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Stochastic completeness and honesty

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Abstract. We show the equivalence of two notions, namely honesty of a semigroup and stochastic completeness of a graph. Honesty occurs in the study of positive perturbations of substochastic semigroups, while stochastic completeness occurs in the study of the heat equation on graphs. We then look at some applications of honesty theory to graphs.

1. Introduction

The two notions of honesty of a semigroup and stochastic completeness of a graph occur in two different areas. Honesty is linked to the study of positive perturbations of substochastic semigroups, i.e. positive semigroups which are contractions on the positive cone of the ordered Banach space they act in, while stochastic completeness occurs in the study of the heat equation on graphs.

In this paper, we are interested in the honesty theory of additive perturbations in the framework of Kato's Perturbation Theorem [2,10,17]. The main idea in Kato's original work in [10] tells us that if A is the generator of a substochastic semigroup on ℓ^1 and B is a positive operator satisfying certain conditions, then there is an extension G of $A+B$ that generates a perturbed substochastic semigroup. The term 'honesty' then occurs in the study of the properties of the perturbed semigroup in Kato's Theorem. We will give the precise technical definition of honesty in Sect. 2; for now, we can simply think of honesty theory as the study of the consistency between the perturbed semigroup and the system it describes in the following sense. A substochastic semigroup on $L^1(\mu)$ is often used to model the time evolution of the states of a system. The nature of the modelled process often requires that the described quantity should be preserved, i.e. the semigroup describing the evolution is conservative (stochastic). However, in some cases, the semigroup turns out not to be conservative even though the modelled system should have this property. This phenomenon is what we will call *dishonesty*. For a system modelled by a strictly substochastic semigroup, we have a loss term representing the loss due to the system. Dishonesty in this case would mean that the described quantity is lost from the system faster than predicted by the loss term.

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The honesty of the perturbed semigroup in Kato's Theorem has been extensively studied, with results going back to Kato's seminal paper [10] where Kato studied the stochasticity of the perturbed semigroup on ℓ^1. Early results include [1] and [17]. More recently, Voigt and Mokhtar-Kharroubi in [13] introduced a more systematic approach to studying the problem on $L^1(\mu)$, that is, via functionals involving resolvent operators. Arlotti et al. in [2] then extended their work to abstract state spaces (real, ordered Banach spaces with generating positive cone on which the norm is additive) and introduced an alternative approach using functionals which are defined using the Dyson–Phillips series representation of the perturbed semigroup instead.

Kato's Theorem on $L^1(\mu)$ and the corresponding honesty theory have applications to many biological problems such as birth and death problems and fragmentation problems. See [3, Chapters 7–9] for an introductory survey of the results in these areas. The theory can also be applied to the transport equation or kinetic theory, for example in [1] and [3, Chapters 10–11]. The extension of Kato's Theorem to abstract state spaces then enabled the application of honesty theory to a non-commutative setting in [2,12].

The concept of stochastic completeness on the other hand occurs in the study of the heat equation on a variety of geometric objects including manifolds [9] and graphs [6,11]. Our paper, however, will only focus on the case involving symmetric weighted graphs.

Stochastic completeness is related to the loss or conservation of heat in the system. The loss of heat is attributed to two reasons: heat loss within the graph from internal factors and heat loss by transport to 'infinity'. The notion of stochastic completeness occurs when we try to differentiate between the two methods of heat loss. More precisely, we say that a graph is stochastically complete if there is no loss of heat to infinity. This study of heat loss is encapsulated mathematically in terms of a Laplacian, L, on the weighted sequence space $\ell^2(V, m)$ and the heat semigroup it generates, $(e^{-tL})_{t\geq 0}$.

Stochastic completeness for graphs has been studied in various settings. For example, Dodziuk and Mathai in [6] studied the bounded Laplacian by assuming a uniform bound on the vertex degree of the graph. Wojciechowski in [18] on the other hand studied locally finite graphs by generalising the approach for the corresponding notion of stochastic completeness on Riemannian manifolds. Keller and Lenz then generalised these results by studying stochastic completeness at infinity of graphs which include a killing term via non-local, regular Dirichlet forms on discrete sets [11]. It is this particular notion of stochastic completeness at infinity which we will investigate in this paper.

These two notions of stochastic completeness and honesty were introduced separately and as far as we know have been studied independently to date. Although Keller and Lenz in [11] acknowledge the strong relation between the work on stochastic completeness of graphs and the work on Markov processes by Feller in [8] and Reuter in [15], which is in fact, where the term honesty originated, there has been no formal attempt so far to link these two concepts together. In this paper, we will show the equivalence of the two notions. It turns out that the heat semigroup is a substochastic semigroup and the Laplacian on graphs can be reformulated as the sum of two operators. This allows us to rephrase the theory of heat equations on graphs in terms of additive perturbations of substochastic semigroups and thus show the equivalence of stochastic completeness and honesty. We will give more precise definitions in Sect. 2, but for now we simply note that (SC_∞) denotes stochastic completeness at infinity and $(e^{-tL_1})_{t\geq 0}$ the heat semigroup on $\ell^1(V, m)$. Then our main result is the following:

THEOREM 1.1. *(SC_∞) of the weighted graph (V, b, c) is equivalent to honesty of the semigroup $(e^{-tL_1})_{t\geq 0}$ on $\ell^1(V, m)$.*

To prove the theorem, we will begin by introducing honesty and stochastic completeness independently in Sect. 2. We then demonstrate how the theory of Laplacians on graphs fits into the framework of Kato's Theorem in Sect. 3. The main aim of this section is to show that the Kato semigroup is in fact equal to the heat semigroup on ℓ^1. This result is in fact the most difficult step towards proving Theorem 1.1. The main difficulty stems from the fact that the Laplacian acts on ℓ^2, while Kato's Theory considers semigroups on ℓ^1. In Sect. 4, we complete the proof of Theorem 1.1. Finally, in Sect. 5, we will discuss some implications of the equivalence we have proven and describe some applications of honesty theory to some examples of graphs.

2. The independent notions of honesty and stochastic completeness

In this paper, a few different notions of positivity will occur. Let X be an ordered Banach space with positive cone X_+. We say that the linear operator A is positive if $Au \in X_+$ for all $u \in D(A) \cap X_+$. If H is a Hilbert space and A a linear operator in H, we say that A is $\mathfrak{H}$-positive if $\langle Au, u\rangle \geq 0$ for all $u \in D(A) \subseteq H$. Finally, if $Q : D(Q) \times D(Q) \subseteq H \times H \to \mathbb{R}$ is a quadratic form, we say that Q is positive if $Q(u, u) \geq 0$ for all $u \in D(Q)$.

Let X be a Banach space and A a linear operator in X. Throughout the paper, we will use A^* to denote the dual of A. A set of operators (A_p) with each operator A_p acting on the Banach space X_p, respectively, will be said to be consistent if they coincide on the intersection of the spaces X_p.

2.1. Honesty theory

In this paper, we will be interested solely in the study of honesty theory of additive perturbations in the framework of Kato's Perturbation Theorem in AL-spaces. The theory we present here will be mostly based on [13].

Let (Ω, μ) be a measure space and $X := L^1(\Omega, \mu)$. In [10, Theorem 1], Kato proved the following theorem (see also [17], [3, Theorem 5.2, Proposition 5.7, Corollary 5.17]).

THEOREM 2.1. *Suppose that the operators A and B with $D(A) \subseteq D(B) \subseteq X$ satisfy:*

(i) *A generates a substochastic semigroup $(U_A(t))_{t\geq 0}$,*
(ii) *$Bu \geq 0$ for $u \in D(A)_+ := D(A) \cap X_+$,*
(iii) *$\int_\Omega (A+B)u\, d\mu \leq 0$ for all $u \in D(A)_+$.*

Then there exists an extension G of $A+B$ that generates a substochastic C_0-semigroup $(V(t))_{t\geq 0}$ on X. The generator G satisfies, for all $\lambda > 0$ and $x \in X$,

$$R(\lambda, G)x = \sum_{k=0}^{\infty} R(\lambda, A)(BR(\lambda, A))^k x.$$

Moreover, $(V(t))_{t\geq 0}$ is the minimal substochastic C_0-semigroup whose generator is an extension of $(A+B)|_D$, where D is any core of A.

Henceforth, we will refer to Theorem 2.1 as Kato's Theorem.

To study the honesty of the perturbed semigroup $(V(t))_{t\geq 0}$, we consider again the operators from Theorem 2.1. We are interested in the functional

$$a_0 : D(G) \to \mathbb{R}, \quad a_0(u) = -\int_\Omega Gu\, \mathrm{d}\mu.$$

It is easy to see that $0 \leq a_0(u) \leq \|Gu\|$ for all $u \in D(G)_+$. We denote the restriction of a_0 to $D(A)$ by a, i.e.

$$a_0|_{D(A)} = a : D(A) \to \mathbb{R}, \quad a(u) = -\int_\Omega Au + Bu\, \mathrm{d}\mu. \tag{2.1}$$

We now use a to define our second functional. Fix $\lambda > 0$ and $u \in X_+$. Since $R(\lambda, A)$ and $BR(\lambda, A)$ are positive, the sequence $R^{(n)}u := \sum_{k=0}^{n} R(\lambda, A)(BR(\lambda, A))^k u$, $n \in \mathbb{N}$ is non-decreasing and in fact converges to $R(\lambda, G)u$. Therefore, we have $a(R^{(n)}u) = a_0(R^{(n)}u) \leq a_0(R(\lambda, G)u)$ for all $n \in \mathbb{N}$, i.e. $(a(R^{(n)}u))_n$ is a bounded, monotone real sequence, which must then be convergent. Taking $u = u^+ - u^- \in X$, $u^+, u^- \in X_+$, we see that this convergence holds for any $u \in X$. Therefore, we can define a new functional on $D(G)$ by

$$\bar{a}_\lambda(R(\lambda, G)u) = \sum_{k=0}^{\infty} a(R(\lambda, A)(BR(\lambda, A))^k u), \quad u \in X.$$

It can be shown [13, Proposition 1.1] that $\bar{a}_\lambda|_{D(A)} = a$ and that the definition of $\bar{a}_\lambda$ is independent of λ. Thus we define $\bar{a} := \bar{a}_\lambda$. From the inequality $a(R^{(n)}u) \leq a_0(R(\lambda, G)u)$ for $u \in X_+$, it follows that $\bar{a}(R(\lambda, G)u) \leq a_0(R(\lambda, G)u)$. This allows us to define a positive functional, $\Delta_\lambda \in X^*$ which will be key in characterising the honesty of the semigroup,

$$\langle \Delta_\lambda, u\rangle = a_0(R(\lambda, G)u) - \bar{a}(R(\lambda, G)u), \quad u \in X. \tag{2.2}$$

To see this, we need the technical definition of honesty as given in [2]. To motivate the definition, consider the following: for any $u \in X_+$ and $t \geq 0$, we have $\int_0^t V(s)u\,\mathrm{d}s \in D(G)$ with $V(t)u - u = G\int_0^t V(s)u\,\mathrm{d}s$. Since the semigroup is positive, we have

$$\|V(t)u\| - \|u\| = -a_0\left(\int_0^t V(s)u\,\mathrm{d}s\right). \tag{2.3}$$

We define honesty to be the following:

DEFINITION 2.2. ([2, Definition 3.8]) The perturbed semigroup $(V(t))_{t\geq 0}$ in Kato's Theorem is said to be honest if and only if

$$\|V(t)u\| - \|u\| = -\bar{a}\left(\int_0^t V(s)u\,\mathrm{d}s\right) \quad \text{for all} \quad t \geq 0, u \in X_+. \tag{2.4}$$

Otherwise, the semigroup is said to be dishonest.

REMARK 2.3. Note that if we have equality in condition (iii) in Kato's Theorem, then $\bar{a} = 0$. Hence, an honest semigroup in this case is simply a stochastic semigroup.

Note that Definition 2.2 tells us that the semigroup is honest if and only if the difference $\|u\| - \|V(t)u\|$, $u \in X_+$ is given by $\bar{a}\left(\int_0^t V(s)u\,\mathrm{d}s\right)$, which is bounded by $a_0\left(\int_0^t V(s)u\,\mathrm{d}s\right)$ so $\bar{a}$ is in some sense the 'minimal' functional. Comparing (2.3) and (2.4), we see that $(V(t))_{t\geq 0}$ is honest if and only if for all $u \in X_+$,

$$a_0\left(\int_0^t V(s)u\,\mathrm{d}s\right) = \bar{a}\left(\int_0^t V(s)u\,\mathrm{d}s\right) \qquad \text{for all} \quad t \geq 0. \tag{2.5}$$

Further calculations (see for example [2, Theorem 3.11]) show that (2.5) holds if and only if $a_0(R(\lambda, G)u) = \bar{a}(R(\lambda, G)u)$ for some $\lambda > 0$. Therefore $(V(t))_{t\geq 0}$ is honest if and only if $\Delta_\lambda = 0$, i.e. no loss occurs.

With these calculations, we see that the functional Δ_λ is a loss functional in the sense that it measures 'how far' a trajectory deviates from an honest one. In fact, this describes the equivalence of (i) and (ii) in Theorem 2.4, which states some well-known characterisations of honesty. The result as stated below can be derived by combining [13, Lemma 1.4, Remarks 1.7] and Definition 2.2.

THEOREM 2.4. *Let* $X = L^1(\Omega, \mu)$, $(V(t))_{t\geq 0}$ *be the perturbed semigroup in Kato's Theorem and* $\lambda > 0$. *The following are equivalent.*

(i) $(V(t))_{t\geq 0}$ *is honest.*
(ii) $\Delta_\lambda = 0$.
(iii) $\lim_{n\to\infty} \|[BR(\lambda, A)]^n u\| = 0$ *for all* $u \in X_+$.
(iv) $G = \overline{A + B}$.

The final result we present in this section is an important property of Δ_λ which will be required later.

PROPOSITION 2.5. ([13, Corollary 1.5]) *Fix* $\lambda > 0$. *If* $\Delta_\lambda \neq 0$, *then* Δ_λ *is the maximal element of* $\{\psi \in X^* : \psi \leq \mathbb{1}, (BR(\lambda, A))^*\psi = \psi\}$.

2.2. Stochastic completeness

In this section, we will present the theory of stochastic completeness on graphs by considering regular Dirichlet forms on discrete sets as carried out by Keller and Lenz in [11]. We begin by introducing Laplacians on graphs.

Let V be a countable set and m a measure on V with full support. We will consider the spaces $\ell^p_m := \ell^p(V, m)$, $1 \leq p < \infty$ defined by

$$\left\{ u : V \to \mathbb{R} \,\middle|\, \sum_{x \in V} m(x)\, |u(x)|^p < \infty \right\}.$$

We will denote by ℓ^∞ the space of bounded functions on V equipped with the supremum norm $\|\cdot\|_\infty$ and $C_c := C_c(V)$ the space of finitely supported functions on V. We will also denote the duality between ℓ^1_m and ℓ^∞ by $\langle u, v\rangle_m = \sum_x m(x)u(x)v(x)$ for all $u \in \ell^\infty$, $v \in \ell^1_m$.

A symmetric weighted graph over V is a pair (b, c) consisting of a map $c : V \to [0, \infty)$ and a map $b : V \times V \to [0, \infty)$ satisfying:

(i) $b(x, x) = 0$ for all $x \in V$,
(ii) $b(x, y) = b(y, x)$ for all $x, y \in V$,
(iii) $\sum_{y \in V} b(x, y) < \infty$ for all $x \in V$.

(V, b, c) then represents a weighted graph with vertex set V and $b(x, y)$ the weight on the edge connecting the point x and y. If $c(x) > 0$, we think of x as connected to the point 'infinity' by an edge with weight $c(x)$ and heat can flow out of the graph to 'infinity' but not vice versa. The map c is also known as the killing term.

For each graph (V, b, c), consider the closed form $Q^M = Q^M_{b,c,m}$ defined on $\ell^2_m \times \ell^2_m$ to $[0, \infty]$ with diagonal given by

$$Q^M(u) = \frac{1}{2} \sum_{x \in V} \sum_{y \in V} b(x, y)(u(x) - u(y))^2 + \sum_{x \in V} c(x)u(x)^2.$$

We denote its restriction to C_c by $Q^C := Q^C_{b,c}$, i.e. $Q^C = Q^M|_{C_c}$. Since Q^M is closed, Q^C is closable. We will denote its closure by $Q = Q_{b,c,m}$ and its domain by $D(Q)$ which is the closure of C_c under the norm $\|\cdot\|_Q := (\|\cdot\|_2^2 + Q(\cdot))^{1/2}$. It is easy to see that Q is also positive. Hence, there exists a unique $\mathfrak{H}$-positive, self-adjoint operator $L = L_{b,c,m}$ with domain $D(Q) = D(L^{1/2})$ and $Q(u) = \left\langle L^{1/2}u, L^{1/2}u \right\rangle$ for $u \in D(Q)$ [5, Theorem 1.2.1]. It turns out that Q is in fact a regular Dirichlet form [11, Theorem 7]. This implies that L generates a positive, contractive semigroup on ℓ^2_m which gives rise to positive, contractive semigroups $(e^{-tL_p})_{t \geq 0}$ on ℓ^p_m for all $p \in [1, \infty]$, strongly continuous for $p \in [1, \infty)$, with generators denoted L_p and $L_2 := L$. From the construction of Q, we can describe the action of the operator L explicitly.

Define the formal Laplacian $\tilde{L}$ on the vector space

$$\tilde{F} := \left\{ u : V \to \mathbb{R} \,\Big|\, \sum_{y \in V} |b(x,y)u(y)| < \infty \quad \text{for all} \quad x \in V \right\}$$

by

$$\tilde{L}u(x) := \frac{1}{m(x)} \sum_{y \in V} b(x,y)(u(x) - u(y)) + \frac{c(x)}{m(x)} u(x)$$

for all $x \in V$. Then for any $p \in [1, \infty]$, $L_p f = \tilde{L} f$ for all $f \in D(L_p)$ [11, Theorem 9].

The notion of subgraphs and their relation to the original graphs will play an important role in this paper. Let (V, b, c) be the weighted graph with measure m and $W \subset V$ with measure m_W the restriction of m to W. A subgraph (W, b_W, c_W) of a weighted graph (V, b, c) is given by a subset W of V and the restriction b_W of b to $W \times W$ and the restriction c_W of c to W. The subgraph (W, b_W, c_W) then gives rise to a regular Dirichlet form $Q_{b_W, c_W, m_W} := \overline{Q^C_{b_W, c_W}}^{\|\cdot\|_{Q_{b_W, c_W, m_W}}}$ on $\ell^2(W, m_W)$ with associated operator L_{b_W, c_W, m_W}.

Let $i_W : \ell^2(W, m_W) \to \ell^2(V, m)$ be the canonical embedding and $p_W : \ell^2(V, m) \to \ell^2(W, m_W)$ the canonical projection. We will see later that it will be more useful to consider the form (defined on $C_c(W)$)

$$\begin{aligned} Q^C_W(u) &= Q(i_W u) \\ &= Q^C_{b_W, c_W}(u) + \sum_{x \in W} d_W(x) u^2(x) \end{aligned}$$

where $d_W(x) := \sum_{y \in V \setminus W} b(x, y)$, with corresponding formal operator

$$\tilde{L}_W u(x) = \frac{1}{m(x)} \left(\sum_{y \in W} b(x,y)(u(x) - u(y)) + \left(\sum_{y \in V \setminus W} b(x,y) + c(x) \right) u(x) \right),$$
$$x \in W.$$

Alternatively, one can view $Q_W := \overline{Q^C_W}^{\|\cdot\|_{Q_W}}$ as the form associated with the weighted graph (W, b^D_W, c^D_W) where $b^D_W = b_W$ and $c^D_W = c_W + d_W$. Hence, a similar argument as above shows that Q_W is a regular Dirichlet form and thus is associated with the operator L_W and the semigroup $(e^{-tL_W})_{t \geq 0}$ on $\ell^2(W, m_W)$. For simplicity of notation, for $f \in \ell^2(V, m)$, we will write $e^{-tL_W} f$ to mean $i_W e^{-tL_W}(p_W f)$ and similarly for the resolvent operators.

The following proposition which tells us that the heat semigroup on a graph can be approximated by heat semigroups on its subgraphs will play an important role in the next section.

PROPOSITION 2.6. *Let (V, b, c) be the weighted graph with measure m and (W_n) an increasing sequence of subsets of V satisfying $V = \bigcup_{n=1}^{\infty} W_n$. Then for any $t \geq 0$ and $f \in \ell^2(V, m)$,*

$$e^{-tL_{W_n}} f \xrightarrow[n\to\infty]{\ell^2(V,m)} e^{-tL} f.$$

Proof. Fix $0 \leq f \in \ell^2(V, m)$. Then [11, Theorem 11(a)] and the dominated convergence theorem implies that $R(\lambda, L_{W_n})f \xrightarrow[n\to\infty]{\ell^2(V,m)} R(\lambda, L)f$. Since the resolvent operators are positive and every $f \in \ell^2(V, m)$ has a decomposition, $f = f^+ - f^-$ with $f^+, f^- \geq 0$, this convergence holds for all $f \in \ell^2(V, m)$. The strong convergence of the semigroups then follows from a Trotter approximation theorem for C_0-semigroups on approximating sequences of Banach spaces [16, Theorem 5.1]. □

Finally, we introduce the concept of stochastic completeness. We begin by introducing a function, formally defined as

$$M_t(x) := e^{-tL}\mathbb{1}(x) + \int_0^t \left(e^{-sL}\frac{c}{m}\right)(x)\,\mathrm{d}s, \quad x \in V. \tag{2.6}$$

Note that this function is well defined if $\frac{c}{m} \in \ell^p_m$ for some $p \in [1, \infty]$ (which may not necessarily hold) and it satisfies $0 \leq M_t \leq \mathbb{1}$. For each $x \in V$, the function $t \mapsto M_t(x)$ is continuous and even differentiable. We would like to determine when the function M_t is equal to $\mathbb{1}$.

THEOREM 2.7. ([11, Theorem 1]) *Let (V, b, c) be a weighted graph and m a measure on V of full support. Then, for any $\lambda > 0$, the function*

$$w_\lambda := \int_0^\infty \lambda e^{-\lambda t}(\mathbb{1} - M_t)\,dt$$

satisfies $0 \leq w_\lambda \leq \mathbb{1}$, solves $(\tilde{L} + \lambda)w_\lambda = 0$ and is the largest non-negative $f \leq \mathbb{1}$ with $(\tilde{L} + \lambda)f \leq 0$. Moreover, the following are equivalent:

(i) *For any $\lambda > 0$, there exists $f \in \ell^\infty \backslash \{0\}$ with $(\tilde{L} + \lambda)f = 0$.*
(ii) *$w_\lambda \neq 0$ for any $\lambda > 0$.*
(iii) *$M_t(x) < 1$ for some $x \in V$ and some $t > 0$.*

This now allows us to define stochastic completeness.

DEFINITION 2.8. ([11, Definition 1.1]) The weighted graph (V, b, c) with measure m of full support is said to satisfy stochastic incompleteness at infinity (SI_∞) if it satisfies one (and thus all) of the equivalent assertions of Theorem 2.7. Otherwise (V, b, c) is said to satisfy stochastic completeness at infinity (SC_∞).

Note that if $c = 0$, i.e. the case of vanishing killing term, then M_t is simply $e^{-tL}\mathbb{1}$. Thus stochastic completeness of the graph is equivalent to the semigroup being stochastic or conservative in this case. The function M_t tells us in fact that a graph is stochastically complete if there is no heat loss to 'infinity'. For full details about the physical interpretation of M_t, see [11, p. 195, Section 7].

3. Laplacians on graphs in Kato's framework

The first step towards showing the equivalence of honesty and stochastic completeness is to demonstrate that the theory of Laplacians on graphs fits into the framework of Kato's Theorem. We begin by reformulating the theory of Laplacians on graphs in terms of Kato's Theorem. Since Kato's Theorem is a result on perturbations, we consider $-\tilde{L}$ as the sum of two operators A, B on ℓ^1_m with

$$Au(x) = -\frac{1}{m(x)}\left(\sum_{y\in V} b(x,y) + c(x)\right)u(x) \quad \text{with} \quad D(A) = \{u \in \ell^1_m : Au \in \ell^1_m\} \tag{3.1}$$

and

$$Bu(x) = \frac{1}{m(x)}\sum_{y\in V} b(x,y)u(y) \quad \text{with} \quad D(B) = D(A). \tag{3.2}$$

Note that for $u \in D(A)_+$ the inequality

$$\|Bu\|_1 = \sum_{x\in V}\sum_{y\in V} b(x,y)u(y) = \sum_{y\in V} u(y)\sum_{x\in V} b(x,y) \leq \|Au\|_1$$

shows that the element Bu defined in (3.2) belongs to ℓ^1_m. For $u \in D(A)$, there exist $u^\pm \in D(A)$ such that $u = u^+ - u^-$, and therefore, (3.2) defines an element $Bu \in \ell^1_m$. Finally, we observe that it follows easily from elementary calculations that $A = \overline{A|_{C_c}}$.

The main result of this section is the following:

THEOREM 3.1. *Let A, B be defined by* (3.1) *and* (3.2)*, respectively. Then the heat semigroup on ℓ^1_m, $(e^{-tL_1})_{t\geq 0}$ coincides with the perturbed semigroup $(V(t))_{t\geq 0}$ derived from A and B in Kato's Theorem.*

First, we show that the decomposition of A, B satisfies Kato's Theorem.

PROPOSITION 3.2. *A and B satisfy the hypotheses of Kato's Theorem, and hence, there exists $G \supseteq A + B$ that generates a C_0-semigroup $(V(t))_{t\geq 0}$ of positive contractions on ℓ^1_m.*

Proof. The operator A is the operator of multiplication with the function $a(x) := -\frac{1}{m(x)}\left(\sum_{y\in V} b(x,y) + c(x)\right)$, and hence, it generates the substochastic semigroup of multiplication with the function $(e^{at})_{t\geq 0}$ on ℓ^1_m (see [7, Section 2.9]).

It remains to consider the operator B. By definition, we have that $D(B) = D(A)$. Moreover, $b(x,y) \geq 0$ for all $x, y \in V$; hence, it follows immediately that B is a positive operator. Additionally, for all $u \in D(A)_+$

$$\langle \mathbb{1}, (A+B)u\rangle_m = \left\langle \mathbb{1}, -\tilde{L}u\right\rangle_m = -\sum_{x\in V} c(x)u(x) \leq 0.$$

Therefore, we can conclude that A and B satisfy the hypotheses of Kato's Theorem and the result follows. □

We now have (potentially) two semigroups on ℓ^1_m, one of which comes from Kato's Theorem, $(V(t))_{t\geq 0}$, while the other originates from considering the theory of Dirichlet forms, $(e^{-tL_1})_{t\geq 0}$. We will show next that the two semigroups coincide.

We begin with some auxiliary information on subgraphs. The reformulation of the set-up of the Laplacian on graphs in terms of Kato's framework allows us to derive new information about the subgraphs of (V, b, c) which will be required later. We begin with the following lemma which gives a condition which ensures that an operator $\tilde{B}$ satisfies the hypotheses of Kato's Theorem.

LEMMA 3.3. *Suppose A, B satisfy the hypotheses of Kato's Theorem and let $\tilde{B}$ with $D(\tilde{B}) \supseteq D(B)$ satisfy $0 \leq \tilde{B}u \leq Bu$ for all $u \in D(A)_+$. Then A, $\tilde{B}$ also satisfy the hypotheses of Kato's Theorem.*

Let A, B denote the operators in Kato's Theorem associated with the weighted graph (V, b, c) and $\tilde{L}_W$ the operator associated with the weighted graph (W, b_W^D, c_W^D) as defined in Sect. 2.2. Note that since $\tilde{L}_W$ is associated with the weighted graph (W, b_W^D, c_W^D), it follows from Proposition 3.2 that the operators

$$A_W u(x) = -\frac{1}{m(x)}\left(\sum_{y\in V} b(x,y) + c(x)\right) u(x), x \in W \quad \text{with}$$
$$D(A_W) = \{u \in \ell^1(W, m_W) : A_W u \in \ell^1(W, m_W)\} \tag{3.3}$$

and

$$B_W u(x) = \frac{1}{m(x)} \sum_{y\in W} b(x,y)u(y), x \in W \quad \text{with domain} \quad D(B_W) = D(A_W) \tag{3.4}$$

satisfy the hypotheses of Kato's Theorem on $\ell^1(W, m_W)$ with the associated Kato subgraph semigroup denoted $(V_W(t))_{t\geq 0}$ and generator G_W.

The following extension of the Kato subgraph semigroup $(V_W(t))_{t\geq 0}$ to $\ell^1(V, m)$ will turn up repeatedly later. Let $\tilde{B}_W$ be the extension of B_W to $\ell^1(V, m)$ defined by

$$\tilde{B}_W u = i_W B_W(p_W u). \tag{3.5}$$

By definition, $\tilde{B}_W$ is positive and $\tilde{B}_W u \leq Bu$ for all $u \in D(A)_+$. Hence it follows from Lemma 3.3 that A, $\tilde{B}_W$ also satisfy Kato's Theorem on $\ell^1(V, m)$ with perturbed semigroup denoted $(\tilde{V}_W(t))_{t\geq 0}$ and generator $\tilde{G}_W$. We will refer to this semigroup as the extended Kato semigroup associated with the subgraph (W, b_W^D, c_W^D).

$(\tilde{V}_W(t))_{t\geq 0}$ is an extension of $(V_W(t))_{t\geq 0}$ in the following sense: From the definitions of A, A_W, B_W, $\tilde{B}_W$, it follows that for all $u \in \ell^1(W, m_W)$,

$$R(\lambda, A)i_W u = i_W R(\lambda, A_W)u, \quad \tilde{B}_W R(\lambda, A)i_W u = i_W B_W R(\lambda, A_W)u. \tag{3.6}$$

Hence by Theorem 2.1 and the continuity of i_W, we have that

$$\begin{aligned} R(\lambda, \tilde{G}_W)i_W u &= \sum_{k=0}^{\infty} R(\lambda, A)(\tilde{B}_W R(\lambda, A))^k i_W u \\ &= i_W \left(\sum_{k=0}^{\infty} R(\lambda, A_W)(B_W R(\lambda, A_W))^k u \right) \\ &= i_W R(\lambda, G_W)u \end{aligned}$$

or equivalently,

$$\tilde{V}_W(t)(i_W u) = i_W V_W(t)u. \tag{3.7}$$

We will also need the following auxiliary lemma. Recall that if T is associated with a Dirichlet form, it generates semigroups of contractions on ℓ^p_m, $p \in [1, \infty]$ [5, Theorem 1.3.3], which we will denote $(U_p(t))_{t \geq 0}$ with generators T_p or simply $(U(t))_{t \geq 0}$ and T wherever they coincide. The following lemma can be proven by simply comparing the Dyson–Phillips series of the perturbed semigroups $(S_1(t))_{t \geq 0}$ and $(S_2(t))_{t \geq 0}$.

LEMMA 3.4. *Suppose $T : D(T) \subset \ell^2_m \to \ell^2_m$ is an operator associated with a Dirichlet form and generates the semigroup $(U(t))_{t \geq 0}$. Let $H_1 \in \mathcal{L}(\ell^1_m)$, $H_2 \in \mathcal{L}(\ell^2_m)$ such that $H_1|_{\ell^1_m \cap \ell^2_m} = H_2|_{\ell^1_m \cap \ell^2_m}$. Then the perturbed semigroups generated by $T_1 + H_1$ on ℓ^1_m, $(S_1(t))_{t \geq 0}$ and $T_2 + H_2$ on ℓ^2_m, $(S_2(t))_{t \geq 0}$ coincide on $\ell^1_m \cap \ell^2_m$.*

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Choose an increasing sequence of finite subsets $F_n \subseteq V$, $n \in \mathbb{N}$, such that $\bigcup_n F_n = V$. Define $b_n(x, y) = (\chi_{F_n \times F_n} b)(x, y)$ where χ_W denotes the indicator function for the set W. Then the operators B_n, $n \in \mathbb{N}$, defined by

$$B_n u(x) = \frac{1}{m(x)} \sum_{y \in V} b_n(x, y)u(y) \quad \text{for all} \quad x \in V$$

are bounded operators in ℓ^1_m and ℓ^2_m. Taking A_2 to be the multiplication operator with maximal domain, i.e.

$$A_2 u(x) = -\frac{1}{m(x)} \left(\sum_{y \in V} b(x, y) + c(x) \right) u(x) \quad \text{with} \quad D(A_2) = \{u \in \ell^2_m : A_2 u \in \ell^2_m\},$$

similar arguments to those in the proof of Proposition 3.2 show that A_2 generates a positive C_0-semigroup of contractions and is self-adjoint. Hence A_2 is associated with a Dirichlet form with consistent set of generators denoted by A_p. Now fix $n \in \mathbb{N}$ and consider the operators $T = T_2, T_1$ and H in Lemma 3.4. Taking $T_2 = A_2, T_1 = A_1$ and $H_1 = H_2 = B_n$, we see that A_1, A_2, B_n satisfy the conditions of Lemma 3.4. Hence, for each $n \in \mathbb{N}$, $t \geq 0$ and $u \in \ell^1_m \cap \ell^2_m$, $U^{(1)}_n(t)u = U^{(2)}_n(t)u$ where $(U^{(1)}_n(t))_{t \geq 0}$ is the semigroup generated by $A_1 + B_n$ on ℓ^1_m and $(U^{(2)}_n(t))_{t \geq 0}$ is the semigroup generated by $A_2 + B_n$ on ℓ^2_m.

Let us consider first the semigroups $(U_n^{(1)}(t))_{t\geq 0}$, $n \in \mathbb{N}$, on ℓ_m^1. Let A and B denote the operators in (3.1) and (3.2). Then clearly, $A_1 = A$. It also follows from the definitions that $B_n u \leq Bu$ for all $u \in D(A)_+$, $n \in \mathbb{N}$. Hence by Lemma 3.3, $(U_n^{(1)}(t))_{t\geq 0}$ is a contractive semigroup for each $n \in \mathbb{N}$. Moreover, for $u \in D(A)_+$,

$$\|Bu - B_n u\|_{\ell_m^1} = \sum_{x\in F_n} \sum_{y\in V\setminus F_n} b(x,y)u(y) + \sum_{x\in V\setminus F_n} \sum_{y\in V} b(x,y)u(y) \to 0 \text{ as } n \to \infty.$$

Therefore, it follows from [17, Proposition 1.6] that for all $t \geq 0$, $U_n^{(1)}(t)$ converges strongly to $V(t)$ on ℓ_m^1 where $(V(t))_{t\geq 0}$ is the semigroup from Kato's construction in Proposition 3.2.

Now consider the semigroups $(U_n^{(2)}(t))_{t\geq 0}$, $n \in \mathbb{N}$, on ℓ_m^2. We will show that for each $t \geq 0$, the sequence $(U_n^{(2)}(t))_{n\in\mathbb{N}}$ converges strongly in ℓ_m^2 to e^{-tL}. Fix $u \in \ell_m^2$ and $t \geq 0$. We have

$$\left\|U_n^{(2)}(t)u - e^{-tL}u\right\|_2 \leq \left\|U_n^{(2)}(t)(u - \chi_{F_n}u)\right\|_2 + \left\|U_n^{(2)}(t)(\chi_{F_n}u) - e^{-tL}u\right\|_2.$$

Note that $\left\|U_n^{(2)}(t)(u - \chi_{F_n}u)\right\|_2 \leq \left\|u - \chi_{F_n}u\right\|_2 \to 0$ as $n \to \infty$, so it remains to consider $\left\|U_n^{(2)}(t)(\chi_{F_n}u) - e^{-tL}u\right\|_2$. We begin by noting that since F_n is finite for all n, $\chi_{F_n}u \in \ell_m^1 \cap \ell_m^2$. Hence $U_n^{(2)}(t)(\chi_{F_n}u) = U_n^{(1)}(t)(\chi_{F_n}u)$. Now for fixed n, consider once again the semigroup $(U_n^{(1)}(t))_{t\geq 0}$. By construction, it follows that $(U_n^{(1)}(t))_{t\geq 0}$ is the extended Kato semigroup associated with the subgraph $(F_n, b_{F_n}^D, c_{F_n}^D)$ described above. Hence by (3.7),

$$\left\|U_n^{(2)}(t)(\chi_{F_n}u) - e^{-tL}u\right\|_2 = \left\|i_{F_n}V_{F_n}(t)(p_{F_n}u) - e^{-tL}u\right\|_2.$$

Finally, since F_n is finite for every n, it follows that $V_{F_n}(t) = e^{-tL_{F_n}}$. Hence

$$\left\|U_n^{(2)}(t)(\chi_{F_n}u) - e^{-tL}u\right\|_2 = \left\|i_{F_n}e^{-tL_{F_n}}(p_{F_n}u) - e^{-tL}u\right\|_2$$

and this converges to 0 by Proposition 2.6. Therefore $\left\|U_n^{(2)}(t)u - e^{-tL}u\right\|_2 \to 0$ as $n \to \infty$ for all $u \in \ell_m^2$.

To complete the proof, we need the following fact which follows from elementary measure theory, namely if $(f_n) \subset \ell_m^1 \cap \ell_m^2$, $g_1 \in \ell_m^1$, $g_2 \in \ell_m^2$ such that $f_n \to g_1$ in ℓ_m^1 and $f_n \to g_2$ in ℓ_m^2, then $g_1 = g_2$. Applying this fact with $f_n = U_n^{(1)}(t)u = U_n^{(2)}(t)u$, $u \in \ell_m^1 \cap \ell_m^2$ and $g_1 = V(t)u$, $g_2 = e^{-tL}u$ for all $t \geq 0$, we can conclude that $V(t)u = e^{-tL}u$ for all $u \in \ell_m^1 \cap \ell_m^2$ and $t \geq 0$. □

One can also prove Theorem 3.1 by applying some results from the theory of quadratic forms instead of approximations by finite subgraphs. We outline briefly this alternative proof.

By applying the fact that the generator G of the Kato semigroup of the graph $(V(t))_{t\geq 0}$ is symmetric and the Riesz–Thorin Interpolation Theorem, we see that

$(V(t))_{t\geq 0}$ can be extended to a set of consistent semigroups on ℓ^p_m, denoted $(V^{(p)}(t))_{t\geq 0}$, with generators G_p. Moreover, Q_{G_2}, the form associated with the generator G_2 is a Dirichlet form. It can then be deduced from [4, Lemma I.4.2.1.1] that $C_c \subseteq D(Q_{G_2})$ and so $Q \subseteq Q_{G_2}$. The reverse inclusion can be deduced from a result relating form domains to the domination of semigroups [14, Proposition 2.23] by using the minimality of the Kato semigroup $(V(t))_{t\geq 0}$.

In contrast to the general case considered in Theorem 3.1, if we impose an extra geometric condition on the space, i.e.

$$\text{For any sequence } (x_n) \subseteq V \text{ with } b(x_n, x_{n+1}) > 0 \text{ for all } n \in \mathbb{N}, \text{ we have } \sum_{n\in\mathbb{N}} m(x_n) = \infty, \tag{GP}$$

then the proof that the Kato semigroup coincides with the heat semigroup simplifies considerably. One of the results which enables this simplification is the theorem that gives the precise form of the generator in this case. In particular, Keller and Lenz show in [11, Theorem 5] that $-L_1$ is in fact the maximal operator in this case, i.e. $-L_1 = -L_1^{\max}$ where $D(L_1^{\max}) = \{u \in \ell^1_m : \tilde{L}u \in \ell^1_m\}$. To complete the proof, simply apply the fact that G is a restriction of the maximal operator $-L_1^{\max}$ (see [10, Lemma 10], [3, Theorem 6.20]) and the fact that if T is the generator of a C_0-semigroup on a Banach space X and $\mathcal{A}$ is a closed extension of T that also generates a C_0-semigroup, then $\mathcal{A} = T$.

4. The equivalence of stochastic completeness and honesty

We can now prove Theorem 1.1. In particular, we will show that $w_\lambda = \Delta_\lambda$ for some, or equivalently, all $\lambda > 0$, where Δ_λ is the functional defined by (2.2) and w_λ is as defined in Theorem 2.7. To do so, we must first give a precise description of $(A+B)^*$. For notational simplicity, let us denote $G_{\min} := (A+B)|_{C_c} = -\tilde{L}|_{C_c}$.

LEMMA 4.1. *Let (V, b, c) be a weighted graph and m a measure with full support. Suppose A, B, $\tilde{L}$, and $G_{\min}$ are as defined above. Then $(A+B)^* = G^*_{\min} = -L^{\max}_\infty$ where*

$$L^{\max}_\infty u = \tilde{L}u \quad \textit{for all} \quad u \in D(L^{\max}_\infty) = \{u \in \ell^\infty : \tilde{L}u \in \ell^\infty\}.$$

Proof. The main step in the proof is to show that $G^*_{\min} = -L^{\max}_\infty$. By definition, $v^* \in D(G^*_{\min})$ if and only if there exists $u^* \in \ell^\infty$ such that $\langle G_{\min}u, v^*\rangle_m = \langle u, u^*\rangle_m$ for all $u \in D(G_{\min})$ and then $G^*_{\min}v^*$ is defined to be u^*. Since $D(G_{\min}) = C_c$ and C_c is the span of the family $(e_x)_{x\in V}$ of standard basis vectors, it suffices to consider the vectors e_x, $x \in V$. Now for all $x \in V$,

$$\langle Ge_x, v^*\rangle_m = -\left(\sum_{y\in V} b(x,y) + c(x)\right) v^*(x) + \sum_{y\in V} b(x,y)v^*(y) = -m(x)\tilde{L}v^*(x).$$

So $v^* \in D(G^*_{\min})$ if and only if $-\tilde{L}v^* \in \ell^\infty$ and in this case, $G^*_{\min}v^* = -\tilde{L}v^*$. Since we require that $-\tilde{L}v^* \in \ell^\infty$, this can be restated as $v^* \in D(G^*_{\min})$ if and only if $v^* \in D(L^{\max}_\infty)$ and in this case $G^*_{\min}v^* = -L^{\max}_\infty v^*$. Therefore $G^*_{\min} = -L^{\max}_\infty$.

To complete the proof, we show that $\overline{G_{\min}} = \overline{A+B}$. In other words, we show that C_c is a core for $\overline{A+B}$. However, this follows since C_c is a core for A and B is an A-bounded operator. The lemma now follows from $G^*_{\min} = \overline{G_{\min}}^* = \overline{A+B}^* = (A+B)^*$. □

The second result we require gives an alternative description of $\mathrm{Ker}(\lambda - (A+B)^*)$ for fixed $\lambda > 0$.

LEMMA 4.2. *Suppose A, B satisfy the conditions of Kato's Theorem and fix* $\lambda > 0$. *Then* $\mathrm{Ker}(\lambda - (A+B)^*) = \mathrm{Ker}(I - (BR(\lambda, A))^*)$.

Proof. Recall first that $\mathrm{Ker}(\lambda - (A+B)^*)$ is the annihilator of $\overline{\mathrm{Im}(\lambda - (A+B))}$. Similarly, $\mathrm{Ker}(I - (BR(\lambda, A))^*)$ is the annihilator of $\overline{\mathrm{Im}(I - BR(\lambda, A))}$. Since $\mathrm{Im}(\lambda - (A+B)) = \mathrm{Im}((\lambda - (A+B))R(\lambda, A)) = \mathrm{Im}(I - BR(\lambda, A))$, it follows that $\mathrm{Ker}(\lambda - (A+B)^*) = \mathrm{Ker}(I - (BR(\lambda, A))^*)$. □

Proof of Theorem 1.1. Fix $\lambda > 0$. We begin by showing that w_λ satisfies the eigenvalue problem $(\lambda - (A+B)^*)w_\lambda = 0$ while Δ_λ satisfies $(\tilde{L} + \lambda)\Delta_\lambda = 0$. From Theorem 2.7, we know that w_λ satisfies $(\tilde{L} + \lambda)w_\lambda = 0$. But w_λ is bounded and $L^{\max}_\infty \subset \tilde{L}$, so this can be equivalently rewritten as $(L^{\max}_\infty + \lambda)w_\lambda = 0$. By Lemma 4.1, this is equivalent to saying $(\lambda - (A+B)^*)w_\lambda = 0$. Noting that Proposition 2.5 and Lemma 4.2 imply that Δ_λ satisfies $(\lambda - (A+B)^*)\Delta_\lambda = 0$, we can reverse this argument with Δ_λ replacing w_λ to prove the second assertion.

Combining Lemma 4.2 and Proposition 2.5, we see that Δ_λ is the maximal element in $\{f \in \ell^\infty_+ : f \leq \mathbb{1}\}$ that satisfies $(\lambda - (A+B)^*)f = 0$ and so $w_\lambda \leq \Delta_\lambda$. Similarly, Theorem 2.7 states that w_λ is the largest non-negative $f \leq \mathbb{1}$ such that $(L^{\max}_\infty + \lambda)f \leq 0$. Hence $\Delta_\lambda \leq w_\lambda$ and so $\Delta_\lambda = w_\lambda$. Therefore, Theorem 2.4 and Theorem 2.7 imply that stochastic completeness at infinity is equivalent to honesty of the Kato semigroup $(V(t))_{t\geq 0}$, which is equal to the heat semigroup $(e^{-tL_1})_{t\geq 0}$ on ℓ^1_m by Theorem 3.1. □

REMARK 4.3. The proof of Theorem 1.1 tells us in fact that (SC_∞) is equivalent to honesty of the Kato semigroup $(V(t))_{t\geq 0}$ described in Proposition 3.2, independently of Theorem 3.1. The role of Theorem 3.1 is to connect (SC_∞) of the given graph to its associated heat semigroup.

Finally, we observe that in order to prove Theorem 1.1, we showed that condition (ii) of Theorem 2.4 is equivalent to the negation of condition (ii) of Theorem 2.7. It turns out that condition (iii) in Theorem 2.7 also follows directly from the definition of honesty as given in Definition 2.2. This follows since standard manipulations show that the function M_t defined in (2.6) can be stated in terms of the functionals of honesty theory as

$$\langle M_t, u\rangle = \langle \mathbb{1}, V(t)u\rangle + \bar{a}\left(\int_0^t V(s)u\,\mathrm{d}s\right), \quad u \in \ell^1_m.$$

5. Applications of honesty in weighted graphs

The equivalence of honesty and stochastic completeness shown in Theorem 1.1 allows us to derive new characterisations of stochastic completeness from Theorem 2.4.

COROLLARY 5.1. *Let (V, b, c) be a weighted graph. If A, B are as defined in (3.1) and (3.2), the following are equivalent:*

(i) (V, b, c) *satisfies* (SC_∞).
(ii) $\lim_{n\to\infty} \|[BR(\lambda, A)]^n u\| = 0$ *for all* $u \in \ell^1(V, m)_+$, *some* $\lambda > 0$.
(iii) $-L_1 = \overline{A + B}$.

We demonstrate how condition (ii) of Corollary 5.1 may be applied to a graph. This example covers the case studied in [18] where infinite but locally finite graphs are considered, under the counting measure and with no killing term.

EXAMPLE 5.2. Consider the infinite but locally finite connected graph $(V, b, 0)$, with m the counting measure and $b(x, y) = 1$ if x is connected to y by an edge and 0 otherwise. We will use $x \sim y$ to denote that x is connected to y by an edge and d_x to denote the degree of x, i.e. the number of edges emanating from x. In this case

$$\tilde{L}u(x) = \sum_{y\sim x}(u(x) - u(y)), \qquad x \in V.$$

PROPOSITION 5.3. *The graph $(V, b, 0)$ is stochastically complete if and only if*

$$\lim_{n\to\infty} \frac{1}{\lambda + d_y} \sum_{x\in V} \sum_{\substack{(i_1,\ldots,i_{n-1})\in V^{n-1}\\ x\sim i_1\sim\cdots\sim i_{n-1}\sim y}} \prod_{k=1}^{n-1} \frac{1}{\lambda + d_{i_k}} = 0$$

for all $y \in V$ and some $\lambda > 0$.

Proof. From Corollary 5.1, we have that $(V, b, 0)$ is stochastically complete if and only if $\lim_{n\to\infty} \|[BR(\lambda, A)]^n u\| = 0$ for all $u \in (\ell^1_m)_+$. Since $BR(\lambda, A)$ is power-bounded [3, p. 147], it suffices to check that $\lim_{n\to\infty} \|[BR(\lambda, A)]^n u\| = 0$ for all u in a dense subset of $(\ell^1_m)_+$. In particular, ℓ^1_m is the closed linear span of the standard basis $(e_x)_{x\in V}$, and hence, $(V, b, 0)$ is stochastically complete if and only if

$$\lim_{n\to\infty} \left\|[BR(\lambda, A)]^n e_x\right\| = 0 \quad \text{for all} \quad x \in V. \tag{5.1}$$

Since $BR(\lambda, A)$ is a positive operator, (5.1) is equivalent to

$$\lim_{n\to\infty} \sum_{x\in V} c^{(n)}_{xy} = 0 \quad \text{for all} \quad y \in V$$

where $c^{(n)}_{xy} = \left\langle e_x, (BR(\lambda, A))^n e_y\right\rangle$, $x, y \in V$.

It remains to calculate $c^{(n)}_{xy}$. Since $b(x, y) = 1$ if and only if $x \sim y$, and $R(\lambda, A)u(x) = \frac{u(x)}{\lambda+d_x}$, we have

$$c_{xy} := c^{(1)}_{xy} = \begin{cases} \frac{1}{\lambda+d_y} & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

Using the fact that $c^{(n+1)}_{xy} = \sum_{i\in V} c_{xi} c^{(n)}_{iy}$ and induction, we find that

$$c^{(n)}_{xy} = \lim_{n\to\infty} \frac{1}{\lambda + d_y} \sum_{\substack{(i_1,\ldots,i_{n-1})\in V^{n-1} \\ x\sim i_1\sim\cdots\sim i_{n-1}\sim y}} \prod_{k=1}^{n-1} \frac{1}{\lambda + d_{i_k}} = 0$$

and the result follows. □

The characterisation of (SC_∞) in Proposition 5.3 is fairly complicated and not easy to apply. Consider for example the special case of model trees, namely trees whose vertex degree is constant on spheres of radius r from a fixed root vertex x_0 (see [18, Section 3.2] for more details). Wojciechowski shows that a model tree satisfies (SC_∞) if and only if $\sum_{r=1}^{\infty} \frac{1}{d_r-1}$ diverges where d_r is the degree of vertices of distance r from x_0 [18, Theorem 3.2.1]. Even in this simple case, it is not clear how the condition in Proposition 5.3 simplifies to this form.

As a second example, let us look at the case when the Laplacian is bounded. Keller and Lenz [11, Remark (a) p. 195] note that if $\tilde{L}$ gives rise to a bounded operator on $\ell^\infty(V)$, then the graph is stochastically complete. Their justification for this is that condition (i) in Theorem 2.7 must fail for λ large enough whenever $\tilde{L}$ is bounded. Corollary 5.1 allows us to derive an alternative proof of this statement. To see this, note first that if $\tilde{L}$ is bounded on ℓ^∞, then by duality, $\tilde{L}$ is bounded on ℓ^1_m. Moreover, the operator A is simply multiplication with a bounded function, and hence, A and B are bounded operators. Thus $A + B$ generates a C_0-semigroup, and so, condition (iii) of Corollary 5.1 implies that the graph satisfies (SC_∞).

Finally, we show that the criteria for (SC_∞) involving subgraphs in [11, Theorem 4] can be derived from the following simple condition for dishonesty.

Recall from Lemma 3.3 that if A, B satisfy Kato's Theorem and $\tilde{B}$ satisfies $0 \le \tilde{B}u \le Bu$ for all $u \in D(A)_+$, then A, $\tilde{B}$ also satisfy Kato's Theorem. Under these assumptions, we can derive the following sufficient condition for dishonesty.

PROPOSITION 5.4. *Suppose A, B satisfy Kato's Theorem with perturbed semigroup $(V(t))_{t\ge0}$. Let $\tilde{B}$ with $D(\tilde{B}) \supseteq D(B)$ satisfy $0 \le \tilde{B}u \le Bu$ for all $u \in D(A)_+$ with perturbed semigroup $(\tilde{V}(t))_{t\ge0}$. If the semigroup $(V(t))_{t\ge0}$ is honest, then so is $(\tilde{V}(t))_{t\ge0}$.*

Proof. Fix $\lambda > 0$. Since both B and $\tilde{B}$ are positive on $D(A)$ and $\tilde{B}u \le Bu$ for all $u \in D(A)_+$, we have by positivity of $R(\lambda, A)$ that $\tilde{B}R(\lambda, A) \le BR(\lambda, A)$. Iterating, we have for all $n \in \mathbb{N}$, $(\tilde{B}R(\lambda, A))^n \le (BR(\lambda, A))^n$ and so $\left\|(\tilde{B}R(\lambda, A))^n u\right\| \le$

$\|(BR(\lambda, A))^n u\|$ for all $u \in X_+$. The result now follows since from Theorem 2.4, we have that $(V(t))_{t\geq 0}$ (resp. $(\tilde{V}(t))_{t\geq 0}$) is honest if and only if for some $\lambda > 0$ and all $u \in X_+$, $\|(BR(\lambda, A))^n u\| \to 0$ (resp. $\|(\tilde{B}R(\lambda, A))^n u\| \to 0$). □

If we consider the extended Kato semigroup associated with the subgraph (W, b_W^D, c_W^D) with operators A, $\tilde{B}_W$ and $(\tilde{V}_W(t))_{t\geq 0}$ as defined in Sect. 3 and observe that from (3.6), for all $u \in \ell^1(W, m_W)$, $\left\|(\tilde{B}_W R(\lambda, A))^n i_W u\right\|_{\ell^1(V,\, m)} = \|(B_W R(\lambda, A_W))^n u\|_{\ell^1(W,\, m_W)}$, we can derive [11, Theorem 4] as a corollary.

COROLLARY 5.5. *Let (V, b, c) be a weighted graph and m a measure on V with full support. If (SC_∞) holds for (V, b, c), then (SC_∞) holds for the weighted graphs (W, b_W^D, c_W^D) for all $W \subseteq V$.*

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