J. Evol. Equ. 15 (2015), 869–878
© 2015 Springer Basel
1424-3199/15/040869-10, *published online* June 3, 2015
DOI 10.1007/s00028-015-0284-4

Journal of Evolution Equations

Dirichlet forms for singular diffusion in higher dimensions

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Abstract. We describe singular diffusion in bounded subsets Ω of $\mathbb{R}^n$ by form methods and characterize the associated operator. We also prove positivity and contractivity of the corresponding semigroup. This results in a description of a stochastic process moving according to classical diffusion in one part of Ω, where jumps are allowed through the rest of Ω.

1. Introduction

The aim of this paper was to present a treatment of multidimensional "singular" diffusion in the framework of Dirichlet forms. Singular diffusion (sometimes called gap diffusion) in one dimension goes back at least to Feller [6] and has a long history, see, e.g., [14] and references therein.

To describe singular diffusion, we consider a suitable measure μ on an open and bounded subset $\Omega \subseteq \mathbb{R}^n$, and let particles move in Ω according to "Brownian motion", where the particles may only be located in the support spt μ of μ. Furthermore, the particles are accelerated or slowed down by the "speed measure" μ. If μ is supported only on a proper subset of Ω, in terms of the stochastic process describing the motion of a particle, this yields a time-changed process (on spt μ), see [10, Section 6.2]. In terms of the Dirichlet form, we may also see that as a trace of the corresponding Dirichlet space [10, Section 6.2].

We want to treat the evolution by constructing the corresponding Dirichlet form. Since the particles moving according to Brownian motion are only located in spt μ, we will interpret the classical Dirichlet form in $L_2(\Omega, \mu)$. There is an abstract generating theorem to find generators associated with forms defined in different spaces in [2]; however, our approach is different in that we consider the form itself in the Hilbert space $L_2(\Omega, \mu)$ (where the generator should act in). We will characterize the generating self-adjoint operator and show that the corresponding C_0-semigroup is submarkovian. The associated process is a jump-diffusion process, with a diffusion part on spt μ and jumps through $\Omega \setminus \mathrm{spt}\, \mu$.

Such singular diffusions in one dimension and the form approach were described in [8,9,17–19], see also, e.g., [16] for form methods. As it turns out in one dimension,

Mathematics Subject Classification: 47D06, 47A07, 35Hxx, 35J70, 60J45
Keywords: Singular diffusion, Dirichlet forms, Submarkovian semigroups, Jump-diffusion process.

functions in the domain of the form (and hence also the operator) have to be affine on the complement of spt μ. Since in one dimension affine functions are exactly the harmonic functions, this will be the right condition occurring in higher dimensions.

In higher dimensions, there are only few results in the literature, see [12,15,17], focussing on the construction of the operator (however under somewhat different assumptions; we will work with capacities).

In Sect. 2, we describe the setup and interpret the classical Dirichlet form in $L_2(\Omega, \mu)$. The generator is characterized in Sect. 3, where also properties of the associated semigroup are proven. In Sect. 4, we apply our result to two different situations. First, we consider singular diffusion supported on a subset of codimension 1. Then we apply our results to diffusion on a fractal domain (we choose the Koch's snowflake here).

2. Dirichlet forms for singular diffusion

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ denote the field of scalars. We write λ^n for the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. We define the classical Dirichlet form τ_0 on Ω by

$$\begin{aligned} D(\tau_0) &:= W^1_{2,0}(\Omega), \\ \tau_0(u, v) &:= \int_\Omega \operatorname{grad} u \cdot \overline{\operatorname{grad} v} \quad (u, v \in D(\tau_0)). \end{aligned}$$

The corresponding form norm $\|\cdot\|_{\tau_0} := \left(\tau_0(\cdot) + \|\cdot\|^2_{L_2(\Omega,\lambda^n)}\right)^{1/2}$ is just the usual W^1_2-norm on Ω, where $\tau_0(u) := \tau_0(u, u)$.

We will provide some notions from potential theory, which will be needed in the following. For an open subset $V \subseteq \Omega$, we define

$$\operatorname{cap}(V) := \inf \left\{ \|u\|^2_{\tau_0};\ u \in D(\tau_0),\ u \geqslant 1\ \lambda^n\text{-a.e. on } V \right\}.$$

For arbitrary $A \subseteq \Omega$, we set

$$\operatorname{cap}(A) := \inf \{\operatorname{cap}(V);\ V \subseteq \Omega \text{ open},\ A \subseteq V\}.$$

Then $\operatorname{cap}(A)$ is called the *capacity* of A. We say that a property holds true *quasi everywhere* (*q.e.*) if there exists $N \subseteq \Omega$ of zero capacity such that the property is satisfied on $\Omega\setminus N$.

Let $(F_k)_{k\in\mathbb{N}}$ be a sequence of closed subsets of Ω satisfying $F_k \subseteq F_{k+1}$ for all $k \in \mathbb{N}$. Then (F_k) is called a *nest* if $\operatorname{cap}(\Omega\setminus F_k) \to 0$. If (F_k) is a nest, then we set

$$C((F_k)) := \left\{u\colon \Omega \to \mathbb{K};\ u|_{F_k} \in C(F_k) \quad (k \in \mathbb{N})\right\}.$$

A function $u\colon \Omega \to \mathbb{K}$ is said to be *quasi-continuous* if there exists a nest (F_k) such that $u \in C((F_k))$. Note that this is equivalent to saying that for any $\varepsilon > 0$, there exists an open subset $U \subseteq \Omega$ such that $\operatorname{cap}(U) < \varepsilon$ and $u|_{\Omega\setminus U} \in C(\Omega\setminus U)$.

PROPOSITION 2.1. (See [10, Theorem 2.1.3]) *Every* $u \in D(\tau_0)$ *admits a q.e. uniquely defined quasi-continuous representative* $\tilde{u}$.

We set (writing $\mathcal{B}(\Omega)$ for the Borel subsets of Ω)

$$M_0(\Omega) := \big\{\mu\colon \mathcal{B}(\Omega) \to [0,\infty];\ \mu\ \sigma\text{-additive}, \\ \mu(N) = 0 \text{ for any Borel set } N \subseteq \Omega \text{ of zero capacity}\big\}.$$

It is easy to see that $\mu \in M_0(\Omega)$ if μ is absolutely continuous with respect to the Lebesgue measure $\lambda^n(\cdot \cap \Omega)$ on Ω. As shown in [4, Theorem 4.1], also the $(n-1)$-dimensional Hausdorff measure on $(n-1)$-dimensional C^1-submanifolds of Ω belongs to $M_0(\Omega)$.

Let $\mu \in M_0(\Omega)$ be a finite measure and $U := \Omega \backslash \operatorname{spt}\mu$. The measure μ may be considered as a "speed measure". Furthermore, we will assume

$$W^1_{2,0}(U) = \left\{u \in W^1_{2,0}(\Omega);\ \tilde{u} = 0\ \mu\text{-a.e.}\right\}, \tag{1}$$

where $\tilde{u}$ is a quasi-continuous representative of u. Note that "$\subseteq$" is trivial; however, "$\supseteq$" does not hold in general, as the following example due to Voigt [20] shows.

EXAMPLE 2.2. We start with a claim: Let $n \geqslant 2$, $\varepsilon > 0$ and $r_0 > 0$. Then there exist $0 < r < r' \leqslant r_0$ and $\varphi \in C^1_c(\mathbb{R}^n)$ such that $\operatorname{spt}\varphi \subseteq B(0, r')$, $\mathbb{1}_{B[0,r]} \leqslant \varphi \leqslant 1$ and $\|\varphi\|_{2,1} \leqslant \varepsilon$. Here $B(y, \rho)$ and $B[y, \rho]$ denote the open and closed balls around y with radius ρ, respectively.

Let $B_+ := \{x \in B(0, 1);\ x_1 > 0\}$. Using the claim, there exist (x^k) in B_+, (r_k) and (r'_k) in $(0,\infty)$ satisfying $r_k < r'_k$ for all $k \in \mathbb{N}$ and (φ_k) in $C^1_c(\mathbb{R}^n)$ such that $\operatorname{spt}\varphi_k \subseteq B(x^k, r'_k) \subseteq B_+$, $\mathbb{1}_{B[x^k,r_k]} \leqslant \varphi_k \leqslant 1$ such that

- the set of accumulation points of (x^k) is exactly $\{x \in B(0, 1);\ x_1 = 0\}$,
- $B(x^k, r'_k) \cap B(x^j, r'_j) = \varnothing$ for all $k, j \in \mathbb{N}$, $k \neq j$,
- $\sum_{k=1}^{\infty} \|\varphi_k\|_{2,1} < \infty$.

Let $K := \overline{\bigcup_{k\in\mathbb{N}} B[x^k, r'_k]}$, $\Omega \supseteq K$ be open and bounded and μ the Lebesgue measure on K. Let $\varphi := \sum_{k\in\mathbb{N}} \varphi_k$ and $\psi \in C^1_c(\mathbb{R}^n)$ such that $\psi = 1$ in a neighborhood of K. Then $\psi - \varphi$ is quasi-continuous and $\psi - \varphi = 1$ on $\{x \in B(0, 1);\ x_1 = 0\}$, a set with positive capacity. On the other hand, $\psi - \varphi = 0\ \mu$-a.e., since $\psi - \varphi = 0$ on $\bigcup_{k\in\mathbb{N}} B[x^k, r_k]$ and the set

$$K \Big\backslash \bigcup_{k\in\mathbb{N}} B\left[x^k, r_k\right] = \{x \in B(0, 1);\ x_1 = 0\}$$

has μ-measure zero. Hence, $\psi - \varphi \in \left\{u \in W^1_{2,0}(\Omega);\ \tilde{u} = 0\ \mu\text{-a.e.}\right\}$, but

$$\psi - \varphi \notin \left\{u \in W^1_{2,0}(\Omega);\ \tilde{u} = 0 \text{ q.e. on } K\right\}.$$

By [11, Theorem 1.13], we observe

$$\left\{u \in W^1_{2,0}(\Omega);\ \tilde{u} = 0 \text{ q.e. on } K\right\} = W^1_{2,0}(\Omega \backslash K).$$

Thus, $\psi - \varphi \notin W^1_{2,0}(\Omega \backslash K)$.

Since Ω is bounded, by Poincaré's inequality we can equip $W_{2,0}^1(\Omega)$ with the inner product

$$(u, v) \mapsto \tau_0(u, v) = \int_\Omega \operatorname{grad} u \cdot \overline{\operatorname{grad} v},$$

inducing a norm, namely $\tau_0(\cdot)^{1/2} = \||\operatorname{grad}(\cdot)|\|_2$, which is equivalent to the usual norm $\|\cdot\|_{2,1}$. We will always equip $W_{2,0}^1(\Omega)$ with this inner product.

PROPOSITION 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $U \subseteq \Omega$ open. Then $W_{2,0}^1(\Omega) = W_{2,0}^1(U) \oplus D_{2,0}^1(U)$, where*

$$D_{2,0}^1(U) := \left\{u \in W_{2,0}^1(\Omega);\ \Delta(u|_U) = 0\right\}.$$

Proof. Let $u \in W_{2,0}^1(\Omega)$. We show that there exists a unique $v \in W_{2,0}^1(U)$ such that

$$0 = \int_U u\Delta\varphi - \int v\Delta\varphi \quad (\varphi \in C_c^\infty(U)).$$

Then $Ju := u - v \in D_{2,0}^1(U)$ and this implies the assertion.

By Poincaré's inequality, we observe that

$$(f, g) \mapsto (f \mid g)_0 := \int_U \operatorname{grad} f \cdot \overline{\operatorname{grad} g}$$

defines an inner product on $W_{2,0}^1(U)$ such that this space becomes a Hilbert space.

Since

$$\left|\int_U u\Delta\varphi\right| = \left|\int_U \operatorname{grad} u \cdot \operatorname{grad} \varphi\right| \leqslant \||\operatorname{grad} u|\|_{L_2(\Omega)} \|\varphi\|_0 \quad \left(\varphi \in C_c^\infty(U)\right),$$

the mapping $\varphi \mapsto -\int_U u\Delta\varphi$ is a continuous linear functional on $W_{2,0}^1(U)$. By Riesz' representation theorem, there exists a unique $v \in W_{2,0}^1(U)$ such that

$$(\varphi \mid \overline{v})_0 = -\int_U u\Delta\varphi \quad \left(\varphi \in C_c^\infty(U)\right).$$

□

Let $J\colon W_{2,0}^1(\Omega) \to D_{2,0}^1(U)$ be the orthogonal projection. Then $\widetilde{Ju} = \tilde{u}\ \mu$-a.e. by (1).

Let $D := \left\{u \in L_2(\Omega, \mu);\ \exists\, v \in W_{2,0}^1(\Omega) : \tilde{v} = u\ \mu\text{-a.e.}\right\}$. Then $\iota\colon D \to D_{2,0}^1(U)$, $\iota(u) := Jv$, where $v \in W_{2,0}^1(\Omega)$ such that $\tilde{v} = u\ \mu$-a.e., is a well-defined linear mapping (again by (1)).

Define

$$D(\tau_D) := D,$$
$$\tau_D(u, v) := \int_\Omega \operatorname{grad} \iota(u)(x) \cdot \overline{\operatorname{grad} \iota(v)(x)}\, \mathrm{d}x = \tau_0(\iota(u), \iota(v)) \quad (u, v \in D(\tau_D)).$$

REMARK 2.4. In fact, we do not need condition (1). One can always work with the decomposition

$$W^1_{2,0}(\Omega) = \left\{u \in W^1_{2,0}(\Omega);\ \tilde{u} = 0\ \mu\text{-a.e.}\right\} \oplus \left\{u \in W^1_{2,0}(\Omega);\ \tilde{u} = 0\ \mu\text{-a.e.}\right\}^{\perp},$$

and define $J\colon D \to \left\{u \in W^1_{2,0}(\Omega);\ \tilde{u} = 0\ \mu\text{-a.e.}\right\}^{\perp}$ as the orthogonal projection. As the previous proposition shows, if (1) is satisfied, the subspace $\left\{u \in W^1_{2,0}(\Omega);\ \tilde{u} = 0\ \mu\text{-a.e.}\right\}^{\perp}$ can be described more explicitly as the space of $W^1_{2,0}(\Omega)$-functions, which are harmonic on $\Omega\backslash \operatorname{spt} \mu$.

THEOREM 2.5. *τ_D is densely defined in $L_2(\Omega, \mu)$, symmetric, nonnegative and closed.*

Proof. τ_D is densely defined since $C_c^\infty(\Omega) \subseteq D(\tau_D)$ is dense in $L_2(\Omega, \mu)$. Symmetry and nonnegativity is clear by definition. To show closedness, let (u_n) in $D(\tau_D)$ be a τ_D-Cauchy sequence, i.e., $\tau_D(u_n - u_m) \to 0$, and $u_n \to u$ in $L_2(\Omega, \mu)$. Since $(\iota(u_n))_n$ is a Cauchy sequence in $W^1_{2,0}(\Omega)$, there exists $v \in W^1_{2,0}(\Omega)$ such that $\iota(u_n) \to v$ in $W^1_{2,0}(\Omega)$. For $\varphi \in C_c^\infty(U)$, we compute

$$0 = \int_U \iota(u_n)\Delta\varphi \to \int_U v\Delta\varphi,$$

i.e., $v \in D^1_{2,0}(U)$.

There exists a subsequence (u_{n_k}) such that $\widetilde{\iota(u_{n_k})} \to \tilde{v}$ q.e. and hence also μ-a.e. Since $\widetilde{\iota(u_n)} = u_n$ μ-a.e., we observe $\tilde{v} = u$ μ-a.e. Hence, $u \in D(\tau_D)$, $\iota(u) = v$ and

$$\tau_D(u_n - u) = \tau_0(\iota(u_n) - v) \to 0. \qquad \square$$

LEMMA 2.6. *The form τ_D is real, i.e., $u \in D(\tau_D)$ implies* $\operatorname{Re} u \in D(\tau_D)$*, and $\tau_D(u, v) \in \mathbb{R}$ for all real $u, v \in D(\tau_D)$.*

We omit the obvious proof of the lemma.

THEOREM 2.7. *Let $F\colon \mathbb{R} \to \mathbb{R}$, $F(x) := (0 \vee x) \wedge 1$ for all $x \in \mathbb{R}$, where $\vee$ and $\wedge$ denote the maximum and minimum, respectively. Let $u \in D(\tau_D)$ be real. Then $F \circ u \in D(\tau_D)$ and $\tau_D(F \circ u) \leqslant \tau_D(u)$.*

Proof. Since $F(x) \leqslant |x|$ for all $x \in \mathbb{R}$, we have $F \circ u \in L_2(\Omega, \mu)$. There exists $v \in W^1_{2,0}(\Omega)$ such that $\tilde{v} = u$ μ-a.e. Clearly, v can be chosen to be real. Then $F \circ u \in W^1_{2,0}(\Omega)$ by the lattice properties of $W^1_{2,0}(\Omega)$. Since F is continuous, by Proposition 2.1 we obtain $\widetilde{F \circ v} = F \circ \tilde{v}$ q.e. and hence also μ-a.e. Thus, $\widetilde{F \circ v} = F \circ u$ μ-a.e. and therefore, $F \circ u \in D(\tau_D)$. Since $\widetilde{F \circ v} = F \circ \tilde{v} = F \circ \widetilde{Jv} = \widetilde{F \circ Jv}$ μ-a.e., by (1) we obtain $J(F \circ v) = J(F \circ Jv)$. Therefore, since J is an orthogonal projection and hence a contraction for the norm $\tau_0(\cdot)^{1/2}$, we obtain

$$\begin{aligned}\tau_D(F \circ u) &= \tau_0(\iota(F \circ u)) = \tau_0(J(F \circ v)) = \tau_0(J(F \circ Jv)) \leqslant \tau_0(F \circ Jv)\\ &\leqslant \tau_0(Jv) = \tau_0(\iota(u)) = \tau_D(u),\end{aligned}$$

where we used the fact that $\tau_0(F \circ w) \leqslant \tau_0(w)$ for any $w \in W^1_{2,0}(\Omega)$. □

THEOREM 2.8. *$C_c^\infty(\Omega)$ is a core for τ_D.*

Proof. Note that [10, Theorem 1.4.2 (ii)] states that $D(\tau_D) \cap L_\infty(\Omega, \mu)$ is a core for τ_D. Thus, it suffices to approximate $u \in D(\tau_D) \cap L_\infty(\Omega, \mu)$. There exists a sequence (φ_l) in $C_c^\infty(\Omega)$ such that $\varphi_l \to \iota(u)$ in $W^1_{2,0}(\Omega)$ (i.e., $\tau_D(\varphi_l - u) \to 0$), $\varphi_l \to \widetilde{\iota(u)}$ q.e. and $M := \sup\left\{\|\varphi_l\|_{\infty,\mathrm{spt}\,\mu};\ l \in \mathbb{N}\right\} < \infty$. Since $\mu \in M_0(\Omega)$, we also have $\varphi_l \to \widetilde{\iota(u)}$ μ-a.e., and since $\widetilde{\iota(u)} = u$ μ-a.e. also $\varphi_l \to u$ μ-a.e. Since $|\varphi_l| \leqslant M\mathbb{1}_\Omega \in L_2(\Omega, \mu)$, Lebesgue's dominated convergence theorem yields $\varphi_l \to u$ in $L_2(\Omega, \mu)$, and therefore, $\varphi_l \to u$ in $D_{\tau_D} = (D(\tau_D), \|\cdot\|_{\tau_D})$. □

REMARK 2.9. In view of Theorem 2.5, Lemma 2.6 and Theorem 2.7, the form τ_D is a symmetric Dirichlet form. Theorem 2.8 assures that τ_D is even regular.

3. Characterization of the operator

Let H be the self-adjoint operator in $L_2(\Omega, \mu)$ associated with τ_D, where Ω and μ are as in the previous section.

DEFINITION. Let $F \in L_{1,\mathrm{loc}}(\Omega; \mathbb{K}^n)$, $g \in L_1(\Omega, \mu)$. Then g is called the *distributional divergence* of F with respect to μ, denoted by $\mathrm{div}_\mu F = g$, if

$$\int_\Omega F(x)\,\mathrm{grad}\,\varphi(x)\,\mathrm{d}x = -\int_\Omega g(x)\varphi(x)\,\mathrm{d}\mu(x) \quad \left(\varphi \in C_c^\infty(\Omega)\right).$$

THEOREM 3.1. *We have*

$$D(H) = \left\{u \in D(\tau_D);\ \mathrm{div}_\mu\,\mathrm{grad}\,\iota(u) \in L_2(\Omega, \mu)\right\},$$
$$Hu = -\,\mathrm{div}_\mu\,\mathrm{grad}\,\iota(u) \quad (u \in D(H)).$$

Proof. First note that for $u \in D(\tau_D)$ and $\varphi \in C_c^\infty(\Omega)$, we have

$$\tau_0(\iota(u), \varphi) = \int_\Omega \mathrm{grad}\,\iota(u) \cdot \overline{\mathrm{grad}\,\varphi} = \int_\Omega \mathrm{grad}\,\iota(u) \cdot \overline{\mathrm{grad}\,\iota(\varphi)} = \tau_D(u, \varphi).$$

Indeed, since $\iota(\varphi) = J\varphi$ and $\varphi - J\varphi \in W^1_{2,0}(U)$, we obtain

$$\int_\Omega \mathrm{grad}\,\iota(u) \cdot \overline{\mathrm{grad}(\varphi - J\varphi)} = 0.$$

Let H_1 be the operator defined by the right-hand side in the theorem. Let $u \in D(H_1)$ and $\varphi \in C_c^\infty(\Omega)$. Then by the above, we have

$$\tau_D(u, \varphi) = \int_\Omega \mathrm{grad}\,\iota(u) \cdot \overline{\mathrm{grad}\,\varphi} = -\int_\Omega \mathrm{div}_\mu\,\mathrm{grad}\,\iota(u)\overline{\varphi}\,\mathrm{d}\mu = (H_1 u \mid \varphi).$$

By continuity and Theorem 2.8, we obtain

$$(H_1 u \mid v) = \tau_D(u, v) \quad (v \in D(\tau_D)).$$

Thus, $u \in D(H)$ and $Hu = H_1 u$.

To show the converse inclusion, let $u \in D(H) \subseteq D(\tau_D)$ and $\varphi \in C_c^\infty(\Omega)$. Then

$$\begin{aligned}\int_\Omega \operatorname{grad} \iota(u) \cdot \overline{\operatorname{grad} \varphi} &= \int_\Omega \operatorname{grad} \iota(u) \cdot \overline{\operatorname{grad} \iota(\varphi)} = \tau_D(u, \varphi) = (Hu \mid \varphi)\\ &= \int_\Omega Hu\overline{\varphi}\, \mathrm{d}\mu.\end{aligned}$$

Hence, $\operatorname{div}_\mu \operatorname{grad} \iota(u)$ exists and $\operatorname{div}_\mu \operatorname{grad} \iota(u) = -Hu \in L_2(\Omega, \mu)$. Thus, $u \in D(H_1)$ and $H_1 u = Hu$. □

REMARK 3.2. The operator H is the multidimensional analog of the operator $-\partial_\mu \partial \iota$ with Dirichlet boundary conditions, see [13,17,18] and also [8,9].

We now focus on properties of the semigroup $(e^{-tH})_{t \geqslant 0}$. A C_0-semigroup $T\colon [0,\infty) \to L(L_2(\Omega, \mu))$ of bounded linear operators in $L_2(\Omega, \mu)$ is called *positive*, if $T(t)f \geqslant 0$ for all $0 \leqslant f \in L_2(\Omega, \mu)$, $t \geqslant 0$. The semigroup is called *submarkovian*, if it is positive and L_∞-contractive, i.e., $f \in L_2(\Omega, \mu)$, $0 \leqslant f \leqslant 1$ implies $0 \leqslant T(t)f \leqslant 1$ for all $t \geqslant 0$.

THEOREM 3.3. *The C_0-semigroup $(e^{-tH})_{t \geqslant 0}$ is submarkovian.*

Proof. By Lemma 2.6, the form τ_D is real. Hence, also the associated operator H and the semigroup $(e^{-tH})_{t \geqslant 0}$ is real. By Theorems 2.5, 2.7 and the Beurling–Deny criteria, the semigroup $(e^{-tH})_{t \geqslant 0}$ is submarkovian. □

REMARK 3.4. In [10, Section 6.2], the traces of Dirichlet forms and associated processes were considered. Our result characterizes the corresponding generating operator H in case of (suitably scaled) Brownian motion on a bounded domain $\operatorname{spt} \mu$, where μ is the corresponding volume measure (i.e., Lebesgue measure). The process may jump through $\Omega \setminus \operatorname{spt} \mu$ however (due to the Dirichlet boundary condition at $\partial\Omega$) gets killed on $\partial\Omega$.

REMARK 3.5. Let us compare our description of the form τ_D and the operator H with the theory of [2]. To this end, let $V := \left\{u \in W^1_{2,0}(\Omega);\ \tilde{u} \in L_2(\Omega, \mu)\right\}$ and equip V with the norm defined by

$$\|u\|_V := \left(\int_\Omega |\operatorname{grad} u|^2 + \int_\Omega |\tilde{u}|^2 \,\mathrm{d}\mu\right)^{1/2} \quad (u \in V).$$

Let $j\colon V \to L_2(\Omega, \mu)$ be defined by $j(u) := \tilde{u}$. Let $a\colon V \times V \to \mathbb{K}$ be defined by

$$a(u, v) := \int_\Omega \operatorname{grad} u \cdot \overline{\operatorname{grad} v} \quad (u, v \in V),$$

i.e., a is the restriction of the classical Dirichlet form τ_0 to V. Then H is also associated with (a, j) as in [2, Theorem 2.1]. One may ask where the projection J of our setup appears in this framework. This may be answered by [2, Proposition 2.3], see also [3, Theorem 8.11].

4. Applications

We will now show two applications. Note that by Remark 2.4, in fact we only need to prove $\mu \in M_0(\Omega)$. However, we will also show "$\supseteq$" in (1) (so that equality in (1) holds).

Note that for an open subset $V \subseteq \mathbb{R}^n$, we have

$$W_{2,0}^1(V) = \left\{u|_V;\ u \in W_2^1(\mathbb{R}^n),\ \tilde{u} = 0 \text{ q.e. on } \partial V\right\},$$

see, e.g., [7, Theorem 2.5] and [5, Theorem 4.2].

EXAMPLE 4.1. Let $n \geqslant 2$, $\Omega := (-1,1)^n \subseteq \mathbb{R}^n$, $\Gamma := \Omega \cap (\mathbb{R}^{n-1} \times \{0\})$ and $\mu := \lambda^{n-1}(\cdot \cap \Gamma)$ be the $(n-1)$-dimensional Lebesgue measure on Γ. Then $\mu \in M_0(\Omega)$ by [4, Theorem 4.1]. We will show the equality in (1). Write $\Omega_+ := \Omega \cap (\mathbb{R}^{n-1} \times (0,\infty))$ and $\Omega_- := \Omega \cap (\mathbb{R}^{n-1} \times (-\infty, 0))$ (Fig. 1).

Let $u \in W_{2,0}^1(\Omega)$, $\tilde{u} = 0$ μ-a.e. There exists (φ^k) in $C_c^\infty(\Omega)$ such that $\varphi^k \to u$ in $W_2^1(\Omega)$ and $\varphi^k \to \tilde{u}$ q.e. Thus, also $\varphi^k(\cdot,0) \to \tilde{u}(\cdot,0) = 0\,\lambda^{n-1}$-a.e.

For $v \in L_2(\Omega)$, let

$$Ev(x) := \begin{cases} v(x) & x \in \Omega, \\ 0 & \mathbb{R}^n \setminus \Omega \end{cases}$$

be the extension of v by zero, and $v_+ := (Ev)|_{\mathbb{R}^{n-1}\times(0,\infty)}$.

We obtain $\varphi_+^k \to u_+$ in $W_2^1(\mathbb{R}^{n-1} \times (0,\infty))$. By [1, Theorem 5.36], there exists a bounded linear trace operator $\operatorname{tr}\colon W_2^1(\mathbb{R}^{n-1} \times (0,\infty)) \to L_2(\mathbb{R}^{n-1})$. Hence, $\operatorname{tr}\varphi_+^k \to \operatorname{tr} u_+$ in $L_2(\mathbb{R}^{n-1})$. Since also $\operatorname{tr}\varphi_+^k = \varphi^k(\cdot,0) \to \tilde{u}_+(\cdot,0) = 0\,\lambda^{n-1}$-a.e. we obtain $\operatorname{tr} u_+ = 0$. By [1, Theorem 5.37], we obtain $u_+ \in W_{2,0}^1(\mathbb{R}^{n-1} \times (0,\infty))$. Two applications of [1, Theorem 5.29] finally yield $u|_{\Omega_+} \in W_{2,0}^1(\Omega_+)$. Analogously, $u|_{\Omega_-} \in W_{2,0}^1(\Omega_-)$, and hence, $u \in W_{2,0}^1(\Omega)$.

Thus, the corresponding stochastic process describes a particle diffusing in the hyperplane and jumping through Ω.

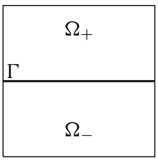

Figure 1. The hypercube Ω divided into two parts Ω_+ and Ω_- by the hyperplane Γ

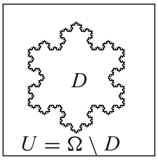

Figure 2. The square Ω and the snowflake D

EXAMPLE 4.2. Let D be the filled (open) Koch's snowflake centered at the origin and $\Omega \subseteq \mathbb{R}^2$ be a large open square centered at the origin such that $\overline{D} \subseteq \Omega$. Let $\mu := \lambda^2(\cdot \cap D)$ be the Lebesgue measure on D (Fig. 2).

Then $\mu \in M_0(\Omega)$. We show equality in (1). Let $u \in W^1_{2,0}(\Omega)$, $\tilde{u} = 0\,\mu$-a.e. By [1, Theorem 5.29], the extension of u by zero yields $u \in W^1_2(\mathbb{R}^2)$. By [7, Theorem 2.5], we observe $\tilde{u} = 0$ q.e. on $\partial\Omega$.

Since $u|_D = 0\,\lambda^2$-a.e., we have $\mathrm{tr}(u|_D) = 0\,\mathcal{H}^d$-a.e. on the boundary of D by [21, Theorem 2], where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure with $d = \frac{\log 4}{\log 3}$. By [5, Corollary 4.5], we thus obtain $\tilde{u} = 0$ q.e. on ∂D.

Hence, for $U := \Omega \backslash D$ we obtain $\tilde{u} = 0$ q.e. on ∂U, which by [7, Theorem 2.5] yields $u \in W^1_{2,0}(U)$.

We can thus describe jump diffusion, where the diffusion takes part on the snowflake D and jumps may occur along its boundary ∂D.

Acknowledgments

C.S. warmly thanks Jürgen Voigt and Hendrik Vogt who led him to the topic. The authors thank the unknown referee for useful comments.

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