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On the trivial solutions for the rotating patch model

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Abstract. In this paper, we study the clockwise simply connected rotating patches for Euler equations. By using the moving plane method, we prove that Rankine vortices are the only solutions to this problem in the class of *slightly* convex domains. We discuss in the second part of the paper the case where the angular velocity $\Omega = \frac{1}{2}$, and we show without any geometric condition that the set of the V-states is trivial and reduced to the Rankine vortices.

1. Introduction

We shall study in this paper some aspects of the vortex motion for the two-dimensional incompressible Euler system which can be written with the vorticity–velocity formulation in the form,

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \quad x \in \mathbb{R}^2, \ t > 0, \\ v = \nabla^{\perp} \Delta^{-1} \omega, \\ \omega(0, x) = \omega_0(x). \end{cases} \tag{1}$$

Here, $\nabla^{\perp} = (-\partial_2, \partial_1)$, $v = (v_1, v_2)$ is the velocity field, and the ω its vorticity given by the scalar $\omega = \partial_1 v_2 - \partial_2 v_1$. The classical theory dealing with the local\global well-posedness of smooth solutions is well developed, and we refer for instance to [1,3].

According to Yudovich result [10], the vorticity equation has a unique global solution in the weak sense provided the initial vorticity ω_0 belongs to $L^1 \cap L^\infty$. This result allows to deal rigorously with the so-called vortex patches which are initial vortices uniformly distributed in a confined region D, that is, $\omega_0 = \chi_D$ the characteristic function of D. Since the vorticity is transported along trajectories, we conclude that the vorticity preserves the vortex patch structure for any positive time. This means that for any $t \geq 0$, $\omega(t) = \chi_{D_t}$, with $D_t = \psi(t, D)$ is the image of D by the flow ψ which satisfies the ordinary differential equation

$$\partial_t \psi(t, x) = v(t, \psi(t, x)), \quad \psi(0, x) = x. \tag{2}$$

The dynamics of the boundary of D_t is in general complex and very difficult to follow. By using the contour dynamics method, we may parametrize the boundary by a function $\gamma_t : \mathbb{T} \to \partial D_t$ satisfying a nonlinear and non-local equation of the following type

$$\partial_t \gamma_t = -\frac{1}{2\pi} \int_{\partial D_t} \log|\gamma_t - \xi| \mathrm{d}\xi.$$

There are few examples known in the literature with explicit dynamics. The first one is Rankine vortex where D is a disc; in this case, the particle trajectories are circles centered at the origin, and therefore, $D_t = D$, $\forall t \geq 0$. The second example is a remarkable one and discovered by Kirchhoff [8] is the ellipses. In this case, the domain D_t does not change its shape and undergoes a perpetual rotation around its barycenter with uniform angular velocity Ω related to the semi-axes a and b through the formula $\Omega = ab/(a+b)^2$. See, for instance, [1, p. 304].

It seems that the ellipses are till now the only explicit example with such properties but whether or not other non-trivial implicit rotating patches exist has been discussed in the last few decades from numerical and theoretical point of view. To be more precise about these structures, we say that $\omega_0 = \chi_D$ is a V-state or a rotating patch if there exists a real number Ω called the angular velocity such that the support of the vorticity $\omega(t) = \chi_{D_t}$ is described by

$$D_t = R_{x_0,\Omega t} D, \quad \forall t \geq 0,$$

with $R_{x_0,\Omega t}$ being the planar rotation with center x_0 and angle Ωt. Deem and Zabusky [4] were the first to reveal numerically the existence of simply connected V-states with the m-fold symmetry for the integer $m = 3, 4, 5$. Recall that a domain D is said to be m-fold symmetric if it is invariant by the dihedral group D_m which is the symmetry group of a regular polygon of m sides. A few years later, Burbea [2] gave an analytic proof by using the bifurcation theory showing the existence of a countable family of V-states with the m-fold symmetry for any $m \geq 2$. They can be identified to one-dimensional branches bifurcating from the Rankine vortex at the simple "eigenvalues" $\big\{\Omega = \frac{m-1}{2m}, m \geq 2\big\}$. See also [6], where the C^∞ boundary regularity of the bifurcated V-states close to the disc was proven. It seems that close to the disc, the bifurcating branches rotate with bounded angular velocities, $\Omega \in]0, \frac{1}{2}[$. It is important to know whether all the V-states possess an angular velocity in this strip. From the implicit function theorem, we know that close to the disc there are no non-trivial V-states associated with $\Omega \notin [0, \frac{1}{2}]$.

In this paper, we give a partial answer to this problem. We shall first show that there is no clockwise rotating patches, that is, $\Omega \leq 0$, but with some geometric constraints. When $\Omega = 0$, this corresponds to stationary patches, and we know from a recent result of Fraenkel [5] in gravitational theory that the discs are the only stationary patches. He used the techniques of moving plane method which can be adapted to our framework only when $\Omega \leq 0$. More precisely, we obtain the following result.

THEOREM 1. *Let D be a C^1 bounded simply connected domain convex or more generally being in the class $\Sigma_{\arccos \frac{1}{\sqrt{5}}}$ introduced in the* Definition 2. *Assume that χ_D*

is a V-state satisfying the Eq. (1) *with the angular velocity* $\Omega < 0$. *Then, necessarily D is a disc.*

The proof uses the moving plane method in the spirit of the papers [5,9]. We start first with reformulating the equation in an integral form by using the strong maximum principle. This can be done by noticing that the stream function ψ associated with the vorticity χ_D is a V-state if and only if the following equation holds true

$$\varphi \triangleq \mu + \frac{1}{2}\Omega|x|^2 - \psi(x) = 0, \quad \forall\, x \in \partial D, \tag{3}$$

with μ a constant. From the maximum principle applied to φ in the domain D, one deduces that $D \subset \{\varphi > 0\}$. To get the equality between the latter sets which is essential for our approach, we need to prove that $\varphi(x) < 0$ for any $x \notin \overline{D}$. It is not clear how to get this result without any geometric constraint on the domain because the function φ is only superharmonic in the complement of D and going to $-\infty$ at infinity and therefore the associated minimum principle is not conclusive. Thus, we should put some geometric constraints on the domain in order to get the desired result. Notice that this kind of problem does not appear when $\Omega = 0$ because the function φ is harmonic, and the maximum principle can be applied. Now, once the domain is fully described by φ, the V-state equation can be reduced to a nonlinear integral equation on φ,

$$\varphi(x) = \mu + \frac{1}{2}\Omega|x|^2 - \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\, H(\varphi(y))\,\mathrm{d}y, \quad \forall x \in \mathbb{R}^2.$$

with $H = \chi_{[0,+\infty[}$ being the Heaviside function. The second step consists in applying the techniques of the moving plane method in order to show that any solution φ is radial and strictly decreasing. Therefore, the conclusion follows from the fact that the boundary ∂D is a level set of φ, and it must be a circle.

When $\Omega \geq 0$, there is a competition between the quadratic potential describing the rotation effect and the gravitational potential. This competition is fruitful in the strip $\Omega \in]0, \frac{1}{2}[$ and leads to non-trivial examples as we have previously quoted concerning the existence of the m-folds. The approach of moving plane method fails for $\Omega \geq \frac{1}{2}$ because we are still in the region where there is an active competition between the two potentials, and we lose the monotonicity of the nonlinear functional. Our second purpose in this paper was to establish a global result for the end point $\Omega = \frac{1}{2}$ which is very special. We shall prove a similar result to Theorem 1 by using different techniques from complex analysis. Our result reads as follows.

THEOREM 2. *Let D be a* C^1 *bounded simply connected domain. Assume that* χ_D *is a V-state satisfying the* Eq. (1) *with the angular velocity* $\Omega = \frac{1}{2}$. *Then, necessarily D is a disc.*

According to the definition (3), we can see that the value $\Omega = \frac{1}{2}$ is very special because in this case, φ is harmonic inside the domain D and vanishes on its boundary.

This leads to an explicit formula for the Newtonian potential in D which is quadratic and turns out to be that of a circle. This is a kind of inverse problem solved in a more general framework in [6]. The proof in our particular situation is more easier, and for the convenience of the reader, we shall give a complete proof of this fact.

REMARK 1. *In* Theorem 2*, we do not assume any geometric property for the V-states, and therefore, it is interesting to see whether we can get rid of the geometric constraints in* Theorem 1*. In the last section, we shall deal with the case* $\Omega = \frac{1}{2}$ *and give a proof for* Theorem 2.

REMARK 2. *In our results, we require the boundary to be at least of class* C^1*. This appears when we move from the equation of the vorticity to the stream function formulation. The regularity could be relaxed to rectifiable boundaries.*

REMARK 3. *Whether there are V-states rotating faster than* $\frac{1}{2}$ *remains open.*

The paper is organized as follows. In Sect. 2, we shall discuss the mathematical model of the V-states. Section 3 is devoted to the clockwise rotating patches and where we prove Theorem 1.

2. Model

Recall that a rotating patch, called also V-state, for Euler equations written in the form (1) is a solution of the type $\omega(t) = \chi_{D_t}$ where the domain D_t rotates uniformly with an angle velocity Ω around its barycenter assumed to be zero, that is,

$$D_t = R_{0,\Omega t} D,$$

with $R_{0,\Omega t}$ being the rotation of center 0 and angle Ωt. According to [2,6] when the boundary of the patch is smooth, this holds true if and only if

$$\big(v(x) - \Omega x^{\perp}\big) \cdot \vec{n}(x) = 0, \quad \forall x \in \partial D, \tag{4}$$

with $\vec{n}(x)$ be the outward-pointing normal unit vector to the boundary, and v be the induced velocity by the patch χ_D which can be recovered from Biot-Savart law in its complex form as follows:

$$v(z) = \frac{i}{2\pi} \int_D \frac{1}{\overline{z} - \overline{y}} \mathrm{d}A(y), \quad z \in \mathbb{C}.$$

Integrating the Eq. (4) yields

$$\psi(z) - \frac{1}{2}\Omega |z|^2 = Cte \triangleq \mu, \quad \forall z \in \partial D, \tag{5}$$

with ψ the stream function associated with the patch χ_D and defined by

$$\psi(z) = \frac{1}{2\pi} \int_D \log|z - y| \, \mathrm{d}A(y), \quad z \in \mathbb{C}.$$

Recall the identity

$$\begin{aligned}\partial_z\psi(z) &= \frac{1}{2}i\overline{v(z)}\\ &= \frac{1}{4\pi}\int_D \frac{1}{z-y}\mathrm{d}A(y).\end{aligned}$$

From the Cauchy-Pompeiu formula, one can write

$$\frac{1}{2\pi i}\int_\Gamma \frac{\overline{z}-\overline{\xi}}{\xi - z}\mathrm{d}\xi = \frac{1}{\pi}\int_D \frac{1}{z-y}\mathrm{d}A(y), \quad z\in\mathbb{C}.$$

Consequently,

$$4\partial_z\psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{z}-\overline{\xi}}{\xi - z}\mathrm{d}\xi, \quad \forall z\in\mathbb{C}. \tag{6}$$

Thus, the Eq. 4 can be written in the following complex form

$$Re\Big\{\Big(2\Omega\overline{z} + \frac{1}{2i\pi}\int_{\partial D}\frac{\overline{\xi}-\overline{z}}{\xi - z}\mathrm{d}\xi\Big)\vec{\tau}(z)\Big\} = 0, \quad \forall\, z\in\partial D,$$

with $\vec{\tau}(z)$ being a tangent unit vector to the boundary at the point z.

3. Clockwise rotating patches

Now, we shall prove the main result stated in Theorem 1. The case $\Omega = 0$ was done in [5], and therefore, we shall focus only on $\Omega < 0$. It will be done in several steps in the spirit of the paper [9]. Firstly, we shall transform the V-states equation into an integral one. To do so, we need some strong geometric conditions on the domains. Secondly, we use the moving plane method to prove that all the solutions of the integral equation are radial symmetric and strictly decreasing. From this result, we can deduce easily Theorem 1.

3.1. Integral equation

Our first goal was to transform the V-states equation into an integral form. This will be done under some geometric constraints on the domains of the V-states. We shall for this purpose need some definitions.

DEFINITION 1. *Let D be a C^1 simply connected domain. Let $x_0\in\partial D$, $\alpha\in[0,\frac{\pi}{2}]$ and define the sector*

$$\mathrm{Sec}_{x_0,\alpha} = \Big\{x\in\mathbb{R}^2\backslash\{x_0\};\ \cos\alpha \le \frac{x-x_0}{\|x-x_0\|}\cdot\vec{\nu}(x_0)\Big\}$$

with $\vec{\nu}(x_0)$ the outward-pointing normal unit vector to the boundary at x_0. We also define the sets

$$D^+_{x_0} = \Big\{x\in D;\ (x-x_0)\cdot\vec{\nu}(x_0)\ge 0\Big\}$$

and

$$D_{x_0}^{-}=\Big\{y=x-2\big[(x-x_0)\cdot\vec{\nu}(x_0)\big]\vec{\nu}(x_0);\ x\in D_{x_0}^{+}\Big\}.$$

Note that the set $D_{x_0}^{+}$ is the part of D located above the tangent line to the boundary at the point x_0 (the orientation is with respect to $\vec{\nu}$). As to the set $D_{x_0}^{-}$, it is the reflection of the set $D_{x_0}^{+}$ with respect to this tangent.

DEFINITION 2. *Let D be a C^1 simply connected bounded domain with zero as barycenter. We say that D belongs to the class Σ_α with $\alpha\in[0,\frac{\pi}{2}]$ if*

(1) For each $x_0\in\partial D$, we have $x_0\cdot\vec{\nu}(x_0)\geq 0$.
(2) For each $x_0\in\partial D$, we have $Sec_{x_0,\alpha}\cap D=\varnothing$.
(3) For each $x_0\in\partial D$, the subset $D_{x_0}^{-}$ introduced in the Definition 1 *is contained in the domain D.*

We shall make some comments.

REMARK 4. *(1) The first assumption in the previous definition means that the barycenter is always below any tangent line to the boundary.*
(2) If a domain belongs to the class Σ_α, then it belongs to the class $\Sigma_{\alpha'}$ for any $\alpha'\in[0,\alpha]$.
(3) Any convex domain belongs to $\Sigma_{\frac{\pi}{2}}$.
(4) Roughly speaking, when α is close to $\frac{\pi}{2}$, a domain in Σ_α is "slightly convex."

Now, we shall write down an integral equation for φ.

PROPOSITION 1. *Let D be a C^1 simply connected domain belonging to the class $\Sigma_{\arccos\frac{1}{\sqrt{5}}}$ introduced in the* Definition 2. *Assume that χ_D is a V-state rotating with a negative angular velocity $\Omega<0$. Let ψ be the stream function of χ_D; then, there exists a constant μ previously defined in* (5) *such that*

$$\varphi(x)=\mu+\frac{1}{2}\Omega|x|^2-\frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\,H(\varphi(y))\,dy,\quad\forall x\in\mathbb{R}^2.\tag{7}$$

with $H=\chi_{[0,+\infty[}$ being the Heaviside function and

$$\varphi(x)\triangleq\mu+\frac{1}{2}\Omega|x|^2-\psi(x).$$

The proof is an immediate consequence of the next lemma which allows to write

$$\psi(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\,H(\varphi(y))\,\mathrm{d}y.$$

The lemma reads as follows.

LEMMA 1. *Let $\Omega<0$ and D be a domain as in the* Proposition 1, *then*

$$D=\big\{x\in\mathbb{R}^2,\varphi(x)>0\big\},\quad\partial D=\big\{x\in\mathbb{R}^2,\varphi(x)=0\big\}.$$

In particular,

$$\forall x\in\mathbb{R}^2,\quad\chi_D(x)=H(\varphi(x)).$$

Proof. The proof will be done in two steps. In the first one, we show that $D \subset \{x \in \mathbb{R}^2, \varphi(x) > 0\}$ which does not require any geometric constraint on the domain, and we use only the maximum principle applied to φ. However, the converse inclusion is more subtle, and it is not clear whether it can be proven without any additional geometric properties for the domain. To get the first inclusion, we observe that φ satisfies in the distribution sense

$$\Delta\varphi = 2\Omega - \chi_D \leq 0, \quad \text{in} \quad \mathbb{R}^2. \tag{8}$$

Taking the restriction of this equation to the domain D, we get in the classical sense the Dirichlet problem

$$\begin{cases} \Delta\varphi < 0 & \text{in} \quad D \\ \varphi = 0, & \partial D. \end{cases} \tag{9}$$

Applying the strong maximum principle, we find

$$\forall x \in D, \quad \varphi(x) > 0$$

and therefore, we obtain the inclusion

$$D \subset \{x \in \mathbb{R}^2, \varphi(x) > 0\}. \tag{10}$$

We should now prove the converse which follows from

$$\mathbb{R}^2 \backslash \overline{D} \subset \{x \in \mathbb{R}^2, \varphi(x) < 0\}. \tag{11}$$

To get this inclusion, we shall first give an elementary proof when the domain D is assumed to be convex which is in fact belongs to the class Σ_α and come back later to the general case. Since ψ has a logarithmic growth at infinity and $\Omega < 0$, then

$$\lim_{|x|\to+\infty} \varphi(x) = -\infty.$$

To get the inclusion, it suffices to show that φ has no critical points outside $\overline{D}$. Assume that there exits $x_0 \notin \overline{D}$ such that $\nabla\varphi(x_0) = 0$. It is easy to see that

$$\begin{aligned} \nabla\varphi(x_0) &= \Omega x_0 - \frac{1}{2\pi}\int_D \frac{x_0 - y}{|x_0 - y|^2}\mathrm{d}y \\ &= 0. \end{aligned}$$

As D is convex, one can find, using the separation theorem, a unit vector e (for example, the outward-pointing normal unit vector) such that

$$(x_0 - y) \cdot e > 0, \quad \forall y \in D.$$

Consequently, we get

$$\begin{aligned} \nabla\varphi(x_0) \cdot e &= \Omega\, x_0 \cdot e - \frac{1}{2\pi}\int_D \frac{(x_0 - y) \cdot e}{|x_0 - y|^2}\mathrm{d}y \\ &< \Omega\, x_0 \cdot e \\ &< 0, \end{aligned}$$

where we have used in the last line the fact that $0 \in D$ and $x_0 \cdot e > 0$. This gives the desired result in the case of convex domains. Now, let us discuss the general case of domains belonging to the class $\Sigma_{\arccos \frac{1}{\sqrt{5}}}$. We claim that for any $x_0 \in \partial D$, the function $g : [0, +\infty) \to \mathbb{R}$ defined by

$$\begin{aligned} g(t) &\triangleq \varphi\big(x_0 + t\vec{\nu}(x_0)\big) \\ &= \mu + \frac{1}{2}\Omega|x_0 + t\vec{\nu}(x_0)|^2 - \psi\big(x_0 + t\vec{\nu}(x_0)\big) \end{aligned}$$

is strictly decreasing. Indeed, by differentiation, we find

$$g'(t) = \Omega(t + x_0 \cdot \vec{\nu}(x_0)) - \nabla\psi\big(x_0 + t\vec{\nu}(x_0)\big) \cdot \vec{\nu}(x_0).$$

Using the assumption (1) of the Definition 2 and the fact that $\Omega < 0$, we obtain

$$\forall t \geq 0, \quad g'(t) < -\nabla\psi\big(x_0 + t\vec{\nu}(x_0)\big) \cdot \vec{\nu}(x_0). \tag{12}$$

By straightforward computations, we get

$$\begin{aligned} \nabla\psi\big(x_0 + t\vec{\nu}(x_0)\big) \cdot \vec{\nu}(x_0) &= \frac{1}{2\pi}\int_D \frac{\big(x_0 + t\vec{\nu}(x_0) - y\big) \cdot \vec{\nu}(x_0)}{|x_0 + t\vec{\nu}(x_0) - y|^2}\mathrm{d}y \\ &= \frac{t}{2\pi}\int_D \frac{1}{|x_0 + t\vec{\nu}(x_0) - y|^2}\mathrm{d}y \\ &\quad + \frac{1}{2\pi}\int_D \frac{\big(x_0 - y\big) \cdot \vec{\nu}(x_0)}{|x_0 + t\vec{\nu}(x_0) - y|^2}\mathrm{d}y. \end{aligned} \tag{13}$$

From the Definition 2, we get the partition $D = D_{x_0}^- \cup D_{x_0}^+ \cup \hat{D}_{x_0}$ with $\hat{D}_{x_0}$ the complement of $D_{x_0}^- \cup D_{x_0}^+$ in D. Therefore, the last integral term may be written in the form

$$\begin{aligned} \int_D \frac{\big(x_0 - y\big) \cdot \vec{\nu}(x_0)}{|x_0 + t\vec{\nu}(x_0) - y|^2}\mathrm{d}y &= \int_{\hat{D}_{x_0}} \frac{\big(x_0 - y\big) \cdot \vec{\nu}(x_0)}{|x_0 + t\vec{\nu}(x_0) - y|^2}\mathrm{d}y \\ &\quad + \int_{D_{x_0}^- \cup D_{x_0}^+} \frac{\big(x_0 - y\big) \cdot \vec{\nu}(x_0)}{|x_0 + t\vec{\nu}(x_0) - y|^2}\mathrm{d}y \\ &\geq \int_{D_{x_0}^- \cup D_{x_0}^+} \frac{\big(x_0 - y\big) \cdot \vec{\nu}(x_0)}{|x_0 + t\vec{\nu}(x_0) - y|^2}\mathrm{d}y \triangleq I_{x_0}, \end{aligned} \tag{14}$$

where we have used the fact that $\big(x_0 - y\big) \cdot \vec{\nu}(x_0) \geq 0$ for any $y \in \hat{D}_{x_0}$. We shall use the change in variables

$$z \in D_{x_0}^- \mapsto y = z - 2\big[(z - x_0) \cdot \vec{\nu}(x_0)\big]\vec{\nu}(x_0) \in D_{x_0}^+$$

which is a diffeomorphism preserving Lebesgue measure, and therefore,

$$\int_{D_{x_0}^+} \frac{\big(x_0 - y\big) \cdot \vec{\nu}(x_0)}{|x_0 + t\vec{\nu}(x_0) - y|^2}\mathrm{d}y = -\int_{D_{x_0}^-} \frac{\big(x_0 - z\big) \cdot \vec{\nu}(x_0)}{|x_0 - t\vec{\nu}(x_0) - z|^2}\mathrm{d}z.$$

Consequently,

$$I_{x_0} = \int_{D_{x_0}^-} (x_0 - y) \cdot \vec{\nu}(x_0) \left(\frac{1}{|x_0 + t\vec{\nu}(x_0) - y|^2} - \frac{1}{|x_0 - t\vec{\nu}(x_0) - y|^2} \right) \mathrm{d}y$$

$$= -4t \int_{D_{x_0}^-} \frac{\left[(x_0 - y) \cdot \vec{\nu}(x_0)\right]^2}{|x_0 + t\vec{\nu}(x_0) - y|^2 |x_0 - t\vec{\nu}(x_0) - y|^2} \mathrm{d}y.$$

Putting together the preceding identity with (14) and (13), we obtain

$$\nabla\psi\big(x_0 + t\vec{\nu}(x_0)\big) \cdot \vec{\nu}(x_0)$$
$$\geq \frac{t}{2\pi} \int_{D_{x_0}^-} \frac{1}{|x_0 + t\vec{\nu}(x_0) - y|^2} \left(1 - 4 \frac{\left[(x_0 - y) \cdot \vec{\nu}(x_0)\right]^2}{|x_0 - t\vec{\nu}(x_0) - y|^2} \right) \mathrm{d}y.$$

Easy computations show that

$$\inf_{t \geq 0} |x_0 - t\vec{\nu}(x_0) - y|^2 = |x_0 - y|^2 - \left[(x_0 - y) \cdot \vec{\nu}(x_0)\right]^2,$$

and therefore,

$$\nabla\psi\big(x_0 + t\vec{\nu}(x_0)\big) \cdot \vec{\nu}(x_0)$$
$$\geq \frac{t}{2\pi} \int_{D_{x_0}^-} \frac{|x_0 - y|^2}{|x_0 + t\vec{\nu}(x_0) - y|^2} \left(\frac{1 - 5\left[\frac{x_0 - y}{|x_0 - y|} \cdot \vec{\nu}(x_0)\right]^2}{|x_0 - y|^2 - \left[(x_0 - y) \cdot \vec{\nu}(x_0)\right]^2} \right) \mathrm{d}y.$$

Recall that for $y \in D_{x_0}^-$

$$\frac{x_0 - y}{|x_0 - y|} \cdot \vec{\nu}(x_0) = \frac{z - x_0}{|z - x_0|} \cdot \vec{\nu}(x_0), \quad \text{with} \quad z = y - 2\left[(y - x_0) \cdot \vec{\nu}(x_0)\right]\vec{\nu}(x_0) \in D_{x_0}^+.$$

According to the definition of $D_{x_0}^+$ and the condition (2) of the Definition 2, one sees that

$$0 \leq \frac{(z - x_0)}{|x_0 - z|} \cdot \vec{\nu}(x_0) \leq \frac{1}{\sqrt{5}}$$

and therefore,

$$\forall y \in D_{x_0}^-, \quad 0 \leq \frac{(x_0 - y)}{|x_0 - y|} \cdot \vec{\nu}(x_0) \leq \frac{1}{\sqrt{5}}.$$

This implies that

$$\forall t \geq 0, \quad \nabla\psi\big(x_0 + t\vec{\nu}(x_0)\big) \cdot \vec{\nu}(x_0) \geq 0.$$

Combining this inequality with (12), one gets

$$\forall t \geq 0, \quad g'(t) < 0.$$

It follows that g is strictly decreasing, and

$$\forall t > 0, \quad g(t) < g(0) = 0.$$

In the last equality, we have used the equation of the V-states, that is, $\varphi(x) = 0,\ \forall\, x \in \partial D$. Coming back to the definition of g, this proves that

$$\forall t > 0, \quad \varphi(x_0 + t\vec{\nu}(x_0)) < 0.$$

According to the condition (2) of the Definition 2, we get

$$\mathbb{R}^2 \backslash \overline{D} = \cup_{t>0, x_0 \in \partial D}]x_0, x_0 + t\vec{\nu}(x_0)].$$

This allows to conclude that

$$\forall x \in \mathbb{R}^2 \backslash \overline{D}, \quad \varphi(x) < 0,$$

and therefore, we get the desired inclusion (11). The proof of the lemma is now complete. □

3.2. Moving plane method

We shall now discuss the symmetry property of any solution of the integral equation (7). This may be done by using the moving plane method in the spirit of the papers [5,9]. Our result reads as follows.

PROPOSITION 2. *Let $\Omega < 0$ and φ be a solution of (7). Then, φ is radial and strictly decreasing with respect to the radial variable r.*

The result of Theorem 1 follows easily from this proposition because the level sets of φ are circles centered at zero and $\partial D \subset \varphi^{-1}(\{0\})$. As shown, Proposition 2 is nothing but the second part (2) of Proposition 5.
Next, we shall establish a list of auxiliary results needed later in the proof of Proposition 5.

Notation: For $\lambda > 0$, we introduce the sets

$$H_\lambda = \Big\{x = (x_1, x_2) \in \mathbb{R}^2, x_1 < \lambda\Big\}, \quad T_\lambda = \Big\{x = (\lambda, x_2) \in \mathbb{R}^2, x_2 \in \mathbb{R}\Big\}$$

, and we define for any $x = (x_1, x_2) \in \mathbb{R}^2$

$$x_\lambda \triangleq (2\lambda - x_1, x_2), \quad \varphi_\lambda(x) \triangleq \varphi(x) - \varphi(x_\lambda).$$

Our main goal now was to prove the following result.

PROPOSITION 3. *There exists $\lambda^\star > 0$ such that for any $\lambda \geq \lambda^\star$,*

$$\varphi_\lambda(x) > 0, \quad \forall x \in H_\lambda.$$

In addition, $\forall\, \lambda_0 > 0$ there exists $R(\lambda_0) > 0$ such that $\forall\, \lambda \geq \lambda_0$

$$\forall\, x \in H_\lambda, \quad |x| \geq R(\lambda_0) \Longrightarrow \varphi_\lambda(x) > 0.$$

Proof. Using Lemma 6 of [9] together with the fact that 0 is the barycenter of D, one may write

$$\forall x \in \mathbb{R}^2\backslash\{0\}, \quad \varphi(x) = \mu + \frac{1}{2}\Omega|x|^2 - \frac{1}{2\pi}|D|\log|x| + h(x),$$

with h being a smooth function outside D and satisfies the asymptotic behavior: There exists $R > 0$ large enough such that

$$\forall |x| \geq R, \quad |h(x)| \leq C|x|^{-2}; \quad |\nabla h(x)| \leq C|x|^{-3}.$$

From the elementary fact $|x_\lambda| > |x|$ for $x \in H_\lambda$, we shall get

$$\begin{aligned}\varphi_\lambda(x) &= \frac{1}{2}\Omega(|x|^2 - |x_\lambda|^2) + \frac{1}{2\pi}|D|\log\big(|x_\lambda|/|x|\big) + h(x) - h(x_\lambda)\\ &> -2\lambda\Omega(\lambda - x_1) + h(x) - h(x_\lambda).\end{aligned}$$

Using the mean value theorem, we obtain for $|x| \geq R$,

$$\begin{aligned}|h(x) - h(x_\lambda)| &\leq |x - x_\lambda| \sup_{y\in[x,x_\lambda]} |\nabla h(y)|\\ &\leq C(\lambda - x_1)|x|^{-3}\\ &\leq C(\lambda - x_1)R^{-3}.\end{aligned}$$

Hence, we get for $x \in H_\lambda$ with $|x| \geq R$

$$\varphi_\lambda(x) > (\lambda - x_1)\big(-CR^{-3} - 2\lambda\Omega\big).$$

We shall now see how to deduce from this inequality the second claim of the proposition, and the first claim is postponed later to the end of the proof. Let $\lambda_0 > 0$ and $\lambda \geq \lambda_0$ and take $x \in H_\lambda$ with $|x| \geq R$. Then, it is plain that

$$\varphi_\lambda(x) > (\lambda - x_1)\big(-CR^{-3} - 2\lambda_0\Omega\big).$$

Now, we choose R such that

$$\lambda_0 = -\frac{C}{2\Omega}R^{-3}$$

which guarantees that $\varphi_\lambda(x) > 0$, and this concludes the proof of the second part of Proposition 3. Now, let us come back to the proof of the first claim of the proposition. For this aim, it suffices to check in the preceding claim that the inequality $\varphi_\lambda(x) > 0$ remains true for any $x \in H_\lambda$ provided that λ is sufficiently large. Now, let $|x| \leq R$ and set

$$M \triangleq \sup_{|x|\leq R} |\varphi(x)|.$$

From the asymptotic behavior of φ, we see that

$$\lim_{|x|\to\infty} \varphi(x) = -\infty.$$

In particular, there exists $A > 0$ such that

$$|x| \geq A \Longrightarrow \varphi(x) < -2M.$$

It is easy to find λ_1 depending only on R and A such that

$$\begin{aligned} \forall\lambda \geq \lambda_1, \quad \forall |x| \leq R \Longrightarrow |x_\lambda| \geq A \\ \Longrightarrow \varphi_\lambda(x) > M. \end{aligned}$$

Set $\lambda^\star = \max\{\lambda_0, \lambda_1\}$; then for any $\lambda \geq \lambda^\star$, we obtain

$$\forall x \in H_\lambda, \quad \varphi_\lambda(x) > 0.$$

This concludes the proof of Proposition 3. □

The next result deals with a continuation principle very useful to prove the strictly monotonicity of φ.

PROPOSITION 4. *Let $\lambda > 0$ and assume that $\varphi_\lambda \geq 0$ in H_λ. Then,*

$$\varphi_\lambda > 0 \quad in \quad H_\lambda \quad and \quad \partial_{x_1}\varphi_\lambda(x) < 0, \ \forall\, x \in T_\lambda.$$

Proof. First, recall that the stream function can be written in the form

$$\psi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\, H(\varphi(y))\,\mathrm{d}y.$$

By using the change in variables $y \mapsto y_\lambda$ which preserves Lebesgue measure, one gets

$$\begin{aligned} \psi(x) &= \frac{1}{2\pi}\int_{H_\lambda} \log|x-y|\, H(\varphi(y))\,\mathrm{d}y + \frac{1}{2\pi}\int_{H_\lambda^c} \log|x-y|\, H(\varphi(y))\,\mathrm{d}y \\ &= \frac{1}{2\pi}\int_{H_\lambda} \log|x-y|\, H(\varphi(y))\,\mathrm{d}y + \frac{1}{2\pi}\int_{H_\lambda} \log|x-y_\lambda|\, H(\varphi(y_\lambda))\,\mathrm{d}y. \end{aligned}$$

Since the reflection with respect to T_λ is an isometry, we get

$$\begin{aligned} \psi(x_\lambda) &= \frac{1}{2\pi}\int_{H_\lambda} \log|x_\lambda-y|\, H(\varphi(y))\,\mathrm{d}y + \frac{1}{2\pi}\int_{H_\lambda} \log|x_\lambda-y_\lambda|\, H(\varphi(y_\lambda))\,\mathrm{d}y \\ &= \frac{1}{2\pi}\int_{H_\lambda} \log|x-y_\lambda|\, H(\varphi(y))\,\mathrm{d}y + \frac{1}{2\pi}\int_{H_\lambda} \log|x-y|\, H(\varphi(y_\lambda))\,\mathrm{d}y. \end{aligned}$$

This implies according to the definition of φ_λ

$$\varphi_\lambda(x) = -2\lambda\Omega(\lambda - x_1) + \frac{1}{2\pi}\int_{H_\lambda} \log\Big(|x-y_\lambda|/|x-y|\Big)\Big[H(\varphi(y)) - H(\varphi(y_\lambda))\Big]\mathrm{d}y.$$

Since H is increasing, and according to the assumption $\varphi_\lambda \geq 0$ in H_λ, one gets

$$H(\varphi(y)) - H(\varphi(y_\lambda)) \geq 0, \quad \forall\, y \in H_\lambda.$$

We combine this with the obvious geometric fact: for $\lambda > 0$,

$$|x - y_\lambda| \geq |x - y|, \quad \forall\, x, y \in H_\lambda.$$

Consequently,

$$\varphi_\lambda(x) \geq -2\lambda\Omega(\lambda - x_1) > 0, \quad \forall x \in H_\lambda.$$

It remains to check that $\partial_{x_1}\varphi_\lambda < 0$ on T_λ. Straightforward computations give for $x \in T_\lambda$ corresponding to $\lambda = x_1$,

$$\begin{aligned}\partial_{x_1} \log\Big(|x - y_\lambda|/|x - y|\Big) &= \frac{x_1 - (2\lambda - y_1)}{|x - y_\lambda|^2} - \frac{x_1 - y_1}{|x - y|^2}\\ &= 2\frac{y_1 - \lambda}{|x - y|^2}\end{aligned}$$

which implies

$$\partial_{x_1} \log\Big(|x - y_\lambda|/|x - y|\Big) < 0, \quad \forall y \in H_\lambda.$$

Therefore, we get for $x \in T_\lambda$

$$\begin{aligned}\partial_{x_1}\varphi_\lambda(x) &= 2\lambda\Omega + \frac{1}{2\pi}\int_{H_\lambda} \partial_{x_1} \log\Big(|x - y_\lambda|/|x - y|\Big)\Big[H(\varphi(y)) - H(\varphi(y_\lambda))\Big]\mathrm{d}y\\ &< 0.\end{aligned}$$

This concludes the proof of Proposition 4. □

Now, we discuss a more precise statement of Proposition 2.

PROPOSITION 5. *The following claims hold true.*

1. *For any $\lambda > 0$, we have*

$$\varphi_\lambda(x) > 0, \quad \forall x \in H_\lambda.$$

2. *The function φ is radial and satisfies for any $r > 0$*

$$\partial_r\varphi(r) < 0.$$

Proof. (1) Define the set

$$I \triangleq \Big\{\lambda > 0, \varphi_\lambda > 0 \quad \text{in} \quad H_\lambda\Big\}.$$

According to Proposition 3, this set is nonempty and contains the interval $[\lambda^\star, \infty[$. Let $(\alpha, \infty[$ be the largest interval contained in I and assume by contradiction that $\alpha > 0$. First, observe that by a continuation principle $\varphi_\alpha \geq 0$ in H_α, and therefore, Proposition 4 implies that

$$\forall x \in H_\alpha, \quad \varphi_\alpha(x) > 0.$$

This means that α belongs to this maximal interval, and thus, it coincides with the closed interval $[\alpha, \infty[$. By the maximality of the interval $[\alpha, +\infty[$, there exist two sequences (α_n) and (x_n) with the following properties

$$0 < \alpha_n < \alpha, \lim_{n\to\infty} \alpha_n = \alpha; \quad x_n \in H_{\alpha_n} \quad \text{and} \quad \varphi_{\alpha_n}(x_n) \le 0.$$

According to Proposition 3, the sequence (x_n) is bounded, and therefore, up to an extraction, we can assume that this sequence converges to some point $\overline{x} \in H_\alpha \cup T_\alpha$. By passing to the limit using the continuity of the map $(\alpha, x) \mapsto \varphi_\alpha(x)$ it is easy to check that

$$\varphi_\alpha(\overline{x}) \le 0$$

and consequently $\overline{x} \in T_\alpha$ because $\varphi_\alpha(x) > 0$ in H_α. Using Proposition 4, we get $\partial_{x_1}\varphi_\alpha(\overline{x}) < 0$. This contradicts the assumption $\varphi_{\alpha_n}(x_n) \le 0$ because it yields

$$\frac{\varphi(x_n^1, x_n^2) - \varphi(2\alpha_n - x_n^1, x_n^2)}{2(\alpha_n - x_n^1)} \le 0.$$

By passing to the limit as n goes to ∞, we find $\partial_{x_1}\varphi(\overline{x}) \ge 0$, which is impossible.

(2) By passing to the limit as $\lambda \to 0$ in the first point of the proposition and using the continuity of φ, one gets

$$\varphi(-x_1, x_2) \ge \varphi(x_1, x_2), \quad \forall x_1 \ge 0.$$

By changing the orientation, we get the reverse inequality, and therefore, we get the equality $\varphi(-x_1, x_2) = \varphi(x_1, x_2)$. This can be implemented for any moving plane (line), and consequently, the solution φ will be invariant by reflection with respect to any line passing by the origin. This means that φ is radial. Now, we can use Proposition 4 by taking $\lambda = x_1 > 0$, leading to

$$\partial_{x_1}\varphi(x_1, 0) < 0.$$

Since φ is radial, we get $\partial_r\varphi(r) < 0$ for any $r > 0$, and therefore, the proof of Proposition 5 is now complete.

□

4. Case $\Omega = \frac{1}{2}$

In this section, we shall be concerned with the proof of Theorem 2 dealing with the special angular velocity $\Omega = \frac{1}{2}$. We will prove the nonexistence of non-trivial V-states rotating with this angular velocity.

Proof. Recall that the boundary of any V-state is described by the equation

$$\varphi \triangleq \mu + \frac{1}{4}|x|^2 - \psi(x) = 0, \quad \forall x \in \partial D.$$

It is easy to see that

$$\Delta\varphi = 1 - \chi_D, \quad x \in \mathbb{R}^2$$

and therefore,

$$\begin{cases} \Delta\varphi = 0 & \text{in } D \\ \varphi = 0, & \partial D. \end{cases} \tag{15}$$

By the uniqueness property of the Cauchy problem, we obtain

$$\forall x \in D, \quad \varphi(x) = 0.$$

This implies that

$$\psi(x) = \mu + \frac{1}{4}|x|^2, \quad \forall x \in \overline{D}$$

and consequently,

$$4\partial_z\psi(z) = \overline{z}, \quad \forall z \in D.$$

By the extension principle for continuous functions, we get

$$\mathcal{C}(\chi_D)(z) = -\overline{z}, \quad \forall z \in \overline{D}.$$

with

$$\mathcal{C}(\chi_D)(z) \triangleq \frac{1}{\pi}\int_D \frac{1}{\xi - z}\mathrm{d}A(\xi)$$

being the Cauchy transform of the domain D. We claim that necessarily the domain is a disc. This was proved in a general framework in [7], and for the convenience of the reader, we shall give a proof in the few next lines. Set

$$G(z) \triangleq z\,\mathcal{C}(\chi_D)(z) \quad \text{and} \quad H \triangleq \operatorname{Im} G(z),$$

defined in $\mathbb{C}\backslash\overline{D}$. The function G is analytic in this domain and has a continuous extension up to the boundary of $\widehat{\mathbb{C}}\backslash D$. Note that $\lim_{z\to\infty} G(z) \in \mathbb{R}$, and consequently, the harmonic function H vanishes on the boundary $\partial D \cup \{\infty\}$. By the maximum principle, this implies that H vanishes everywhere in $\mathbb{C}\backslash D$. Therefore, G is constant everywhere and in particular in ∂D, that is,

$$|z|^2 = Cte, \quad \forall\, z \in \partial D.$$

This proves that ∂D is a disc. □

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