J. Evol. Equ. 14 (2014), 387–401
1424-3199/14/020387-15, *published online* February 7, 2014
DOI 10.1007/s00028-014-0219-5

Journal of Evolution Equations

Area-preserving evolution of nonsimple symmetric plane curves

XIAO-LIU WANG AND LING-HUA KONG

Abstract. The area-preserving nonlocal flow in the plane is investigated for locally convex closed curves, which may be nonsimple. For highly symmetric convex curves, the flows converge to m-fold circles, while for Abresch–Langer type curves, the convergence to m-fold circles happens if and only if the enclosed algebraic area is positive.

1. Introduction

In this paper, we study a nonlocal version of the curve shortening flow. Let us start by reviewing results on the curve shortening flow. For any closed plane curve X_0, one may consider its evolution under the rule

$$\frac{\partial X}{\partial t} = kN, \quad X(\cdot, t) = X_0(\cdot), \tag{1.1}$$

where N and k denote the inner unit normal and curvature of the curve $X(\cdot, t)$, respectively. The curve shortening problem has been studied intensively. First, it was proved by Gage and Hamilton [13] that every simple closed convex curve is shrunk to a point in finite time under (1.1), and furthermore, after normalizing the enclosing area of the curve at every instant to be constant, the normalized curve tends to a circle. Next, Grayson [14] shows that every simple closed curve stays simple and evolves smoothly to a convex curve in finite time. For immersed curves, it is not hard to show that a unique solution for (1.1) exists for small time. However, simple examples show that it may develop some singularity before it shrinks to a point. Thus, the study turns to the classification of singularities, see for instance, Angenent [3], Oaks [24], or examining concrete examples such cardioid-shaped curves in Angenent and Velazquez [4]. In the other direction, there are results for locally convex, immersed closed curves with some symmetries. For instance, a class of highly symmetric curves consisting of n-fold rotational symmetry and total curvature of $2m\pi$ with $n > 2m$ was introduced in Epstein and Gage [10] (see definition in Section 5 of [10]), and it was shown that

Mathematics Subject Classification (2010): Primary 53C44, 35B40; Secondary 35K59, 37B25
Keywords: Mean curvature flow, nonlocal parabolic equation, asymptotic behavior.
This work is partially supported by the National Natural Science Foundation of China 11101078, 11171064 and the Education Department Program of Liaoning Province L2010068.

any such curve evolves under (1.1) to a point which is an asymptotic m-fold circle in finite time. When m and n satisfy the relation $n/2 < m < n/\sqrt{2}$, Abresch and Langer [1] discovered that there are self-similar solutions to (1.1) (which are called to be 'Abresch–Langer curves' in this paper), and they proposed some saddle point property for these solutions (see [8] for more references). This property was later confirmed in Au [5], see also Wang [29].

As every simple closed curve evolves to a round point under the curve shortening flow, it is expected that the isoperimetric ratio would behave nicely under the evolution. In fact, Gage [11] showed that the isoperimetric ratio decreases under the flow for convex curves. Unfortunately, the ratio may increase when the initial curve is not convex. To look for a modified flow which decreases the isoperimetric ratio, Gage introduced a nonlocal version of the curve shortening flow in [12]

$$\frac{\partial X}{\partial t} = \left(k - \frac{2m\pi}{L(t)}\right)N, \quad X(\cdot, 0) = X_0(\cdot), \tag{1.2}$$

where m and L denote the winding number of the immersed closed curve and its perimeter at time t, respectively. It is not hard to show that this modified flow exists for small time, preserves local convexity and the enclosing signed area, and decreases the perimeter and hence the isoperimetric ratio. It is shown in Gage [12] that a simple closed convex curve is evolved into the circle smoothly as time goes to infinity. Singularity formation for (1.2) is studied in Escher and Ito [9]. It is worthwhile to study (1.2) as parallel to (1.1).

In this paper, we first consider (1.2) for initial curves in the class of highly symmetric convex curves. Denote by $\mathcal{H}_{m,n}$ the highly symmetric convex curves with n-fold rotational symmetry and total curvature of $2m\pi$ $(n > 2m)$. We have

THEOREM 1.1. *For $X_0 \in \mathcal{H}_{m,n}$, $n > 2m$, the flow (1.2) exists for all time and it converges to an m-fold circle smoothly as time goes to infinity.*

Next, we turn to the class of Abresch–Langer curves. We would like to put them in a more general setting, that is, Abresch–Langer type curves. Before giving the definition, we define the support function of locally convex closed curve X (parameterized by its tangent angle $\theta \in [0, 2m\pi]$) to be

$$h(\theta) = < X(\theta), -N(\theta) >,$$

where $-N$ is the outward normal vector $(\sin\theta, -\cos\theta)$ or $(\cos(\theta - \pi/2), \sin(\theta - \pi/2))$. The relationship between locally convex, closed curves and their support functions is contained in Proposition 2.1 of [7].

The Abresch–Langer type curves, denoted by $\mathcal{A}_{m,n}$, are defined to be locally convex smooth curves, which have n-fold rotational symmetry and total curvature of $2m\pi$ with $n < 2m$ (m and n are mutually prime). In addition, we require that their support functions h and curvature functions k are symmetric with respect to $\theta = 0$ and $\theta = m\pi/n$, and both strictly decreasing in $(0, m\pi/n)$. The examples of $\mathcal{A}_{m,n}$ are illustrated in Fig. 1.

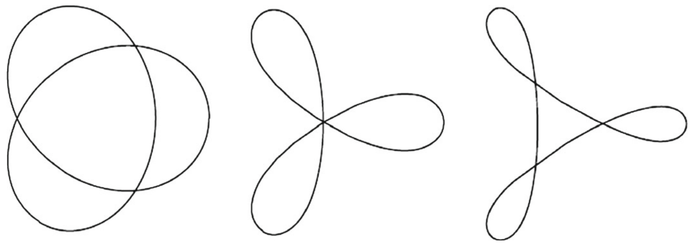

Figure 1. Abresh–Langer type curves

Before stating the results for $\mathcal{A}_{m,n}$, we note that if the curvature blows up at a finite time, it means that the maximal existence time of the flow must be finite. When the maximal existence time is infinite, the curvature may be uniformly bounded or blow up at infinite time.

THEOREM 1.2. *Assume that $X_0 \in \mathcal{A}_{m,n}$ whose enclosed algebraic area is A_0. The following conclusions hold.*

(1) *When A_0 is positive, the flow under (1.2) exists for all time and converges to an m-fold circle smoothly as time goes to infinity.*
(2) *When $A_0 < 0$, the curvature of the flow blows up at finite time.*
(3) *When $A_0 = 0$, the curvature of the flow blows up at the maximal existence time.*

It is concluded that for X_0 in $\mathcal{H}_{m,n}$, the asymptotic shape of its evolution is the same under (1.2) and curve shortening flow. For X_0 in $\mathcal{A}_{m,n}$, the asymptotic shape under (1.2) is some different from that under curve shortening flow, where the singularities can also appear even if $A_0 > 0$ (see [5]).

In the following, we reformulate our problem via support functions and curvature functions. One may find the details in [7]. Since each curve $X(u,t)$ is strictly convex, each point on it has a unique tangent and one can use the tangent angle $\theta \in [0, 2m\pi]$ to parameterize it. Generally speaking, θ is a function of u and t. In order to make θ independent of time t, one can attain that by adding a tangential component to the velocity vector $\partial X/\partial t$, which does not affect the geometric shape of the evolving curve (see, for example, [12]). Then, evolution equations can be expressed in the coordinates of θ and t.

Denote by $h(\theta,t)$ the support function of $X(\theta,t)$. Then, Problem (1.2) can be reformulated as the following initial value problem for $h(\theta,t)$,

$$\begin{cases} h_t = -(h + h_{\theta\theta})^{-1} + 2m\pi L^{-1}(t), & (\theta,t) \in [0,2m\pi] \times (0,T), \\ h(\theta,0) = h_0, & \theta \in [0,2m\pi] \end{cases} \tag{1.3}$$

where h_0 denotes the support function of initial curve X_0.

If we denote the curvature function of $X(\theta,t)$ by $k(\theta,t)$, Problem (1.2) can be reformulated by

$$\begin{cases} k_t = k^2(k_{\theta\theta} + k - 2m\pi L^{-1}(t)), & (\theta,t) \in [0,2m\pi] \times (0,T), \\ k(\theta,0) = k_0, & \theta \in [0,2m\pi] \end{cases} \tag{1.4}$$

where k_0 denotes the curvature of initial curve X_0. We note that above problems are both periodic in θ. Throughout this paper, Problem (1.4) will be frequently used and sometime we also need recur to Problem (1.3).

In the end of this section, we say more about nonlocal flow. As an interesting variant of the popular curve shortening flow, the nonlocal curvature flow, arising in many application fields [27], such as phase transitions and image processing, has received much attention in recent years. Generally, the normal speed takes the form of

$$V = [F(k(u,t)) - \lambda(t)]N, \tag{1.5}$$

where $F(k)$ is a given function of curvature satisfying $F'(z) > 0$ for all z in its domain and $\lambda(t)$ is a function of time, which may depend on certain global quantities of $X(.,t)$, say enclosed area $A(t)$, length $L(t)$ or others. When $F(k) = k$, Problem (1.5) is usually called k-type nonlocal flow problem. Except for the area-preserving flow considered here, there are other k-type flows. For example, Ma and Zhu [21] studied a length-preserving flow, and Jiang and Pan [16] studied a nonlocal flow increasing the area of evolving curves and decreasing their length. They all obtained the same convergence as that in [12]. When $F(k) = k^{-1}$, Problem (1.5) is called $1/k$-type nonlocal flow problem and has been investigated by Pan and Yang [25] and Pan and Zhang [26]. Recently, the generalized case $F(k) = k^{\alpha}$ ($\alpha \neq 0$) is also considered, see [19]. In the higher dimensional case, people also consider nonlocal flows. For example, there are Huisken's volume preserving mean curvature flow [15] and McCoy's surface area-preserving mean curvature flow [23]. We also refer the readers to [9] and [6] for some other types of area-preserving flow.

The rest of our paper is organized as the following. In Sect. 2, the sufficient condition on the curvature is established for the convergence of flow. Then, we prove Theorem 1.1 and 1.2 in Sects. 3 and 4, respectively, where the key point is to establish time-independent upper bound on the evolving curves' curvature.

2. A general convergence result

A sufficient condition on the curvature for the convergence of flow is given as follows.

THEOREM 2.1. *If the flow under (1.2) exists for all time and the curvature of evolving curves are time-independently bounded from above, the flow must converge to an m-fold circle smoothly as $t \to \infty$.*

In order to prove Theorem 2.1, some lemmas are prepared. From now on, we set $I = [-m\pi/n, m\pi/n]$ and $\lambda(t) = 2m\pi L^{-1}(t)$.

LEMMA 2.2. *Consider the flow under (1.2). If the curvature of evolving curves is time-independently bounded from above, it is also time-independently bounded from below.*

Proof. We first claim: the curvature k of $X(., t)$ satisfies

$$\sup_{I\times[0,t]} (k_\theta^2 + k^2) \le \max\left\{ \sup_{I\times[0,t]} k^2, \quad \sup_{I\times\{0\}} (k_\theta^2 + k^2)\right\} \tag{2.1}$$

for all $t \in [0, T)$. The proof is analogous to Lemma I1.12 in Andrews [2]. Let $\Phi = (k_\theta)^2 + k^2$ and let $t > 0$ be fixed. Suppose at $(\theta_0, t_0) \in I \times [0, t]$ we have $\Phi(\theta_0, t_0) = \sup_{I\times[0,t]}(k_\theta^2 + k^2)$. Then, we may assume $t_0 > 0$ (otherwise we are done). We claim that $k_\theta(\theta_0, t_0) = 0$. If not, then at (θ_0, t_0) the following properties

$$\begin{cases} k_{\theta\theta} + k = 0, \\ \Phi_{\theta\theta} = 2k_\theta(k_{\theta\theta\theta} + k_\theta) \le 0, \\ \frac{\partial\Phi}{\partial t} = 2k^2 k_\theta(k_{\theta\theta\theta} + k_\theta) - 4\lambda(t)kk_\theta^2 - 2\lambda(t)k^3 \ge 0, \end{cases}$$

give a contradiction. Hence, $k_\theta(\theta_0, t_0) = 0$, and we conclude

$$\sup_{I\times[0,t]} (k_\theta^2 + k^2) = k^2(\theta_0, t_0) \le \sup_{I\times[0,t]} k^2.$$

By (2.1) and the uniform upper bound of the curvature, there is a constant C independent of time such that

$$|k_\theta(\theta, t)| \le C \quad \text{for all} \quad (\theta, t) \in I \times [0, T). \tag{2.2}$$

This implies

$$\left|\log \frac{k(\theta_2, t)}{k(\theta_1, t)}\right| = \left|\int_{\theta_1}^{\theta_2} \frac{k_\theta(\theta, t)}{k(\theta, t)}\, \mathrm{d}\theta\right| \le CL(t) \le CL(0)$$

for all $t \in [0, T)$ and any $\theta_1, \theta_2 \in I$. In particular, we have **Harnack-type estimate**:

$$\frac{k_{\max}(t)}{k_{\min}(t)} \le e^{CL(0)}.$$

Also, by

$$\frac{2m\pi}{k_{\max}(t)} \le \int_0^{2m\pi} \frac{1}{k(\theta, t)}\, \mathrm{d}\theta = L(t) \le L(0),$$

we have

$$k_{\min}(t) \ge k_{\max}(t)e^{-CL(0)} \ge \frac{2m\pi}{L(0)} e^{-CL(0)}.$$

The proof is done. □

If the two-side bound on k is obtained, it will yield the smooth estimates for k.

LEMMA 2.3. *Consider the flow under (1.2). If the curvature of evolving curves is time-independently bounded from above, for every $l \in \mathbf{Z}^+$, there exists a constant C_l depending only on X_0 such that*

$$\sup_{I\times[0,T)} |k^{(l)}(\theta,t)| \le C_l.$$

Proof. Recall that $k(\theta,t)$ satisfies

$$k_t = k^2 k_{\theta\theta} + k^3 - \lambda(t)k^2.$$

Therefore, the estimate in Lemma 2.2 guarantees that the above equation is uniformly parabolic. In addition, during the proof of Lemma 2.2, we also know that k_θ has the uniform estimate independent of time. Then, by standard Schauder estimate, we can obtain the uniform estimate for $k_{\theta\theta}$. So we can regard the above equation as a linear parabolic equation

$$k_t = ak_{\theta\theta} + bk, a = k^2, b = k^2 - \lambda(t)k,$$

Then, by using the same techniques in the proof of Theorem 8 of [20], we can obtain the estimates independent of time for all higher-order derivatives of k in t and θ. This can be also done via the regularity estimates for linear parabolic equation, see [17], [18]. □

We can complete the proof of Theorem 2.1 now.

Proof of Theorem 2.1. Take the Lyapunov functional to be

$$\mathcal{F}(t) = L^2.$$

□

Then, $\mathcal{F}'(t) = 2LL'(t) \le 0$. For any $t > 0$, $\int_0^t \mathcal{F}'(t)\,\mathrm{d}t = \mathcal{F}(t) - \mathcal{F}(0) \ge -\mathcal{F}(0)$, and hence

$$\int_0^\infty \mathcal{F}'(t)\,\mathrm{d}t > -\infty. \tag{2.3}$$

We claim that

$$\mathcal{F}'(t) \to 0, t \to \infty. \tag{2.4}$$

Suppose by contrary it does not hold. Then, there is a constant $C_0 > 0$ and a sequence $\{\widetilde{t_i}\}_{i=1}^\infty$ with $\{\widetilde{t_i}\} \to \infty$ as $i \to \infty$, such that $\mathcal{F}'(\widetilde{t_i}) \le -C_0$. Recall the regularity results in Lemma 2.3. Therefore, we can find a $\rho_0 > 0$ (independent of $\widetilde{t_i}$), such that $|\mathcal{F}'(t)| \ge \frac{C_0}{2}$, $t \in [\widetilde{t_i}, \widetilde{t_i} + \rho_0]$, and so

$$\int_{\widetilde{t_i}}^{\widetilde{t_i}+\rho_0} \mathcal{F}'(t)\,\mathrm{d}t \le -\frac{C_0\rho_0}{2}. \tag{2.5}$$

On the other hand, (2.3) implies that $\lim_{i\to\infty} \int_{t_i}^\infty \mathcal{F}'(t)\,\mathrm{d}t = 0$, a contradiction with (2.5). Thus, our claim (2.4) holds.

For any sequence $\{t_j\}_{j=1}^\infty$ with $\{t_j\}\to\infty$ as $j\to\infty$, $\{k(\theta,t_j)\}_{j=1}^\infty$ is equi-continuous, which can be observed by the fact that all the derivatives of $k(\theta,t)$are uniformly bounded. By Arzela–Ascoli Theorem, we can take out a subsequence $\{t_{j_k}\}_{k=1}^{\infty}$, such that $\{k(\theta, t_{j_k})\}_{k=1}^{\infty}$ converges smoothly to k^*, with k^* smooth and strictly positive. Since

$$\mathcal{F}'(t_{j_k}) = 2\left[\left(\int_0^{2m\pi} \mathrm{d}\theta\right)^2 - \int_0^{2m\pi} \frac{\mathrm{d}\theta}{k(t_{j_k})} \int_0^{2m\pi} k(t_{j_k})\,\mathrm{d}\theta\right] \to 0,\ k \to \infty,$$

we take the limit to obtain

$$4m^2\pi^2 = \left(\int_0^{2m\pi} k^*(\theta)\mathrm{d}\theta\right)\left(\int_0^{2m\pi} \frac{\mathrm{d}\theta}{k^*(\theta)}\right).$$

Hölder inequality tells us that k^* is a constant. This means that the limit curve is an m-fold circle.

3. Proof of Theorem 1.1

In this section, we restrict ourselves to the evolution of curves in $\mathcal{H}_{m,n}$ and prove Theorem 1.1. The ideas are basically motivated by [12] and [28]. By using the support function, we improve the curvature's estimate to be time-independent, so that we can show the convergence of flow by Theorem 2.1.

For curves in $\mathcal{H}_{m,n}$, Epstein and Gage [10] showed that their support functions must be positive by choosing the origin to be symmetric center and the following Bonnesen-type inequalities hold.

LEMMA 3.1. *Let $X \in \mathcal{H}_{m,n}$. There holds*
(1) $rL - A - m\pi r^2 \geq 0$ *for* $r \in [r_{\mathrm{in}}, r_{\mathrm{out}}]$,
(2) $L^2 - 4m\pi A \geq m^2\pi^2(r_{\mathrm{out}} - r_{\mathrm{in}})^2$
where r_{in} and r_{out} denote, respectively, the radii of the largest inscribed circle and the smallest circumscribed circle of the curve.

Let h denote the support function relative to the center of symmetry. It is evident that $r_{\mathrm{in}} \leq h \leq r_{\mathrm{out}}$. From this fact and the inequality (1) in Lemma 3.1, it follows that $hL - A - \pi m h^2 \geq 0$. By integrating it with respect to the arc length s and using the fact that $\int_X h\,\mathrm{d}s = 2A$, we have

$$\int_X h^2\,\mathrm{d}s \leq \frac{LA}{m\pi}. \tag{3.1}$$

Then, we use the Hölder inequality to obtain

$$L = \int_0^L hk\,\mathrm{d}s \leq \left(\int_0^L h^2\,\mathrm{d}s\right)^{1/2}\left(\int_0^L k^2\,\mathrm{d}s\right)^{1/2}.$$

With (3.1), we deduce that

LEMMA 3.2. *Let $X \in \mathcal{H}_{m,n}$. There holds*

$$\int_X k^2\,\mathrm{d}s \geq \frac{m\pi L}{A}. \tag{3.2}$$

Note that (3.2) can be regarded as the generalization of Gage's inequality (see [11]). Using (3.2), we can compute the decay rate of the isoperimetric deficiency for the evolving curves to obtain

LEMMA 3.3. *If $X_0 \in \mathcal{H}_{m,n}$, then for curves $X(\theta, t)$ evolving according to (1.2), we have*

$$\frac{m^2\pi^2}{A}(r_{\text{out}} - r_{\text{in}})^2 \leq \frac{L^2}{A} - 4m\pi \leq C_1 e^{-C_2 t},\ \forall\, t \in (0, T), \tag{3.3}$$

where C_1 and C_2 only depend on initial data X_0.

The global existence of the flow is proved via considering the parabolic equation (1.4) satisfied by k. The proof just follows the lines of [12]. According to [12], we define the *median curvature k^** as

$$k^*(t) = \sup\{\beta : k(\theta, t) > \beta \text{ on some interval of length } \pi\}.$$

For $X_0 \in \mathcal{H}_{m,n}$ with enclosed algebraic area A_0 and length L_0, we have the estimate

$$k^*(t) \leq L_0/(2A_0),\quad t \in [0, T).$$

Note that the rotational symmetry of X_0 guarantees that k_0 is $2m\pi/n$-periodic in θ and so is $k(\theta, t)$ in view of the parabolic equation (1.4). Since $2m\pi/n < \pi$, we have

$$k^*(t) = k_{\min}(t).$$

The length of the curve is given by

$$L = \int_0^{2m\pi} \frac{\mathrm{d}\theta}{k(\theta, t)} < \int_0^{2m\pi} \frac{\mathrm{d}\theta}{k_{\min}(t)} = \frac{2m\pi}{k_{\min}(t)}.$$

Hence, $k^*(t) \leq 2m\pi/L(t)$. As $L(t)^2 \geq 4m\pi A(t)$, we have $k^*(t) \leq L(t)/(2A(t)) \leq L_0/(2A_0)$.

Subsequently, mimicking the proof of Proposition 3.6 in Gage [12], we can show the integral $\int_0^{2m\pi} \log k(\theta, t)\,\mathrm{d}\theta$ has an upper bound $C(X_0, T)$ on $[0, T)$. In the original arguments, we only need change the integral interval to be $[0, 2m\pi]$ and use the isoperimetric inequality for nonsimple closed curves. To go further, we need to estimate the L^2-norm of k_θ, which can be done according to Lemma 3.4 and Corollary 3.5 in [12]. In fact, we have

$$\int_0^{2m\pi} (k_\theta)^2\,\mathrm{d}\theta \leq 2m\pi M^2 + DM + C$$

where $M = \sup_{[0,2m\pi]\times[0,T)} k(\theta, t)$ and the constants C, D only depend on the initial curve. Now, we can convert the upper bound for $\int_0^{2m\pi} \log k(\theta, t)\,\mathrm{d}\theta$ to be a bound for $\sup_{[0,2m\pi]\times[0,T)} k(\theta, t)$.

Let (θ_1, t_1) be a point such that $k(\theta_1, t_1) = \mathrm{d}M$ with $0 < d < 1$. Then

$$\begin{aligned} k(\theta_1, t_1) - k(\theta, t_1) &= \int_\theta^{\theta_1} k_\theta \, \mathrm{d}\theta \le \left(\int_0^{2m\pi} (k_\theta)^2 \right)^{1/2} (\theta_1 - \theta)^{1/2} \\ &\le (2m\pi M^2 + DM + C)^{1/2} (\theta_1 - \theta)^{1/2} \\ &\le MC_1 (\theta_1 - \theta)^{1/2} \end{aligned}$$

for some constant C_1 (only depending on initial curve) since we can assume $M > 1$. It means that

$$k(\theta, t_1) \ge \mathrm{d}M - MC_1(\theta_1 - \theta)^{1/2}.$$

So we can estimate

$$\begin{aligned} &\int_0^{2m\pi} \log k(\theta, t) \, \mathrm{d}\theta \\ &= \int_{|\theta - \theta_1| \le (\frac{d}{2C_1})^2} \log k(\theta, t) \, \mathrm{d}\theta + \int_{|\theta - \theta_1| \ge (\frac{d}{2C_1})^2} \log k(\theta, t) \, \mathrm{d}\theta \\ &\ge \log \left(\frac{\mathrm{d}M}{2} \right) \cdot \frac{\mathrm{d}^2}{2C_1^2} + \left(2m\pi - \frac{\mathrm{d}^2}{2C_1^2} \right) \log(k_{\min}(0) e^{-\mu t_1}) \\ &\ge C_2 \log M + C_3 - C_4 t_1. \end{aligned}$$

where the estimate $k_{\min}(t) \ge k_{\min}(0) e^{-\mu t_1}$ is from the proof of Lemma 3.1 in [12], and the constants $C_2(> 0), C_3, C_4(> 0), \mu(> 0)$ only depend on initial curve. Thus, an upper bound for M is deduced, which may depend on T. We can conclude the global existence result of the flow as follows.

LEMMA 3.4. *Under (1.2) with $X_0 \in \mathcal{H}_{m,n}$, the flow exists for all time and the evolving curves also belong to $\mathcal{H}_{m,n}$.*

Apart from the proof line of Gage [12], we improve the upper bound of k to be time-independent, in order to obtain a more detailed proof of the flow's convergence than Gage [12]. To go ahead, we need two-side positive bound for support function h.

LEMMA 3.5. *Consider Problem (1.2) with $X_0 \in \mathcal{H}_{m,n}$. The support function h of $X(., t)$ satisfies*

$$0 < r_0 \le h(\theta, t) \le R_0, \quad (\theta, t) \in I \times [0, T)$$

for some constants r_0 and R_0.

Proof. Note that r_{out} obviously has positive upper and lower bound. After the global existence of flow is established in Lemma 3.4, we take the limit in (3.3) to obtain

$$r_{\text{out}} - r_{\text{in}} \to 0, \quad t \to \infty.$$

It implies r_{in} also has a positive lower bound. The proof is done just by recalling the inequality $r_{\text{in}} \le h(\theta, t) \le r_{\text{out}}$, which is true because of the symmetry of $X(., t)$. $\square$

LEMMA 3.6. *Consider Problem (1.2) with $X_0 \in \mathcal{H}_{m,n}$. The curvature function k of $X(., t)$ satisfies*

$$k(\theta, t) \leq M_1, \quad \forall\, (\theta, t) \in I \times [0, T),$$

for some constant M_1 independent of time T.

Proof. The method is originally from Tso [28]. For convenience, we write $\lambda(t) = 2m\pi/L(t)$. Fix a $t \in (0, T)$. Consider the quantity $Q = k/(h-a)$ where $h \geq 2a$. According to Lemma 3.5, one can choose $a = r_0/2$. Let the maximum of Q over $I \times [0, t]$ be attained at (θ_0, t_0), $t_0 > 0$. At the point (θ_0, t_0), we have

$$\frac{\partial Q}{\partial \theta} = 0, \quad \frac{\partial Q}{\partial t} \geq 0, \quad \text{and} \quad \frac{\partial^2 Q}{\partial \theta^2} \leq 0.$$

A direct computation shows that

$$\begin{aligned} Q_\theta &= \frac{k_\theta}{h-a} - \frac{k h_\theta}{(h-a)^2}, \\ Q_{\theta\theta} &= \frac{k_{\theta\theta}}{h-a} - \frac{2h_\theta}{h-a} Q_\theta - \frac{k h_{\theta\theta}}{(h-a)^2} \\ &= \frac{k_{\theta\theta}}{h-a} - \frac{2h_\theta}{h-a} Q_\theta - \frac{1-kh}{(h-a)^2}, \end{aligned}$$

and

$$\begin{aligned} Q_t &= \frac{k_t}{h-a} - \frac{k h_t}{(h-a)^2} \\ &= \frac{k^2 k_{\theta\theta}}{h-a} + \frac{k^3 - \lambda(t) k^2}{h-a} - \frac{\lambda(t) k - k^2}{(h-a)^2} \end{aligned}$$

where the last inequality is due to the equations for k and h. Substituting $Q_{\theta\theta}$ and Q_θ, we have

$$\frac{\partial Q}{\partial t} = k^2 Q_{\theta\theta} + \frac{2k^2 h_\theta Q_\theta}{h-a} + \frac{2k^2}{(h-a)^2} - \frac{ak^3}{(h-a)^2} - \lambda(t)\left(\frac{k^2}{h-a} + \frac{k}{(h-a)^2}\right).$$

Since

$$\begin{aligned} 0 \leq \frac{\partial Q}{\partial t} &\leq \frac{2k^2}{(h-a)^2} - \frac{ak^3}{(h-a)^2} \\ &\leq -Q^2[a^2 Q - 2] \end{aligned}$$

(where the inequality $h - a \geq a > 0$ is used), we deduce that $Q(\theta_0, t_0) \leq \frac{2}{a^2}$. When the maximum of Q attains at the initial time, we have $Q \leq \max_I Q(\theta, 0)$. Hence, $Q \leq \max\left\{\frac{2}{a^2}, \max_I Q(\theta, 0)\right\} := M$. That is $\frac{k}{h-a} \leq M$. It follows that

$$k \leq M(h-a) \leq M(R_0 - r_0/2).$$

□

Now, the proof of Theorem 1.1 can be completed.

Proof of Theorem 1.1. Since the time-independent upper bound for k has been obtained in Lemma 3.6, we can use Theorem 2.1 to draw the conclusion. $\square$

4. Proof of Theorem 1.2

In this section, we consider the evolution of curves in $\mathcal{A}_{m,n}$ and prove Theorem 1.2. When $A_0 \le 0$, the conclusion is obvious. Since the flow is area-preserving, it certainly cannot converge to an m-fold circle, which has positive algebraic area. As a result, we must have curvature blow-up at maximal existence time. Furthermore, by Proposition 9 in [9], the life span of the flow must be finite when $A_0 < 0$. When $A_0 = 0$, it is remarked in [9] that whether the life span is finite or not is still an open problem.

We only need deal with the case of $A_0 > 0$. We point out that the proof of Theorem 1.1 relies much on the use of the Bonnesen-type inequalities. But unfortunately, we know little about the relevant results for general nonsimple locally convex curves, in particular, the curves in $\mathcal{A}_{m,n}$. However, the 'good' shape of curves guarantees that the estimates for curvature are feasible.

Two lemmas are prepared to give some information about the shape of the evolving curves starting some $X_0 \in \mathcal{A}_{m,n}$.

LEMMA 4.1. *Consider Problem (1.2) with $X_0 \in \mathcal{A}_{m,n}$. Let $h(\theta, t)$ be support function of $X(., t)$ and $k(\theta, t)$ be curvature function of $X(., t)$. Then,*

(a) *both of h and k are $2m\pi/n$-periodic in θ, and symmetric with respect to $\theta = 0$ and $\theta = m\pi/n$;*
(b) *both of h and k always attain their maximum at $\theta = 0$. h_θ and k_θ are negative on $(0, m\pi/n)$.*

Proof. It is easy to observe that (a) holds. We only prove (b). By differentiating the equation in (1.3), we see that the function $u = h_\theta$ satisfies a parabolic equation

$$u_t = a(\theta, t)u_{\theta\theta} + b(\theta, t)u, \quad (\theta, t) \in I \times [0, T)$$

where $a(\theta, t) = b(\theta, t) = k^2$ and $T > 0$. According to the Sturm comparison principle (see [22]), the number of zeroes of u is nonincreasing in time. Note that at $t = 0$, the function

$$u(\theta, 0) = \frac{\partial}{\partial\theta} h_0(\theta)$$

has exactly 2 zeros in I (a circle). Hence, the number of zeroes of $u(\theta, t)$ cannot exceed two for all $t \in [0, T)$. On the other hand, by the reflectional symmetry of Problem (1.3) with respect to the axis $\theta = 0$ and $\theta = m\pi/n$, we see that $u(\theta, t)$ must vanish at $\theta = 0$ and $m\pi/n$ for every $t \in [0, T)$. So we conclude that $u(\theta, t)$ does not change its sign on $(-m\pi/n, 0)$ and $(0, m\pi/n)$ for all $t \in [0, T)$. Then, the conclusion for h follows. The conclusion for k can be proved similarly. $\square$

LEMMA 4.2. *Consider Problem (1.2) with $X_0 \in \mathcal{A}_{m,n}$. The support function h of $X(.,t)$ satisfies*

$$h_0(m\pi/n) \le h(\theta,t) \le h_0(0), \quad (\theta,t) \in I \times [0,T).$$

Proof. We claim that for any time $t \in [0,T)$

$$h_t < 0 \text{ at } \theta = 0; \; h_t > 0 \text{ at } \theta = m\pi/n.$$

Indeed, since $k(m\pi/n,t) \le k(\theta,t) \le k(0,t)$ by Lemma 4.1, we have

$$\frac{2m\pi}{k(0,t)} < L = \int_0^{2m\pi} \frac{\mathrm{d}\theta}{k(\theta,t)} < \frac{2m\pi}{k(m\pi/n,t)}.$$

Hence,

$$k(m\pi/n,t) < \frac{2m\pi}{L} < k(0,t).$$

Then, the claim is true in view of the equation $h_t = 2m\pi/L - k$. The proof is done. □

For $X_0 \in \mathcal{A}_{m,n}$, if its support function h_0 is positive everywhere, then by the above lemma we have two-side positive bound for h. It permits us to argue as in Sect. 3 to show the convergence of the flow. But this will fail when h_0 is nonpositive at $m\pi/n$. So we employ the method of Au [5] to overcome the difficulty.

LEMMA 4.3. *Consider Problem (1.2) with $X_0 \in \mathcal{A}_{m,n}$ with $A_0 > 0$. There exists a constant M independent of time T such that*

$$k(\theta,t) \le M, \quad \forall\, (\theta,t) \in I \times [0,T).$$

Proof. After suitable coordinates are chosen, the leave of evolving curve X with $\theta \in (-\delta,\delta)$ (δ is suitably small) is a graph $u(x,t)$ over some interval, say $(-R(t),R(t))$. More precisely, we may express $X(\theta,t) = (x,u(x,t))$ with $x \in [-R(t),R(t)]$. Furthermore, we claim that the interval $(-R(t),R(t))$ can be independent of time t. Indeed, when $A_0 > 0$, it is easy to observe that the area $\tilde{A}(t)$ enclosed by each leave satisfies $\tilde{A}(t) > A_0/n$. Set the horizontal width of the leave is $d(t)$ and the vertical width is $l(t)$ (see Fig. 2). Obviously, $\tilde{A}(t) \le d(t)l(t)$. This means

$$d(t) \ge \tilde{A}(t)/l(t) \ge A_0/(nl(t)).$$

Recall that $l(t) \le h(0,t) \le h_0(0)$. Hence, our claim is true.

Now, we take the interval to be $(-R,R)$. A calculation shows that (see [7]) u satisfies

$$u_t = \frac{u_{xx}}{1+u_x^2} - \frac{2m\pi}{L(t)}\sqrt{1+u_x^2}, \quad (x,t) \in (-R,R) \times (0,T). \tag{4.1}$$

Note that u is a concave function. Therefore, we have the estimate

$$\begin{aligned}\sup\{|u_x| : x \in (-R/2,R/2)\} &\le \frac{2}{R}\operatorname{osc}\{u(x,t) : x \in (-R,R)\} \\ &\le \frac{2}{R}h(0,t) \le \frac{2}{R}h_0(0).\end{aligned}$$

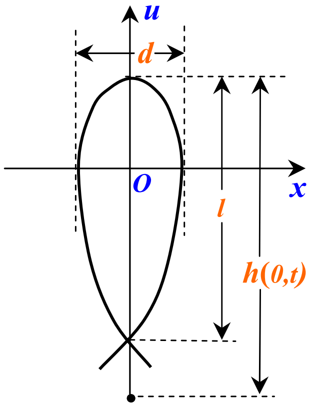

Figure 2. A leave represented by a graph

Differentiating (4.1), we know $w = u_x$ satisfies

$$w_t = \frac{w_{xx}}{1+w^2} - \frac{2ww_x^2}{(1+w^2)^2} - \frac{2m\pi}{L(t)}\frac{ww_x}{\sqrt{1+w^2}}, \quad (x,t) \in (-R,R) \times (0,T).$$

Since we have the estimate $|w| \le \frac{2}{R}h_0(0)$ on $(-R/2, R/2) \times (0,T)$, by the interior gradient estimate for quasilinear parabolic equations (see Theorem 11.18 in [18]), we obtain

$$|u_{xx}| = |w_x| \le C(R), \quad (x,t) \in (-R/4, R/4) \times (t - R/4, t),$$

with $t \in (R/4, T)$. This gives a time-independent upper bound for $k = u_{xx}/(1+u_x^2)^{3/2}$. □

Now, the proof of Theorem 1.2 can be completed.

Proof of Theorem 1.2. For the case of $A_0 > 0$, since we have obtained the time-independent upper bound for k in Lemma 4.3, the convergence is an immediate result of Theorem 2.1. The conclusion for the case of $A_0 \le 0$ is obvious as noted in the beginning of this section. □

For the asymptotic shape of the flow in Theorem 1.2(2), it can be observed such a lemma holds.

LEMMA 4.4. *For $X_0 \in \mathcal{A}_{m,n}$, the flow under (1.2) exists as long as the area of each leave of evolving curves is positive.*

Proof. Define $k^*(t)$ as in Sect. 3. We claim that $k^*(t)$ is bounded as long as the area of each leave is positive. If $M < k^*(t)$, then $k(\theta,t) > M$ on some interval $(\theta_0, \theta_0 + \pi)$. This implies that each leave lies between paralleled lines whose distance is given by

$$\int_{\theta_0}^{\theta_0+\pi} \frac{\sin(\theta-\theta_0)}{k(\theta,t)}\,\mathrm{d}\theta \le \frac{2}{M}$$

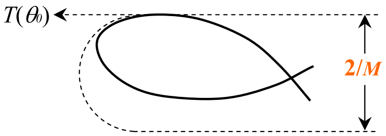

Figure 3. A leave of evolving curve

(see Fig. 3). The diameter is bounded by $L(t)/(2n)$, and the area of each leave (denoted by $\tilde{A}(t)$) is bounded by the width times the diameter, that is, $\tilde{A}(t) < L(t)/M$. We have $M \leq L(t)/\tilde{A}(t)$. Since M can be chosen arbitrarily close to $k^*(t)$, we have $k^*(t) \leq L(t)/\tilde{A}(t)$. So our claim is true. Then, arguing as in Sect. 3, we can use the boundedness of $k^*(t)$ to show global existence of the flow. □

Lemma 4.4 tells us that if the flow exists only for finite time, say T^*, then the area of each leave must be zero at $t = T^*$. When this happens, if $A_0 < 0, n$ cusps are formed; if $A_0 = 0$, the flow evolves into a point.

Acknowledgments

The authors would like to thank Prof. Kaiseng Chou for his constant encouragement and stimulating discussions during the preparation of this work.

REFERENCES

[1] U. Abresch & J. Langer, The normalized curve shortening flow and homothetic solutions, J. Differential Geom. 23(1986) 175–196.
[2] B. Andrews, Evolving convex curves, Calc. Var. Partial Differential Equations 7(1998) 315–371.
[3] S. Angenent, On the formation of singularities in the curve shortening flow, J. Differential Geom. 33(1991) 601–633.
[4] S.B. Angenent, J. J. L. Velázquez, Asymptotic shape of cusp singularities in curve shortening, Duke Math. J. 77 (1995) 71–110.
[5] T.K. Au, On the saddle point property of Abresch–Langer curves under the curve shortening flow, Comm. Anal. Geom. 18(2010) 1–21.
[6] K.S. Chou, A blow-up criterion for the curve shortening flow by surface diffusion, Hokkaido mathematical journal 32(2003) 1–19.
[7] K.S. Chou, X.P. Zhu, The Curve Shortening Problem. Chapman & Hall/CRC, Boca Raton, FL, 2001.
[8] C.L. Epstein, M.I. Weinstein, A stable manifold theorem for the curve shortening equation, Comm. Pure Appl. Math. 40(1987) 119–139.
[9] J. Escher, K. Ito, Some dynamic properties of volume preserving curvature driven flows, Mathematische Annalen 333(2005) 213–230.
[10] C.L. Epstein, M.E. Gage, The curve shortening flow, in: Wave Motion: Theory, Modelling, and Computation, Proc. Conf. Hon. 60th Birthday P.D.Lax, Publ. Math. Sci. Res. Inst. 7(1987) 15–59.
[11] M.E. Gage, An isoperimetric inequality with applications to curve shortening, Duke Math. J. 50(1983) 1225–1229.
[12] M.E. Gage, On an area-preserving evolution equation for plane curves, Nonlinear problems in geometry (Mobile, Ala., 1985), 51–62, Contemp. Math., 51, Amer. Math. Soc., Providence, RI, 1986.

[13] M.E. Gage, R. Hamilton, The heat equation shrinking convex plane curves, J. Differential Geom. 23(1986) 69–96.
[14] M.A. Grayson, The heat equation shrinks embedded plane curves to round points, J. Differential Geom. 26(1987) 285–314.
[15] G. Huisken, The volume preserving mean curvature flow, J. Reine Angew. Math. 382(1987) 34–48.
[16] L.S. Jiang, S.L. Pan, On a non-local curve evolution problem in the plane, Comm. Anal. Geom. 16(2008) 1–26.
[17] O. A. Ladyženskaja & V. A Solonnikov & N. N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type. Translations of Mathematical Monographs, Vol. **23** American Mathematical Society, Providence, R.I. 1967.
[18] G.M. Lieberman, Second Order Parabolic Differential Equations. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
[19] Y.C. Lin, D.H. Tsai, Application of Andrews and Green-Osher inequalities to nonlocal flow of convex plane curves, J. of Evolution Equations, 2012.
[20] Y.C. Lin, D.H. Tsai, On a simple maximum principle technique applied to equations on the circle, J. Diff. Eq. 245(2008) 377–391.
[21] L. Ma, A.Q. Zhu, On a length preserving curve flow, Monatshefte Für Mathematik, 165(2012) 57–78.
[22] X.Y. Chen, H. Matano, Convergence, asymptotic periodicity, and finite point blow-up in one-dimensional semilinear heat equations, J. Diff. Eq. 78(1989) 160–190.
[23] J. McCoy, The surface area preserving mean curvature flow, Asian J. of Math., 7 (2003) 7–30.
[24] J.A. Oaks, Singularities and self-intersections of curves evolving on surfaces, Indiana Univ. Math. J. 43 (1994) 959–981.
[25] S.L. Pan, J.N. Yang, On a non-local perimeter-preserving curve evolution problem for convex plane curves, Manuscripta Math. 127(2008) 469–484.
[26] S.L. Pan, H. Zhang, On a curve expanding flow with a nonlocal term, Comm. Contemp. Math. 12(2010) 815–829.
[27] G. Sapiro, A. Tannenbaum, Area and length preserving geometric invariant scale-spaces, Pattern Analysis and Machine Intelligence, IEEE Transactions, 17(1995) 67–72.
[28] K. Tso, Deforming a hypersurface by its Gauss-Kronecker curvature, Comm. Pure Appl. Math. 38 (1985) 867–882.
[29] X.L. Wang, The stability of m-fold circles in the curve shortening problem, Manuscripta Math. 134 (2011) 493–511.

X.-L. Wang
Department of Mathematics,
Southeast University,
Nanjing 210096,
People's Republic of China
E-mail: xlwang.seu@gmail.com

L.-H. Kong
School of Science,
Dalian Ocean University,
No. 52 Heishijiao Street, Dalian
116023, People's Republic of China
E-mail: klh1222@163.com