (0,T) \times \mathbb{R}. \tag{2.27}$$

In particular, we have

$$\delta \left| \int_0^t\!\!\int_{\mathbb{R}} P_\delta \partial_{tx}^2 P_\delta \mathrm{d}s\mathrm{d}x \right| \leq C(T), \quad 0 < t < T. \tag{2.28}$$

Proof. Integrating the second equation of (2.1) on $(-\infty, x)$, for (2.3), we have that

$$\int_{-\infty}^{x} u_\delta(t,y)\mathrm{d}y = P_\delta(t,x) - \delta\partial_x P_\delta(t,x). \tag{2.29}$$

Differentiating (2.29) with respect to t, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{x} u_\delta(t,y)\mathrm{d}y = \int_{-\infty}^{x} \partial_t u_\delta(t,y)\mathrm{d}y = \partial_t P_\delta(t,x) - \delta\partial_{tx}^2 P_\delta(t,x). \tag{2.30}$$

It follows from an integration of the first equation of (2.1) on $(-\infty, x)$ and (2.23) that

$$\int_{-\infty}^{x} \partial_t u_\delta(t,y)\mathrm{d}y + f(u_\delta(t,x)) - \partial_x u_\delta(t,x) = \gamma F_\delta(t,x). \tag{2.31}$$

Due to (2.30) and (2.31), we have

$$\partial_t P_\delta(t,x) - \delta\partial_{tx}^2 P_\delta(t,x) = \gamma F_\delta(t,x) - f(u_\delta(t,x)) + \partial_x u_\delta(t,x). \tag{2.32}$$

Multiplying (2.32) by $P_\delta - \delta\partial_x P_\delta$, we have

$$\begin{aligned}(\partial_t P_\delta - \delta\partial_{tx}^2 P_\delta)(P_\delta - \delta\partial_x P_\delta) = \gamma F_\delta(P_\delta - \delta\partial_x P_\delta) \\ - f(u_\delta)(P_\delta - \delta\partial_x P_\delta) \\ + \partial_x u_\delta(P_\delta - \delta\partial_x P_\delta).\end{aligned} \tag{2.33}$$

Integrating (2.33) on $(0,x)$, we have

$$\int_0^x \partial_t P_\delta P_\delta \mathrm{d}y - \delta\int_0^x \partial_t P_\delta \partial_x P_\delta \mathrm{d}y$$

$$
\begin{aligned}
&- \delta \int_0^x P_\delta \partial_{tx}^2 P_\delta \mathrm{d}y + \delta^2 \int_0^x \partial_{tx}^2 P_\delta \partial_x P_\delta \mathrm{d}y \\
&= \gamma \int_0^x F_\delta P_\delta \mathrm{d}y - \gamma\delta \int_0^x F_\delta \partial_x P_\delta \mathrm{d}y \\
&\quad - \int_0^x f(u_\delta) P_\delta \mathrm{d}y + \delta \int_0^x f(u_\delta) \partial_x P_\delta \mathrm{d}y \\
&\quad + \int_0^x \partial_x u_\delta P_\delta \mathrm{d}y - \delta \int_0^x \partial_x u_\delta \partial_x P_\delta \mathrm{d}y.
\end{aligned} \tag{2.34}
$$

We observe that

$$
- \delta \int_0^x \partial_x P_\delta \partial_t P_\delta \mathrm{d}y = -\delta P_\delta \partial_t P_\delta + \delta P_\delta(t,0) \partial_t P_\delta(t,0) + \delta \int_0^x P_\delta \partial_{tx}^2 P_\delta \mathrm{d}y. \tag{2.35}
$$

Therefore, (2.34) and (2.35) give

$$
\begin{aligned}
&\int_0^x \partial_t P_\delta P_\delta \mathrm{d}y + \delta^2 \int_0^x \partial_{tx}^2 P_\delta \partial_x P_\delta \mathrm{d}y \\
&= \delta P_\delta \partial_t P_\delta - \delta P_\delta(t,0) \partial_t P_\delta(t,0) + \gamma \int_0^x F_\delta P_\delta \mathrm{d}y \\
&\quad - \gamma\delta \int_0^x F_\delta \partial_x P_\delta \mathrm{d}y - \int_0^x f(u_\delta) P_\delta \mathrm{d}y + \delta \int_0^x f(u_\delta) \partial_x P_\delta \mathrm{d}y \\
&\quad + \int_0^x \partial_x u_\delta P_\delta \mathrm{d}y - \delta \int_0^x \partial_x u_\delta \partial_x P_\delta \mathrm{d}y.
\end{aligned} \tag{2.36}
$$

Sending $x \to -\infty$, for (2.3), we get

$$
\begin{aligned}
&\int_0^{-\infty} \partial_t P_\delta P_\delta \mathrm{d}y + \delta^2 \int_0^{-\infty} \partial_{tx}^2 P_\delta \partial_x P_\delta \mathrm{d}y \\
&= -\delta P_\delta(t,0) \partial_t P_\delta(t,0) + \gamma \int_0^{-\infty} F_\delta P_\delta \mathrm{d}y \\
&\quad - \gamma\delta \int_0^{-\infty} F_\delta \partial_x P_\delta \mathrm{d}y - \int_0^{-\infty} f(u_\delta) P_\delta \mathrm{d}y \\
&\quad + \delta \int_0^{-\infty} f(u_\delta) \partial_x P_\delta \mathrm{d}y + \int_0^{-\infty} \partial_x u_\delta P_\delta \mathrm{d}y \\
&\quad - \delta \int_0^{-\infty} \partial_x u_\delta \partial_x P_\delta \mathrm{d}y,
\end{aligned} \tag{2.37}
$$

while sending $x \to \infty$,

$$
\begin{aligned}
&\int_0^{\infty} \partial_t P_\delta P_\delta \mathrm{d}y + \delta^2 \int_0^{\infty} \partial_{tx}^2 P_\delta \partial_x P_\delta \mathrm{d}y \\
&= -\delta P_\delta(t,0) \partial_t P_\delta(t,0) + \gamma \int_0^{\infty} F_\delta P_\delta \mathrm{d}y - \gamma\delta \int_0^{\infty} F_\delta \partial_x P_\delta \mathrm{d}y
\end{aligned}
$$

$$- \int_0^\infty f(u_\delta) P_\delta \mathrm{d}y + \delta \int_0^\infty f(u_\delta) \partial_x P_\delta \mathrm{d}y$$
$$+ \int_0^\infty \partial_x u_\delta P_\delta \mathrm{d}y - \delta \int_0^\infty \partial_x u_\delta \partial_x P_\delta \mathrm{d}y. \tag{2.38}$$

Since

$$\int_{\mathbb{R}} P_\delta \partial_t P_\delta \mathrm{d}x = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} P_\delta^2 \mathrm{d}x,$$
$$\delta^2 \int_{\mathbb{R}} \partial_{tx}^2 P_\delta \partial_x P_\delta \mathrm{d}x = \frac{\delta^2}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} (\partial_x P_\delta)^2 \mathrm{d}x,$$

it follows from (2.37) and (2.38) that

$$\begin{aligned}\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} P_\delta^2 \mathrm{d}x + \frac{\delta^2}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} (\partial_x P_\delta)^2 \mathrm{d}x \\ = \gamma \int_{\mathbb{R}} F_\delta P_\delta \mathrm{d}x - \gamma \delta \int_{\mathbb{R}} F_\delta \partial_x P_\delta \mathrm{d}x \\ - \int_{\mathbb{R}} f(u_\delta) P_\delta \mathrm{d}x + \delta \int_{\mathbb{R}} f(u_\delta) \partial_x P_\delta \mathrm{d}x \\ + \int_{\mathbb{R}} \partial_x u_\delta P_\delta \mathrm{d}x - \delta \int_{\mathbb{R}} \partial_x u_\delta \partial_x P_\delta \mathrm{d}x.\end{aligned} \tag{2.39}$$

Due to (2.15) and (2.23),

$$\begin{aligned}2\gamma \int_{\mathbb{R}} F_\delta P_\delta \mathrm{d}x = 2\gamma \int_{\mathbb{R}} F_\delta \partial_x F_\delta \mathrm{d}x = \gamma (F_\delta(t, \infty))^2 \\ = \gamma \left(\int_{\mathbb{R}} P_\delta(t, x) \mathrm{d}x \right)^2 = 0.\end{aligned} \tag{2.40}$$

(2.39) and (2.40) give

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{\mathbb{R}} P_\delta^2 \mathrm{d}x + \delta^2 \int_{\mathbb{R}} (\partial_x P_\delta)^2 \mathrm{d}x \right) \\ = -2\gamma\delta \int_{\mathbb{R}} F_\delta \partial_x P_\delta \mathrm{d}x - 2 \int_{\mathbb{R}} f(u_\delta) P_\delta \mathrm{d}x \\ + 2\delta \int_{\mathbb{R}} f(u_\delta) \partial_x P_\delta \mathrm{d}x + 2 \int_{\mathbb{R}} \partial_x u_\delta P_\delta \mathrm{d}x \\ - 2\delta \int_{\mathbb{R}} \partial_x u_\delta \partial_x P_\delta \mathrm{d}x.\end{aligned} \tag{2.41}$$

Thanks to (2.3), (2.15) and (2.23),

$$- 2\delta\gamma \int_{\mathbb{R}} \partial_x P_\delta F_\delta \mathrm{d}x = 2\delta\gamma \int_{\mathbb{R}} P_\delta \partial_x F_\delta \mathrm{d}x = 2\delta\gamma \int_{\mathbb{R}} P_\delta^2 \mathrm{d}x \le 2\gamma \int_{\mathbb{R}} P_\delta^2 \mathrm{d}x, \tag{2.42}$$

while for (2.3),

$$2 \int_{\mathbb{R}} \partial_x u_\delta P_\delta \mathrm{d}x = - 2 \int_{\mathbb{R}} u_\delta \partial_x P_\delta \mathrm{d}x. \tag{2.43}$$

Hence, from (1.12), (2.42) and (2.43), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} P_\delta^2 \mathrm{d}x + \delta^2 \int_{\mathbb{R}} (\partial_x P_\delta)^2 \mathrm{d}x\right)\\
&\quad\le 2\gamma \int_{\mathbb{R}} P_\delta^2 \mathrm{d}x - 2\int_{\mathbb{R}} f(u_\delta) P_\delta \mathrm{d}x + 2\delta \int_{\mathbb{R}} f(u_\delta)\partial_x P_\delta \mathrm{d}x\\
&\qquad - 2\int_{\mathbb{R}} u_\delta \partial_x P_\delta \mathrm{d}x - 2\delta \int_{\mathbb{R}} \partial_x u_\delta \partial_x P_\delta \mathrm{d}x\\
&\quad\le 2\gamma \int_{\mathbb{R}} P_\delta^2 \mathrm{d}x + 2\left|\int_{\mathbb{R}} f(u_\delta) P_\delta \mathrm{d}x\right| + 2\delta \left|\int_{\mathbb{R}} f(u_\delta)\partial_x P_\delta \mathrm{d}x\right|\\
&\qquad + 2\left|\int_{\mathbb{R}} u_\delta \partial_x P_\delta \mathrm{d}x\right| + 2\delta \left|\int_{\mathbb{R}} \partial_x u_\delta \partial_x P_\delta \mathrm{d}x\right|\\
&\quad\le 2\gamma \int_{\mathbb{R}} P_\delta^2 \mathrm{d}x + 2\int_{\mathbb{R}} |f(u_\delta)||P_\delta| \mathrm{d}x + 2\delta \int_{\mathbb{R}} |f(u_\delta)||\partial_x P_\delta| \mathrm{d}x\\
&\qquad + 2\int_{\mathbb{R}} |u_\delta||\partial_x P_\delta| \mathrm{d}x + 2\delta \int_{\mathbb{R}} |\partial_x u_\delta||\partial_x P_\delta| \mathrm{d}x\\
&\quad\le 2\gamma \int_{\mathbb{R}} P_\delta^2 \mathrm{d}x + 2C_0 \int_{\mathbb{R}} |P_\delta| u_\delta^2 \mathrm{d}x + 2C_0\delta \int_{\mathbb{R}} |\partial_x P_\delta| u_\delta^2 \mathrm{d}x\\
&\qquad + 2\int_{\mathbb{R}} |u_\delta||\partial_x P_\delta| \mathrm{d}x + 2\delta \int_{\mathbb{R}} |\partial_x u_\delta||\partial_x P_\delta| \mathrm{d}x.
\end{aligned}$$

From the Young inequality,

$$2\int_{\mathbb{R}} |\partial_x P_\delta||u_\delta| \le \|\partial_x P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$\begin{aligned}
2\delta \int_{\mathbb{R}} |\partial_x u_\delta||\partial_x P_\delta| \mathrm{d}x &= \int_{\mathbb{R}} \left|\frac{\partial_x u_\delta}{\sqrt{\gamma}}\right| |2\sqrt{\gamma}\delta \partial_x P_\delta| \mathrm{d}x\\
&\le \frac{1}{2\gamma} \|\partial_x u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\delta^2\gamma \|\partial_x P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} G(t) - 2\gamma G(t) \le \|u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ 2C_0 \int_{\mathbb{R}} |P_\delta| u_\delta^2 \mathrm{d}x\\
&+ 2C_0\delta \int_{\mathbb{R}} |\partial_x P_\delta| u_\delta^2 \mathrm{d}x + \|\partial_x P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&+ \frac{1}{2\gamma} \|\partial_x u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned} \tag{2.44}$$

where

$$G(t) = \|P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} + \delta^2 \|\partial_x P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{2.45}$$

We observe that, from (2.10),

$$2C_0 \int_{\mathbb{R}} |P_\delta| u_\delta^2 \mathrm{d}x \le C_0 e^{2\gamma t} \|P_\delta\|_{L^\infty(I_{T,1})}, \tag{2.46}$$

where $I_{T,1}$ is defined in (2.27). Since $0 < \delta < 1$, it follows from (2.10) and (2.11) that

$$2C_0\delta\int_{\mathbb{R}}|\partial_x P_\delta|u_\delta^2\mathrm{d}x \le 2C_0\delta\,\|\partial_x P_\delta(t,\cdot)\|_{L^\infty(\mathbb{R})}\,\|u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} \le 2\sqrt{\delta}C_0e^{3\gamma t} \le C_0e^{3\gamma t}. \tag{2.47}$$

Again by (2.11), we have that

$$\|\partial_x P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C_0e^{2\gamma t}. \tag{2.48}$$

Therefore, (2.10), (2.47) and (2.48) give

$$\frac{\mathrm{d}}{\mathrm{d}t}G(t) - 2\gamma G(t) \le C_0\left(\|P_\delta\|_{L^\infty(I_{T,1})}+1\right)e^{2\gamma t} + C_0e^{3\gamma t} + \frac{1}{2\gamma}\|\partial_x u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

The Gronwall Lemma, (2.2), (2.10) and (2.45) give

$$\begin{aligned}
&\|P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} + \delta^2\|\partial_x P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad\le \|P_0\|^2_{L^2(0,\infty)}e^{2\gamma t} + \left(\|P_\delta\|_{L^\infty(I_{T,1})}+1\right)te^{2\gamma t} + C_0te^{3\gamma t}\\
&\qquad+\frac{e^{2\gamma t}}{2\gamma}\int_0^t e^{-2\gamma s}\|\partial_x u_\delta(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\quad\le \|P_0\|^2_{L^2(0,\infty)}e^{2\gamma t} + \left(\|P_\delta\|_{L^\infty(I_{T,1})}+1\right)te^{2\gamma t} + C_0te^{3\gamma t} + C_0e^{2\gamma t}.
\end{aligned}$$

Hence,

$$\|P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} + \delta^2\|\partial_x P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T)\left(\|P_\delta\|_{L^\infty(I_{T,1})}+1\right). \tag{2.49}$$

Due to (2.11), (2.49) and the Hölder inequality,

$$\begin{aligned}
P_\delta^2(t,x) &\le 2\int_{\mathbb{R}}|P_\delta||\partial_x P_\delta|\mathrm{d}x \le 2\|P_\delta(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x P_\delta(t,\cdot)\|_{L^2(\mathbb{R})}\\
&\le 2\sqrt{C(T)\left(\|P_\delta\|_{L^\infty(I_{T,1})}+1\right)}\sqrt{C_0}e^{\gamma t} \le C(T)\left(\|P_\delta\|_{L^\infty(I_{T,1})}+1\right).
\end{aligned}$$

Therefore,

$$\|P_\delta\|^2_{L^\infty(I_{T,1})} - C(T)\|P_\delta\|_{L^\infty(I_{T,1})} - C(T) \le 0,$$

which gives (2.24). (2.25) and (2.26) follow from (2.24) and (2.49).

Let us show that (2.28) holds. Multiplying (2.32) by P_δ, an integration on $\mathbb{R}$ and (2.40) give

$$\begin{aligned}
2\delta\int_{\mathbb{R}}\partial^2_{tx}P_\delta P_\delta\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\|P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\gamma\int_{\mathbb{R}}F_\delta P_\delta\mathrm{d}x\\
&\quad+2\int_{\mathbb{R}}f(u_\delta)P_\delta\mathrm{d}x - 2\int_{\mathbb{R}}\partial_x u_\delta P_\delta\mathrm{d}x\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\|P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\int_{\mathbb{R}}f(u_\delta)P_\delta\mathrm{d}x - 2\int_{\mathbb{R}}\partial_x u_\delta P_\delta\mathrm{d}x.
\end{aligned}$$

An integration on $(0, t)$ gives

$$2\delta \int_0^t\int_{\mathbb{R}} \partial_{tx}^2 P_\delta P_\delta \mathrm{d}x = \|P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} - \|P_{\varepsilon,\delta,0}\|^2_{L^2(\mathbb{R})} + 2\int_0^t\int_{\mathbb{R}} f(u_\delta)P_\delta \mathrm{d}x - 2\int_0^t\int_{\mathbb{R}} \partial_x u_\delta P_\delta \mathrm{d}x.$$

It follows from (1.12), (2.10), (2.24) and (2.25) that

$$\begin{aligned} 2\delta \left|\int_0^t\int_{\mathbb{R}} \partial_{tx}^2 P_\delta P_\delta \mathrm{d}s\mathrm{d}x\right| &\le \|P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|P_{\varepsilon,\delta,0}\|^2_{L^2(\mathbb{R})} \\ &\quad + 2\int_0^t\int_{\mathbb{R}} |f(u_\delta)||P_\delta|\mathrm{d}s\mathrm{d}x \\ &\quad + 2\int_0^t\int_{\mathbb{R}} |\partial_x u_\delta||P_\delta|\mathrm{d}s\mathrm{d}x \\ &\le \|P_{\delta,0}\|^2_{L^2(\mathbb{R})} + 2C(T)\int_0^t\int_{\mathbb{R}} u_\delta^2 \mathrm{d}s\mathrm{d}x \\ &\quad + 2\int_0^t\int_{\mathbb{R}} |\partial_x u_\delta||P_\delta|\mathrm{d}s\mathrm{d}x + C(T) \\ &\le \|P_{\delta,0}\|^2_{L^2(\mathbb{R})} + C(T) \\ &\quad + 2\int_0^t\int_{\mathbb{R}} |\partial_x u_\delta||P_\delta|\mathrm{d}s\mathrm{d}x. \end{aligned}$$

Observe that, thanks to (2.10),

$$\begin{aligned} &\int_0^t \|\partial_x u_\delta(s,\cdot)\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\ &\quad \le e^{2\gamma t}\int_0^t e^{-2\gamma s}\|\partial_x u_\delta(s,\cdot)\|^2_{L^2(\mathbb{R})} \mathrm{d}s \le C(T). \end{aligned} \tag{2.50}$$

Due to the Young inequality,

$$\begin{aligned} &2\int_{\mathbb{R}} |\partial_x u_\delta||P_\delta|\mathrm{d}s\mathrm{d}x \\ &\quad \le \|P_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\quad \le C(T) + \|\partial_x u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})}. \end{aligned} \tag{2.51}$$

Then, from (2.50) and (2.51), we have that

$$\begin{aligned} &2\int_0^t\int_{\mathbb{R}} |P_\delta||\partial_x u_\delta|\mathrm{d}s\mathrm{d}x \\ &\quad \le \int_0^t \|P_\delta(s,\cdot)\|^2_{L^2(\mathbb{R})} \mathrm{d}s + \int_0^t \|\partial_x u_\delta(s,\cdot)\|^2_{L^2(\mathbb{R})} \mathrm{d}s \le C(T). \end{aligned}$$

Therefore,

$$2\delta \left|\int_0^t\int_{\mathbb{R}} P_\delta \partial_{tx}^2 P_\delta \mathrm{d}s\mathrm{d}x\right| \le \|P_{\varepsilon,0}\|^2_{L^2(\mathbb{R})} + C(T),$$

which gives (2.28). □

LEMMA 2.6. *Let $T > 0$. Then,*

$$\|u_\delta\|_{L^\infty(I_{T,1})} \leq \|u_0\|_{L^\infty(\mathbb{R})} + C(T), \tag{2.52}$$

where $I_{T,1}$ is defined in (2.27).

Proof. Due to (2.1) and (2.24),

$$\partial_t u_\delta + \partial_x f(u_\delta) - \partial_{xx}^2 u_\delta \leq \gamma C(T).$$

Since the map

$$\mathcal{F}(t) := \|u_0\|_{L^\infty(\mathbb{R})} + \gamma C(T)t,$$

solves the equation

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \gamma C(T)$$

and

$$\max\{u_\delta(0,x), 0\} \leq \mathcal{F}(t), \quad (t,x) \in I_{T,1},$$

the comparison principle for parabolic equations implies that

$$u_\delta(t,x) \leq \mathcal{F}(t), \quad (t,x) \in I_{T,1}.$$

In a similar way we can prove that

$$u_\delta(t,x) \geq -\mathcal{F}(t), \quad (t,x) \in I_{T,1}.$$

Therefore,

$$|u_\delta(t,x)| \leq \|u_0\|_{L^\infty(\mathbb{R})} + \gamma C(T)t \leq \|u_0\|_{L^\infty(\mathbb{R})} + C(T),$$

which gives (2.52). □

LEMMA 2.7. *Let $T > 0$ and $0 < \delta < 1$. We have that*

$$\|\partial_x u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} + \int_0^t \left\|\partial_{xx}^2 u_\delta(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq C(T). \tag{2.53}$$

Proof. Let $0 < t < T$. Multiplying (2.1) by $-\partial_{xx}^2 u_\delta$, we have

$$\begin{aligned} &-\partial_{xx}^2 u_\delta \partial_t u_\delta + (\partial_{xx}^2 u_\delta)^2 \\ &\quad = -\gamma P_\delta \partial_{xx}^2 u_\delta - f'(u_\delta)\partial_x u_\delta \partial_{xx}^2 u_\delta. \end{aligned} \tag{2.54}$$

Since

$$-\int_{\mathbb{R}} \partial_{xx}^2 u_\delta \partial_t u_\delta \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}} (\partial_x u_\delta)^2\right),$$

integrating (2.54) on $\mathbb{R}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} (\partial_x u_\delta)^2 \mathrm{d}x\right) + 2\int_{\mathbb{R}} (\partial_{xx}^2 u_\delta)^2 \mathrm{d}x$$

$$
\begin{aligned}
&= -2\gamma \int_{\mathbb{R}} P_\delta \partial_{xx}^2 u_\delta \mathrm{d}x \\
&\quad - 2 \int_{\mathbb{R}} f'(u_\delta) \partial_x u_\delta \partial_{xx}^2 u_\delta \mathrm{d}x.
\end{aligned}
$$

Due to (2.10), (2.25), (2.52) and the Young inequality,

$$
\begin{aligned}
&- 2\gamma \int_{\mathbb{R}} P_\delta \partial_{xx}^2 u_\delta \mathrm{d}x \\
&\quad \le 2\gamma \left| \int_{\mathbb{R}} P_\delta \partial_{xx}^2 u_\delta \mathrm{d}x \right| \\
&\quad \le 2 \int_{\mathbb{R}} \left| \sqrt{2}\gamma P_\delta \right| \left| \frac{\partial_{xx}^2 u_\delta}{\sqrt{2}} \right| \mathrm{d}x \\
&\quad \le 2\gamma^2 \left\| P_\delta(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \left\| \partial_{xx}^2 u_\delta(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\quad \le C(T) + \frac{1}{2} \left\| \partial_{xx}^2 u_\delta(t,\cdot) \right\|_{L^2(\mathbb{R})}^2, \\
&- 2 \int_{\mathbb{R}} f'(u_\delta) \partial_x u_\delta \partial_{xx}^2 u_\delta \mathrm{d}x \\
&\quad \le 2 \left| \int_{\mathbb{R}} f'(u_\delta) \partial_x u_\delta \partial_{xx}^2 u_\delta \mathrm{d}x \right| \\
&\quad \le 2 \int_{\mathbb{R}} \left| \sqrt{2} f'(u_\delta) \partial_x u_\delta \right| \left| \frac{\partial_{xx}^2 u_\delta}{\sqrt{2}} \right| \mathrm{d}x \\
&\quad \le 2 \int_{\mathbb{R}} (f'(u_\delta))^2 (\partial_x u_\delta^2) + \frac{1}{2} \int_{\mathbb{R}} (\partial_{xx}^2 u_\delta)^2 \mathrm{d}x \\
&\quad \le 2 \left\| f' \right\|_{L^\infty(I_{T,2})}^2 \left\| \partial_x u_\delta(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \left\| \partial_{xx}^2 u_\delta(t,\cdot) \right\|_{L^2(\mathbb{R})}^2,
\end{aligned}
$$

where

$$
I_{T,2} = \left(- \left\| u_0 \right\|_{L^\infty(\mathbb{R})} - C(T), \left\| u_0 \right\|_{L^\infty(\mathbb{R})} + C(T) \right). \tag{2.55}
$$

Therefore,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(\left\| \partial_x u_\delta(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \right) + 2 \left\| (\partial_{xx}^2 u_\delta(t,\cdot)) \right\|_{L^2(\mathbb{R})}^2 \\
&\quad \le \left\| \partial_{xx}^2 u_\delta(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \left\| f' \right\|_{L^\infty(I_{T,2})}^2 \left\| \partial_x u_\delta(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T),
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(\left\| \partial_x u_\delta(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \right) + \left\| \partial_{xx}^2 u_\delta(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\quad \le \left\| f' \right\|_{L^\infty(I_{T,2})}^2 \left\| \partial_x u_\delta(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T).
\end{aligned}
$$

An integration on $(0, t)$ and (2.2) give

$$\begin{aligned}&\|\partial_x u_\delta(t,\cdot)\|^2_{L^2(\mathbb{R})} + \int_0^t \left\|\partial^2_{xx} u_\delta(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s\\ &\quad\le 2\left\|f'\right\|^2_{L^\infty(I_{T,2})} \int_0^t \|\partial_x u_\delta(s,\cdot)\|^2_{L^2(\mathbb{R})} \mathrm{d}s + C(T).\end{aligned} \tag{2.56}$$

(2.53) follows from (2.50) and (2.56). □

LEMMA 2.8. *Let $T > 0$ and $0 < \delta < 1$. We have that*

$$\|\partial_x u_\delta\|_{L^\infty(I_{T,1})} \le C(T), \tag{2.57}$$

where $I_{T,1}$ is defined in (2.27). Moreover,

$$\left\|\partial^2_{xx} u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \left\|\partial^3_{xxx} u_\delta(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \le C(T). \tag{2.58}$$

Proof. Let $0 < t < T$. Multiplying (2.1) by $\partial^4_{xxxx} u_\delta$, we have

$$\begin{aligned}&\partial^4_{xxxx} u_\delta \partial_t u_\delta - \partial^4_{xxxx} u_\delta \partial^2_{xx} u_\delta\\ &\quad= \gamma P_\delta \partial^4_{xxxx} u_\delta - f'(u_\delta)\partial_x u_\delta \partial^4_{xxxx} u_\delta.\end{aligned} \tag{2.59}$$

Since

$$\begin{aligned}\int_{\mathbb{R}} \partial^4_{xxxx} u_\delta \partial_t u_\delta \mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}} (\partial^2_{xx} u_\delta)^2 \mathrm{d}x\right),\\ -\int_{\mathbb{R}} \partial^4_{xxxx} u_\delta \partial^2_{xx} u_\delta \mathrm{d}x &= \int_{\mathbb{R}} (\partial^3_{xxx} u_\delta)^2 \mathrm{d}x,\\ \gamma \int_{\mathbb{R}} P_\delta \partial^4_{xxxx} u_\delta \mathrm{d}x &= -\gamma \int_{\mathbb{R}} \partial_x P_\delta \partial^3_{xxx} u_\delta \mathrm{d}x,\\ -\int_{\mathbb{R}} f'(u_\delta)\partial_x u_\delta \partial^4_{xxxx} u_\delta \mathrm{d}x &= \int_{\mathbb{R}} f''(u_\delta)(\partial_x u_\delta)^2 \partial^3_{xxx} u_\delta \mathrm{d}x\\ &\quad+ \int_{\mathbb{R}} f'(u_\delta)\partial^2_{xx} u_\delta \partial^3_{xxx} u_\delta \mathrm{d}x,\end{aligned}$$

integrating (2.54) on $\mathbb{R}$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} (\partial^2_{xx} u_\delta)^2 \mathrm{d}x\right) + 2\int_{\mathbb{R}} (\partial^3_{xxx} u_\delta)^2 \mathrm{d}x\\ &\quad= -2\gamma \int_{\mathbb{R}} \partial_x P_\delta \partial^3_{xxx} u_\delta \mathrm{d}x\\ &\qquad+ 2\int_{\mathbb{R}} f''(u_\delta)(\partial_x u_\delta)^2 \partial^3_{xxx} u_\delta \mathrm{d}x\\ &\qquad+ 2\int_{\mathbb{R}} f'(u_\delta)\partial^2_{xx} u_\delta \partial^3_{xxx} u_\delta \mathrm{d}x.\end{aligned}$$

Due to (2.11), (2.52), (2.53) and the Young inequality,

$$\begin{aligned}
&-2\gamma\int_{\mathbb{R}}\partial_x P_\delta\partial_{xxx}^3 u_\delta \mathrm{d}x\\
&\quad\le 2\gamma\left|\int_{\mathbb{R}}\partial_x P_\delta\partial_{xxx}^3 u_\delta \mathrm{d}x\right|\\
&\quad\le 2\int_{\mathbb{R}}\left|\sqrt{3}\gamma\partial_x P_\delta\right|\left|\frac{\partial_{xxx}^3 u_\delta}{\sqrt{3}}\right|\mathrm{d}x\\
&\quad\le 3\gamma^2\left\|\partial_x P_\delta(t.\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{1}{3}\left\|\partial_{xxx}^3 u_\delta(t.\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)+\frac{1}{3}\left\|\partial_{xxx}^3 u_\delta(t.\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2\int_{\mathbb{R}} f''(u_\delta)(\partial_x u_\delta)^2\partial_{xxx}^3 u_\delta \mathrm{d}x\\
&\quad\le 2\left|\int_{\mathbb{R}} f''(u_\delta)(\partial_x u_\delta)^2\partial_{xxx}^3 u_\delta \mathrm{d}x\right|\\
&\quad\le 2\int_{\mathbb{R}}\left|\sqrt{3}f''(u_\delta)(\partial_x u_\delta)^2\right|\left|\frac{\partial_{xxx}^3 u_\delta}{\sqrt{3}}\right|\mathrm{d}x\\
&\quad\le 3\int_{\mathbb{R}}(f''(u_\delta))^2(\partial_x u_\delta)^4\mathrm{d}x+\frac{1}{3}\left\|\partial_{xxx}^3 u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le 3\left\|f''\right\|^2_{L^\infty(I_{T,2})}\left\|\partial_x u_\delta\right\|^2_{L^\infty(I_{T,1})}\left\|\partial_x u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{1}{3}\left\|\partial_{xxx}^3 u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le 3\left\|f''\right\|^2_{L^\infty(I_{T,2})}C(T)\left\|\partial_x u_\delta\right\|^2_{L^\infty(I_{T,1})}\\
&\qquad+\frac{1}{3}\left\|\partial_{xxx}^3 u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2\int_{\mathbb{R}} f'(u_\delta)\partial_{xx}^2 u_\delta\partial_{xxx}^3 u_\delta \mathrm{d}x\\
&\quad\le 2\left|\int_{\mathbb{R}} f'(u_\delta)\partial_{xx}^2 u_\delta\partial_{xxx}^3 u_\delta \mathrm{d}x\right|\\
&\quad\le 2\int_{\mathbb{R}}\left|\sqrt{3}f'(u_\delta)\partial_{xx}^2 u_\delta\right|\left|\frac{\partial_{xxx}^3 u_\delta}{\sqrt{3}}\right|\mathrm{d}x\\
&\quad\le 3\int_{\mathbb{R}}(f'(u_\delta))^2(\partial_{xx}^2 u_\delta)^2\mathrm{d}x+\frac{1}{3}\left\|\partial_{xxx}^3 u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le 3\left\|f'\right\|^2_{L^\infty(I_{T,2})}\left\|\partial_{xx}^2 u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{1}{3}\left\|\partial_{xxx}^3 u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where $I_{T,1}$ is defined in (2.27) and $I_{T,2}$ is defined in (2.55). Therefore,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\partial_{xx}^2 u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)+2\left\|\partial_{xxx}^3 u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le\left\|\partial_{xxx}^3 u_\delta(t.\cdot)\right\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

$$+3\left\|f''\right\|^2_{L^\infty(I_{T,2})}C(T)\left\|\partial_x u_\delta\right\|^2_{L^\infty(I_{T,1})}$$
$$+3\left\|f'\right\|^2_{L^\infty(I_{T,2})}\left\|\partial^2_{xx}u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T),$$

that is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\partial^2_{xx}u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)+\left\|\partial^3_{xxx}u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T)\left\|\partial_x u_\delta\right\|^2_{L^\infty(I_{T,1})}+C(T)\\
&\quad+C(T)\left\|\partial^2_{xx}u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

An integration on $(0,t)$, (2.2) and (2.53) give

$$\begin{aligned}
&\left\|\partial^2_{xx}u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\int_0^t\left\|\partial^3_{xxx}u_\delta(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\le\left(C(T)\left\|\partial_x u_\delta\right\|^2_{L^\infty(I_{T,1})}+C(T)\right)\int_0^t\mathrm{d}s\\
&\quad+C(T)\int_0^t\left\|\partial^2_{xx}u_\delta(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\le C(T)\left\|\partial_x u_\delta\right\|^2_{L^\infty(I_{T,1})}+C(T).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\|\partial^2_{xx}u_\delta(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\int_0^t\left\|\partial^3_{xxx}u_\delta(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\le C(T)\left(1+\left\|\partial_x u_\delta\right\|^2_{L^\infty(I_{T,1})}\right).
\end{aligned}\tag{2.60}$$

Due to (2.53), (2.60) and the Hölder inequality,

$$\begin{aligned}
(\partial_x u_\delta(t,x))^2&\le 2\int_{\mathbb{R}}|\partial_x u_\delta||\partial^2_{xx}u_\delta|\mathrm{d}x\\
&\le 2\left\|\partial_x u_\delta(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial^2_{xx}u_\delta(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\le C(T)\sqrt{\left(1+\left\|\partial_x u_\delta\right\|^2_{L^\infty(I_{T,1})}\right)}.
\end{aligned}$$

Then,

$$\left\|\partial_x u_\delta\right\|^4_{L^\infty(I_{T,1})}-C(T)\left\|\partial_x u_\delta\right\|^2_{L^\infty(I_{T,1})}-C(T)\le 0,$$

which gives (2.57).
(2.58) follows from (2.57) and (2.60). □

Arguing as in [5], we obtain the following result

LEMMA 2.9. *Let $T>0$, $\ell>2$ and $0<\delta<1$. For each $t\in(0,T)$,*

$$\partial^\ell_x u_\delta(t,\cdot)\in L^2(\mathbb{R}).\tag{2.61}$$

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

We begin by proving the following result

LEMMA 3.1. *Let* $T > 0$. *Assume* (1.9), (1.10), (1.11) *and* (1.12). *Then there exist*

$$u \in L^{\infty}((0,T)\times\mathbb{R}) \cap C((0,T); H^{\ell}(\mathbb{R})), \quad \ell > 2, \tag{3.1}$$

$$P \in L^{\infty}((0,T)\times\mathbb{R}) \cap L^{2}((0,T)\times\mathbb{R}), \tag{3.2}$$

where u *is a classical solution of the Cauchy problem of* (1.8).

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}$ be any convex C^2 entropy function, and $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q' = f'\eta'$. By multiplying the first equation in (2.1) with $\eta'(u)$ and using the chain rule, we get

$$\partial_t \eta(u_\delta) + \partial_x q(u_\delta) = \underbrace{\partial_{xx}^2 \eta(u_\delta)}_{=:\mathcal{L}_{1,\delta}} \underbrace{-\eta''(u_\delta)\,(\partial_x u_\delta)^2}_{=:\mathcal{L}_{2,\delta}} \underbrace{+\gamma\eta'(u_\delta)P_\delta}_{=:\mathcal{L}_{3,\delta}},$$

where $\mathcal{L}_{1,\delta}$, $\mathcal{L}_{2,\delta}$, $\mathcal{L}_{3,\delta}$ are distributions.

Let us show that

$$\{\mathcal{L}_{1,\delta}\}_\delta \text{ is compact in } H^{-1}((0,T)\times\mathbb{R}),\ T > 0. \tag{3.3}$$

Since

$$\partial_{xx}^2 \eta(u_\delta) = \partial_x(\eta'(u_\delta)\partial_x u_\delta),$$

we have to prove that

$$\{\eta'(u_\delta)\partial_x u_\delta\}_\delta \text{ is bounded in } L^2((0,T)\times\mathbb{R}),\ T > 0, \tag{3.4}$$

$$\{\eta''(u_\delta)(\partial_x u_\delta)^2 + \eta'(u_\delta)\partial_{xx}^2 u_\delta\}_\delta \text{ is bounded in } L^2((0,T)\times\mathbb{R}),\ T > 0. \tag{3.5}$$

We begin by proving that (3.4) holds. Thanks to Lemmas 2.3 and 2.6,

$$\begin{aligned}
\left\|\eta'(u_\delta)\partial_x u_\delta\right\|^2_{L^2((0,T)\times\mathbb{R})} &\le \left\|\eta'\right\|^2_{L^\infty(I_{T,2})} \int_0^T \|\partial_x u_\delta(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \\
&\le \left\|\eta'\right\|^2_{L^\infty(I_{T,2})} e^{2\gamma T} \int_0^T e^{-2\gamma s}\,\|\partial_x u_\delta(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \\
&\le \frac{1}{2}\left\|\eta'\right\|^2_{L^\infty(I_{T,2})} e^{2\gamma T}\,\|u_0\|^2_{L^2(\mathbb{R})} \le C(T),
\end{aligned}$$

where $I_{T,2}$ is defined in (2.55).

We claim that

$$\{\eta''(u_\delta)(\partial_x u_\delta)^2\}_\delta \text{ is bounded in } L^2((0,T)\times\mathbb{R}). \tag{3.6}$$

Due to Lemmas 2.3, 2.6, 2.8

$$\begin{aligned}&\left\|\eta''(u_\delta)(\partial_x u_\delta)^2\right\|^2_{L^2((0,T)\times\mathbb{R})}\\&\quad\le \left\|\eta''\right\|^2_{L^\infty(I_{T,2})}\int_0^T\!\!\int_{\mathbb{R}}(\partial_x u_\delta(s,x))^4\mathrm{d}s\mathrm{d}x\\&\quad\le \left\|\eta''\right\|^2_{L^\infty(I_{T,2})}\left\|\partial_x u_\delta\right\|^2_{L^\infty(I_{T,1})}\int_0^T\left\|\partial_x u_\delta(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\&\quad\le \frac{1}{2}\left\|\eta''\right\|^2_{L^\infty(I_{T,2})}\left\|\partial_x u_\delta\right\|^2_{L^\infty(I_{T,1})}e^{2\gamma T}\left\|u_0\right\|^2_{L^2(\mathbb{R})}\le C(T),\end{aligned}$$

where $I_{T,1}$ is defined in (2.27).

We claim that

$$\{\eta'(u_\delta)\partial^2_{xx}u_\delta\}_\delta \text{ is bounded in } L^2((0,T)\times\mathbb{R}). \tag{3.7}$$

Thanks to Lemmas 2.6 and 2.7,

$$\begin{aligned}\left\|\eta'(u_\delta)\partial^2_{xx}u_\delta\right\|^2_{L^2((0,T)\times\mathbb{R})}&\le \left\|\eta'\right\|^2_{L^\infty(I_{T,2})}\int_0^T\left\|\partial^2_{xx}u_\delta(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\&\le \left\|\eta'\right\|^2_{L^\infty(I_{T,2})}C(T)\le C(T).\end{aligned}$$

(3.6) and (3.7) give (3.5).
Therefore, (3.3) follows from (3.4) and (3.5).

We have that

$$\{\mathcal{L}_{2,\delta}\}_{\delta>0} \text{ is bounded in } L^1((0,T)\times\mathbb{R}).$$

Due to Lemmas 2.3, 2.6,

$$\begin{aligned}\left\|\eta''(u_\delta)(\partial_x u_\delta)^2\right\|_{L^1((0,T)\times\mathbb{R})}&\le \left\|\eta''\right\|_{L^\infty(I_{T,2})}\int_0^T\left\|\partial_x u_\delta(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\&\le \left\|\eta'\right\|^2_{L^\infty(I_{T,2})}e^{2\gamma T}\int_0^T e^{-2\gamma s}\left\|\partial_x u_\delta(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\&\le \frac{\left\|\eta'\right\|^2_{L^\infty(I_{T,2})}e^{2\gamma T}}{2}\left\|u_0\right\|^2_{L^2(\mathbb{R})}\le C(T).\end{aligned}$$

We have that

$$\{\mathcal{L}_{3,\delta}\}_{\delta>0} \text{ is bounded in } L^1_{loc}((0,T)\times\mathbb{R}).$$

Let K be a compact subset of $(0,T)\times\mathbb{R}$. By Lemmas 2.5 and 2.6,

$$\begin{aligned}\left\|\gamma\eta'(u_\delta)P_\delta\right\|_{L^1(K)}&=\gamma\int_K|\eta'(u_\varepsilon)||P_\varepsilon|\mathrm{d}t\mathrm{d}x\\&\le\gamma\left\|\eta'\right\|_{L^\infty(I_{T,2})}\left\|P_\varepsilon\right\|_{L^\infty(I_{T,1})}|K|.\end{aligned}$$

Therefore, Murat's Lemma [21] implies that

$$\{\partial_t\eta(u_\delta)+\partial_x q(u_\delta)\}_{\delta>0} \text{ lies in a compact subset of } H^{-1}_{\mathrm{loc}}((0,\infty)\times\mathbb{R}). \tag{3.8}$$

The L^∞ bound stated in Lemma 2.6, (3.8) and the Tartar's compensated compactness method [27] give the existence of a subsequence $\{u_{\delta_k}\}_{k\in\mathbb{N}}$ and a limit function $u \in L^\infty((0,T)\times\mathbb{R})$ such that

$$u_{\delta_k} \to u \text{ a.e. and in } L^p_{loc}((0,T)\times\mathbb{R}), \quad 1 \le p < \infty. \tag{3.9}$$

Hence,

$$u_{\delta_k} \to u \quad \text{in } L^\infty((0,T)\times\mathbb{R}). \tag{3.10}$$

Moreover, for convexity, we have

$$\begin{aligned} &\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2e^{2\gamma t}\int_0^t e^{-2\gamma s}\|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \le C(T),\\ &\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \int_0^t \left\|\partial^2_{xx}u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le C(T),\\ &\left\|\partial^2_{xx}u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \left\|\partial^3_{xxx}u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le C(T). \end{aligned} \tag{3.11}$$

We need only to observe that

$$\begin{aligned} &2e^{2\gamma t}\int_0^t e^{-2\gamma s}\|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\\ &\quad\le 2e^{2\gamma t}\liminf_k \int_0^t e^{-2\gamma s}\left\|\partial_x u_{\delta_k}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le C(T),\\ &\int_0^t \left\|\partial^2_{xx}u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le \liminf_k \int_0^t \left\|\partial^2_{xx}u_{\delta_k}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le C(T),\\ &\int_0^t \left\|\partial^3_{xxx}u(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le \liminf_k \int_0^t \left\|\partial^3_{xxx}u_{\delta_k}(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le C(T). \end{aligned}$$

Moreover, it follows from convexity and Lemma 2.9 that

$$\partial_x^\ell u(t,\cdot) \in L^2(\mathbb{R}), \quad \ell > 2, \quad t \in (0,T). \tag{3.12}$$

Therefore, (3.10), (3.11) and (3.12) give (3.1). (3.2) follows from Lemma 2.5.

Finally, we prove that

$$\int_{-\infty}^x u(t,y)\mathrm{d}y = P(t,x), \quad \text{a.e. in} \quad (t,x) \in I_{T,1}. \tag{3.13}$$

Integrating the second equation of (2.1) on $(-\infty, x)$, for (2.3), we have that

$$\int_{-\infty}^x u_{\delta_k}(t,y)\mathrm{d}y = P_{\delta_k}(t,x) - \delta_k\partial_x P_{\delta_k}(t,x). \tag{3.14}$$

We show that

$$\delta\partial_x P_\delta(t,x) \to 0 \text{ in } L^\infty((0,T)\times\mathbb{R}), \quad T > 0 \text{ as } \delta \to 0. \tag{3.15}$$

It follows from (2.11) that

$$\delta \left\| \partial_x P_\delta \right\|_{L^\infty((0,T)\times\mathbb{R})} \le \sqrt{\delta} e^{\gamma t} \left\| u_{\varepsilon,0} \right\|_{L^2(\mathbb{R})} = \sqrt{\delta} C(T) \to 0,$$

that is (3.15).

Therefore, (3.13) follows from (3.1), (3.2), (3.14) and (3.15). The proof is done. □

LEMMA 3.2. *Let $u(t,x)$ be a classical solution of* (1.7)*, or* (1.8)*. Then,*

$$\int_{\mathbb{R}} u(t,x)\mathrm{d}x = 0, \quad t \ge 0, \tag{3.16}$$

Proof. Differentiating (1.8) with respect to x, we have

$$\partial_x(\partial_t u + \partial_x f(u) - \partial^2_{xx} u) = \gamma u. \tag{3.17}$$

Since u is a smooth solution of (1.8), an integration over $\mathbb{R}$ gives (3.16). □

We are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.1 gives the existence of a classical solution of (1.7), or (1.8), while Lemma 3.2 says that the solution has zero mean.

Let us show that $u(t,x)$ is unique and (1.14) holds. Let $u,\ v$ be two classical solutions of (1.7), or (1.8), that is

$$\begin{cases} \partial_t u + f'(u)\partial_x u = \gamma P^u + \partial^2_{xx} u, & t > 0, x \in \mathbb{R}, \\ \partial_x P^u = u, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

$$\begin{cases} \partial_t v + f'(v)\partial_x v = \gamma P^v + \partial^2_{xx} v, & t > 0, x \in \mathbb{R}, \\ \partial_x P^v = v, & t > 0, x \in \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}. \end{cases}$$

Then, the function

$$\omega(t,x) = u(t,x) - v(t,x) \tag{3.18}$$

is solution of the following Cauchy problem

$$\begin{cases} \partial_t \omega + f'(u)\partial_x u - f'(v)\partial_x v = \gamma \Omega + \partial^2_{xx}\omega, & t > 0,\ x \in \mathbb{R}, \\ \partial_x \Omega = \omega, & t > 0,\ x \in \mathbb{R}, \\ \omega(0,x) = u_0(x) - v_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.19}$$

where

$$\begin{aligned} \Omega(t,x) &= P^u(t,x) - P^v(t,x) \\ &= \int_{-\infty}^x u(t,y)\mathrm{d}y - \int_{-\infty}^x v(t,y)\mathrm{d}y \\ &= \int_{-\infty}^x (u(t,y) - v(t,y))\mathrm{d}y = \int_{-\infty}^x \omega(t,y)\mathrm{d}y. \end{aligned} \tag{3.20}$$

It follows from Lemma 3.2 and (3.20) that

$$\Omega(t,\infty)=\int_{\mathbb{R}}u(t,y)\mathrm{d}y-\int_{\mathbb{R}}v(t,y)\mathrm{d}y=0. \tag{3.21}$$

Observe that, from (3.18),

$$\begin{aligned}f'(u)\partial_x u-f'(v)\partial_x v&=f'(u)\partial_x u-f'(u)\partial_x v+f'(u)\partial_x v-f'(v)\partial_x v\\&=f'(u)\partial_x(u-v)+(f'(u)-f'(v))\partial_x v\\&=f'(u)\partial_x\omega+(f'(u)-f'(v))\partial_x v.\end{aligned}$$

Therefore, the first equation of (3.19) is equivalent to the following one:

$$\partial_t\omega+f'(u)\partial_x\omega+(f'(u)-f'(v))\partial_x v=\gamma\Omega+\partial_{xx}^2\omega. \tag{3.22}$$

Moreover, since u and v are in $L^\infty((0,T)\times\mathbb{R})$, we have that

$$\left|f'(u(t,x))-f'(v(t,x))\right|\le C(T)|u(t,x)-v(t,x)|,\quad (t,x)\in(0,T)\times\mathbb{R}, \tag{3.23}$$

where

$$C(T)=\sup_{(0,T)\times\mathbb{R}}\left\{|f''(u)|+|f''(v)|\right\}. \tag{3.24}$$

Therefore, (3.18) and (3.23) give

$$\left|f'(u(t,x))-f'(v(t,x))\right|\le C(T)|\omega(t,x)|,\quad (t,x)\in(0,T)\times\mathbb{R}. \tag{3.25}$$

Multiplying (3.22) by ω, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\omega^2\mathrm{d}x&=2\int_{\mathbb{R}}\omega\partial_t\omega\mathrm{d}x\\&=2\int_{\mathbb{R}}\omega\partial_{xx}^2\omega\mathrm{d}x-2\int_{\mathbb{R}}\omega f'(u)\partial_x\omega\mathrm{d}x\\&\quad-2\int_{\mathbb{R}}\omega(f'(u)-f'(v))\partial_x v\mathrm{d}x+2\gamma\int_{\mathbb{R}}\Omega\omega\mathrm{d}x\\&=-2\int_{\mathbb{R}}(\partial_x\omega)^2\mathrm{d}x+\int_{\mathbb{R}}\omega^2 f''(u)\partial_x u\mathrm{d}x\\&\quad-2\int_{\mathbb{R}}\omega_\varepsilon(f'(u)-f'(v))\partial_x v\mathrm{d}x+2\gamma\int_{\mathbb{R}}\Omega\omega\mathrm{d}x.\end{aligned}$$

It follows from the second equation of (3.19) and Lemma 3.2 that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\quad\le\int_{\mathbb{R}}\omega^2|f''(u)||\partial_x u|\mathrm{d}x+2\int_{\mathbb{R}}|\omega||(f'(u)-f'(v))||\partial_x v|\mathrm{d}x.\end{aligned} \tag{3.26}$$

Since $u(t,\cdot),\ v(t,\cdot) \in H^{\ell}(\mathbb{R})$, $\ell > 2$, for each $t \in (0,T)$, then

$$\partial_x u(t,\cdot), \partial_x v(t,\cdot) \in H^{\ell-1}(\mathbb{R}) \subset L^{\infty}(\mathbb{R}), \quad t \in (0,T). \tag{3.27}$$

Therefore, thanks to (3.23), (3.24), (3.26) and (3.27),

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

The Gronwall Lemma gives

$$\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2e^{C(T)t}\int_0^s e^{-C(T)s}\|\partial_x\omega(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le e^{C(T)t}\|\omega_0\|^2_{L^2(\mathbb{R})}. \tag{3.28}$$

Hence, (1.14) follows from (3.18), (3.19) and (3.28). □

REFERENCES

[1] J. Boyd. Ostrovsky and Hunters generic wave equation for weakly dispersive waves: matched asymptotic and pseudospectral study of the paraboloidal travelling waves (corner and near-corner waves). *Euro. Jnl. of Appl. Math.*, 16(1):65–81, 2005.

[2] J. Canosa and J. Gazdag. The Korteweg-de Vries-Burgers equation *Journal of Computational Physics*, vol. 23, no. 4, 393–403, 1977.

[3] G. M. Coclite and L. di Ruvo. Wellposedness of bounded solutions of the non-homogeneous initial boundary value problem for the Ostrovsky–Hunter equation. To appear on *J. Hyperbolic Differ. Equ.*, 12:221–248, 2015.

[4] G. M. Coclite and L. di Ruvo. Wellposedness results for the Short Pulse Equation. To appear on *Z. Angew. Math. Phys.*, 66:1529–1557, 2015.

[5] G. M. Coclite, H. Holden, and K. H. Karlsen. Wellposedness for a parabolic-elliptic system. *Discrete Contin. Dyn. Syst.*, 13(3):659–682, 2005.

[6] R. Grimshaw and D. E. Pelinovsky. Global existence of small-norm solutions in the reduced Ostrovsky equation. *Discr. Cont. Dynam. Syst. A*, 34:557–566, 2014.

[7] L. di Ruvo. Discontinuous solutions for the Ostrovsky–Hunter equation and two phase flows. *Phd Thesis, University of Bari*, 2013. www.dm.uniba.it/home/dottorato/dottorato/tesi/.

[8] G. Gui and Y. Liu. On the Cauchy problem for the Ostrovsky equation with positive dispersion. *Comm. Part. Diff. Eqs.*, 32(10–12):1895–1916, 2007.

[9] Z.S Fend and Q.G. Meng Burgers-Korteweg-de Vries equation and its traveling solitary waves. *S. in China Series A: Mathem. Springer-Verlag*, 50(3):412–422, 2007.

[10] J. Hunter. Numerical solutions of some nonlinear dispersive wave equations. Computational solution of nonlinear systems of equations (Fort Collins, CO, 1988) *Lectures in Appl. Math.*, 26, Amer. Math. Soc., Providence, RI, 301–316, 1990.

[11] J. Hunter and K. P. Tan. Weakly dispersive short waves *Proceedings of the IVth international Congress on Waves and Stability in Continuous Media*, Sicily, 1987.

[12] D. J. Korteweg, and G. de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philosophical Magazine*, vol. 39, no. 240, 422–443, 1895.

[13] S. Levandosky and Y. Liu. Stability of solitary waves of a generalized Ostrovsky equation. *SIAM J. Math. Anal.*, 38(3):985–1011, 2006.

[14] S. Levandosky and Y. Liu. Stability and weak rotation limit of solitary waves of the Ostrovsky equation. *Discr. Cont. Dyn. Syst. B*, 7(7):793–806, 2007.

[15] F. Linares and A. Milanes. Local and global well-posedness for the Ostrovsky equation. *J. Diff. Eqs.*, 222(2):325–340, 2006.

[16] Y. Liu. On the stability of solitary waves for the Ostrovsky equation. *Quart. Appl. Math.*, 65(3):571–589, 2007.
[17] Y. Liu, D. Pelinovsky, and A. Sakovich. Wave breaking in the Ostrovsky–Hunter equation. *SIAM J. Math. Anal.* 42(5):1967–1985, 2010.
[18] Y. Liu and V. Varlamov. Cauchy problem for the Ostrovsky equation. *Discr. Cont. Dyn. Syst.*, 10(3):731–753, 2004.
[19] Y. Liu and V. Varlamov. Stability of solitary waves and weak rotation limit for the Ostrovsky equation. *J. Diff. Eqs.*, 203(1):159–183, 2004.
[20] A. J. Morrison, E. J. Parkes, and V. O. Vakhnenko. The N loop soliton solutions of the Vakhnenko equation. *Nonlinearity*, 12(5):1427–1437, 1999.
[21] F. Murat. L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$. *J. Math. Pures Appl. (9)*, 60(3):309–322, 1981.
[22] L. A. Ostrovsky. Nonlinear internal waves in a rotating ocean. *Okeanologia*, 18:181–191, 1978.
[23] E. J. Parkes. Explicit solutions of the reduced Ostrovsky equation. *Chaos, Solitons and Fractals*, 31(3):602–610, 2007.
[24] E. J. Parkes and V. O. Vakhnenko. The calculation of multi-soliton solutions of the Vakhnenko equation by the inverse scattering method. *Chaos, Solitons and Fractals*, 13(9):1819–1826, 2002.
[25] Y. A. Stepanyants. On stationary solutions of the reduced Ostrovsky equation: periodic waves, compactons and compound solitons. *Chaos, Solitons and Fractals*, 28(1):193–204, 2006.
[26] J.J. Shu The Proper analytical solution of the Korteweg-de Vries-Burgers equation. *J. of Physics A-Mathem. and General*, 20(2):49–56, 1987.
[27] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, pages 136–212. Pitman, Boston, Mass., 1979.
[28] K. Tsugawa. Well-posedness and weak rotation limit for the Ostrovsky equation. *J. Differential Equations* 247(12):3163–3180, 2009.

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