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Weighted energy estimates for p-evolution equations in SG classes

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Abstract. We prove energy estimates for linear p-evolution equations in weighted Sobolev spaces under suitable assumptions on the behavior at infinity of the coefficients with respect to the space variables. As a consequence, we obtain well posedness for the related Cauchy problem in the Schwartz spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$.

1. Introduction

Let us start by considering the Cauchy problem

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = f(t,x) & (t,x) \in [0,T] \times \mathbb{R}, \\ u(0,x) = g(x) & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

$D = -i\partial$, where $P(t,x,D_t,D_x)$ is a differential evolution operator of the form

$$P(t,x,D_t,D_x) = D_t + a_p(t)D_x^p + \sum_{j=0}^{p-1} a_j(t,x)D_x^j, \tag{1.2}$$

with $p \in \mathbb{N}$, $p \geq 2$, $a_p \in C([0,T];\mathbb{R})$ and $a_j \in C([0,T];\mathcal{B}^\infty)$, where $\mathcal{B}^\infty$ stands for the class of complex valued $C^\infty(\mathbb{R}_x)$ functions with uniformly bounded derivatives.

Operators of the form above are usually referred to as "p-evolution operators"; the condition that $a_p(t)$ is real valued means that the principal symbol of P (in the sense of Petrowski) has the real characteristic $\tau = -a_p(t)\xi^p$; by the Lax-Mizohata theorem, this is a necessary condition to have a unique solution in Sobolev spaces of the Cauchy problem (1.1) in a neighborhood of $t = 0$, for any $p \geq 1$. Notice that in the case $p = 1$, operator (1.2) is strictly hyperbolic; in the case $p = 2$, operators of the form (1.2) with real characteristics are usually called "Schrödinger-type evolution operators," being the Schrödinger operator the most relevant model in the class.

A wide literature concerning the well posedness of problem (1.1) in Sobolev spaces exists for $p = 1, 2$. For general $p \geq 2$, many results are known when the coefficients $a_j(t,x)$ are real valued, see for instance [1–3, 11, 13, 15]. When the coefficients $a_j(t,x)$

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are complex valued for some $1 \leq j \leq p-1$, then we know from [8,20] that some decay conditions for $|x| \to \infty$ must be required on the imaginary part of the coefficients in order to obtain H^∞ well posedness. In the papers [20,21], Ichinose has given necessary and sufficient conditions for the case $p = 2$, $x \in \mathbb{R}$. Kajitani and Baba [22] then proved that, for $p = 2$ and $a_2(t)$ constant, $x \in \mathbb{R}^n$, the Cauchy problem (1.1) is H^∞ well posed if

$$\operatorname{Im} a_1(t, x) = \mathcal{O}(|x|^{-\sigma}), \;\; \sigma \geq 1, \;\; \text{as } |x| \to \infty, \tag{1.3}$$

uniformly with respect to $t \in [0, T]$. Second-order equations with $p = 2$ and decay conditions as $|x| \to \infty$ have been considered, for example, in [12,16]. Cicognani and Colombini [14] treated the case $p = 3$ proving H^∞ well posedness under the conditions

$$\begin{aligned} &|\operatorname{Im} a_2| \leq C a_3(t)\langle x\rangle^{-1}, \\ &|\operatorname{Im} a_1| + |\operatorname{Re} \partial_x a_2| \leq C a_3(t)\langle x\rangle^{-1/2}. \end{aligned} \tag{1.4}$$

Recently, Ascanelli et al. [6] extended the results of [14] and [22] to the case $p \geq 4$, giving sufficient conditions for H^∞ well posedness of the Cauchy problem for the operator (1.2); results in [6] have then been generalized to pseudo-differential systems in [5] and to higher order equations in [4]; semi-linear three-evolution equations have been then studied in [7]. Recently, in [8], a necessary condition of decay at infinity for the coefficients of (1.2) with arbitrary $p \geq 2$ has been given.

In this paper, we want to consider the Cauchy problem (1.1) when $P(t, x, D_t, D_x)$ is an evolution operator of the form

$$P(t, x, D_t, D_x) = D_t + a_p(t, D_x) + \sum_{j=0}^{p-1} a_j(t, x, D_x), \tag{1.5}$$

$p \in \mathbb{N}$, $p \geq 2$, where a_j are pseudo-differential operators with symbols a_j of order j for $0 \leq j \leq p$, and for every $t \in [0, T]$, $x, \xi \in \mathbb{R}$ we have: $a_p(t, \xi) \in \mathbb{R}$, $a_j(t, x, \xi) \in \mathbb{C}$ for $0 \leq j \leq p-1$.

In [5], it has been proved the following:

THEOREM 1.1. *The Cauchy problem (1.1) for the operator (1.5) is H^∞ well posed under the assumptions:*

$$|\partial_\xi a_p(t, \xi)| \geq C_p |\xi|^{p-1} \qquad \forall t \in [0, T], \; |\xi| \gg 1, \tag{1.6}$$

for some $C_p > 0$ and

$$|\operatorname{Im} \partial_\xi^\alpha a_j(t, x, \xi)| \leq C_\alpha \langle x\rangle^{-\frac{j}{p-1}} \langle \xi\rangle_h^{j-\alpha}, \;\; 1 \leq j \leq p-1 \tag{1.7}$$

$$|\operatorname{Im} \partial_\xi^\alpha D_x a_j(t, x, \xi)| \leq C_\alpha \langle x\rangle^{-\frac{j-1}{p-1}} \langle \xi\rangle_h^{j-\alpha}, \;\; 2 \leq j \leq p-1 \tag{1.8}$$

$$|\operatorname{Im} \partial_\xi^\alpha D_x^\beta a_j(t, x, \xi)| \leq C_\alpha \langle x\rangle^{-\frac{j-[\beta/2]}{p-1}} \langle \xi\rangle_h^{j-\alpha}, \;\; 1 \leq \left[\frac{\beta}{2}\right] \leq j-1, \; 3 \leq j \leq p-1 \tag{1.9}$$

for all $(t, x, \xi) \in [0, T] \times \mathbb{R}^2$ *and for some* $C_\alpha > 0$*, where* $\langle\cdot\rangle_h = \sqrt{h^2 + |\cdot|^2}$, $h \geq 1$*. More precisely, there exists* $\sigma > 0$ *such that for all* $f \in C([0, T]; H^s)$ *and* $g \in H^s$*, there is a unique solution* $u \in C([0, T]; H^{s-\sigma})$ *of* (1.1), (1.5) *which satisfies the following energy estimate:*

$$\|u(t, \cdot)\|^2_{s-\sigma} \leq C_s \left(\|g\|^2_s + \int_0^t \|f(\tau, \cdot)\|^2_s \, d\tau \right) \qquad \forall t \in [0, T], \tag{1.10}$$

for some $C_s > 0$.

Formula (1.10) shows that the Cauchy problem (1.1), (1.5) is H^∞ well posed with loss of σ derivatives, in the sense that the solution is less regular than the Cauchy data. This phenomenon, which is usual in the theory of degenerate hyperbolic equations, appears so also in the theory of non-degenerate p-evolution equations for $p \geq 2$ and has been yet observed in [13,21,22]. Notice that assumptions (1.6)–(1.9) are consistent with the conditions in (1.3), (1.4). The loss of derivatives appearing in (1.10) for the solution of (1.1) is explicitly computed in [5], and it can be avoided by slightly strengthening the sole assumption (1.7) for $j = p - 1$. Formula (1.10) gives so an accurate information about the regularity of the solution, but, in spite of the very precise decay conditions on the coefficients, it does not say anything about the behavior of the solution as $|x| \to \infty$.

This suggests us to change the setting of the Cauchy problem (1.1) to gain the possibility of giving similar precise information on the behavior of the solution for $|x| \to \infty$; namely, one could try to obtain energy estimates in suitable weighted Sobolev spaces and well posedness in the Schwartz spaces $\mathcal{S}(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$.

Results of the above type have been proved for strictly hyperbolic equations ($p = 1$) by Cordes [17]; we also recall similar results when the coefficients are not Lipschitz continuous in t, see [9,10]. The natural framework consists in dealing with pseudo-differential operators with symbols in the classes $SG^{m_1,m_2} = SG^{m_1,m_2}(\mathbb{R}^2)$, with $m_j \in \mathbb{R}$, $j = 1, 2$, defined as the class of all functions $p(x, \xi) \in C^\infty(\mathbb{R}^2)$ satisfying the following estimates:

$$\|p\|_{\alpha,\beta} := \sup_{(x,\xi)\in\mathbb{R}^2} \langle\xi\rangle^{-m_1+\alpha} \langle x\rangle^{-m_2+\beta} |\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| < \infty \tag{1.11}$$

for every $\alpha, \beta \in \mathbb{N}$. We refer to [17,18,24,25] for a detailed calculus for this class. In the following, we shall prove energy estimates in the weighted Sobolev spaces $H_{s_1,s_2}(\mathbb{R})$, $s_j \in \mathbb{R}$, $j = 1, 2$, defined as the space of all $u \in \mathcal{S}'(\mathbb{R})$ satisfying the following condition:

$$\|u\|_{s_1,s_2} = \|\langle x\rangle^{s_2} \langle D\rangle^{s_1} u\|_{L^2} < \infty, \tag{1.12}$$

where we denote by $\langle D\rangle^{s_1}$ the Fourier multiplier with symbol $\langle\xi\rangle^{s_1}$. It is worth to recall that for $s_2 = 0$ we recapture the standard Sobolev spaces and that the following identities hold:

$$\bigcap_{s_1,s_2\in\mathbb{R}} H_{s_1,s_2}(\mathbb{R}) = \mathcal{S}(\mathbb{R}), \qquad \bigcup_{s_1,s_2\in\mathbb{R}} H_{s_1,s_2}(\mathbb{R}) = \mathcal{S}'(\mathbb{R}). \tag{1.13}$$

Moreover, we recall that $\mathcal{S}(\mathbb{R})$ is dense in $H_{s_1,s_2}(\mathbb{R})$ for any $s_1, s_2 \in \mathbb{R}$.

The main result of the paper is the following.

THEOREM 1.2. *Let $P(t, x, D_t, D_x)$ be an operator of the form (1.5) and assume that the following conditions hold:*

$$a_p \in C([0, T]; SG^{p,0}), \tag{1.14}$$

$$|\partial_\xi a_p(t, \xi)| \geq C_p |\xi|^{p-1} \qquad \forall t \in [0, T],\ |\xi| \gg 1,\ \textit{with } C_p > 0, \tag{1.15}$$

$$a_j \in C([0, T]; SG^{j,-j/(p-1)}), \quad j = 0, \dots, p-1. \tag{1.16}$$

Then, the Cauchy problem (1.1) is well posed in $\mathcal{S}(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$. More precisely, there exists $\sigma > 0$ such that for all $s_1, s_2 \in \mathbb{R}$, $f \in C([0, T]; H_{s_1,s_2}(\mathbb{R}))$ and $g \in H_{s_1,s_2}(\mathbb{R})$, there is a unique solution $u \in C([0, T]; H_{s_1,s_2-\sigma}(\mathbb{R}))$ which satisfies the following energy estimate:

$$\|u(t, \cdot)\|^2_{s_1,s_2-\sigma} \leq C \left(\|g\|^2_{s_1,s_2} + \int_0^t \|f(\tau, \cdot)\|^2_{s_1,s_2}\, d\tau \right) \qquad \forall t \in [0, T], \tag{1.17}$$

for some $C = C(s_1, s_2) > 0$.

REMARK 1.3. The energy estimate (1.17) shows that under our assumptions, we can obtain well posedness in $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ *without any loss of derivatives* for the solution u of (1.1), paying this with a modification of the rate of decay/growth at infinity of the solution with respect to the Cauchy data. The solution of (1.1) has so the same regularity as the Cauchy data; but, if we start from data with a prescribed decay at infinity, then a loss of decay appears in the solution; similarly, if the data have a fixed polynomial growth at infinity, then the solution presents a stronger growth. Moreover, the solution exists uniquely and the precise value of σ is computed, see formulas (3.4) and (3.13). We address the reader to Remark 3.8 and to the examples at the end of Sect. 3 for further comments on the phenomenon of the loss with respect to the second Sobolev index.

REMARK 1.4. The proof of Theorem 1.2 is in part inspired by [6], but it takes advantage of the fact that, in the new framework we are considering, we can admit initial data with polynomial growth with respect to the space variable. On the other hand, the assumptions on the coefficients given in our paper are stronger than the ones used in [5,6]. Hence, one can modify the approach and define the functions λ_{p-k} in the following by (2.14) for $1 \leq k \leq p-1$ (hence also for $k = 1$) as in [6] and repeat readily the argument of the proof using the estimates of Lemma 2.1 in [6] instead of Lemma 2.5 in the case $k = 1$. In this way, we are able to prove that there exists $\sigma' > 0$ such that for all $s_1, s_2 \in \mathbb{R}$, $f \in C([0, T]; H_{s_1,s_2}(\mathbb{R}))$ and $g \in H_{s_1,s_2}(\mathbb{R})$, there is a unique solution $u \in C([0, T]; H_{s_1-\sigma',s_2}(\mathbb{R}))$ which satisfies the following energy estimate:

$$\|u(t, \cdot)\|^2_{s_1-\sigma',s_2} \leq C_s \left(\|g\|^2_{s_1,s_2} + \int_0^t \|f(\tau, \cdot)\|^2_{s_1,s_2}\, d\tau \right) \qquad \forall t \in [0, T], \tag{1.18}$$

for some $C = C(s_1, s_2) > 0$. We do not prove here this alternative result, the proof being a repetition of the one of Theorem 1.1 in [6] in our functional setting.

REMARK 1.5. If the condition (1.16) with $j = p - 1$ is strengthened into

$$a_{p-1} \in C([0, T]; SG^{p-1,-(1+\epsilon)}),$$

for any $\epsilon > 0$, then the Cauchy problem (1.1) is well posed in $\mathcal{S}(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$ without loss of derivatives and without modification of the behavior at infinity.

REMARK 1.6. We observe that the assumption (1.16) in Theorem 1.2 can be slightly weakened without changing the argument of the proof. Namely, we can replace the condition (1.16) with the following

$$\mathrm{Re}\, a_j \in C([0, T]; SG^{j,0}), \; \mathrm{Im}\, a_j \in C([0, T]; SG^{j,-j/(p-1)}), \quad 0 \leq j \leq p-1. \tag{1.19}$$

The argument of the proof remains essentially the same, but it involves more complicate notation. For this reason, we prefer to present our main result using the more simple assumption (1.16). We refer to Remark 3.7 at the end of the paper for some comments on the more refined result. Finally, we observe that if $a_j(t, x, D_x)$ are differential operators, assumptions (1.14), (1.15), (1.19) are consistent with the ones given in [6,14,22] for the corresponding case $a_p(t) > C_p > 0\ \forall t \in [0, T]$.

2. Preliminaries

In this section, we collect some basic notions on SG classes of pseudo-differential operators and prove some preliminary results which will be used in the proof of Theorem 1.2 in the next section.

2.1. SG-pseudo-differential operators

We first recall that SG classes can be regarded as a particular case of general Hörmander classes, see [19, Chapter XVIII]. A specific calculus in different functional settings can be found in [17,18,24,25]. Here, we recall only some basic facts which will be used in the proof of our result. In general, fixed $d \in \mathbb{N}\setminus\{0\}$, the space $SG^{m_1,m_2}(\mathbb{R}^{2d})$ is the space of all functions $p(x, \xi) \in C^\infty(\mathbb{R}^{2d})$ satisfying the following estimates:

$$\|p\|_{\alpha,\beta} := \sup_{(x,\xi)\in\mathbb{R}^{2d}} \langle\xi\rangle^{-m_1+|\alpha|}\langle x\rangle^{-m_2+|\beta|}|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| < \infty \tag{2.1}$$

for every $\alpha, \beta \in \mathbb{N}$. We can associate with every $p \in SG^{m_1,m_2}(\mathbb{R}^{2d})$ a pseudo-differential operator defined by

$$Pu(x) = p(x, D)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} p(x, \xi)\hat{u}(\xi)\, d\xi. \tag{2.2}$$

The operator $p(x, D)$ is a linear continuous map $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ which extends to a continuous map $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. Concerning the action of these operators on weighted Sobolev spaces, we have that if $p \in SG^{m_1,m_2}(\mathbb{R}^{2d})$, then the map

$$p(x, D) : H_{s_1,s_2}(\mathbb{R}^d) \to H_{s_1-m_1,s_2-m_2}(\mathbb{R}^d)$$

is continuous for every $s_1, s_2 \in \mathbb{R}$, where the space $H_{s_1,s_2}(\mathbb{R}^d)$ is obviously defined in arbitrary dimension as the space of all $u \in \mathcal{S}'(\mathbb{R}^d)$ satisfying (1.12). We also recall the following result concerning the composition and the adjoint of SG operators.

PROPOSITION 2.1. Let $p \in SG^{m_1,m_2}(\mathbb{R}^{2d})$ and $q \in SG^{m_1',m_2'}(\mathbb{R}^{2d})$. Then, there exists a symbol $s \in SG^{m_1+m_1',m_2+m_2'}(\mathbb{R}^{2d})$ such that $p(x, D)q(x, D) = s(x, D) + R$ where R is a smoothing operator $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. Moreover, s has the following asymptotic expansion

$$s(x, \xi) \sim \sum_{\alpha} \alpha!^{-1} \partial_\xi^\alpha p(x, \xi) D_x^\alpha q(x, \xi)$$

i.e., for every $N \geq 1$, we have

$$s(x, \xi) - \sum_{|\alpha|<N} \alpha!^{-1} \partial_\xi^\alpha p(x, \xi) D_x^\alpha q(x, \xi) \in SG^{m_1+m_1'-N,m_2+m_2'-N}(\mathbb{R}^{2d}).$$

PROPOSITION 2.2. Let $p \in SG^{m_1,m_2}(\mathbb{R}^{2d})$ and let P^* be the L^2-adjoint of $p(x, D)$. Then, there exists a symbol $p^* \in SG^{m_1,m_2}(\mathbb{R}^{2d})$ such that $P^* = p^*(x, D)+ R'$, where R' is a smoothing operator $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. Moreover, p^* has the following asymptotic expansion

$$p^*(x, \xi) \sim \sum_{\alpha} \alpha!^{-1} \partial_\xi^\alpha D_x^\alpha \overline{p(x, \xi)}$$

i.e., for every $N \geq 1$, we have

$$p^*(x, \xi) - \sum_{|\alpha|<N} \alpha!^{-1} \partial_\xi^\alpha D_x^\alpha \overline{p(x, \xi)} \in SG^{m_1-N,m_2-N}(\mathbb{R}^{2d}).$$

We also recall the definition of the class $S^m(\mathbb{R}^{2d})$, $m \in \mathbb{R}$, defined as the space of all symbols $p(x, \xi) \in C^\infty(\mathbb{R}^{2d})$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|}, \qquad (x, \xi) \in \mathbb{R}^{2d}$$

for every $\alpha, \beta \in \mathbb{N}^d$. It is important for the sequel to notice that

$$SG^{m_1,m_2}(\mathbb{R}^{2d}) \subset S^{m_1}(\mathbb{R}^{2d}) \tag{2.3}$$

for any $m_1, m_2 \in \mathbb{R}$ with $m_2 \leq 0$ and that the operators with symbols in $S^0(\mathbb{R}^{2d})$ map continuously $H_{s_1,s_2}(\mathbb{R}^d)$ to itself for every $s_1, s_2 \in \mathbb{R}$.

In the proof of Theorem 1.2, we shall also use the sharp Gårding inequality applied to SG operators. This result is known as a particular case of [19, Theorem 18.6.14]. However, for our purposes, we also need a precise estimate of the order of the remainder with respect to ξ, which has been proved only for symbols in the Hörmander classes $S^m(\mathbb{R}^{2d})$, see [23, Theorem 4.2]. Nevertheless, we have to observe that the operators we shall consider have negative order with respect to x. Hence, in view of the inclusion (2.3), we can base the proof of this result on the classical sharp Gårding inequality for standard Hörmander symbols and estimate the order of the remainder term with respect to ξ by looking at its classical asymptotic expansion. Namely, we have the following result.

THEOREM 2.3. *Let* $m_1 \geq 0, m_2 \leq 0$, $a \in SG^{m_1,m_2}(\mathbb{R}^{2d})$ *with* $\operatorname{Re} a(x,\xi) \geq 0$. *Then, there exist pseudo-differential operators* $Q = q(x,D)$, $\tilde{R} = \tilde{r}(x,D)$ *and* $R_0 = r_0(x,D)$ *with symbols, respectively,* $q \in SG^{m_1,m_2}(\mathbb{R}^{2d})$, $\tilde{r} \in SG^{m_1-1,m_2}(\mathbb{R}^{2d})$ *and* $r_0 \in S^0(\mathbb{R}^{2d})$ *such that*

$$a(x,D) = q(x,D) + \tilde{r}(x,D) + r_0(x,D) \tag{2.4}$$

$$\operatorname{Re}\langle q(x,D)u,u\rangle \geq 0 \qquad \forall u \in \mathcal{S}(\mathbb{R}^d). \tag{2.5}$$

Proof. Since $m_2 \leq 0$, then $SG^{m_1,m_2}(\mathbb{R}^{2d}) \subset S^{m_1}(\mathbb{R}^{2d})$. Hence, the classical Gårding inequality gives the existence of two symbols q and r such that $a(x,D) = q(x,D) + r(x,D)$ and $q(x,D)$ satisfies (2.5). Let us now consider the asymptotic expansion of the remainder term $r(x,D)$. By Theorem 4.2 in [23], we have that

$$r(x,\xi) \sim \psi_1(\xi) D_x a(x,\xi) + \sum_{|\alpha+\beta|\geq 2} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta a(x,\xi), \tag{2.6}$$

for some real-valued functions $\psi_1, \psi_{\alpha,\beta}$ with $\psi_1 \in SG^{-1,0}(\mathbb{R}^{2d})$ and $\psi_{\alpha,\beta} \in SG^{(|\alpha|-|\beta|)/2,0}(\mathbb{R}^{2d})$. In particular, we have that

$$r(x,\xi) = \psi_1(\xi) D_x a(x,\xi) + \sum_{2\leq|\alpha+\beta|\leq 2m_1-1} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta a(x,\xi) + r_0(x,\xi),$$

for a symbol $r_0 \in S^0(\mathbb{R}^{2d})$. Moreover, it is easy to verify that $\psi(\xi) D_x a(x,\xi) \in SG^{m_1-1,m_2-1}(\mathbb{R}^{2d})$ and that

$$\sum_{2\leq|\alpha+\beta|\leq 2m_1-1} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta a(x,\xi) \in SG^{m_1-1,m_2}(\mathbb{R}^{2d}).$$

Then, we have that

$$\tilde{r}(x,\xi) := \psi_1(\xi) D_x a(x,\xi) + \sum_{2\leq|\alpha+\beta|\leq 2m_1-1} \psi_{\alpha,\beta}(\xi)\partial_\xi^\alpha D_x^\beta a(x,\xi) \in SG^{m_1-1,m_2}(\mathbb{R}^{2d}).$$

This concludes the proof. □

REMARK 2.4. In the sequel of the paper, we will often replace the weight function $\langle\xi\rangle$ with $\langle\xi\rangle_h = (h^2 + |x|^2)^{1/2}$ for some $h \geq 1$ to prove our results. It is clear that this modification does not change the definition of the class $SG^{m_1,m_2}(\mathbb{R}^{2d})$ and of the spaces $H^{s_1,s_2}(\mathbb{R}^d)$, and their properties.

2.2. Changes of variables and conjugations

The idea of the proof of Theorem 1.2 is to prove an energy estimate in $L^2(\mathbb{R})$ for the operator

$$iP = \partial_t + ia_p(t, D_x) + \sum_{j=0}^{p-1} ia_j(t, x, D_x) = \partial_t + A(t, x, D_x). \tag{2.7}$$

We have

$$\begin{aligned} \frac{d}{dt}\|u\|_0^2 &= 2\,\mathrm{Re}\langle \partial_t u, u\rangle = 2\,\mathrm{Re}\langle iPu, u\rangle - 2\,\mathrm{Re}\langle Au, u\rangle \\ &\le \|f\|_0^2 + \|u\|_0^2 - 2\,\mathrm{Re}\langle Au, u\rangle. \end{aligned} \tag{2.8}$$

Notice that $2\,\mathrm{Re}\langle Au, u\rangle = \langle (A+A^*)u, u\rangle$, with A^* the formal adjoint of A, and $A+A^*$ is an operator with symbol in $SG^{p-1,-1}$, hence with positive order with respect to ξ. This implies that the desired energy estimate is not straightforward, and in order to obtain it, we need to transform the Cauchy problem (1.1) into an equivalent one of the form

$$\begin{cases} P_\lambda u_\lambda = f_\lambda \\ u_\lambda(0, x) = g_\lambda, \end{cases} \tag{2.9}$$

where $P_\lambda = D_t - iA_\lambda$ and $\mathrm{Re}\, A_\lambda(t, x, \xi) \ge 0$; then, we apply Theorem 2.3 to obtain the estimate from below

$$\mathrm{Re}\langle A_\lambda v, v\rangle \ge -c||v||_0^2$$

for $v \in \mathcal{S}(\mathbb{R})$ and for some positive constant c. This, computing as in (2.8), will give an L^2 energy estimate for the solution u_λ of the Cauchy problem (2.9). The operator P_λ will be the result of $p-1$ conjugations of P with operators of the form $e^{\lambda_{p-k}(x,D_x)}$, $k = 1, \ldots, p-1$, namely:

$$\begin{aligned}(iP)_\lambda := (e^{\lambda_1(x,D_x)})^{-1} \cdots (e^{\lambda_{p-2}(x,D_x)})^{-1}(e^{\lambda_{p-1}(x,D_x)})^{-1}(iP)e^{\lambda_{p-1}(x,D_x)}e^{\lambda_{p-2}(x,D_x)} \\ \cdots e^{\lambda_1(x,D_x)}. \end{aligned} \tag{2.10}$$

Here and in the following, we shall denote by $e^{\pm\lambda_{p-k}(x,D_x)}$, $k = 1, \cdots, p-1$, the operators with symbols $e^{\pm\lambda_{p-k}(x,\xi)}$ and the functions λ_{p-k} will be chosen such that:

- $\lambda_{p-k}(x, \xi)$ are real valued, $1 \le k \le p-1$;
- $e^{\lambda_{p-1}(x,\xi)} \in SG^{0,M_{p-1}}$ for some $M_{p-1} > 0$ and $e^{\lambda_{p-k}(x,\xi)} \in SG^{0,0}$ for $2 \le k \le p-1$;
- the operator $e^{\lambda_{p-k}(x,D_x)}$ is invertible for every $1 \le k \le p-1$ and the principal part of $(e^{\lambda_{p-k}(x,D_x)})^{-1}$ is $e^{-\lambda_{p-k}(x,D_x)}$;
- the operator

$$\begin{aligned} A_\lambda := (e^{\lambda_1(x,D_x)})^{-1} \cdots (e^{\lambda_{p-2}(x,D_x)})^{-1}(e^{\lambda_{p-1}(x,D_x)})^{-1} \\ (iA)e^{\lambda_{p-1}(x,D_x)}e^{\lambda_{p-2}(x,D_x)} \cdots e^{\lambda_1(x,D_x)} \end{aligned}$$

  is such that $\mathrm{Re}\langle A_\lambda v, v\rangle \ge -c||v||_0^2\ \forall v(t,\cdot) \in \mathcal{S}(\mathbb{R})$.

After the transformation of the problem (1.1) into (2.9) with P_λ defined by (2.10) and f_λ and g_λ given by

$$f_\lambda = (e^{\lambda_1(x,D_x)})^{-1} \cdots (e^{\lambda_{p-1}(x,D_x)})^{-1} f, \quad g_\lambda = (e^{\lambda_1(x,D_x)})^{-1} \cdots (e^{\lambda_{p-1}(x,D_x)})^{-1} g, \tag{2.11}$$

we will obtain an energy estimate in $L^2(\mathbb{R})$ for the new variable

$$u_\lambda(t,x) = (e^{\lambda_1(x,D_x)})^{-1} \cdots (e^{\lambda_{p-2}(x,D_x)})^{-1} (e^{\lambda_{p-1}(x,D_x)})^{-1} u(t,x) \tag{2.12}$$

which will yield to an estimate of the form (1.17) for the solution u of (1.1).

Let us now define the functions λ_j. We set

$$\lambda_{p-1}(x,\xi) := M_{p-1}\omega\left(\frac{\xi}{h}\right)\int_0^x \frac{1}{\langle y\rangle}dy, \tag{2.13}$$

and for $2 \le k \le p-1$

$$\lambda_{p-k}(x,\xi) := M_{p-k}\omega\left(\frac{\xi}{h}\right)\langle\xi\rangle_h^{-k+1}\int_0^x \langle y\rangle^{-\frac{p-k}{p-1}}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^{p-1}}\right)dy\,, \tag{2.14}$$

where $M_{p-1}, M_{p-2}, \ldots, M_1$ are positive constants to be chosen later on, $\omega \in C^\infty(\mathbb{R})$ is such that

$$\omega(\xi) = \begin{cases} 0 & |\xi| \le 1 \\ \operatorname{sgn}(\partial_\xi a_p(t,\xi)) & |\xi| \ge R \end{cases} \tag{2.15}$$

for some $R > 1$, and $\psi \in C_0^\infty(\mathbb{R})$ is such that $0 \le \psi(y) \le 1\ \forall y \in \mathbb{R}$, $\psi(y) = 1$ for $|y| \le \frac{1}{2}$, $\psi(y) = 0$ for $|y| \ge 1$. Notice that assumption (1.15) ensures the existence of $R > 0$ such that for every fixed ξ with $|\xi| > R$ the sign of the function $\partial_\xi a_p(t,\xi)$ remains constant for every $t \in [0,T]$, then ω is well defined and does not depend on t.

Definition (2.14) is in part inspired by [5,6]; more precisely, the symbols λ_{p-k} in (2.14) are exactly the same as in [5], while the symbol λ_{p-1} in (2.13) is new. It can be considered only in the framework of the SG calculus, where symbols with polynomial growth in x can be handled. The setting we are using allows to construct a transformation with a "stronger" λ_{p-1} with respect to [5] still remaining in (weighted) Sobolev spaces.

LEMMA 2.5. The function λ_{p-1} defined by (2.13) satisfies the following estimates:

$$|\lambda_{p-1}(x,\xi)| \le M_{p-1}(1+\ln\langle x\rangle) \tag{2.16}$$

$$|\partial_\xi^\alpha\partial_x^\beta\lambda_{p-1}(x,\xi)| \le M_{p-1}C_{\alpha,\beta}\langle x\rangle^{-\beta}\langle\xi\rangle_h^{-\alpha} \quad \alpha \ge 0, \beta \ge 1, \tag{2.17}$$

$$|\partial_\xi^\alpha\lambda_{p-1}(x,\xi)| \le M_{p-1}C'_{\alpha,R}\langle\xi\rangle_h^{-\alpha}\left(1+\ln\langle x\rangle\chi_{E_{h,R}}(\xi)\right), \quad \alpha \ge 1, \tag{2.18}$$

with positive constants $C, C_{\alpha,\beta}, C'_{\alpha,R}$, where $\chi_{E_{h,R}}$ is the characteristic function of the set $E_{h,R} = \{\xi \in \mathbb{R} \mid h \le |\xi| \le hR\}$.

Proof. A simple explicit computation of the integral in (2.13) gives

$$|\lambda_{p-1}(x,\xi)| \leq M_{p-1}\log(2\langle x\rangle) < M_{p-1}(1+\ln\langle x\rangle). \tag{2.19}$$

For $\beta \geq 1$, we have

$$|\partial_x^\beta \lambda_{p-1}(x,\xi)| = M_{p-1}\left|\omega\left(\frac{\xi}{h}\right)\partial_x^{\beta-1}\langle x\rangle^{-1}\right| \leq M_{p-1}C_\beta\langle x\rangle^{-\beta}, \tag{2.20}$$

and for $\alpha, \beta \geq 1$:

$$\begin{aligned}|\partial_\xi^\alpha\partial_x^\beta \lambda_{p-1}(x,\xi)| = M_{p-1}\left|\omega^{(\alpha)}\left(\frac{\xi}{h}\right)h^{-\alpha}\partial_x^{\beta-1}\langle x\rangle^{-1}\right| &\leq M_{p-1}C_{\alpha,\beta}h^{-\alpha}\langle x\rangle^{-\beta}\\ &\leq M_{p-1}C_{\alpha,\beta}\langle\xi\rangle_h^{-\alpha}\langle x\rangle^{-\beta}.\end{aligned} \tag{2.21}$$

Finally, for $\alpha \geq 1$:

$$\begin{aligned}|\partial_\xi^\alpha \lambda_{p-1}(x,\xi)| &= M_{p-1}\left|\omega^{(\alpha)}\left(\frac{\xi}{h}\right)h^{-\alpha}\int_0^x \frac{1}{\langle y\rangle}dy\right|\\ &\leq M_{p-1}C_\alpha h^{-\alpha}\ln\langle x\rangle\chi_{E_{h,R}}(\xi)\\ &\leq M_{p-1}C_{\alpha,R}\langle\xi\rangle_h^{-\alpha}\ln\langle x\rangle\chi_{E_{h,R}}(\xi),\end{aligned} \tag{2.22}$$

since $h^{-1} \leq \langle R\rangle\langle\xi\rangle_h^{-1}$ on $E_{h,R}$. □

LEMMA 2.6. Let $\lambda_{p-k}, k = 2,\ldots,p-1$ be defined by (2.14). Then, for every $\alpha, \beta \in \mathbb{N}$, there exists a constant $C_{k,\alpha,\beta} > 0$ such that

$$\begin{aligned}|\partial_\xi^\alpha\partial_x^\beta \lambda_{p-k}(x,\xi)| &\leq C_{k,\alpha,\beta}M_{p-k}\langle x\rangle^{\frac{k-1}{p-1}-\beta}\langle\xi\rangle_h^{-\alpha-k+1}\chi_\xi(x)\\ &\leq C_{k,\alpha,\beta}M_{p-k}\langle x\rangle^{-\beta}\langle\xi\rangle_h^{-\alpha},\end{aligned} \tag{2.23}$$

where $\chi_\xi(x)$ denotes the characteristic function of the set $\left\{x \in \mathbb{R} \mid \langle x\rangle \leq \langle\xi\rangle_h^{p-1}\right\}$. In particular, we have that $\lambda_{p-k} \in SG^{0,0}$ for $2 \leq k \leq p-1$.

Proof. See [6, Lemma 2.1]. □

From the estimates proved in Lemma 2.5 and 2.6, we obtain, by simply applying the Faà di Bruno formula, the following estimates for the symbols $e^{\pm\lambda_{p-k}(x,\xi)}$. We leave the details of the proof to the reader.

LEMMA 2.7. Let $\lambda_{p-k}, k = 1,\ldots,p-1$ be defined by (2.13) and (2.14). Then,

$$|e^{\pm\lambda_{p-1}(x,\xi)}| \leq K\langle x\rangle^{M_{p-1}}, \tag{2.24}$$

$$|\partial_\xi^\alpha e^{\pm\lambda_{p-1}(x,\xi)}| \leq C_\alpha[1+\ln\langle x\rangle\chi_{E_{h,R}}(\xi)]^\alpha\langle\xi\rangle_h^{-\alpha}e^{\pm\lambda_{p-1}(x,\xi)},\ \alpha \geq 1, \tag{2.25}$$

$$|\partial_x^\beta e^{\pm\lambda_{p-1}(x,\xi)}| \leq C_\beta\langle x\rangle^{-\beta}e^{\pm\lambda_{p-1}(x,\xi)},\ \beta \geq 1, \tag{2.26}$$

$$|\partial_\xi^\alpha \partial_x^\beta e^{\pm\lambda_{p-1}(x,\xi)}| \leq C_{\alpha,\beta}\left[1+\ln\langle x\rangle \chi_{E_{h,R}}(\xi)\right]^\alpha \langle x\rangle^{-\beta}\langle\xi\rangle_h^{-\alpha} e^{\pm\lambda_{p-1}(x,\xi)}, \qquad \alpha,\beta\geq 1, \quad (2.27)$$

$$|\partial_\xi^\alpha \partial_x^\beta e^{\pm\lambda_{p-k}(x,\xi)}| \leq C_{\alpha,\beta,k}\langle x\rangle^{-\beta}\langle\xi\rangle_h^{-\alpha} e^{\pm\lambda_{p-k}(x,\xi)}, \qquad 2\leq k\leq p-1, \alpha,\beta\in\mathbb{N}, \quad (2.28)$$

for some positive constants $K, C_\alpha, C_\beta, C_{\alpha,\beta}, C_{\alpha,\beta,k}$.

The next two results state the invertibility of the operators $e^{\lambda_{p-k}(x,D_x)}$ for $k = 1,\ldots,p-1$.

LEMMA 2.8. Let $\lambda_{p-1}(x,\xi)$ be defined by (2.13). Then, there exists $h_1\geq 1$ such that for $h\geq h_1$ the operator $e^{\lambda_{p-1}(x,D)}$ is invertible and

$$(e^{\lambda_{p-1}(x,D_x)})^{-1} = e^{-\lambda_{p-1}(x,D_x)}(I+R_{p-1}) \quad (2.29)$$

where I is the identity operator and R_{p-1} has principal symbol given by

$$r_{p-1,-1}(x,\xi)\in SG^{-1,0}.$$

Proof. By Proposition 2.1, it follows that

$$e^{\lambda_{p-1}(x,D_x)}e^{-\lambda_{p-1}(x,D_x)} = I - r_{p-1,-1}(x,D) + r_{p-1,-2}(x,D),$$

where

$$r_{p-1,-1}(x,\xi) = \partial_\xi\lambda_{p-1}(x,\xi)D_x\lambda_{p-1}(x,\xi)$$

and

$$r_{p-1,-2}(x,\xi)\sim\sum_{m\geq 2}\frac{1}{m!}\partial_\xi^m e^{\lambda_{p-1}(x,\xi)}D_x^m e^{-\lambda_{p-1}(x,\xi)}.$$

By (2.17) and (2.18)

$$r_{p-1,-1}(x,\xi)\in SG^{-1,-1+\varepsilon} \quad \textit{for every} \quad \varepsilon>0$$

and

$$r_{p-1,-2}(x,\xi)\in SG^{-2,-2+\varepsilon} \quad \textit{for every} \quad \varepsilon>0.$$

More precisely,

$$|\partial_\xi^\alpha D_x^\beta r_{p-1,-1}(x,\xi)| \leq \sum_{\alpha_1+\alpha_2=\alpha}\Bigg(c_{\alpha_1,\alpha_2,\beta}|\partial_\xi^{\alpha_1+1}\lambda_{p-1}|\cdot|\partial_\xi^{\alpha_2}D_x^{\beta+1}\lambda_{p-1}| + \sum_{\beta_1+\beta_2=\beta,\ \beta_1\neq 0} c_{\alpha_1,\alpha_2,\beta_1,\beta_2}|\partial_\xi^{\alpha_1+1}D_x^{\beta_1}\lambda_{p-1}|\cdot|\partial_\xi^{\alpha_2}D_x^{\beta_2+1}\lambda_{p-1}|\Bigg)$$

$$\leq C_{\alpha,\beta,R}\langle\xi\rangle_h^{-\alpha}\langle x\rangle^{-\beta}\langle\xi\rangle_h^{-1}(\ln\langle x\rangle+1)\langle x\rangle^{-1}$$
$$\leq 2C_{\alpha,\beta,R}\cdot h^{-1}\langle\xi\rangle_h^{-\alpha}\langle x\rangle^{-\beta},\quad \forall\alpha,\beta\geq 0. \tag{2.30}$$

Setting $r_{p-1}(x,\xi):=r_{p-1,-1}(x,\xi)-r_{p-1,-2}(x,\xi)$, we have also

$$|\partial_\xi^\alpha D_x^\beta r_{p-1}(x,\xi)|\leq C'_{\alpha,\beta,R}\cdot h^{-1}\langle\xi\rangle_h^{-\alpha}\langle x\rangle^{-\beta},\qquad \forall\alpha,\beta\geq 0 \tag{2.31}$$

and for some $C'_{\alpha,\beta,R}>0$; this means that for h large enough, operator $I-R_{p-1}$ is invertible by Neumann series and $\sum_{n=0}^{+\infty}R_{p-1}^n$ is the inverse operator. Similar considerations hold for $e^{-\lambda_{p-1}(x,D_x)}e^{\lambda_{p-1}(x,D_x)}$; thus, $e^{-\lambda_{p-1}}\sum_{n=0}^{+\infty}R_{p-1}^n$ is a left and right inverse for $e^{\lambda_{p-1}(x,D_x)}$. The lemma is then proved. □

LEMMA 2.9. Let $\lambda_{p-k}(x,\xi)$, $k=2,\ldots,p-1$ be defined by (2.14). Then, for every k, there exists $h_k\geq 1$ such that for $h\geq h_k$ the operator $e^{\lambda_{p-k}(x,D)}$ is invertible and

$$(e^{\lambda_{p-k}(x,D_x)})^{-1}=e^{-\lambda_{p-k}(x,D_x)}(I+R_{p-k}) \tag{2.32}$$

where I is the identity operator and R_{p-k} has principal symbol

$$r_{p-k,-k}(x,\xi)=\partial_\xi\lambda_{p-k}(x,\xi)D_x\lambda_{p-k}(x,\xi)\in SG^{-k,-\frac{p-k}{p-1}}.$$

Proof. The construction of the inverse is completely analogous to the one of Lemma 2.8. Moreover, by (2.23), we have that

$$|r_{p-k,-k}(x,\xi)|=|\partial_\xi\lambda_{p-k}(x,\xi)D_x\lambda_{p-k}(x,\xi)|\leq C_kM_{p-k}^2\langle x\rangle^{2\frac{k-1}{p-1}-1}\langle\xi\rangle_h^{-2k+1}\chi_\xi(x)$$
$$\leq C_kM_{p-k}^2\langle x\rangle^{-\frac{p-k}{p-1}}\langle\xi\rangle_h^{-k}$$
$$\leq C_kM_{p-k}^2h^{-1} \tag{2.33}$$

since on the support of $\chi_\xi(x)$, we have $\langle x\rangle\leq\langle\xi\rangle_h^{p-1}$. The derivatives of $r_{p-k,-k}$ can be estimated similarly. Thus, for h large enough, we obtain (2.32), and from (2.33), we have $r_{p-k,-k}\in SG^{-k,-\frac{p-k}{p-1}}$. □

3. The proof of Theorem 1.2

The proof of Theorem 1.2 needs some preparation. As announced in Section 2, we shall reduce the Cauchy problem (1.1) to the problem (2.9), where the operator P_λ is defined by (2.10) and the functions f_λ, g_λ and u_λ are given, respectively, by (2.11) and (2.12). We first prove the following result.

PROPOSITION 3.1. Let P_λ be defined by (2.10). Then, we have:

$$(iP)_\lambda=\partial_t+ia_p(t,D_x)+\sum_{\ell=1}^{p-1}Q_{p-\ell}(t,x,D_x)+r_0(t,x,D_x),$$

for some operators $Q_{p-\ell}$ with symbols $Q_{p-\ell}(t,x,\xi) \in SG^{p-\ell,0}$, satisfying

$$\mathrm{Re}\langle Q_{p-\ell}v, v\rangle \geq 0 \qquad v \in \mathcal{S}(\mathbb{R}),\ 1 \leq \ell \leq p-1$$

and for some operator r_0 with symbol in S^0.

Proposition 3.1 will be proved in $p-1$ steps each of them corresponding to a conjugation with an operator of the form $e^{\lambda_{p-k}}, k = 1, \ldots, p-1$. We know by (2.23) and (2.24) that the operators $e^{\pm\lambda_{p-1}(x,D_x)}$ have order $(0, M_{p-1})$, while the other operators $e^{\pm\lambda_{p-k}(x,D_x)}$ have order $(0,0)$ for $k = 2, \ldots, p-1$. In the proof of the energy estimate (1.17), the first conjugation will play an essential role since it determines the loss of H_{s_1,s_2} regularity; the others all work similar to each other. This is the reason why we shall organize the proof of Proposition 3.1 as follows. We present in detail the first two transformations, and then, we argue by induction. Each of these steps corresponds to a different lemma. Before this, we give a preliminary result which states for an operator with symbol $a(x,\xi) \in SG^{m_1,0}$ the form of the composed operator $e^{-\lambda_i(x,D_x)}a(x,D_x)e^{\lambda_i(x,D_x)}$, $i = 1, \ldots, p-1$, with λ_i defined as in the previous section.

LEMMA 3.2. Let $a \in SG^{m_1,0}$ and let $\lambda_i, i = 1, \ldots, p-1$ be defined by (2.13), (2.14). Then, the symbol of $e^{-\lambda_i(x,D_x)}a(x,D_x)e^{\lambda_i(x,D_x)}$, $i = 1, \ldots, p-1$ is given by:

$$\left(e^{-\lambda_i}ae^{\lambda_i}\right)(x,\xi) = a + \sum_{\alpha=1}^{m_1-1}\frac{1}{\alpha!}\partial_\xi^\alpha a \cdot e^{-\lambda_i}\cdot D_x^\alpha e^{\lambda_i} + \sum_{\gamma=1}^{m_1-1}\frac{1}{\gamma!}\partial_\xi^\gamma e^{-\lambda_i}D_x^\gamma(ae^{\lambda_i})$$
$$+\sum_{\gamma=1}^{m_1-2}\sum_{\alpha=1}^{m_1-1}\frac{1}{\alpha!\gamma!}\partial_\xi^\gamma e^{-\lambda_i}D_x^\gamma(\partial_\xi^\alpha a\cdot D_x^\alpha e^{\lambda_i}) + r_0 \tag{3.1}$$

with $r_0 \in SG^{0,0}$. Moreover, the symbol

$$r(x,\xi) = \sum_{\gamma=1}^{m_1-1}\frac{1}{\gamma!}\partial_\xi^\gamma e^{-\lambda_i}D_x^\gamma(ae^{\lambda_i})$$
$$+\sum_{\gamma=1}^{m_1-2}\sum_{\alpha=1}^{m_1-1}\frac{1}{\alpha!\gamma!}\partial_\xi^\gamma e^{-\lambda_i}D_x^\gamma(\partial_\xi^\alpha a\cdot D_x^\alpha e^{\lambda_i}) \in SG^{m_1-1,0}.$$

Proof. The proof easily follows by Proposition 2.1. □

REMARK 3.3. If we assume that $a \in S^{m_1}$ instead of $a \in SG^{m_1,0}$, then Lemma 3.2 still holds with $r \in S^{m_1-1}$.

LEMMA 3.4. Let h be as in Lemma 2.8 and consider, for $h \geq h_1$, the operator $(iP)_1 = (e^{\lambda_{p-1}(x,D_x)})^{-1}(iP)e^{\lambda_{p-1}(x,D_x)}$. There exist operators $Q_{p-1}(t,x,D_x)$, $a_{j,1}(t,x,D_x)$ and $r_1(t,x,D_x)$ with symbols $Q_{p-1}(t,x,\xi) \in SG^{p-1,-1}$, $a_{j,1}(t,x,\xi) \in SG^{j,-j/(p-1)}$, $1 \leq j \leq p-2$ and $r_1(t,x,\xi) \in S^0$ such that

$$(iP)_1 = \partial_t + ia_p(t, D_x) + Q_{p-1}(t, x, D_x) + \sum_{j=1}^{p-2} ia_{j,1}(t, x, D_x) + r_1(t, x, D_x),$$

and

$$\mathrm{Re}\langle Q_{p-1}(t, x, D_x)v, v\rangle \geq 0,\ \forall v \in \mathcal{S}(\mathbb{R}).$$

Proof. We first notice that by Lemma 2.8, we have

$$\begin{aligned}
(iP)_1(t, x, D_x) &= \partial_t + \sum_{j=0}^{p} (e^{\lambda_{p-1}(x,D_x)})^{-1} ia_j e^{\lambda_{p-1}(x,D_x)} \\
&= \partial_t + \sum_{j=0}^{p} e^{-\lambda_{p-1}(x,D_x)} ia_j e^{\lambda_{p-1}(x,D_x)} \\
&\quad + \sum_{j=0}^{p} e^{-\lambda_{p-1}(x,D_x)} iR_{p-1}a_j e^{\lambda_{p-1}(x,D_x)} \\
&= \partial_t + e^{-\lambda_{p-1}(x,D_x)} ia_p e^{\lambda_{p-1}(x,D_x)} \\
&\quad + \sum_{j=1}^{p-1} e^{-\lambda_{p-1}(x,D_x)} i(a_j + R_{p-1}a_{j+1}) e^{\lambda_{p-1}(x,D_x)} + s_0 \\
&= \partial_t + e^{-\lambda_{p-1}(x,D_x)} ia_p e^{\lambda_{p-1}(x,D_x)} \\
&\quad + \sum_{j=1}^{p-1} e^{-\lambda_{p-1}(x,D_x)} i\tilde{a}_j e^{\lambda_{p-1}(x,D_x)} + s_0
\end{aligned} \tag{3.2}$$

for some $s_0(t, x, D_x)$ of order (0, 0), and with new operators $\tilde{a}_j = a_j + R_{p-1}a_{j+1}$ having symbol $\tilde{a}_j(t, x, \xi) \in SG^{j,-j/(p-1)}$. We now apply formula (3.1), observing that in the case $i = p - 1$, the term r vanishes for $|\xi| \geq hR$ since it is a sum of products with at least one ξ-derivative of λ_{p-1} appearing in each factor, see (2.18). Then, in particular, the term r has order (0, 0) since it is compactly supported in ξ and with order 0 in x. Hence, we obtain for the related operators:

$$\begin{aligned}
(iP)_1 &= \partial_t + ia_p + \sum_{\alpha=1}^{p-1} \frac{1}{\alpha!} \partial_\xi^\alpha ia_p \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} \\
&\quad + \sum_{j=1}^{p-1} \left(i\tilde{a}_j + \sum_{\alpha=1}^{j-1} \frac{1}{\alpha!} \partial_\xi^\alpha i\tilde{a}_j \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} \right) + s_0 \\
&= \partial_t + ia_p + \partial_\xi a_p \partial_x \lambda_{p-1} + i\tilde{a}_{p-1} + \sum_{\alpha=2}^{p-1} \frac{1}{\alpha!} \partial_\xi^\alpha ia_p \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} \\
&\quad + \sum_{j=1}^{p-2} i\tilde{a}_j + \sum_{j=1}^{p-1} \sum_{\alpha=1}^{j-1} \frac{1}{\alpha!} \partial_\xi^\alpha i\tilde{a}_j \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} + \tilde{s}_0
\end{aligned} \tag{3.3}$$

for a term $\tilde{s}_0$ with symbol in $SG^{0,0}$. Now we consider the real part of the terms of order $p-1$ with respect to ξ in (3.3) and use (1.15), (1.16), (2.15), (2.20) for $|\xi| \geq hR$ to get:

$$\begin{aligned} \operatorname{Re}\left(i\tilde{a}_{p-1} + \partial_\xi a_p \partial_x \lambda_{p-1}\right) &= -\operatorname{Im}\tilde{a}_{p-1} + \partial_\xi a_p \partial_x \lambda_{p-1} \\ &= -\operatorname{Im} a_{p-1} + M_{p-1}|\partial_\xi a_p(t,\xi)|\langle x\rangle^{-1} \\ &\geq \left(2^{-(p-1)/2} C_p M_{p-1} - C\right)\langle \xi\rangle_h^{p-1}\langle x\rangle^{-1}, \end{aligned}$$

where we used that $\tilde{a}_j = a_j$ for $|\xi| > hR$, and we used also the inequality: $\langle a\rangle_h \leq \sqrt{2}|a|$ for $a \in \mathbb{R}$, $|a| \geq h$. We thus obtain

$$\operatorname{Re}\left(i\tilde{a}_{p-1} + \partial_\xi a_p \partial_x \lambda_1\right) \geq 0$$

if we choose the constant M_{p-1} large enough:

$$M_{p-1} \geq 2^{(p-1)/2} C/C_p. \tag{3.4}$$

We can then apply Theorem 2.3 to the operator $i\tilde{a}_{p-1} + \partial_\xi a_p \partial_x \lambda_{p-1}$ and obtain that there exist pseudo-differential operators $Q_{p-1}(t,x,D_x)$, $\tilde{R}_{p-2}(t,x,D_x)$ and R_0 with symbols $Q_{p-1}(t,x,\xi) \in SG^{p-1,-1}$, $\tilde{R}_{p-2}(t,x,\xi) \in SG^{p-2,-1}$, $R_0(t,x,\xi) \in S^0$ such that

$$\operatorname{Re}\langle Q_{p-1}(t,x,D_x)v, v\rangle \geq 0, \ \forall v \in \mathcal{S}(\mathbb{R})$$

and

$$i\tilde{a}_{p-1} + \partial_\xi a_p \partial_x \lambda_{p-1} = Q_{p-1} + \tilde{R}_{p-2} + R_0.$$

Finally, we estimate the last three terms in the right-hand side of (3.3). We observe that for $2 \leq \alpha \leq p-1$, we have

$$|\partial_\xi^\alpha a_p \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}}| \leq C_p' \langle\xi\rangle_h^{p-\alpha}\langle x\rangle^{-1-\alpha} \leq C_p'\langle\xi\rangle_h^{p-2}\langle x\rangle^{-\frac{p-2}{p-1}}, \tag{3.5}$$

since $\alpha + 1 \geq 3 > (p-2)/(p-1)$. Similarly, we can estimate the derivatives of the above symbol observing that the worst case occurs when ξ derivatives of order γ fall on the term $e^{-\lambda_{p-1}}$. By (2.27), this produces a term of type $(\ln\langle x\rangle)^\gamma$ in the estimates but since

$$(\ln\langle x\rangle)^\gamma\langle x\rangle^{-1-\alpha} \leq (\ln\langle x\rangle)^\gamma\langle x\rangle^{-3} \leq C\langle x\rangle^{-\frac{p-2}{p-1}}$$

for every $\gamma > 0$, we conclude that $\partial_\xi^\alpha a_p \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} \in SG^{p-2,-(p-2)/(p-1)}$. By similar arguments, we obtain for $1 \leq \alpha \leq j-1$ that $\partial_\xi^\alpha \tilde{a}_j \cdot e^{-\lambda_{p-1}} \cdot D_x^\alpha e^{\lambda_{p-1}} \in SG^{j-1,-(j-1)/(p-1)}$. Hence, we can gather the last three terms in the right-hand side of (3.3) and the remainder term $\tilde{R}_{p-2}$ obtained from the application of Theorem 2.3 and conclude that

$$(iP)_1 = \partial_t + ia_p(t,D_x) + Q_{p-1}(t,x,D_x) + \sum_{j=1}^{p-2} ia_{j,1}(t,x,D_x) + r_1(t,x,D)$$

for some $a_{j,1} \in SG^{j,-j/(p-1)}$ and $r_1 \in S^0$. Lemma 3.4 is proved. □

To perform the second transformation, we need to compute the operator

$$(iP)_2 = (e^{\lambda_{p-2}(x,D_x)})^{-1}(iP)_1 e^{\lambda_{p-2}(x,D_x)}.$$

By Lemma 2.9, for $h \geq h_2$, there exists an operator R_{p-2} with principal symbol

$$r_{p-2,-2}(x,\xi) = \partial_\xi \lambda_{p-2}(x,\xi) D_x \lambda_{p-2}(x,\xi) \in SG^{-2,-(p-2)/(p-1)}$$

such that:

$$(iP)_2 = e^{-\lambda_{p-2}(x,D_x)}(I + R_{p-2})(iP)_1 e^{\lambda_{p-2}(x,D_x)}.$$

We have the following result.

LEMMA 3.5. Let h_1, h_2 be as in Lemmas 2.8 and 2.9 and let $h \geq \max\{h_1, h_2\}$. Then, there exist pseudo-differential operators $Q_{p-2}(t,x,D_x)$, $a_{j,2}(t,x,D_x)$ and $r_2(t,x,D_x)$ with symbols $Q_{p-2}(t,x,\xi) \in SG^{p-2,0}$, $a_{j,2}(t,x,\xi) \in SG^{j,-j/(p-1)}$ for $1 \leq j \leq p-3$, $r_2(t,x,\xi) \in S^0$ such that:

$$\begin{aligned}(iP)_2 = \partial_t + ia_p(t,D_x) + Q_{p-1}(t,x,D_x) + Q_{p-2}(t,x,D_x)\\ + \sum_{j=1}^{p-3} ia_{j,2}(t,x,D_x) + r_2(t,x,D_x),\end{aligned}$$

and

$$\mathrm{Re}\langle Q_{p-2}(t,x,D_x)v, v\rangle \geq 0,\ \forall v \in \mathcal{S}(\mathbb{R}).$$

Proof. By Lemma 3.4, we have (omitting the notation (t,x,D_x)):

$$\begin{aligned}(iP)_2 &= \partial_t + e^{-\lambda_{p-2}}\left(ia_p + Q_{p-1} + \sum_{j=1}^{p-2} ia_{j,1} + r_1\right)e^{\lambda_{p-2}}\\ &\quad + e^{-\lambda_{p-2}}\left(iR_{p-2}a_p + R_{p-2}Q_{p-1} + \sum_{j=1}^{p-2} iR_{p-2}a_{j,1} + R_{p-2}r_1\right)e^{\lambda_{p-2}},\end{aligned} \tag{3.6}$$

where $a_{j,1}(t,x,D)$ have symbols in $SG^{j,-j/(p-1)}$, and r_1 has a symbol in S^0. Now we observe that $iR_{p-2}a_p \in SG^{p-2,-(p-2)/(p-1)}$, $R_{p-2}Q_{p-1} \in SG^{p-3,-(p-3)/(p-1)}$ and $R_{p-2}ia_{j,1} \in SG^{j-2,-(j-2)/(p-1)}$ for every $j = 1,\ldots,p-2$. Then, we can write (3.6) as follows:

$$(iP)_2 = \partial_t + e^{-\lambda_{p-2}(x,D_x)}\left(ia_p + Q_{p-1} + \sum_{j=1}^{p-2} i\tilde{a}_{j,1}\right)e^{\lambda_{p-2}(x,D_x)} + s_0$$

for some s_0 with symbol in S^0 and with new symbols $\tilde{a}_{j,1} \in SG^{j,-j/(p-1)}$. Here, we have considered the composition $e^{\lambda_{p-2}}(R_{p-2}r_1)e^{\lambda_{p-2}}$ using Remark 3.3. It is important to underline that the operator $i\tilde{a}_{p-2,1}$ has the symbol:

$$i\tilde{a}_{p-2,1}(t,x,\xi) = ia_{p-2,1}(t,x,\xi) + ia_p(t,\xi)\partial_\xi\lambda_{p-2}(t,x,\xi)D_x\lambda_{p-2}(t,x,\xi), \tag{3.7}$$

where the first term in the right-hand side depends only on the constant M_{p-1} and the second only on M_{p-2} which will be chosen later on in this second transformation. But now, by formula (3.1), we get:

$$\begin{aligned}
(iP)_2 = {} & \partial_t + ia_p + Q_{p-1} + i\tilde{a}_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2} + \sum_{j=1}^{p-3} i\tilde{a}_{j,1} \\
& + \sum_{\alpha=2}^{p-1} \frac{1}{\alpha!}\partial_\xi^\alpha ia_p \cdot e^{-\lambda_{p-2}} \cdot D_x^\alpha e^{\lambda_{p-2}} - ia_p\partial_\xi\lambda_{p-2}D_x\lambda_{p-2} \\
& + \sum_{\gamma=2}^{p-1} \frac{1}{\gamma!}\partial_\xi^\gamma e^{-\lambda_{p-2}} \cdot \left(ia_p D_x^\gamma e^{\lambda_{p-2}}\right) \\
& + \sum_{\gamma=1}^{p-2}\sum_{\alpha=1}^{p-1} \frac{1}{\alpha!\gamma!}\partial_\xi^\gamma e^{-\lambda_{p-2}} D_x^\gamma \left(\partial_\xi^\alpha a_p D_x^\alpha e^{\lambda_{p-2}}\right) \\
& + \sum_{\alpha=1}^{p-2} \frac{1}{\alpha!}\partial_\xi^\alpha Q_{p-1} \cdot e^{-\lambda_{p-2}} \cdot D_x^\alpha e^{\lambda_{p-2}} \\
& + \sum_{j=1}^{p-2}\sum_{\alpha=1}^{j-1} \frac{1}{\alpha!}\partial_\xi^\alpha i\tilde{a}_{j,1} \cdot e^{-\lambda_{p-2}} \cdot D_x^\alpha e^{\lambda_{p-2}} \\
& + \sum_{\gamma=1}^{p-2} \frac{1}{\gamma!}\partial_\xi^\gamma e^{-\lambda_{p-2}} D_x^\gamma \left(Q_{p-1}e^{\lambda_{p-2}}\right) \\
& + \sum_{\alpha=1}^{p-2}\sum_{\gamma=1}^{p-3} \frac{1}{\alpha!\gamma!}\partial_\xi^\gamma e^{-\lambda_{p-2}} D_x^\gamma (\partial_\xi^\alpha Q_{p-1} D_x^\alpha e^{\lambda_{p-2}}) \\
& + \sum_{j=1}^{p-2}\sum_{\gamma=1}^{j-1} \frac{1}{\gamma!}\partial_\xi^\gamma e^{-\lambda_{p-2}} D_x^\gamma (i\tilde{a}_{j,1}e^{\lambda_{p-2}}) \\
& + \sum_{j=1}^{p-2}\sum_{\alpha=1}^{j-1}\sum_{\gamma=1}^{j-2} \frac{1}{\alpha!\gamma!}\partial_\xi^\gamma e^{-\lambda_{p-2}} D_x^\gamma (\partial_\xi^\alpha (i\tilde{a}_{j,1}) D_x^\alpha e^{\lambda_{p-2}}) \\
& + s_0.
\end{aligned}$$

By estimating as before the orders of the terms appearing in the above formula, using (2.23) and by (3.7), we obtain that

$$(iP)_2 = \partial_t + ia_p + Q_{p-1} + \partial_\xi a_p \partial_x \lambda_{p-2} + ia_{p-2,1} + \sum_{j=1}^{p-3} i\tilde{\tilde{a}}_{j,1} + \tilde{s}_0, \quad (3.8)$$

for some operators $\tilde{\tilde{a}}_{j,1}(t,x,D)$ with symbols $\tilde{\tilde{a}}_{j,1}(t,x,\xi) \in SG^{j,-j/(p-1)}$, and with a term $\tilde{s}_0$ with symbol in S^0. Now we want to apply Theorem 2.3 to the operator $ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2}$, namely to the term of order $p-2$ with respect to ξ. We observe that for $|\xi| \geq hR$, we have:

$$\begin{aligned}
&\mathrm{Re}(ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2}) = -\mathrm{Im} a_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2} \\
&\quad = -\mathrm{Im} a_{p-2,1} + M_{p-2} \partial_\xi a_p \omega \left(\frac{\xi}{h}\right) \langle x\rangle^{-\frac{p-2}{p-1}} \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right) \langle\xi\rangle_h^{-1} \\
&\quad = -\mathrm{Im} a_{p-2,1} + M_{p-2} |\partial_\xi a_p(t,\xi)| \langle\xi\rangle_h^{-1} \langle x\rangle^{-\frac{p-2}{p-1}} \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right) \\
&\quad \geq -C\langle\xi\rangle_h^{p-2} \langle x\rangle^{-\frac{p-2}{p-1}} + M_{p-2} C_p |\xi|^{p-1} \langle\xi\rangle_h^{-1} \langle x\rangle^{-\frac{p-2}{p-1}} \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right) \\
&\quad \geq -C\langle\xi\rangle_h^{p-2} \langle x\rangle^{-\frac{p-2}{p-1}} + M_{p-2} C_p 2^{-(p-1)/2} \langle\xi\rangle_h^{p-2} \langle x\rangle^{-\frac{p-2}{p-1}} \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right) \\
&\quad = \left(-C + M_{p-2} C_p 2^{-(p-1)/2}\right) \langle\xi\rangle_h^{p-2} \langle x\rangle^{-\frac{p-2}{p-1}} \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right) \\
&\qquad -C\langle\xi\rangle_h^{p-2} \langle x\rangle^{-\frac{p-2}{p-1}} \left(1 - \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\right) \\
&\quad \geq \left(-C + M_{p-2} C_p 2^{-(p-1)/2}\right) \langle\xi\rangle_h^{p-2} \langle x\rangle^{-\frac{p-2}{p-1}} \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right) - C',
\end{aligned}$$

where $C = C(M_{p-1})$ is a constant depending on the already chosen $M_{p-1} > 0$ and, in the last inequality, we used the fact that on the support of $1 - \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)$, we have $\langle\xi\rangle_h^{p-2} \langle x\rangle^{-\frac{p-2}{p-1}} \leq C'$. Choosing the constant $M_{p-2} \geq C(M_{p-1}) 2^{(p-1)/2}/C_p$, we obtain that

$$\mathrm{Re}(ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2}) \geq -C' \quad \text{for} \quad |\xi| \geq hR.$$

Then, we can apply Theorem 2.3 to the symbol $ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2} + C' \in SG^{p-2,0}$. There exist operators $Q_{p-2}(t,x,D)$, $\tilde{R}_{p-3}$, R_0 such that

$$\begin{aligned}
&ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2} + C' = Q_{p-2} + \tilde{R}_{p-3} + R_0, \\
&\mathrm{Re}\langle Q_{p-2} v, v\rangle \geq 0 \quad \forall v \in \mathcal{S}(\mathbb{R}).
\end{aligned}$$

It is now crucial to observe that $Q_{p-2}(t,x,\xi) \in SG^{p-3,0}$, $R_0 \in S^0$, whereas by (2.6) we have $\tilde{R}_{p-3} \in SG^{p-3,-(p-3)/(p-1)}$ since the constant C' does not appear in the expression of the symbol. As a matter of fact, we have

$$\tilde{r}_{p-3}(x,\xi) = \psi_1(\xi) D_x(ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2}) + \sum_{|\alpha+\beta|\geq 2} \psi_{\alpha,\beta}(\xi) \partial_\xi^\alpha D_x^\beta (ia_{p-2,1} + \partial_\xi a_p \partial_x \lambda_{p-2}),$$

which is in $SG^{p-3,-(p-3)/(p-1)}$. The proposition is proved. □

We now prove by induction on $n \geq 2$ the following:

LEMMA 3.6. Given $n \in \mathbb{N}$, $n \geq 2$, we can find a constant $h_n \geq 1$ such that for $h \geq h_n$ there exist pseudo-differential operators $Q_{p-\ell}(t,x,D_x)$, $1 \leq \ell \leq n$, $a_{j,n}(t,x,D_x)$, $1 \leq j \leq p-n-1$, and $r_n(t,x,D_x)$ with symbols $Q_{p-\ell}(t,x,\xi) \in SG^{p-\ell,0}$, $a_{j,n}(t,x,\xi) \in SG^{j,-j/(p-1)}$, $r_n(t,x,\xi) \in S^0$ such that the operator

$$(iP)_n =: (e^{\lambda_{p-n}(x,D_x)})^{-1} \cdots (e^{\lambda_{p-2}(x,D_x)})^{-1} (e^{\lambda_{p-1}(x,D_x)})^{-1} \times (iP) e^{\lambda_{p-1}(x,D_x)} e^{\lambda_{p-2}(x,D_x)} \cdots e^{\lambda_{p-n}(x,D_x)}$$

can be written in the form:

$$(iP)_n = \partial_t + ia_p(t,D_x) + \sum_{\ell=1}^{n} Q_{p-\ell}(t,x,D_x) + \sum_{j=1}^{p-n-1} ia_{j,n}(t,x,D_x) + r_n(t,x,D_x),$$

and

$$\mathrm{Re}\langle Q_{p-\ell}(t,x,D_x)v, v\rangle \geq 0, \ \forall v \in \mathcal{S}(\mathbb{R}), \ 1 \leq \ell \leq n. \tag{3.9}$$

Proof. For $n = 2$, this is exactly the statement of Lemma 3.5. Let us suppose that for $h \geq h_{n-1}$, it holds

$$(iP)_{n-1} = \partial_t + ia_p(t,D_x) + \sum_{\ell=1}^{n-1} Q_{p-\ell}(t,x,D_x) + \sum_{j=1}^{p-n} ia_{j,n-1}(t,x,D_x) + r_{n-1}(t,x,D_x), \tag{3.10}$$

for some pseudo-differential operators $Q_{p-\ell}(t,x,D_x)$, $1 \leq \ell \leq n-1$, $a_{j,n-1}(t,x,D_x)$, $1 \leq j \leq p-n$, and $r_{n-1}(t,x,D_x)$ with symbols $Q_{p-\ell}(t,x,\xi) \in SG^{p-\ell,0}$, $a_{j,n-1}(t,x,\xi) \in SG^{j,-j/(p-1)}$, $r_{n-1}(t,x,\xi) \in S^0$ such that

$$\mathrm{Re}\langle Q_{p-\ell}(t,x,D_x)v, v\rangle \geq 0, \ \forall v \in \mathcal{S}(\mathbb{R}), \ 1 \leq \ell \leq n-1,$$

and consider the operator

$$(iP)_n = (e^{\lambda_{p-n}(x,D_x)})^{-1} (iP)_{n-1} e^{\lambda_{p-n}(x,D_x)}.$$

Lemma 2.9 gives, for h large enough, say $h \geq \tilde{h}_n$, that

$$(iP)_n = e^{-\lambda_{p-n}(x,D_x)} (I + R_{p-n}) (iP)_{n-1} e^{\lambda_{p-n}(x,D_x)},$$

with a pseudo-differential operator R_{p-n} having the principal symbol

$$r_{p-n,-n}(x,\xi) = \partial_\xi \lambda_{p-n}(x,\xi) D_x \lambda_{p-n}(x,\xi) \in SG^{-n,-(p-n)/(p-1)}.$$

Thus, for $h \geq h_n = \max\{\tilde{h}_n, h_{n-1}\}$ and by (3.10), we have (omitting (t, x, D_x) in the notation):

$$(iP)_n = \partial_t + e^{-\lambda_{p-n}} \left(ia_p + \sum_{\ell=1}^{n-1} Q_{p-\ell} + \sum_{j=1}^{p-n} ia_{j,n-1} + r_{n-1} \right) e^{\lambda_{p-n}}$$
$$+ e^{-\lambda_{p-n}} \left(iR_{p-n}a_p + \sum_{\ell=1}^{n-1} R_{p-n} Q_{p-\ell} + \sum_{j=1}^{p-n} iR_{p-n}a_{j,n-1} + R_{p-n}r_{n-1} \right) e^{\lambda_{p-n}}.$$

Now we notice that $R_{p-n}Q_{p-\ell} \in SG^{p-n-\ell,-(p-n)/(p-1)} \subset SG^{p-n,-(p-n)/(p-1)}$ since $1 \leq \ell \leq n-1$, $iR_{p-n}a_p \in SG^{p-n,-(p-n)/(p-1)}$, and $R_{p-n}ia_{j,n-1} \in SG^{j-n,-(p-n)/(p-1)} \subset SG^{j-n,-(j-n)/(p-1)}$, $j = 1, \ldots, p-n$. So we can write $(iP)_n$ as follows:

$$(iP)_n = \partial_t + e^{-\lambda_{p-n}(x,D_x)} \left(ia_p + \sum_{\ell=1}^{n-1} Q_{p-\ell} + \sum_{j=1}^{p-n} i\tilde{a}_{j,n-1} \right) e^{\lambda_{p-n}(x,D_x)} + s_0$$

for some s_0 with symbol in S^0, and with new symbols $\tilde{a}_{j,n-1} \in SG^{j,-j/(p-1)}$. Again, we need to notice that at level $p-n$ we get:

$$i\tilde{a}_{p-n,n-1}(t,x,\xi) = ia_{p-n,n-1}(t,x,\xi) + ia_p(t,\xi)\partial_\xi \lambda_{p-n}(t,x,\xi) D_x \lambda_{p-n}(t,x,\xi)$$

where the first term in the right-hand side depends on the constants $M_{p-1}, \ldots, M_{p-n+1}$ chosen before and the second one depends only on the constant M_{p-n} which will be chosen later on; this does not cause any trouble in the choice of the constant M_{p-n} because, again by (3.1), we get:

$$(iP)_n = \partial_t + ia_p + \sum_{\ell=1}^{n-1} Q_{p-\ell} + i\tilde{a}_{p-n,n-1} + \partial_\xi a_p \partial_x \lambda_{p-n} + \sum_{j=1}^{p-n-1} i\tilde{a}_{j,n-1}$$
$$+ \sum_{\alpha=2}^{p-n} \frac{1}{\alpha!} \partial_\xi^\alpha ia_p \cdot e^{-\lambda_{p-n}} \cdot D_x^\alpha e^{\lambda_{p-n}} - ia_p \partial_\xi \lambda_{p-n} D_x \lambda_{p-n}$$
$$+ \sum_{\gamma=2}^{p-n} \frac{1}{\gamma!} \partial_\xi^\gamma e^{-\lambda_{p-n}} \cdot (ia_p D_x^\gamma e^{\lambda_{p-n}})$$
$$+ \sum_{\gamma=1}^{p-n} \sum_{\alpha=1}^{p-n+1} c_{\alpha,\gamma} \partial_\xi^\gamma e^{-\lambda_{p-n}} \partial_\xi^\alpha a_p D_x^{\alpha+\gamma} e^{\lambda_{p-n}}$$
$$+ \sum_{\ell=1}^{n-1} \sum_{\alpha=1}^{p-\ell-n} \frac{1}{\alpha!} \partial_\xi^\alpha Q_{p-\ell} \cdot e^{-\lambda_{p-n}} \cdot D_x^\alpha e^{\lambda_{p-n}}$$

$$+\sum_{j=1}^{p-n}\sum_{\alpha=1}^{j-1}\frac{1}{\alpha!}\partial_\xi^\alpha i\tilde{a}_{j,n-1}\cdot e^{-\lambda_{p-n}}\cdot D_x^\alpha e^{\lambda_{p-n}}$$

$$+\sum_{\ell=1}^{n-1}\sum_{\gamma=1}^{p-\ell}\frac{1}{\gamma!}\partial_\xi^\gamma e^{-\lambda_{p-n}}D_x^\gamma(Q_{p-\ell}e^{\lambda_{p-n}})$$

$$+\sum_{\ell=1}^{n-1}\sum_{\alpha=1}^{p-n}\sum_{\gamma=1}^{p-\ell-1}\frac{1}{\alpha!\gamma!}\partial_\xi^\gamma e^{-\lambda_{p-n}}D_x^\gamma(\partial_\xi^\alpha Q_{p-\ell}D_x^\alpha e^{\lambda_{p-n}})$$

$$+\sum_{j=1}^{p-n}\sum_{\gamma=1}^{j-1}\frac{1}{\gamma!}\partial_\xi^\gamma e^{-\lambda_{p-n}}D_x^\gamma(i\tilde{a}_{j,n-1}e^{\lambda_{p-n}})$$

$$+\sum_{j=1}^{p-n}\sum_{\alpha=1}^{j-1}\sum_{\gamma=1}^{j-2}\frac{1}{\gamma!}\partial_\xi^\gamma e^{-\lambda_{p-n}}D_x^\gamma(\partial_\xi^\alpha(i\tilde{a}_{j,n-1})D_x^\alpha e^{\lambda_{p-n}})+\tilde{t}_0$$

$$=\partial_t+ia_p+\sum_{\ell=1}^{n-1}Q_{p-\ell}+\left(\partial_\xi a_p\partial_x\lambda_{p-n}+ia_{p-n,n-1}\right)+\sum_{j=1}^{p-n-1}i\tilde{\tilde{a}}_{j,n-1}+\tilde{s}_0, \tag{3.11}$$

for some operators $\tilde{\tilde{a}}_{j,n-1}(t,x,D_x)$ with symbols $\tilde{\tilde{a}}_{j,n-1}(t,x,\xi)\in SG^{j,-j/(p-1)}$, $1\le j\le p-n-1$, depending on $M_{p-1},\ldots,M_{p-n+1}$ and where $\tilde{s}_0$ is a term containing operators with symbol in S^0.

As done in the second transformation, we now look at the real part of the terms of order $p-n$ with respect to ξ in (3.11); for $|\xi|\ge hR$, we have by (1.16),(2.14) and for a positive constant $C=C(M_{p-1},\ldots,M_{p-n+1})$:

$$\begin{aligned}
\mathrm{Re}(ia_{p-n,n-1}+\partial_\xi a_p\partial_x\lambda_{p-n}) &= -\mathrm{Im}a_{p-n,n-1}\\
&\quad+M_{p-n}\partial_\xi a_p\omega\left(\frac{\xi}{h}\right)\langle x\rangle^{-\frac{p-n}{p-1}}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\langle\xi\rangle_h^{-n+1}\\
&\ge -C\langle\xi\rangle_h^{p-n}\langle x\rangle^{-\frac{p-n}{p-1}}+M_{p-n}C_p|\xi|^{p-1}\langle\xi\rangle_h^{-n+1}\langle x\rangle^{-\frac{p-n}{p-1}}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)\\
&\ge(-C+M_{p-n}C_p2^{-(p-1)/2})\langle\xi\rangle_h^{p-n}\langle x\rangle^{-\frac{p-n}{p-1}}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^{p-1}}\right)-C'\ge-C'
\end{aligned}$$

if we choose $M_{p-n}\ge C(M_{p-1},\ldots,M_{p-n+1})2^{(p-1)/2}/C_p$. An application of Theorem 2.3 to the symbol $ia_{p-n,n-1}+\partial_\xi a_p\partial_x\lambda_{p-n}+C'\in SG^{p-n,0}$ gives then the existence of operators $Q_{p-n}(t,x,D)$, $\tilde{R}_{p-n-1}(t,x,D)$, $R_0(t,x,D)$ with symbols, respectively, $Q_{p-n}(t,x,\xi)\in SG^{p-n,0}$, $\tilde{R}_{p-n-1}\in SG^{p-n-1,-(p-n)/(p-1)}$, $R_0\in S^0$ such that

$$\begin{aligned}
&ia_{p-n,n-1}+\partial_\xi a_p\partial_x\lambda_{p-n}+C'=Q_{p-n}+\tilde{R}_{p-n-1}+R_0,\\
&\mathrm{Re}\langle Q_{p-n}v,v\rangle\ge 0\quad\forall v\in\mathcal{S}(\mathbb{R}).
\end{aligned}$$

Then, from (3.11), we finally obtain

$$
\begin{aligned}
(iP)_n &= \partial_t + ia_p + \sum_{\ell=1}^{n-1} Q_{p-\ell} + Q_{p-n} + \tilde{R}_{p-n-1} + \sum_{j=1}^{p-n-1} i\tilde{\tilde{a}}_{j,n-1} + s_0 + R_0 \\
&= \partial_t + ia_p + \sum_{\ell=1}^{n} Q_{p-\ell} + \sum_{j=1}^{p-n-1} ia_{j,n} + r_n
\end{aligned}
$$

for some operators $a_{j,n}$, r_n with symbols $a_{j,n} \in SG^{j,-j/(p-1)}$, $r_n \in S^0$ and where $Q_{p-\ell}$ satisfy (3.9) for $1 \leq \ell \leq n$. The lemma is proved. □

Proposition 3.1 follows directly from Lemmas 3.4, 3.5 and 3.6.

Proof of Theorem 1.2. By the change of variable (2.12), the Cauchy problem (1.1) in the unknown $u(t,x)$ is reduced to the Cauchy problem (2.9) for the unknown $u_\lambda(t,x)$, where P_λ is defined by (2.10) and f_λ and g_λ are defined by (2.11). Now we apply Proposition 3.1 to derive an energy estimate for the solution to the Cauchy problem (2.9). For every $v \in C^1([0,T], \mathcal{S}(\mathbb{R}))$, we have

$$
\begin{aligned}
\frac{d}{dt}\|v\|_0^2 &= 2\,\mathrm{Re}\langle \partial_t v, v\rangle \\
&= 2\,\mathrm{Re}\langle (iP_\lambda)v, v\rangle - 2\sum_{\ell=1}^{p-1} \mathrm{Re}\langle Q_{p-\ell}v, v\rangle - 2\,\mathrm{Re}\langle r_0 v, v\rangle \\
&\leq C(\|P_\lambda v\|_{L^2}^2 + \|v\|_{L^2}^2).
\end{aligned}
$$

By standard arguments from the energy method, we deduce that, for all $s = (s_1, s_2) \in \mathbb{R}^2$ and every $v \in C^1([0,T], \mathcal{S}(\mathbb{R}))$, the following estimate holds

$$
\|v(t,\cdot)\|_{s_1,s_2}^2 \leq c' \left(\|v(0,\cdot)\|_{s_1,s_2}^2 + \int_0^t \|P_\lambda v(\tau,\cdot)\|_{s_1,s_2}^2 d\tau \right) \quad \forall t \in [0,T], \tag{3.12}
$$

for some $c' > 0$. The energy estimate (3.12) can be extended by a density argument to a function $v \in C^1([0,T], H_{s_1,s_2}(\mathbb{R}))$ for every $s_1, s_2 \in \mathbb{R}$. This implies that if $f_\lambda \in C([0,T], H_{s_1,s_2}(\mathbb{R}))$, $g_\lambda \in H_{s_1,s_2}(\mathbb{R})$, then the Cauchy problem (2.9) has a unique solution $u_\lambda(t,x) \in C^1([0,T], H_{s_1,s_2}(\mathbb{R})) \cap C([0,T], H_{s_1+p,s_2}(\mathbb{R}))$ satisfying (3.12). By the relation (2.12) between u and u_λ and by Lemma 2.7, we obtain existence and uniqueness of a solution u of (1.1). Moreover, from (2.11), (2.12) and (3.12), we get

$$
\begin{aligned}
\|u\|_{s_1,s_2-2M_{p-1}}^2 &\leq c_1\|u_\lambda\|_{s_1,s_2-M_{p-1}}^2 \leq c_2\left(\|g_\lambda\|_{s_1,s_2-M_{p-1}}^2 + \int_0^t \|f_\lambda\|_{s_1,s_2-M_{p-1}}^2 d\tau\right) \\
&\leq c_3\left(\|g\|_{s_1,s_2}^2 + \int_0^t \|f\|_{s_1,s_2}^2 d\tau\right)
\end{aligned} \tag{3.13}
$$

for some $c_1, c_1, c_3 > 0$. This gives the energy estimate (1.17) and proves well posedness in $\mathcal{S}, \mathcal{S}'$ of the Cauchy problem (1.1). □

REMARK 3.7. As outlined in the Introduction, Theorem 1.2 can be proved replacing the assumption (1.16) by the weaker condition (1.19) and repeating readily the argument of the proof above. For the sake of brevity, we leave the details to the reader. We restrict ourselves to observe that the assumption (1.19) is sufficient for the application of Theorem 2.3 in the proofs of Lemmas 3.4, 3.5, 3.6 where only the imaginary parts of the symbols $a_j(t, x, \xi)$ are involved and that every step can be performed identically with the only difference that the symbols $a_{j,\ell}$, $\ell = 1, \ldots, p-1$ appearing in the above lemmas are now such that $\mathrm{Re}\, a_{j,\ell} \in SG^{j,0}$ and $\mathrm{Im}\, a_{j,\ell} \in SG^{j,-j/(p-1)}$. Nevertheless, this is sufficient to obtain the assertion of Proposition 3.1.

REMARK 3.8. As usual in this type of problems, it is natural to wonder whether the loss in the second Sobolev index may really appear or it is due to a lack of sharpness in the method of the proof. Here, we want to present some examples where this type of phenomenon appears although we can show a modification of the behavior of the solution only with respect to either the initial datum $g(x)$ or the function $f(t, x)$. An example where the solution exhibits a loss of decay/increase of growth with respect to both g and f seems to be difficult to construct, and at this moment, it is out of reach. Nevertheless, the examples below show that this behavior of the solution is not surprising in our setting. We stress the fact that in all the following examples, no loss of derivatives appears in the solution.

EXAMPLE 1. Consider the Cauchy problem

$$\begin{cases} D_t u + D_x^2 u + \frac{i}{\langle x\rangle} D_x u = f(t, x) \\ u(0, x) = g(x). \end{cases} \tag{3.14}$$

The function $u(t, x) = x(1 + tx)$ solves (3.14) with Cauchy data $g(x) = x$ and

$$f(t, x) = -ix^2 + (1 + 2tx)\langle x\rangle^{-1} - 2t.$$

We observe that both u and f belong to $C([0, T], H_{s_1,s_2})$ for every $s_1 \in \mathbb{R}$, $s_2 < -5/2$, while $g \in H_{s_1,s_2}$ for every $s_1 \in \mathbb{R}$ and for $s_2 < -3/2$. Hence, in this case, we have a loss in the second Sobolev index with respect to the initial datum: The solution presents a stronger growth with respect to g.

EXAMPLE 2. The function $u(t, x) = (t + x)^2$ solves the Cauchy problem (3.14) with Cauchy data $g(x) = x^2$ and

$$f(t, x) = -2it - 2ix - 2 + 2t\langle x\rangle^{-1} + 2x\langle x\rangle^{-1}.$$

In this example $u \in C([0, T], H_{s_1,s_2})$ and $g \in H_{s_1,s_2}$ for every $s_1 \in \mathbb{R}$, $s_2 < -5/2$, while $f \in C([0, T], H_{s_1,s_2})$ for every $s_1 \in \mathbb{R}$, $s_2 < -3/2$. We have so a loss in the second Sobolev index with respect to f.

EXAMPLE 3. The function $u(t, x) = t\langle x\rangle^k$ with $k \in \mathbb{Z}$ solves the Cauchy problem (3.14) with Cauchy data $g(x) = 0$ and

$$f(t, x) = -i\langle x\rangle^k + ktx\langle x\rangle^{k-3} - kt\langle x\rangle^{k-2} - k(k-2)tx^2\langle x\rangle^{k-4}.$$

We observe that the solution belongs to $C([0,T], H_{s_1,s_2})$ for every $s_1 \in \mathbb{R}, s_2 < -k - 1/2$, as well as f. In this case, we have an *infinite* loss in the second Sobolev index with respect to the initial datum but the same decay/growth with respect to f as $|x| \to \infty$.

REFERENCES

[1] R. Agliardi, *Cauchy problem for p-evolution equations*, Bull. Sci. Math. 126, n. 6 (2002), 435–444.

[2] R. Agliardi, *Cauchy problem for evolution equations of Schrödinger type*, J. Differential Equations 180, n.1 (2002), 89–98.

[3] R. Agliardi, L. Zanghirati, *Cauchy problem for nonlinear p-evolution equations*, Bull. Sci. Math. 133 (2009), 406–418.

[4] A. Ascanelli, C. Boiti, *Cauchy problem for higher order p-evolution equations*, J. Differential Equations 255 (2013), 2672–2708.

[5] A. Ascanelli, C. Boiti, *Well-posedness of the Cauchy problem for p-evolution systems of pseudo-differential operators.*, J. Pseudo-Differ. Oper. Appl. 4 (2013), 113–143.

[6] A. Ascanelli, C. Boiti, L. Zanghirati, *Well-posedness of the Cauchy problem for p-evolution equations*, J. Differential Equations 253 (2012), 2765–2795.

[7] A. Ascanelli, C. Boiti, L. Zanghirati, *Well posedness for semi-linear 3-evolution equations in Sobolev spaces*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 60, n.1 (2014), 5–21.

[8] A. Ascanelli, C. Boiti, L. Zanghirati, *A Necessary condition for H^∞ Well-Posedness of p-evolution equations* (2014), submitted, http://arxiv.org/abs/1406.6183.

[9] A. Ascanelli, M. Cappiello, *Log-Lipschitz regularity for SG hyperbolic systems*, J. Differential Equations 230 (2006), 556–578.

[10] A.Ascanelli, M.Cappiello: *Hölder continuity in time for SG hyperbolic systems*, J. Differential Equations 244 (2008), 2091–2121.

[11] A. Ascanelli, M. Cicognani, *Schrödinger equations of higher order*, Math. Nachr. 280, n.7 (2007), 717–727.

[12] A. Ascanelli, M. Cicognani, F. Colombini, *The global Cauchy problem for a vibrating beam equation*, J. Differential Equations 247 (2009), 1440–1451.

[13] M. Cicognani, F. Colombini, *Sharp regularity of the coefficients in the Cauchy problem for a class of evolution equations*, Differ. Integral. Equ. 16 (2003), 1321–1344.

[14] M. Cicognani, F. Colombini, *The Cauchy problem for p-evolution equations.*, Trans. Amer. Math. Soc. 362, n. 9 (2010), 4853–4869.

[15] M. Cicognani, F. Hirosawa, M. Reissig, *Loss of regularity for p-evolution type models*, J. Math. Anal. Appl. 347 (2008), 35–58.

[16] M. Cicognani, M. Reissig, *On Schrödinger type evolution equations with non-Lipschitz coefficients*, Ann. Mat. Pura Appl. 190, n.4 (2011), 645–665.

[17] H.O. Cordes, *The technique of pseudo-differential operators*, Cambridge Univ. Press, 1995.

[18] Y.V. Egorov and B.-W. Schulze, *Pseudo-differential operators, singularities, applications*, Operator Theory: Advances and Applications 93, Birkhäuser Verlag, Basel, 1997.

[19] L. Hörmander, *The analysis of linear partial differential operators, III*, Springer-Verlag, 1985.

[20] W. Ichinose, *Some remarks on the Cauchy problem for Schrödinger type equations* Osaka J. Math. 21 (1984), 565–581.

[21] W. Ichinose, *Sufficient condition on H^∞ well-posedness for Schrödinger type equations*, Comm. Partial Differential Equations, 9, n.1 (1984), 33–48.

[22] K. Kajitani, A. Baba, *The Cauchy problem for Schrödinger type equations*, Bull. Sci. Math. 119 (1995), 459–473.

[23] H. Kumano-Go, *Pseudo-differential operators* The MIT Press, Cambridge, London, 1982.

[24] C. Parenti, *Operatori pseudodifferenziali in $\mathbb{R}^n$ e applicazioni*, Ann. Mat. Pura Appl. 93 (1972), 359–389.

[25] E. Schrohe, *Spaces of weighted symbols and weighted Sobolev spaces on manifolds.* In "Pseudo-differential Operators", Proceedings Oberwolfach 1986. H. O. Cordes, B. Gramsch and H. Widom editors, Springer LNM, 1256 New York, 360–377 (1987).

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