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An inhomogeneous nonlocal diffusion problem with unbounded steps

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Abstract. We consider the following nonlocal equation

$$\int J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}dy-u(x)=0 \qquad x\in\mathbb{R},$$

where J is an even, compactly supported, Hölder continuous kernel with unit integral and g is a continuous positive function. Our main concern will be with unbounded functions g, contrary to previous works. More precisely, we study the influence of the growth of g at infinity on the integrability of positive solutions of this equation, therefore determining the asymptotic behavior as $t\to+\infty$ of the solutions to the associated evolution problem in terms of the growth of g.

1. Introduction

In the present paper we will be concerned with the nonlocal evolution problem

$$\begin{cases} u_t(x,t)=\displaystyle\int J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\mathrm{d}y-u(x,t), & x\in\mathbb{R},\ t>0\\ u(x,0)=u_0(x) & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where J is a Hölder continuous function which we assume nonnegative, even, supported in the unit interval $[-1,1]$ and with $\int_{\mathbb{R}} J(x)\mathrm{d}x=1$. For simplicity we will also assume throughout that $J>0$ in $(-1,1)$. The function g is continuous and positive in $\mathbb{R}$, but no additional conditions at infinity will be assumed for the moment.

Problem (1.1) was introduced in [8] as a generalization of the well-studied case $g=1$. If $u(x,t)$ is thought of as the density of a population, then the term

$$J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}$$

represents the probability that an individual which lies in the position y reaches the position x. Since J is supported in the interval $(-1,1)$, this means that position x can only be reached from those y which verify $|x-y|<g(y)$, so that $g(y)$ can be viewed as the "step" of the jump of individuals standing at position y and introduces

a heterogeneity in the dispersal of the species (see also [14,16,17,25,28,29] for other nonlocal models considering heterogeneities).

While a lot of previous works have considered problem (1.1) with $g = 1$ (cf. for instance [1–7,13,18,19,21–23,27,30] and references therein), the problem with a general g has not received as much attention as far as we know. We refer the reader to [9,10,12,14,15,26], where typically g is a bounded function. Therefore our main purpose here will be to analyze the asymptotic behavior of solutions of (1.1) when g is unbounded. Closely related to the present paper is [11], where the asymptotic behavior of solutions of (1.1) was studied for some special unbounded functions g.

It is easily seen by an application of semigroup theory that problem (1.1) admits a unique solution u for every $u_0 \in L^1(\mathbb{R})$, which in addition verifies $\|u(\cdot,t)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}$ for every $t > 0$ (see [9,11]). Moreover, the asymptotic behavior of this solution strongly depends on the positive solutions of the stationary counterpart, that is

$$\int J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}\mathrm{d}y - u(x) = 0, \qquad x \in \mathbb{R}, \tag{1.2}$$

and more precisely on its positive supersolutions w. It was proved in [9] (see also [11]) that:

(a) If $w \in L^1(\mathbb{R})$ is a positive supersolution, then w is actually a solution of (1.2), and every positive solution of (1.1) verifies

$$u(\cdot,t) \to \frac{\|u_0\|_{L^1(\mathbb{R})}}{\|w\|_{L^1(\mathbb{R})}}\, w \quad \text{in } L^1(\mathbb{R})$$

as $t \to +\infty$.

(b) If there exists a positive supersolution $w \notin L^1(\mathbb{R})$, then every positive solution of (1.1) verifies $u(\cdot,t) \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $t \to +\infty$. In particular, (1.2) does not have solutions in $L^1(\mathbb{R})$.

Our objective in the present paper will be twofold: on the one hand, we will find conditions on g which allow us to obtain positive solutions of (1.2), while on the other hand we will discuss whether these solutions are actually in $L^1(\mathbb{R})$, thereby providing a description of the asymptotic behavior of solutions of problem (1.1).

The main novelty in this work is that we will be mainly dealing with unbounded functions g. To begin with, we will consider the simpler case where g satisfies:

$$\limsup_{|s|\to\infty} \frac{g(s)}{|s|} < 1. \tag{1.3}$$

The importance of this condition is due to the fact that the integral in (1.2) is performed in a bounded set for every $x \in \mathbb{R}$. It ensures the existence of solutions, although they do not necessarily belong to $L^1(\mathbb{R})$. One of our first objectives will be then to determine some additional conditions on g which would let us know whether positive solutions lie in $L^1(\mathbb{R})$ or not.

When g verifies (1.3), a solution of (1.2) is a function $u \in C(\mathbb{R})$ which verifies the equation for every $x \in \mathbb{R}$.

THEOREM 1. *Assume $g \in C(\mathbb{R})$ is positive and satisfies (1.3). Then there exists a positive solution $p \in C(\mathbb{R})$ of (1.2). Moreover, assume there exists a nondecreasing, continuous, positive function h verifying*

$$\limsup_{s\to+\infty} \frac{h(\lambda s)}{h(s)} < \infty \tag{1.4}$$

for some $\lambda > 1$ and such that

$$C_1 h(|s|) \le g(s) \le C_2 h(|s|) \qquad \textit{if } |s| \ge s_0 \tag{1.5}$$

for some $C_1, C_2, s_0 > 0$. Then

(a) *If $\int \frac{1}{g^2} < \infty$, then $p \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\lim_{|x|\to\infty} p(x) = 0$. In addition, if q is any other positive solution of (1.2), there exists $\mu > 0$ such that $q = \mu p$.*

(b) *If $\limsup_{|s|\to\infty} \frac{g(s)}{|s|} < \frac{1}{2}$, h is differentiable for large $|s|$ and $h'(s)$ is bounded, then $p \notin L^1(\mathbb{R})$, provided that $\int \frac{1}{g^2} = \infty$. Besides, no positive solution of (1.2) belongs to $L^1(\mathbb{R})$.*

REMARKS 1. (a) An example for the function h is $h(s) = s^\alpha$ for $s > 0$ and some $\alpha > 0$. Then part (a) will hold for $\alpha \in (\frac{1}{2}, 1]$, while part (b) will be applicable if $\alpha \le \frac{1}{2}$.

(b) The existence of positive solutions in case (a) of Theorem 1 follows also when condition (1.3) is relaxed to $g(s) \le |s|$ for large $|s|$. See Remark 2 in Sect. 2.

(c) It is not hard to show that if h is nondecreasing and (1.4) is verified for some $\lambda > 1$, then it is also verified for every $\lambda > 0$.

The natural question as to the necessity of condition (1.3) arises. We will see that, under some conditions of a different type, problem (1.2) can also admit positive solutions. However, the problem becomes more delicate without (1.3), since the integration in (1.2) can be made in an unbounded set, therefore causing additional complications in our proofs. In particular, the concept of solution has to be modified, since now the behavior of the solutions at infinity is relevant. Accordingly, we say that u is a solution of (1.2) if $u \in C(\mathbb{R})$, $\frac{u}{g} \in L^1(\mathbb{R})$ and the equation is verified for every $x \in \mathbb{R}$.

Our next result is aimed at functions g not verifying (1.3). It turns out that the finiteness of the integral $\int \frac{1}{g}$ is determinant regarding existence of positive solutions in $L^1(\mathbb{R})$.

THEOREM 2. *Assume $g \in C(\mathbb{R})$ is positive. Then*

(a) If $\int \frac{1}{g} = \infty$ and $\liminf_{|s|\to\infty} \frac{g(s)}{|s|} > 1$, there exists a positive solution $p \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Moreover, $\lim_{|x|\to\infty} p(x) = 0$.

(b) If $\int \frac{1}{g} < \infty$ and $\lim_{|s|\to\infty} g(s) = +\infty$, then there exists a positive supersolution $w \in C(\mathbb{R})$ of (1.2). Moreover, $w \notin L^1(\mathbb{R})$ and no solutions of (1.2) in $L^1(\mathbb{R})$ exist.

At the sight of Theorems 1 and 2, one could wonder what happens with the integrability of the solutions when the function g grows linearly at infinity, but not verifying the condition (1.3). In this regard, Theorem 2(a) provides with positive solutions, but does not give any hint as to their integrability.

To analyze further this subject, we will consider next functions g verifying $g(s) = \lambda|s|$ for large $|s|$ and some $\lambda > 0$. It turns out that the integrability of solutions strongly depends on the value of λ and even on the kernel J. This subtle matter will be studied in Sect. 3. As a hint of what is to be found there, we have the next result:

THEOREM 3. *Assume $g(s) = \lambda|s|$ for $|s| \geq s_0 > 0$ and some $\lambda > 0$. There exist $\lambda_0, \lambda_\infty > 0$, depending only on the kernel J, such that $\sqrt{2} < \lambda_0 \leq \lambda_\infty$ and with the property that for $\lambda < \lambda_0$ all positive solutions of* (1.2) *belong to $L^1(\mathbb{R})$, while no positive solutions in $L^1(\mathbb{R})$ exist when $\lambda > \lambda_\infty$.*

It is worthy of mention that $\lambda_0 = \lambda_\infty$ is to be expected depending on the kernel J, but it is possible to exhibit kernels for which $\lambda_0 < \lambda_\infty$. In this case, for values $\lambda \in (\lambda_0, \lambda_\infty)$, the solutions may or may not belong to $L^1(\mathbb{R})$ (see precise details in Sect. 3). This result is somehow contrary to intuition, since one would expect in principle that if the solutions do not belong to $L^1(\mathbb{R})$ for some $\bar{\lambda}$, they should not belong to $L^1(\mathbb{R})$ for $\lambda \geq \bar{\lambda}$.

It is also relevant to point out that solutions u of (1.2) verifying $\frac{u}{g} \in L^1(\mathbb{R})$ belong to $L^\infty(\mathbb{R})$, with $\lim_{|x|\to+\infty} u(x) = 0$, Indeed

$$u(x) \leq \|J\|_\infty \int_{D_x} \frac{u(y)}{g(y)} \mathrm{d}y \to 0 \tag{1.6}$$

as $x \to +\infty$, where $D_x = \{y \in \mathbb{R} : |x-y| < g(y)\}$. It also follows from (1.6) that, if g is bounded from below and $u \in L^1(\mathbb{R})$, then $u \in L^\infty(\mathbb{R})$ with $\lim_{|x|\to+\infty} u(x) = 0$. On the other hand, it can also be proved that solutions of (1.2) in $L^1(\mathbb{R})$ are always one-signed (cf. the proof of Lemma 3 in [9]).

As a final comment, it is interesting to illustrate our results with a family of positive functions $g \in C(\mathbb{R})$ verifying $g(s) = \lambda|s|^q$, for $|s| \geq s_0 > 0$ and some $q > 0, \lambda > 0$ (the definition of g in $|s| < s_0$ is of no importance regarding existence and integrability of solutions). We have:

- When $0 < q \leq \frac{1}{2}$, there exists a positive solution which is not in $L^1(\mathbb{R})$.
- For $\frac{1}{2} < q < 1$, there are positive solutions in $L^1(\mathbb{R})$.
- If $q = 1$, there exists a positive solution for every $\lambda > 0$. Morever, there are values $\lambda_0, \lambda_\infty$ with $\sqrt{2} < \lambda_0 \leq \lambda_\infty$ and such that for $0 < \lambda < \lambda_0$ the solutions belong to $L^1(\mathbb{R})$, while they do not when $\lambda > \lambda_\infty$. For some kernels we have $\lambda_0 < \lambda_\infty$ and in the interval $\lambda \in (\lambda_0, \lambda_\infty)$ there are values for which solutions belong to $L^1(\mathbb{R})$ and others where they do not.
- When $q > 1$, there are no solutions in $L^1(\mathbb{R})$ (actually there are no solutions *at all*, see [11]).

The rest of the paper is organized as follows: in Sect. 2 we prove Theorems 1 and 2 through a sequence of lemmas. Section 3 is devoted to the study of the case where $g(s) = \lambda|s|$ for large $|s|$ and some $\lambda > 0$, and in particular to prove Theorem 3.

2. Proof of Theorems 1 and 2

Before coming to the actual proof of the theorems, we will consider some lemmas which deal with estimates for a sequence of approximate solutions. Remember that we are always assuming that $g \in C(\mathbb{R})$ is positive. We first truncate g to be bounded and bounded away from zero: let $N > 1$ and define:

$$g_N(s) = \max\left\{\min\{g(s), N\}, \frac{1}{N}\right\},$$

so that $\frac{1}{N} \le g_N(s) \le N$. We can apply Theorem 4.3 in [8] to obtain a positive bounded solution $p_N \in C(\mathbb{R})$ of problem (1.2) with g replaced by g_N. We normalize p_N so that

$$\int_0^{\max g_N} \int_{x-w}^{x+w} p_N(s) \int_{\frac{w}{g_N(s)}}^{1} J(z)\mathrm{d}z\mathrm{d}s\mathrm{d}w = 1, \quad x \in \mathbb{R}. \tag{2.1}$$

We remark that the quantity in the left-hand side of (2.1) is constant for positive solutions of (1.2); therefore, this normalization is possible (see for instance Proposition 2.1 in [10]). Let us see that local bounds for p_N can be obtained. We begin with L^1 local bounds.

LEMMA 4. *For every $M > 0$, there exist $C, N_0 > 0$ such that, if $N \ge N_0$ then*

$$\int_{-M}^{M} p_N(s)\mathrm{d}s \le C.$$

Proof. Let $M > 0$ and choose $\eta \in (0, \min\{1, \max g_N\})$ small enough. By (2.1):

$$\begin{aligned} 1 &\ge \int_{\frac{\eta}{2}}^{\eta} \int_{x-w}^{x+w} p_N(s) \int_{\frac{w}{g_N(s)}}^{1} J(z)\mathrm{d}z\mathrm{d}s\mathrm{d}w \\ &\ge \int_{\frac{\eta}{2}}^{\eta} \int_{x-\frac{\eta}{2}}^{x+\frac{\eta}{2}} p_N(s) \int_{\frac{\eta}{g_N(s)}}^{1} J(z)\mathrm{d}z\mathrm{d}s\mathrm{d}w. \end{aligned}$$

If $|x| \le M$, we have $(x - \frac{\eta}{2}, x + \frac{\eta}{2}) \subset (-M - \frac{\eta}{2}, M + \frac{\eta}{2}) \subset (-M - 1, M + 1)$. Since g is bounded from below in $[-M - 1, M + 1]$, the same is true for g_N if N is large enough. Therefore $g_N(s) \ge g_0$ in $[-M - 1, M + 1]$ for some $g_0 > 0$, so that

$$\int_{\frac{\eta}{g_N(s)}}^{1} J(z)\mathrm{d}z \ge \int_{\frac{\eta}{g_0}}^{1} J(z)\mathrm{d}z =: C_0 > 0,$$

if η is diminished as to have $\eta < g_0$. Hence

$$1 \geq C_0 \int_{\frac{\eta}{2}}^{\eta} \int_{x-\frac{\eta}{2}}^{x+\frac{\eta}{2}} p_N(s) \mathrm{d}s \mathrm{d}w = \frac{\eta C_0}{2} \int_{x-\frac{\eta}{2}}^{x+\frac{\eta}{2}} p_N(s) \mathrm{d}s.$$

Since the interval $[-M, M]$ can be covered with a finite number of intervals of length η, this last inequality proves the lemma. □

The next bounds we are going to consider will be handy when proving the integrability of solutions in Theorem 1, part (a). They are slightly more accurate than those in Lemma 4.

LEMMA 5. *Assume $g(s) \geq C_1 h(|s|)$, for $|s| \geq s_0 > 0$, where $C_1 > 0$ and h is positive and nondecreasing. Let $x \geq s_0$ and $\delta \in (0, \min\{1/2, C_1/2\})$. Then there exists $C = C(\delta)$ and $N_0 = N_0(\delta, x)$ such that*

$$\int_{x}^{x+\delta h(x)} p_N(s) \mathrm{d}s \leq \frac{C}{h(x)} \tag{2.2}$$

for $N \geq N_0$.

Proof. For large enough N (depending on x) we have $2\delta h(x) \leq \max g_N$. Then by (2.1):

$$\begin{aligned} 1 &\geq \int_{\delta h(x)}^{2\delta h(x)} \int_{x-w}^{x+w} p_N(s) \int_{\frac{w}{g_N(s)}}^{1} J(z) \mathrm{d}z \mathrm{d}s \mathrm{d}w \\ &\geq \int_{\delta h(x)}^{2\delta h(x)} \int_{x}^{x+\delta h(x)} p_N(s) \int_{\frac{2\delta h(x)}{g_N(s)}}^{1} J(z) \mathrm{d}z \mathrm{d}s \mathrm{d}w. \end{aligned}$$

Observe that g is bounded from below, therefore $g_N(s) = \min\{g(s), N\}$ for large N. Since $h(x) \leq N$ if N is large enough, it can be easily checked that $2\delta h(x)/g_N(s) \leq \theta := \max\{2\delta, 2\delta/C_1\} < 1$. Therefore

$$\begin{aligned} 1 &\geq \int_{\delta h(x)}^{2\delta h(x)} \int_{x}^{x+\delta h(x)} p_N(s) \int_{\theta}^{1} J(z) \mathrm{d}z \mathrm{d}s \mathrm{d}w \\ &= C\delta h(x) \int_{x}^{x+\delta h(x)} p_N(s) \mathrm{d}s, \end{aligned}$$

for some $C > 0$ depending on δ, which shows (2.2). The proof of the lemma is concluded. □

Our last lemma is of a different nature and deals with the solution of a certain initial value problem associated with an auxiliary differential equation.

LEMMA 6. *Assume $h \in C(\mathbb{R})$ is positive, nondecreasing for $s \geq s_0 > 0$ and verifies (1.4) and $h(s) \leq s$ for $s \geq s_0$. For $A \in (0, 1)$, let v be the solution of the equation*

$$\begin{cases} v' = Ah(v) & t > 0 \\ v(0) = v_0, \end{cases} \tag{2.3}$$

where $v_0 \geq s_0$ and $' = d/dt$. Then for every $\delta > 0$, we can choose A small enough such that v is defined in $[0, +\infty)$, $\lim_{t\to+\infty} v(t) = +\infty$ and

$$\frac{v(t+1) - v(t)}{h(v(t))} \leq \delta \qquad \textit{for } t > 0.$$

Proof. Notice that the quantity

$$S(A) = \sup_{t \geq s_0} \frac{h(e^A t)}{h(t)}$$

is finite by (1.4) and Remark 1(c). We also have that S is nondecreasing as a function of A. Thus we may choose A small so that $AS(A) \leq AS(1) \leq \delta$.

Next observe that (2.3) can be explicitly solved to give

$$\int_{v_0}^{v(t)} \frac{\mathrm{d}s}{h(s)} = At, \qquad t > 0. \tag{2.4}$$

Since $h(s) \leq s$ for $s \geq s_0$, we see that the integral on the left-hand side of (2.4) diverges at infinity, so that v is defined in $[0, +\infty)$ and $\lim_{t\to+\infty} v(t) = +\infty$. We also obtain from (2.4) that

$$\log\left(\frac{v(t+1)}{v(t)}\right) = \int_{v(t)}^{v(t+1)} \frac{\mathrm{d}s}{s} \leq \int_{v(t)}^{v(t+1)} \frac{\mathrm{d}s}{h(s)} = A,$$

therefore $v(t+1) \leq e^A v(t)$ so that $h(v(t+1)) \leq h(e^A v(t))$. Hence

$$\begin{aligned} v(t+1) - v(t) &= \int_t^{t+1} Ah(v(s))\mathrm{d}s \leq Ah(v(t+1)) \\ &\leq Ah(e^A v(t)) \leq AS(A)h(v(t)) \\ &\leq \delta h(v(t)), \end{aligned}$$

as was to be proved. □

Finally, let us proceed to the proof of our main results.

Proof of Theorem 1. Observe first that the integration in (1.2) takes place in the set $D_x = \{y \in \mathbb{R} : |x - y| < g(y)\}$. If $\varepsilon > 0$ is chosen small enough, by condition (1.3), we have $g(s) \leq (1 - \varepsilon)|s|$ if $|s| \geq s_1$, for some $s_1 > 0$ which depends on ε. It is then easily seen that $g_N(s) \leq (1 - \varepsilon)|s|$ if $|s| \geq s_1$ for large N.

Now take $x > 0$ (the case $x < 0$ being analyzed similarly). If $y \in D_x$ verifies $|y| \geq s_1$, then $y - (1 - \varepsilon)|y| < x < y + (1 - \varepsilon)|y|$, so that $y \geq 0$ follows, hence $\varepsilon y < x < (2 - \varepsilon)y$. We deduce

$$D_x \subset (-s_1, s_1) \cup \left(\frac{1}{2 - \varepsilon}x, \frac{1}{\varepsilon}x\right). \tag{2.5}$$

Thus for every $K > 0$, there exists $M > 0$, depending on K such that $D_x \subset [-M, M]$ for every $x \in [0, K]$. Letting $g_0 = \inf_{[-M,M]} g$, we obtain

$$p_N(x) = \int_{D_x} J\left(\frac{x-y}{g_N(y)}\right) \frac{p_N(y)}{g_N(y)} \mathrm{d}y \leq \frac{\|J\|_\infty}{g_0} \int_{-M}^{M} p_N(y) \mathrm{d}y \leq C$$

by Lemma 4 (from now on, we will use the letter C to denote different positive constants). This inequality can be extended to $x \in [-K, K]$. On the other hand, if $x, z \in [-K, K]$:

$$\begin{aligned} |p_N(x) - p_N(z)| &\leq \int_{D_x \cup D_z} \left| J\left(\frac{x-y}{g_N(y)}\right) - J\left(\frac{z-y}{g_N(y)}\right) \right| \frac{p_N(y)}{g_N(y)} \mathrm{d}y \\ &\leq L|x-z|^\alpha \int_{-M}^{M} \frac{p_N(y)}{g_N(y)^{1+\alpha}} \mathrm{d}y \\ &\leq \frac{L}{g_0^{1+\alpha}} |x-z|^\alpha \int_{-M}^{M} p_N(y) \mathrm{d}y \leq C|x-z|^\alpha, \end{aligned}$$

where $L > 0, \alpha \in (0, 1)$ stand for the Hölder constant and exponent of J, respectively. We deduce that $\{p_N\}_{N>0}$ is equicontinuous and uniformly bounded in $[-K, K]$, for every $K > 0$. Thus by Arzelá-Ascoli's theorem and a diagonal argument, we obtain the existence of a sequence $N_k \to +\infty$ such that $p_{N_k} \to p$ uniformly on compact sets of $\mathbb{R}$ (we will continue to denote p_{N_k} as p_N for simplicity).

Since D_x is a bounded set for every $x \in \mathbb{R}$ by (2.5), we may pass to the limit in the equation verified by p_N to see that p is a nonnegative solution of (1.2). Our next task will be to prove that p is nontrivial. For this purpose, we come again to the normalization (2.1), setting $x = 0$, and rewrite it using Fubini's theorem

$$\begin{aligned} 1 = &\int_0^{\max g_N} \int_s^{\max g_N} p_N(s) \int_{\frac{w}{g_N(s)}}^{1} J(z) \mathrm{d}z \mathrm{d}w \mathrm{d}s \\ &+ \int_{-\max g_N}^{0} \int_{-s}^{\max g_N} p_N(s) \int_{\frac{w}{g_N(s)}}^{1} J(z) \mathrm{d}z \mathrm{d}w \mathrm{d}s. \end{aligned} \tag{2.6}$$

If we use again that $g_N(s) \leq (1-\varepsilon)|s|$ if $|s| \geq s_1$, we obtain $w/g_N(s) \geq 1/(1-\varepsilon) > 1$ when $0 < s \leq w \leq \max g_N$; therefore, the integration with respect to w in the first integral in (2.6) is reduced to the interval $[0, s_1]$. Similarly, in the second integral the integration reduces to $[-s_1, 0]$. Noticing in addition that $g_N = g$ in $[-s_1, s_1]$ if N is large enough, we obtain from (2.6) that

$$1 = \int_{-s_1}^{s_1} \left(\int_{|s|}^{\max_{[-s_1,s_1]} g} \int_{\frac{w}{g(s)}} J(z) \mathrm{d}z \mathrm{d}w \right) p_N(s) \mathrm{d}s. \tag{2.7}$$

Since $p_N \to p$ uniformly on compact sets of $\mathbb{R}$, we may pass to the limit in (2.7) to obtain that p is nontrivial. According to the maximum principle, it follows that $p > 0$

in $\mathbb{R}$. We include a sketch of proof of this fact for completeness: assume $x_0 \in \mathbb{R}$ is such that $p(x_0) = 0$. Then

$$\int J\left(\frac{x_0 - y}{g(y)}\right) \frac{p(y)}{g(y)} \mathrm{d}y = 0.$$

Since g is positive in $\mathbb{R}$, this implies that $p \equiv 0$ in a neighborhood of x_0. Now a rather standard connectedness argument shows that $p \equiv 0$ in $\mathbb{R}$. Since p is nontrivial, this is not possible, so that $p > 0$ in $\mathbb{R}$.

Our next step is to prove part (a). Fix $\delta \in (0, \min\{1/2, C_1/2\})$. We may pass to the limit as $N \to +\infty$ in (2.2) in the statement of Lemma 5 to obtain that

$$\int_y^{y+\delta h(y)} p(s)\mathrm{d}s \le \frac{C}{h(y)} \tag{2.8}$$

for every $y \ge s_0$, where $C = C(\delta)$ is independent of y. Now fix $x \ge s_0$ and define recursively the sequence $x_0 = x$, $x_{n+1} = x_n + \delta h(x_n)$. We claim that $x_n \ge v(n)$, where v is the function given in Lemma 6 with $v(0) = s_0$ and A sufficiently small. This is proved by induction. Clearly $x_0 \ge v(0)$; assume $x_n \ge v(n)$ for some n. Then

$$x_{n+1} = x_n + \delta h(x_n) \ge v(n) + \delta h(v(n)) \ge v(n+1)$$

by Lemma 6. Thus $x_n \ge v(n)$ for every $n \in \mathbb{N}$, and in particular $x_n \to +\infty$ as $n \to +\infty$. Using (2.8) we have

$$\int_x^{x_{n+1}} p(s)\mathrm{d}s = \sum_{k=0}^{n} \int_{x_k}^{x_{k+1}} p(s)\mathrm{d}s \le C \sum_{k=0}^{n} \frac{1}{h(x_k)} \le C \sum_{k=0}^{n} \frac{1}{h(v(k))}.$$

Finally, observe that $h(v(k))$ is nondecreasing, and

$$\int^{\infty} \frac{\mathrm{d}s}{h(v(s))} = \frac{1}{A} \int^{\infty} \frac{\mathrm{d}t}{h(t)^2} < +\infty,$$

so that the series $\sum_{k=0}^{\infty} \frac{1}{h(v(k))}$ converges and

$$\int_x^{x_{n+1}} p(s)\mathrm{d}s \le C,$$

where C does not depend on n. Letting $n \to +\infty$ we have $p \in L^1(0, +\infty)$. Arguing similarly for negative x we obtain $p \in L^1(\mathbb{R})$.

To conclude the proof of this part, assume there exists another solution q of (1.2). Choose $\mu > 0$ such that $q(0) = \mu p(0)$. It is not difficult to check that the function $w = \min\{q, \mu p\}$ is a supersolution of (1.2), which is in $L^1(\mathbb{R})$. Therefore w is a solution of (1.2) (cf. Remark 1 in [11]). Now consider the set $\Omega := \{x \in \mathbb{R} : q(x) = \mu p(x)\}$, which is clearly a closed set. It is also open, for if $x_0 \in \Omega$, then

$$\begin{aligned} w(x_0) = q(x_0) &= \int J\left(\frac{x_0 - y}{g(y)}\right) \frac{q(y)}{g(y)} \mathrm{d}y \\ &\ge \int J\left(\frac{x_0 - y}{g(y)}\right) \frac{w(y)}{g(y)} \mathrm{d}y = w(x_0), \end{aligned}$$

so that $q \equiv w$ in a neighborhood of x_0, and similarly $\mu p \equiv w$ in a neighborhood of x_0. Hence Ω is open. Since $0 \in \Omega$, it follows that $\Omega = \mathbb{R}$ and so $q = \mu p$ in $\mathbb{R}$.

To prove part (b), we take a large positive x and denote by $f(x)$ the largest solution of the equation $w = g(x + w)$ [which exists thanks to condition (1.3)]. It is to be noted that the condition $\limsup_{s\to+\infty} \frac{g(s)}{|s|} < \frac{1}{2}$ implies

$$\limsup_{x\to+\infty} \frac{f(x)}{x} < 1. \tag{2.9}$$

Then, from (2.1),

$$1 = \int_0^{f(x)} \int_{x-w}^{x+w} p_N(s) \int_{\frac{w}{g_N(s)}}^{1} J(z)\mathrm{d}z\mathrm{d}s\mathrm{d}w,$$

and passing to the limit as $N \to +\infty$:

$$\begin{aligned} 1 &= \int_0^{f(x)} \int_{x-w}^{x+w} p(s) \int_{\frac{w}{g(s)}}^{1} J(z)\mathrm{d}z\mathrm{d}s\mathrm{d}w \\ &\leq \int_0^{f(x)} \int_{x-w}^{x+w} p(s)\mathrm{d}s\mathrm{d}w \leq f(x) \int_{x-f(x)}^{x+f(x)} p(s)\mathrm{d}s. \end{aligned}$$

Thus, if x is large enough:

$$\begin{aligned} \int_{x-f(x)}^{x+f(x)} \frac{p(s)}{g(s)}\mathrm{d}s &\geq \frac{1}{C_2 h(x+f(x))} \int_{x-f(x)}^{x+f(x)} p(s)\mathrm{d}s \\ &\geq \frac{C_1}{C_2 f(x) g(x+f(x))} = \frac{C_1}{C_2 f(x)^2}, \end{aligned} \tag{2.10}$$

since $f(x) = g(x + f(x))$. On the other hand, choose $\delta \in (0, 1)$ and denote $\widetilde{D}_x = \{y \in \mathbb{R} : |x - y| \leq C_1 \delta h(y)\}$. Then we have

$$p(x) = \int J\left(\frac{x-y}{g(y)}\right) \frac{p(y)}{g(y)}\mathrm{d}y \geq c_\delta \int_{\widetilde{D}_x} \frac{p(y)}{g(y)}\mathrm{d}y,$$

where $c_\delta = \inf_{|z|\leq\delta} J(z) > 0$. Observe that, when δ is small enough, and using the hypothesis that h' is bounded, the set $\widetilde{D}_x$ can be expressed as $\widetilde{D}_x = [h_1(x), h_2(x)]$, where $h_1(x), h_2(x)$ are, respectively, the unique solutions of the equations $y + \delta C_1 h(y) = x$, and $y - \delta C_1 h(y) = x$ (we remark for its immediate use that both functions h_1, h_2 are increasing). Therefore:

$$p(x) \geq c_\delta \int_{h_1(x)}^{h_2(x)} \frac{p(y)}{g(y)}\mathrm{d}y. \tag{2.11}$$

Next define the sequence $\{x_k\}$ by $x_1 = x$ and $x_{k+1} = h_1^{-1}(h_2(x_k))$, for $k = 1, \ldots, \ell$, where ℓ is a positive integer to be chosen. It follows from (2.11) that

$$\sum_{k=0}^{\ell-1} p(H^k(x)) \geq c_\delta \int_{h_1(x)}^{h_2(x_\ell)} \frac{p(y)}{g(y)}\mathrm{d}y, \tag{2.12}$$

where we have set $H = h_1^{-1} \circ h_2$. Also observe that

$$h_2(x_k) - h_1(x_k) = \delta C_1 h(h_1(x_k)) + h(h_2(x_k))$$
$$\geq 2C_1\delta h(h_1(x_k)) \geq 2C_1\delta h(h_1(x))$$

so that we obtain the estimate $h_2(x_\ell) - h_1(x) \geq 2C_1\delta\ell h(h_1(x))$ for the length of the interval of integration in (2.12). Taking into account that $h(x) = O(x)$ for large x, we further deduce

$$x = h_1(x) + \delta C_1 h(h_1(x)) \leq (1 + \delta C C_1)h_1(x).$$

for some $C > 0$. Hence

$$h_2(x_\ell) - h_1(x) \geq 2C_1\delta\ell h\left(\frac{1}{1+\delta C C_1}x\right). \tag{2.13}$$

On the other hand, condition (2.9) allows to choose $z = z(x)$ so that $z - f(z) = h_1(x)$ if x is large enough. Moreover, there exists $\eta > 0$ such that

$$\eta \leq 1 - \frac{f(z)}{z} = \frac{h_1(x)}{z} = \frac{x - C_1\delta h(h_1(x))}{z} \leq \frac{x}{z}.$$

This in turn implies

$$f(z) = g(z + f(z)) \leq C_2 h(z + f(z)) \leq C_2 h(2z) \leq C_2 h\left(\frac{2}{\eta}x\right) \tag{2.14}$$

for large x. Combining (2.13), (2.14) and hypothesis (1.4) [and also Remark 1(c)], we deduce that for a suitably large ℓ, independent of x, the inequality

$$h_2(x_\ell) - h_1(x) \geq 2f(z)$$

holds. Then, since $z - f(z) = h_1(x)$, we arrive from (2.12), taking into account that $[z - f(z), z + f(z)] \subset [h_1(x), h_2(x_\ell)]$ and using again (2.14), at

$$\sum_{k=0}^{\ell-1} p(H^k(x)) \geq c_\delta \int_{z-f(z)}^{z+f(z)} \frac{p(y)}{g(y)}\mathrm{d}y \geq \frac{C_1 c_\delta}{C_2 f(z)^2} \geq \frac{C_1 c_\delta}{C_2^2 h(\frac{2}{\eta}x)^2}.$$

Take now $b > a \gg 1$. Integrating the last inequality in (a, b):

$$\sum_{k=0}^{\ell-1} \int_a^b p(H^k(x))\mathrm{d}x \geq \frac{C_1 c_\delta}{C_2^2} \int_a^b \frac{1}{h(\frac{2}{\eta}x)^2}\mathrm{d}x = \frac{C_1 c_\delta \eta}{2C_2^2} \int_{\frac{2a}{\eta}}^{\frac{2b}{\eta}} \frac{1}{h(y)^2}\mathrm{d}y. \tag{2.15}$$

To conclude the proof, we first notice that $H' \geq 1$ for large values of x. Indeed, $x = h_1(x) + \delta C_1 h(h_1(x))$ implies that $h_1^{-1}(x) = x + \delta C_1 h(x)$, so that $H(x) = h_1^{-1}(h_2(x)) = h_2(x) + \delta C_1 h(h_2(x)) = 2h_2(x) - x$ and $H' = 2h_2' - 1$ follows. In addition, $h_2(x) = x + \delta C_1 h(h_2(x))$, so that $h_2'(x) = 1 + \delta C_1 h'(h_2(x))h_2'(x) \geq 1$, thus $H' \geq 1$. Now:

$$\int_a^b p(H(x))\mathrm{d}x = \int_{H(a)}^{H(b)} \frac{p(y)}{H'(H^{-1}(y))}\mathrm{d}y \leq \int_{H(a)}^{H(b)} p(y)\mathrm{d}y,$$

and similarly

$$\int_a^b p(H^k(x))\mathrm{d}x \leq \int_{H^k(a)}^{H^k(b)} p(y)\mathrm{d}y.$$

From (2.15) we finally obtain

$$\int_{H(a)}^{H^{\ell-1}(b)} p(y)\mathrm{d}y \geq C \int_{\frac{2a}{\eta}}^{\frac{2b}{\eta}} \frac{1}{h(y)^2}\mathrm{d}y$$

for some $C > 0$. Letting $b \to +\infty$, since $1/h^2 \notin L^1$ we deduce that $p \notin L^1(\mathbb{R})$ either.

Finally, if there were another solution in $L^1(\mathbb{R})$, then a similar argument as in the proof of part (a) would imply that $p \in L^1(\mathbb{R})$. Thus problem (1.2) does not admit positive solutions in $L^1(\mathbb{R})$ in this case. This concludes the proof. $\square$

REMARK 2. It is not hard to see that the proof that the solutions p_N converge locally uniformly to a nonnegative, nontrivial function also works if (1.3) is replaced by $g(s) \leq |s|$ for large $|s|$. Arguing as in the final part of the proof of Theorem 2 below, it can also be seen that p is a positive solution of (1.2).

Proof of Theorem 2. We begin with part (a). The existence of a positive solution of (1.2) follows by a completely different method than the one used before. Choose and fix a function $\phi \in C(\mathbb{R})$ with $\phi > 0$ and

$$\int \frac{\phi(y)}{g(y)}\mathrm{d}y < +\infty.$$

For $K > 0$, consider the problem

$$u(x) - \int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}\mathrm{d}y = \int_{|y|\geq K} J\left(\frac{x-y}{g(y)}\right)\frac{\phi(y)}{g(y)}\mathrm{d}y \tag{2.16}$$

for $x \in [-K, K]$. It is not hard to show that the operator

$$T_K u(x) = \int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}\mathrm{d}y, \qquad x \in [-K, K],$$

is compact in $C([-K, K])$, since g is positive and bounded in $[-K, K]$. Moreover, its spectral radius is less than one. Indeed, if $r = \mathrm{spr}(T_K) \geq 1$, by Krein–Rutman's theorem (cf. [24]) we would obtain a function $u \in C([-K, K])$, $u > 0$ such that $T_K u = ru$. Integrating in $[-K, K]$ and using Fubini's theorem we would have:

$$\int_{-K}^{K} u(y) \left(r - \frac{1}{g(y)} \int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right) \mathrm{d}x \right) \mathrm{d}y = 0,$$

which is impossible, since the integrand is a nonnegative function which is not identically zero.

Therefore there exists a positive solution $u_K \in C[-K, K]$ of (2.16). Define

$$p_K(x) := \begin{cases} \dfrac{u_K(x)}{u_K(0)}, & |x| \le K \\ \dfrac{\phi(x)}{u_K(0)}, & |x| > K. \end{cases}$$

We deduce that $p_K > 0$ in $[-K, K]$, $p_K(0) = 1$ and

$$p_K(x) = \int J\left(\frac{x-y}{g(y)}\right) \frac{p_K(y)}{g(y)} \mathrm{d}y, \quad |x| \le K.$$

Our immediate task will be to obtain bounds for the functions p_K. If $\varepsilon > 0$ is sufficiently small, by our hypotheses on g, there exists $M > 0$ such that $g(s) \ge (1+\varepsilon)|s|$ if $|s| \ge M$. Then

$$\begin{aligned} 1 = p_K(0) &= \int J\left(\frac{y}{g(y)}\right) \frac{p_K(y)}{g(y)} \mathrm{d}y \ge \int_{|y| \ge M} J\left(\frac{y}{g(y)}\right) \frac{p_K(y)}{g(y)} \mathrm{d}y \\ &\ge \inf_{|z| \le \frac{1}{1+\varepsilon}} J(z) \int_{|y| \ge M} \frac{p_K(y)}{g(y)} \mathrm{d}y, \end{aligned}$$

so that

$$\int_{|y| \ge M} \frac{p_K(y)}{g(y)} \mathrm{d}y \le C,$$

for some positive constant C which does not depend on K. Let $g_0 = \inf_{\mathbb{R}} g$. In particular:

$$\int_{M}^{M + \frac{1}{4} g_0} \frac{p_K(y)}{g(y)} \mathrm{d}y \le C.$$

Denoting $g_1 = \max_{[-M - \frac{1}{4} g_0, M + \frac{1}{4} g_0]} g$, it easily follows that

$$\int_{M}^{M + \frac{1}{4} g_0} p_K(y) \mathrm{d}y \le C g_1.$$

Whence it exists $x_0 \in [M, M + \frac{1}{4} g_0]$ such that $p_K(x_0) \le \frac{4 C g_1}{g_0}$. Thus

$$\int J\left(\frac{x_0 - y}{g(y)}\right) \frac{p_K(y)}{g(y)} \mathrm{d}y \le \frac{4 C g_1}{g_0}.$$

Observe that in the set $Q := \{y \in \mathbb{R} : |x_0 - y| \le \frac{1}{2} g_0\}$ the inequality $|x_0 - y| \le g(y)$ holds, so that Q is contained in the region of integration. Moreover, $J\left(\frac{x_0-y}{g(y)}\right) \ge \inf_{|z|\le\frac{1}{2}} J(z) > 0$ in Q. Therefore:

$$\int_{x_0-\frac{1}{2}g_0}^{x_0+\frac{1}{2}g_0} \frac{p_K(y)}{g(y)} \mathrm{d}y \le \frac{4Cg_1}{g_0 \inf_{|z|\le\frac{1}{2}} J(z)}.$$

Now, the interval $[x_0 - \frac{1}{2} g_0, x_0 + \frac{1}{2} g_0]$ contains $[M - \frac{1}{4} g_0, M]$, so that

$$\int_{M-\frac{1}{4}g_0}^{M} \frac{p_K(y)}{g(y)} \mathrm{d}y \le \frac{4Cg_1}{g_0 \inf_{|z|\le\frac{1}{2}} J}.$$

We can repeat this process finitely many times to arrive at

$$\int \frac{p_K(y)}{g(y)} \mathrm{d}y \le C, \tag{2.17}$$

for some positive constant C not depending on K. Arguing as in the proof of Theorem 1, we obtain that for some sequence $K_n \to +\infty$, $p_{K_n} \to p$ uniformly on compact sets of $\mathbb{R}$, where $p \ge 0$, $p(0) = 1$. To pass to the limit in the equation verified by p_{K_n}, we fix $x \in \mathbb{R}$, take $R > |x|$ and notice that for sufficiently large n:

$$\int_{-R}^{R} J\left(\frac{x-y}{g(y)}\right) \frac{p_{K_n}(y)}{g(y)} \mathrm{d}y \le p_{K_n}(x).$$

Hence letting first $n \to +\infty$ and then $R \to +\infty$:

$$\int J\left(\frac{x-y}{g(y)}\right) \frac{p(y)}{g(y)} \mathrm{d}y \le p(x).$$

Thus p is a supersolution of (1.2). In particular, $p > 0$ in $\mathbb{R}$. We claim that, since $\int \frac{1}{g} = +\infty$, p is a solution of (1.2). For this aim, we remark that by (2.17) and with a similar procedure to the one just used, we achieve $\frac{p}{g} \in L^1(\mathbb{R})$. Fix $x, z \in \mathbb{R}$. Then for large enough $M, K > 0$:

$$p_K(x) - p_K(z) = \int_{-M}^{M} \left(J\left(\frac{x-y}{g(y)}\right) - J\left(\frac{z-y}{g(y)}\right)\right) \frac{p_K(y)}{g(y)} \mathrm{d}y + \gamma_{K,M}$$

where

$$\gamma_{K,M} = \int_{|y|\ge M} \left(J\left(\frac{x-y}{g(y)}\right) - J\left(\frac{z-y}{g(y)}\right)\right) \frac{p_K(y)}{g(y)} \mathrm{d}y.$$

Since $g(y) \ge |y|$ for large $|y|$ it also follows that $g(y) \ge M$ if $|y| \ge M$, so that

$$|\gamma_{K,M}| \le L|x-z|^\alpha \int_{|y|\ge M} \frac{p_K(y)}{g(y)^{1+\alpha}} \mathrm{d}y \le \frac{L|x-z|^\alpha}{M^\alpha} \int \frac{p_K(y)}{g(y)} \mathrm{d}y \le \frac{C}{M^\alpha}.$$

Setting $K = K_n$, and passing to a subsequence if necessary we have $\gamma_{K_n,M} \to \overline{\gamma}_M$, which verifies $|\overline{\gamma}_M| \le \frac{C}{M^\alpha}$. Moreover:

$$p(x) - p(z) = \int_{-M}^{M} \left(J\left(\frac{x-y}{g(y)}\right) - J\left(\frac{z-y}{g(y)}\right)\right) \frac{p(y)}{g(y)} \mathrm{d}y + \overline{\gamma}_M,$$

and letting $M \to +\infty$ we deduce, since $x, z \in \mathbb{R}$ were arbitrary that there exists a constant $A \in \mathbb{R}$ such that

$$p(x) = \int J\left(\frac{x-y}{g(y)}\right) \frac{p(y)}{g(y)} \mathrm{d}y + A, \qquad \text{for } x \in \mathbb{R}.$$

We have already seen that p is a supersolution of (1.2), so that $A \ge 0$. Also, $p(x) \ge A$ in $\mathbb{R}$. Hence, since $p/g \in L^1(\mathbb{R})$:

$$+\infty > \int \frac{p(y)}{g(y)} \mathrm{d}y \ge A \int \frac{1}{g(y)} \mathrm{d}y,$$

and $A = 0$ is implied by the condition $\frac{1}{g} \notin L^1(\mathbb{R})$. Then p is a positive solution of (1.2).

Finally, notice that $\lim_{|x|\to+\infty} p(x) = 0$ by (1.6) in the Introduction, therefore $p \in L^\infty(\mathbb{R})$. This concludes the proof of part (a).

For part (b) we will follow closely the proof of Theorem 1.2 in [11]. Notice that under the hypothesis that $\frac{1}{g} \in L^1(\mathbb{R})$, the operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$Th(x) = \int J\left(\frac{x-y}{g(y)}\right) \frac{h(y)}{g(y)} \mathrm{d}y, \quad x \in \mathbb{R}, \tag{2.18}$$

is a Hilbert–Schmidt operator. Indeed, by Fubini's theorem:

$$\begin{aligned} \int\int J\left(\frac{x-y}{g(y)}\right)^2 \frac{1}{g(y)^2} \mathrm{d}y\mathrm{d}x &= \int \frac{1}{g(y)^2} \int J\left(\frac{x-y}{g(y)}\right)^2 \mathrm{d}x\mathrm{d}y \\ &= \int J(z)^2 \mathrm{d}z \int \frac{1}{g(y)} \mathrm{d}y < +\infty. \end{aligned}$$

Therefore T is compact in $L^2(\mathbb{R})$. We claim that the spectral radius r of T is less than one. Assume for a contradiction that $r \ge 1$. Then Krein–Rutman's theorem implies the existence of a nontrivial $q \in L^2(\mathbb{R})$, $q \ge 0$, such that $T^* q = rq$, that is,

$$\int J\left(\frac{x-y}{g(x)}\right) \frac{q(y)}{g(x)} \mathrm{d}y = rq(x), \quad x \in \mathbb{R}. \tag{2.19}$$

We claim that $q \in C(\mathbb{R})$ and $\lim_{|x|\to\infty} q(x) = 0$. Indeed,

$$\begin{aligned} r|g(x)q(x) - g(z)q(z)| &\le \int_D \left| J\left(\frac{x-y}{g(x)}\right) - J\left(\frac{z-y}{g(z)}\right)\right| q(y)\mathrm{d}y \\ &\le L \int_D \left|\frac{x-y}{g(x)} - \frac{z-y}{g(z)}\right|^\alpha q(y)\mathrm{d}y \\ &\le L \left(\int_D \left|\frac{x-y}{g(x)} - \frac{z-y}{g(z)}\right|^{2\alpha} \mathrm{d}y\right)^{\frac{1}{2}} \|q\|_{L^2}, \end{aligned}$$

where $D = B(x, g(x)) \cup B(z, g(z))$. If x, z are taken in a compact set, then $D \subset [-M, M]$ for some $M > 0$, so that

$$\left| \frac{x-y}{g(x)} - \frac{z-y}{g(z)} \right| \leq \frac{M|g(z) - g(x)| + |xg(z) - zg(x)|}{g(x)g(z)}.$$

Whence:

$$r|g(x)q(x) - g(z)q(z)| \leq (2M)^{\frac{1}{2}} L \|q\|_{L^2} \frac{(M|g(z) - g(x)| + |xg(z) - zg(x)|)^{\alpha}}{g(x)^{\alpha} g(z)^{\alpha}}.$$

It follows that if $z \to x$ then $g(z)q(z) \to g(x)q(x)$, that is, gq is continuous in $\mathbb{R}$, hence so is q. On the other hand, using Jensen's inequality in (2.19):

$$r^2 q(x)^2 \leq \frac{1}{g(x)} \int J\left(\frac{x-y}{g(x)}\right) q(y)^2 \mathrm{d}y \leq \frac{\|J\|_{\infty} \|q\|_{L^2}^2}{g(x)} \to 0,$$

since $\lim_{|s| \to +\infty} g(s) = +\infty$. Therefore $q \in L^{\infty}(\mathbb{R})$, and it attains its maximum at some point $x_0 \in \mathbb{R}$. Then

$$r\|q\|_{\infty} = \int J\left(\frac{x_0 - y}{g(x_0)}\right) \frac{q(y)}{g(x_0)} \mathrm{d}y \leq \|q\|_{\infty}.$$

We deduce $r = 1$ and $q \equiv \|q\|_{\infty}$ in $B(x_0, g(x_0))$. A connectedness argument gives that necessarily $q \equiv \|q\|_{\infty}$ in $\mathbb{R}$, which is impossible.

Thus $r < 1$. Now choose an arbitrary nonnegative, nontrivial $\xi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \cap C(\mathbb{R})$. There exists a unique solution of the equation $w - Tw = \xi$, given by the von Neumann series $w = \sum_{n=0}^{\infty} T^n \xi \geq 0$. In addition, since $\xi \in C(\mathbb{R})$ we obtain easily that $w = \xi + Tw \in C(\mathbb{R})$ as well. Thus $w \geq Tw$, that is, w is a supersolution of (1.2) and by the strong maximum principle $w > 0$ in $\mathbb{R}$. Using that $\int Tv = \int v$ for every $v \in L^1(\mathbb{R})$, we also deduce that $w \notin L^1(\mathbb{R})$.

Finally, assume there exists a solution $p \in L^1(\mathbb{R})$ of (1.2). Since $g(x) \geq g_0 > 0$ in $\mathbb{R}$, we obtain

$$p(x) \leq \frac{\|J\|_{\infty}}{g_0} \|p\|_{L^1}$$

so that $p \in L^{\infty}(\mathbb{R})$, and in particular $p \in L^2(\mathbb{R})$, which is not possible since the spectral radius of T is strictly less than one in $L^2(\mathbb{R})$. The proof is concluded. □

REMARK 3. It is worthy of mention that the operator T defined in (2.18) is well defined and compact in $L^q(\mathbb{R})$ for every $q > 1$ under the condition $\frac{1}{g} \in L^1(\mathbb{R})$. Indeed, the inequality

$$|Th(x)| \leq \|J\|_{\infty} \|h\|_{\infty} \int \frac{1}{g(y)} \mathrm{d}y$$

shows that $T : L^{\infty}(\mathbb{R}) \to L^{\infty}(\mathbb{R})$, so that by interpolation T is defined in $L^q(\mathbb{R})$ for every $q \geq 1$. Moreover, T is compact in $L^2(\mathbb{R})$, hence Theorem 4.2.14 in [20] implies the compactness of T in $L^q(\mathbb{R})$ for every $q \geq 2$. Thus T^* is compact in L^q for every $q \in (1, 2)$, and the same theorem gives that T^* is compact in L^q for every $q > 1$, so that T is also compact in $L^q(\mathbb{R})$ for every $q > 1$.

However, T is not compact in $L^1(\mathbb{R})$. Observe that $\|Tu\|_{L^1} = \|u\|_{L^1}$ for every nonnegative $u \in L^1(\mathbb{R})$. Thus $\|T\| = 1$, and it follows similarly that $\|T^k\| = 1$ for every $k \in \mathbb{N}$. Thus the spectral radius of T is one. If T were compact, by Krein–Rutman's theorem we would have a positive solution of (1.2), which is not possible by Theorem 2, part (b).

3. An insight into the critical case

This final section is devoted to analyze what could be termed as a "critical case" in the subject of existence of positive solutions in $L^1(\mathbb{R})$ of the problem:

$$\int J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}\mathrm{d}y - u(x) = 0, \qquad x \in \mathbb{R}. \tag{1.2}$$

More precisely, we will assume throughout that $g(s) = \lambda|s|$ for some $\lambda > 0$ and all sufficiently large $|s|$, and we will study the influence of the parameter λ in the question of integrability of solutions. Observe that, by Theorems 1 and 2 [cf. also Remark 1(b)], there are always positive solutions of (1.2) for these functions g whenever $\lambda > 0$.

As we have seen in the previous sections, it turns out that the value $\lambda = 1$ should play a special role. This is easily seen by considering the domain of integration $D_x := \{y : \ |x-y| < g(y)\}$. For all sufficiently large positive x, we have

$$\begin{aligned}
D_x &= \left(\frac{1}{1+\lambda}x, \frac{1}{1-\lambda}x\right) && \text{if } \lambda < 1\\
D_x &= \left(\frac{1}{1+\lambda}x, +\infty\right) && \text{for } \lambda = 1\\
D_x &= \left(-\infty, -\frac{1}{\lambda-1}x\right) \cup \left(\frac{1}{\lambda+1}x, +\infty\right) && \text{when } \lambda > 1.
\end{aligned} \tag{3.1}$$

Thus we see that D_x is a finite interval for $\lambda < 1$ and becomes infinite when λ crosses the value 1 (of course, a similar analysis can be done when x is negative). This explains why in some of our reasonings below the cases $\lambda < 1$ and $\lambda > 1$ have to be treated separately.

In order to understand problem (1.2) when $g(s) = \lambda|s|$ for large $|s|$, it is convenient first to consider the case where $g(s) = \lambda|s|$ for $s \in \mathbb{R}$, that is,

$$u(x) = \int J\left(\frac{x-y}{\lambda|y|}\right)\frac{u(y)}{\lambda|y|}\mathrm{d}y, \qquad x \in \mathbb{R}\backslash\{0\}. \tag{3.2}$$

Observe that this equation has to be considered in $\mathbb{R}\backslash\{0\}$, the reason being that there is a singularity introduced by the fact that the function $\lambda|s|$ vanishes at $s = 0$. However, it can be checked that the problem is meaningful.

We look for a solution of (3.2) of the form $u(x) = |x|^{-\alpha}$, $x \in \mathbb{R}\backslash\{0\}$, for some $\alpha > 0$. In what follows we are only dealing in detail with the case $\lambda > 1$, since the case $0 < \lambda \leq 1$ is slightly simpler. For $x > 0$ we have

$$\begin{aligned}\int J\left(\frac{x-y}{\lambda|y|}\right)\frac{|y|^{-\alpha}}{\lambda|y|}\mathrm{d}y &= \frac{1}{\lambda}\int_{\frac{1}{\lambda+1}x}^{\infty} J\left(\frac{x-y}{\lambda y}\right)y^{-\alpha-1}\mathrm{d}y \\ &+\frac{1}{\lambda}\int_{-\infty}^{\frac{1}{\lambda-1}x} J\left(\frac{x-y}{-\lambda y}\right)(-y)^{-\alpha-1}\mathrm{d}y. \end{aligned} \tag{3.3}$$

In the first integral of the right-hand side of (3.3), we make the substitution $t = \frac{x-y}{\lambda y}$, to obtain:

$$\int_{\frac{1}{\lambda+1}x}^{\infty} J\left(\frac{x-y}{\lambda y}\right)y^{-\alpha-1}\mathrm{d}y = \lambda x^{-\alpha}\int_{-\frac{1}{\lambda}}^{1} J(t)(1+\lambda t)^{\alpha-1}\mathrm{d}t,$$

and with the same change of variable

$$\int_{-\infty}^{\frac{1}{\lambda-1}x} J\left(\frac{x-y}{-\lambda y}\right)(-y)^{-\alpha-1}\mathrm{d}y = \lambda x^{-\alpha}\int_{-1}^{-\frac{1}{\lambda}} J(s)|\lambda s+1|^{\alpha-1}\mathrm{d}s,$$

where we have used the symmetry of J. Thus we see that $u = |x|^{-\alpha}$ will be a solution of the equation in (3.2) in $x > 0$ if and only if

$$\int_{-1}^{1} J(t)|1+\lambda t|^{\alpha-1}\mathrm{d}t = 1.$$

A similar argument works for $x < 0$. Thus it makes sense to consider the function F_λ defined by

$$F_\lambda(\alpha) = \int_{-1}^{1} J(t)|1+\lambda t|^{\alpha-1}\mathrm{d}t - 1, \qquad \alpha > 0 \tag{3.4}$$

and to look for the zeros of this function, which will provide with special solutions of (3.2). They will turn out to be determinant for the problem (1.2).

It will be proved in Lemma 8 below that F_λ has precisely two zeros, counting multiplicities. One of them is $\alpha = 1$ and the other one will be denoted by $\alpha(\lambda)$ (when $\alpha = 1$ is a double zero we simply set $\alpha(\lambda) = 1$).

Then we have:

THEOREM 7. *Assume $g \in C(\mathbb{R})$, $g > 0$ verifies $g(s) = \lambda|s|$ for $|s| \geq s_0 > 0$, where $\lambda > 0$. Then:*

(a) If $\alpha(\lambda) > 1$, then all positive solutions p of (1.2) lie in $L^1(\mathbb{R})$ and there exist $C_1, C_2 > 0$ such that

$$\frac{C_1}{|x|^{\alpha(\lambda)}} \leq p(x) \leq \frac{C_1}{|x|^{\alpha(\lambda)}} \qquad \textit{for large } |x|. \tag{3.5}$$

(b) When $\alpha(\lambda) < 1$, the positive solutions of (1.2) do not belong to $L^1(\mathbb{R})$.

The most important property with regard to F_λ is:

LEMMA 8. *The function F_λ is strictly convex and belongs to $C^\infty(0,+\infty)$. Moreover,* $\lim_{\alpha\to+\infty} F_\lambda(\alpha)=+\infty$*, and either F_λ has exactly two positive zeros, $\alpha=1$ and $\alpha=\alpha(\lambda)\neq 1$ or F_λ vanishes only at $\alpha=1$ with $F'_\lambda(1)=0$. When $\lambda\le 1$, we have $\alpha(\lambda)=2$.*

REMARK 4. As we have already quoted, when F_λ vanishes only at $\alpha=1$, we will set $\alpha(\lambda)=1$.

Proof of Lemma 8. The assertion about the smoothness of F_λ is more or less immediate (in spite of the fact that F_λ has a singularity at $t=-\frac{1}{\lambda}$ when $\lambda\ge 1$ and $\alpha<1$, the singularity is integrable since $\alpha>0$). Moreover,

$$F''_\lambda(\alpha)=\int_{-1}^{1} J(t)|1+\lambda t|^{\alpha-1}(\log|1+\lambda t|)^2\mathrm{d}t>0,$$

so that F_λ is a strictly convex function. As a consequence, F_λ has at most two zeros. The fact that one of these zeros is precisely $\alpha=1$ follows because J has unit integral. Also, choosing $\delta\in(0,\frac{1}{2})$ small enough:

$$F_\lambda(\alpha)\ge\int_{\delta}^{2\delta} J(t)(1+\lambda t)^{\alpha-1}\mathrm{d}t\ge(\inf_{|z|\le 2\delta} J(z))\delta(1+\lambda\delta)^{\alpha-1}-1\to+\infty$$

as $\alpha\to+\infty$, and when $\lambda>1$:

$$\begin{aligned}F_\lambda(\alpha)&\ge\int_{-\frac{1}{\lambda}}^{0} J(t)(1+\lambda t)^{\alpha-1}\mathrm{d}t-1\ge\inf_{|z|\le\frac{1}{\lambda}} J(z)\int_{-\frac{1}{\lambda}}^{0}(1+\lambda t)^{\alpha-1}\mathrm{d}t-1\\&=\frac{\inf_{|z|\le\frac{1}{\lambda}} J(z)}{\lambda\alpha}-1\to+\infty\qquad\text{as }\alpha\to 0+.\end{aligned}$$

Thus we deduce that, for $\lambda>1$, either F_λ vanishes exactly twice, or it vanishes only at $\alpha=1$, which is a global minimum for F_λ. On the other hand, if $\lambda\le 1$, we have for $t\in(-1,0)$, $1+\lambda t\ge 1+t\ge 0$, and of course the same inequality is true when $t\in(0,1)$. Therefore

$$F_\lambda(\alpha)=\int_{-1}^{1} J(t)(1+\lambda t)^{\alpha-1}\mathrm{d}t-1.$$

Since J is symmetric, it follows that $F_\lambda(2)=0$. Hence by the strict convexity of F_λ, we deduce that it only vanishes at $\alpha=1,2$. Whence $\alpha(\lambda)=2$ in this case, the proof is concluded. □

REMARK 5. The presence of the solution $u(x)=\frac{1}{|x|}$, which is not in $L^1(\mathbb{R})$ is somehow surprising, but it is only due to the fact that the weight $\lambda|s|$ vanishes at zero. As Theorem 7 shows, the only important point in order to decide whether the positive solutions of (1.2) are in $L^1(\mathbb{R})$ or not is the relation between the values $\alpha(\lambda)$ and 1.

Proof of Theorem 7. (a) Assume $\alpha(\lambda) > 1$ and let $u_\lambda = |x|^{-\alpha(\lambda)}$, $x \neq 0$. Choose $N \gg s_0$ and take $A > 0$ such that $p < Au_\lambda$ in $|x| \leq N$. Define

$$v = \min\{p, Au_\lambda\}.$$

It is to be noted that v is well-defined since $v = p$ in $|x| \leq N$, while both functions make sense if $|x| > N$ and are solutions of (1.2) if N is adequately large. With this in mind, it is easily seen that v is a supersolution of (1.2). Since $v \in L^1(\mathbb{R})$, because $\alpha(\lambda) > 1$, it follows that v is in fact a solution. Arguing as in the proof of Theorem 1(a) (a little bit of extra care is needed because of the singularity at zero), we obtain that p has to be a multiple of v. Therefore $p = v \in L^1(\mathbb{R})$ and

$$p(x) \leq \frac{A}{|x|^{\alpha(\lambda)}} \qquad \text{for large } |x|,$$

which shows the upper bound in (3.5). The lower bound is obtained in a similar way: choose $B > 0$ such that $p \geq Bu_\lambda$ in $1 \leq |x| \leq N$. The function

$$w(x) = \max\{p(x), Bu_\lambda(x)\chi_{|x|\geq 1}\}$$

is a subsolution in $L^1(\mathbb{R})$, therefore a solution of (1.2) (here $\chi_{|x|\geq 1}$ denotes the characteristic function of the set $\{x \in \mathbb{R} : |x| \geq 1\}$). Thus $w = p$, so that

$$p(x) \geq \frac{B}{|x|^{\alpha(\lambda)}} \qquad \text{for } |x| \geq 1.$$

(b) Finally assume $\alpha(\lambda) < 1$, and for the sake of contradiction suppose $p \in L^1(\mathbb{R})$. Take N as before and define

$$z = \begin{cases} Du_\lambda(x), & |x| \leq N \\ \min\{p(x), Du_\lambda(x)\}, & |x| > N, \end{cases}$$

where D is such that $Du_\lambda(x) < p(x)$ on $|x| = N$. It can be checked that z is a supersolution of (3.2). Since $z \in L^1(\mathbb{R})$ [because $\alpha(\lambda) < 1$] it also follows that z is actually a solution of (3.2). However, any two solutions of (3.2) in $L^1(\mathbb{R})$ are linearly dependent, so we deduce $z = Du_\lambda$. In particular, $Du_\lambda(x) \leq p(x)$ if $|x| > N$, which is a contradiction since we assume $p \in L^1(\mathbb{R})$ while $u_\lambda \notin L^1(\mathbb{R})$. Thus $p \notin L^1(\mathbb{R})$, as we wanted to show. This finishes the proof. □

To conclude this section, we return to the question on whether $\alpha(\lambda) > 1$ or $\alpha(\lambda) < 1$ for a given value of λ. In order to study this matter, we analyze the function

$$G(\lambda) = F'_\lambda(1) = \int_{-1}^{1} J(t) \log|1 + \lambda t|\, dt.$$

It is clear that $G(\lambda) < 0$ implies $\alpha(\lambda) > 1$, while $G(\lambda) > 0$ and $G(\lambda) = 0$ mean $\alpha(\lambda) < 1$ and $\alpha(\lambda) = 1$, respectively. Thus the important point is to precisely determine the zeros of the function G and the regions where it is positive or negative. This turns out to be a very subtle question, which depends even on the kernel J.

We can prove the following:

LEMMA 9. *There exist* $\lambda_0, \lambda_\infty > 0$ *such that* $\sqrt{2} < \lambda_0 \le \lambda_\infty$ *and with the property that* $G < 0$ *in* $(0, \lambda_0)$ *and* $G > 0$ *in* $(\lambda_\infty, +\infty)$.

Proof. Observe first that

$$G(\lambda) = \int_{-1}^{1} J(t) \log \left| \lambda \left(t + \frac{1}{\lambda} \right) \right| \mathrm{d}t = \log \lambda + \int_{-1}^{1} J(t) \log \left| t + \frac{1}{\lambda} \right| \mathrm{d}t$$

Therefore $G(\lambda) - \log \lambda$ converges to a finite value as $\lambda \to +\infty$. Thus $\lim_{\lambda \to +\infty} G(\lambda) = +\infty$. Since G is continuous, the proof will be concluded if we show that $G(\lambda) < 0$ for $\lambda \in (0, \sqrt{2}]$. Write:

$$G(\lambda) = \int_{-1}^{-\frac{1}{\lambda}} J(t) \log |1 + \lambda t| \mathrm{d}t + \int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} J(t) \log |1 + \lambda t| \mathrm{d}t$$

$$+ \int_{\frac{1}{\lambda}}^{1} J(t) \log |1 + \lambda t| \mathrm{d}t =: I_1 + I_2 + I_3.$$

For the first of these integrals we have

$$I_1 = \int_{\frac{1}{\lambda}}^{1} J(t) \log(\lambda t - 1) \mathrm{d}t$$

so that

$$I_1 + I_3 = \int_{\frac{1}{\lambda}}^{1} J(t) \log(\lambda^2 t^2 - 1) \mathrm{d}t < 0,$$

since $\lambda^2 t^2 - 1 \le \lambda^2 - 1 \le 1$, and the logarithm becomes negative. The remaining integral can be dealt with in a similar fashion:

$$I_2 = \int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} J(t) \log(1 + \lambda t) \mathrm{d}t = \int_{0}^{\frac{1}{\lambda}} J(t) \log(1 - \lambda^2 t^2) \mathrm{d}t < 0.$$

Therefore $G(\lambda) < 0$ if $0 < \lambda \le \sqrt{2}$. The proof is concluded. □

Proof of Theorem 3. It is a straightforward consequence of Theorem 7 and Lemma 9. □

REMARK 6. Numerical evidence shows that for many kernels J the equality $\lambda_0 = \lambda_\infty$ holds (actually, the function G looks like that in Fig. 1).

However, this is not always the case, and it would be interesting to find sufficient conditions which imply this equality.

We will construct next a kernel with $\lambda_0 < \lambda_\infty$. Let H be a C^1, even, compactly supported, nonnegative, even function with unit integral. For small $\varepsilon > 0$ define

$$J_\varepsilon(x) = \frac{1}{6\varepsilon} \left(2H\left(\frac{x + x_2}{\varepsilon} \right) + H\left(\frac{x + x_1}{\varepsilon} \right) + H\left(\frac{x - x_1}{\varepsilon} \right) + 2H\left(\frac{x - x_2}{\varepsilon} \right) \right),$$

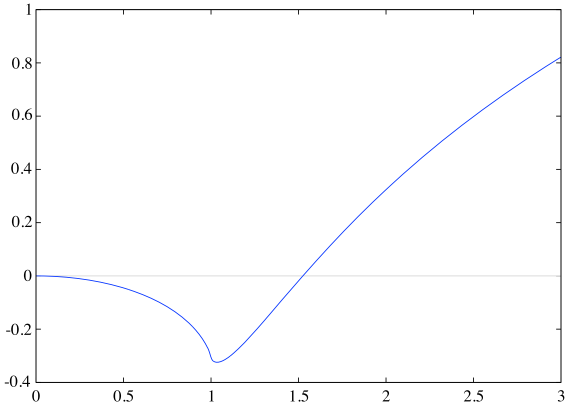

Figure 1. Graph of the function G for an approximately constant kernel

where $x_1 = \frac{1}{4}, x_2 = \frac{3}{4}$. Then J_ε is also a C^1, compactly supported, nonnegative, even function with unit integral. Consider the function

$$G(\lambda, \varepsilon) = \int_{-1}^{1} J_\varepsilon(t) \log|1 + \lambda t| \mathrm{d}t, \qquad \lambda > 0, \ \varepsilon > 0, \ \varepsilon \sim 0.$$

Performing the change of variables $s = 1 + \lambda t$, it is not difficult to see that G is continuous with respect to both variables λ, ε when ε is small enough, and continuously differentiable with respect to λ for $\lambda \neq \frac{16}{4}, 16$. Moreover,

$$G(\lambda, 0) = \frac{1}{6} \log\left(\left(\frac{9\lambda^2}{16} - 1 \right)^2 \left| \frac{\lambda^2}{16} - 1 \right| \right).$$

By using standard calculus, it is possible to show that $G(\lambda, 0)$ vanishes exactly three times, at some values μ_1, μ_2, μ_3 verifying $\frac{16}{9} < \mu_1 < \mu_2 < 16 < \mu_3$. Moreover, these are simple zeros of $G(\lambda, 0)$. Therefore, we may apply the implicit function theorem to obtain that the equation $G(\lambda, \varepsilon) = 0$ can be solved in a neighborhood of μ_i, $i = 1, 2, 3$ for small ε. Hence there exist three functions $\lambda_i(\varepsilon)$, $i = 1, 2, 3$ such that $G(\lambda_i(\varepsilon), \varepsilon) = 0$ and $\lambda_1(\varepsilon) < \lambda_2(\varepsilon) < \lambda_3(\varepsilon)$ for small enough ε. This entails that for the kernels J_ε we always have $\lambda_0 < \lambda_\infty$ for small enough ε. Let us mention in passing that the kernel J_ε can be constructed strictly positive in $(-1 + \varepsilon, 1 - \varepsilon)$, by simply adding a suitable function H_ε of order ε^γ for some $\gamma > 1$ and small ε.

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