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Highly time-oscillating solutions for very fast diffusion equations

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Abstract. This work is concerned with the fast diffusion equation

$$u_t = \nabla \cdot \left(u^{m-1}\nabla u\right) \qquad (\star)$$

with prescribed positive data on a smoothly bounded domain $\Omega \subset \mathbb{R}^n,\ n \geq 3$ and positive $m < 1$. We consider solutions with boundary data $u = a > 0$ and initial data $u_0(x) \geq a$ that are continuous for $x \neq 0 \in \Omega$ and have a singularity at $x = 0$. By skilfully choosing the behaviour of u_0 near 0 and under the further condition $m < (n-2)/n$, we construct global in-time solutions $u(x,t)$ that oscillate as $t \to \infty$ between divergence to infinity at times $t_{2i} \to \infty$ and convergence to a at times $t_{2i-1} \to \infty$. This happens locally uniformly in x.

0. Introduction

We consider the Dirichlet problem

$$\begin{cases} u_t = \nabla \cdot \left(u^{m-1}\nabla u\right), & x \in \Omega,\ t > 0, \\ u(x,t) = a, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Note that for $m = 1$, this reduces to studying the classical heat equation that has a well-known theory. Here, we consider the exponent range $m < 1$ which corresponds to what is called the fast diffusion equation (FDE). Data and solutions are assumed to be positive, but they are not necessarily finite everywhere. This problem, which has been widely studied in recent decades, generates an evolution process with a number of remarkable properties. Our goal here is to investigate one of such peculiar features that can shortly be described as follows: the presence of a certain type of singular behaviour of the initial data at just one isolated point may lead to a solution that undergoes very large oscillations in time whose amplitude can be made to fit any sequential pattern (at suitable times). To make notation easier, we take the isolated point to be the origin, $0 \in \Omega$.

In order to present our results in more detail, let us recall some known facts of the theory. First of all, for every $0<m<1$, the initial value problem for the FDE is known

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to be solvable in the sense of weak solutions for integrable data, and it generates a semigroup in the space in $L^1(\Omega)$ if for instance zero boundary data are prescribed or the domain is the whole space, cf. Bénilan's thesis [1] or [2]. But the existence theory can be widely extended; thus, Herrero and Pierre showed in [9] that one can take as initial data any unbounded locally integrable data, which is certainly not possible in the heat equation. Moreover, we can also take functions that are not locally integrable, e.g., Radon measures, cf. [6,10]. We may even take nonnegative Borel measures with locally infinite measure, [5], though in that case it may happen that the solution will continue to be singular at some points for positive times. Permanent singularities are important for us in what follows.

An important exponent comes up in the already mentioned studies, $m_c := (n-2)/n$, in dimensions $n \geq 3$. Indeed, when $m < m_c$, a number of more curious properties happens and many of them are described in the monographs [7,13]. One of the peculiar aspects of the equation in this range of exponents is the lack of strong smoothing effect, by which we mean that data in L^1, or even L^p with small $p > 1$, do not produce solutions that are bounded in time since they may exhibit singular points for positive times. In a recent paper, [14], we have studied in close detail the long-term effect of having a certain point singularity in the initial data when $m < 1$. Under the assumption that u_0 is singular at just one point, say at $x = 0 \in \Omega$, and that it behaves like a power function there, $u_0(x) \sim |x|^{-\gamma}$ for small $|x|$, we have shown that the behaviour in time of the singularity depends on the value of γ as we explain just below. Let us finally point out that the constant boundary data $u = a > 0$ are taken for simplicity so as to concentrate our attention on the effect of the isolated singularity.

PROBLEM WITH SINGULAR DATA. The study of [14] motivated us to look for the possibility of finding solutions with an oscillating behaviour in time due to the presence of a unique isolated singularity of the initial data u_0 at one point, say $x = 0$. We take m in the range $0 < m < m_c$ that is commonly referred to as the *very fast diffusion* range. We shall assume that the positive function $u_0 \in C^2(\overline{\Omega}\backslash\{0\})$ has an isolated singularity at $x = 0$ by requiring that $u_0(x) \to +\infty$ as $x \to 0$.

As to such data, we recall the result [13] and [14] in more detail. If the singularity has the shape of an inverse power of $|x|$,

$$u_0(x) \sim |x|^{-\gamma} \quad \text{as } x \to 0 \tag{0.2}$$

with some $\gamma > 0$, then the size of γ decides between various types of behaviour of the solution:

- If $\gamma < \frac{2}{1-m}$, then the singularity is immediately smoothed out by the evolution; more precisely, the solution is smooth in $\overline{\Omega}$ for all positive times.
- If $\gamma = \frac{2}{1-m}$, then there exists a positive but finite *blow-down time* T such that $u(\cdot, t)$ keeps the singularity at $x = 0$ for all $t < T$ but becomes smooth afterwards, for $t > T$.

- In the case $\frac{2}{1-m} < \gamma < \frac{n-2}{m}$, the phenomenon of *infinite-time blow-down* occurs: the singularity persists for all times but disappears in the limit $t \to \infty$ in that any member of the ω-limit set of u is a bounded function.
- If $\gamma = \frac{n-2}{m}$, then the solution keeps its isolated singularity not only for all finite times but also in the limit $t \to \infty$.
- Finally, when $\gamma > \frac{n-2}{n}$, the singularity remains located at the origin for all times, and in the limit $t \to \infty$ the solution *grows up* everywhere in the sense that it tends to $+\infty$ for all $x \in \Omega$.

In fact, it was proved in [14] that generalizations to the case when

$$\underline{a}|x|^{-\gamma_1} \le u_0(x) \le \overline{a}|x|^{-\gamma_2} \quad \text{for all } x \in \Omega\backslash\{0\} \tag{0.3}$$

are possible, provided that both γ_1 and γ_2 lie both in the one of the subregions indicated above. For instance, infinite-time blow-down occurs if $\frac{2}{1-m} < \gamma_1 \le \gamma_2 < \frac{n-2}{m}$, and solutions grow up if $\gamma_1 > \frac{n-2}{m}$.

Main results. In the present paper, we discuss an effect that may arise when the initial data yet satisfy (0.3), but with γ_1 and γ_2 belonging to *different* ranges of the above mentioned list, thus allowing u_0 to oscillate rather widely *in space* near $x = 0$, we shall see that this may lead to large oscillations *in time* of the corresponding solution of (0.1). To be more precise, we can state our main result as follows.

THEOREM 1. *Suppose that* $0 \in \Omega$, $0 < m < m_c$ *and* $a > 0$. *Then, there exists* $u_0 \in C^\infty(\overline{\Omega}\backslash\{0\})$ *such that* $u_0|_{\partial\Omega} = a$, $u_0 \ge a$ *in* Ω *and such that* (0.1) *has a unique global singular classical solution* u *with the following property: there exists an increasing sequence of times* $t_k \to \infty$ *such that*

$$u(\cdot, t_{2k}) \to \infty \quad \textit{locally uniformly in } \Omega\backslash\{0\}, \tag{0.4}$$

and

$$u(\cdot, t_{2k-1}) \to a \quad \textit{locally uniformly in } \Omega\backslash\{0\} \tag{0.5}$$

as $k \to \infty$. *Moreover, for any* γ_1 *and* γ_2 *satisfying*

$$\frac{2}{1-m} < \gamma_1 < n < \frac{n-2}{m} < \gamma_2, \tag{0.6}$$

these initial data u_0 *can be chosen in such a way that* (0.3) *is valid with certain positive constants* $\underline{a}$ *and* $\overline{a}$.

Some precedents. A number of papers have treated the complicated or chaotic long-term behaviour of nonlinear parabolic equations, due to special properties of the data or the equation. Thus, the case of the heat equation and porous medium equation was treated by Vazquez-Zuazua in [15] and subsequently by Cazenave-Dickstein-Weissler, [4]. The equation is posed in the whole space, and the origin of the oscillations for large times is the oscillatory behaviour of the initial data as $|x| \to \infty$. We can even take nonnegative and bounded solutions and data.

Similarly, some oscillating solutions for diffusion equations with sources were detected in the presence of nondegenerate diffusion and supercritical reaction terms for $\Omega = \mathbb{R}^n$, [11], and for degenerate diffusion even in bounded domains with smooth initial data, see [16] and [17]. Similar results can be expected for other equations such as first-order conservation laws or viscous fluid problems. Some results are given in [15].

In another direction, in [3] Carrillo and the first author consider generalized porous medium equations of the form $u_t = \Delta\Phi(u)$, also posed in the whole space, and show oscillatory behaviour of the asymptotic ω-limit (for a rescaling of the solution) due to the oscillating behaviour of the nonlinearity $\Phi(u)$ only near $u = 0$. This solved in the negative a conjecture about simple attractors (of Barenblatt type) for such equations.

In contrast with these cases, our oscillatory asymptotics depends only on the behaviour of the initial data near a single singular point. The consequence to derive is that (any neighbourhood of) *the singularity may contain very sophisticated information.*

1. Preliminaries

The standard way to construct solutions of Problem (0.1) proceeds by approximation, using the solutions u_ε of the following family of regularized problems:

$$\begin{cases} u_{\varepsilon t} = \nabla \cdot \left(u_\varepsilon^{m-1} \nabla u_\varepsilon\right), & x \in \Omega,\ t > 0, \\ u_\varepsilon(x,t) = a, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\varepsilon \in (0, 1/a)$ and $u_{0\varepsilon} := \min\{u_0, (1/\varepsilon)\}$. Since $\inf_{x\in\Omega} u_0(x) > 0$, it follows that $u_{0\varepsilon}$ is positive and belongs to $W^{1,\infty}(\Omega)$ and satisfies $u_{0\varepsilon} = a$ on $\partial\Omega$ for such ε, whence standard parabolic theory ensures that Problem (1.1) possesses a unique global positive classical solution. It is easy to see by comparison that the u_ε are ordered and

$$u_\varepsilon \nearrow u_p \text{ in } \overline{\Omega} \times [0, \infty) \tag{1.2}$$

holds for some limit function u_p attaining values in $(0, +\infty]$. This function is commonly called the *proper solution* of (0.1). The term was coined in [8] in the study of blow-up problems in reaction-diffusion. There is large literature on proper solutions for blow-up problems, and they were used for this equation in our previous paper [14].

It will be essential to our approach to know that the solutions we are dealing with depend continuously on the initial data in an appropriate sense. In order to demonstrate that this is true for the above proper solutions (cf. Lemma 8 below), let us consider a further solution property that is shared by u_p in our case but much easier to verify for a given function.

DEFINITION 1. *Let* $T \in (0, \infty]$. *By a singular classical solution of* (0.1) *in* $\Omega \times (0, T)$ (*with singularity at the origin*) *we mean a function*

$$u \in C^0(\overline{\Omega} \times [0, T); (0, +\infty]) \cap C^{2,1}((\Omega \backslash \{0\}) \times (0, T))$$

that solves $u_t = \nabla \cdot (u^{m-1}\nabla u)$ *classically in* $(\Omega\backslash\{0\}) \times (0, T)$ *and that satisfies the initial and boundary conditions* $u|_{\partial\Omega} = a$ *and* $u|_{t=0} = u_0$ *as well as the singularity condition* $u(0, t) = +\infty$ *for all* $t \in (0, T)$ *in the sense of limit as* $x \to 0$.

In the case $T = \infty$, u *is said to be a global singular classical solution of Problem* (0.1).

REMARK. *Our class of singular classical solutions is a special subclass of the set of extended continuous solutions introduced in* [5].

We now observe that when (0.3) holds with sufficiently large γ_1, the proper solution indeed solves (0.1) in the latter sense. This is a consequence of the fact that under this assumption the singularity at $x = 0$ persists for all times:

LEMMA 2. *Suppose that* (0.3) *holds for some* $0 < \underline{a} \le \overline{a}$ *and* $\frac{2}{1-m} < \gamma_1 \le \gamma_2 < \infty$. *Then,* u_p *is a global singular classical solution of* (0.1). *Moreover, for all* $T > 0$, *there exist* $c_0(T) > 0$ *and* $c_1(T) > 0$ *such that*

$$c_0(T)|x|^{-\gamma_1} \le u_p(x, t) \le c_1(T)|x|^{-\gamma_2} \quad \textit{for all } x \in \overline{\Omega}\backslash\{0\} \quad \textit{and } t \in [0, T]. \tag{1.3}$$

If $\gamma_2 < \frac{n-2}{m}$, *then* $c_1(T)$ *can be chosen independent of* T.

Proof. See [14, Lemma 3.3, Lemma 3.9, Corollary 3.10] $\square$

1.1. Uniqueness of singular classical solutions

As an important step towards the proof of the property of continuous dependence on u_0, we shall make sure in this section that singular classical solutions are unique and hence always coincide with u_p, still provided that (0.3) holds with $\gamma_1 > \frac{2}{1-m}$.

To begin with, we provide a lower bound which would be trivial for bounded classical solutions.

LEMMA 3. *Let* $T \in (0, \infty]$ *and* u *be a singular classical solution of* (0.1) *in* $\Omega \times (0, T)$. *Then,* $u \ge c := \inf_{x\in\Omega} u_0(x)$ *in* $\Omega \times (0, T)$.

Proof. For $\delta \in (0, 1)$, we let $w(x, t) := u(x, t) - c - \delta t$ for $(x, t) \in \overline{\Omega} \times [0, T)$ and assume that for some $T' \in (0, T)$, w were not nonnegative in $\Omega \times (0, T')$. Then, there would exist $x_0 \in \overline{\Omega}$ and $t_0 \in [0, T']$ such that $w(x_0, t_0) = \min_{(x,t)\in\overline{\Omega}\times[0,T']} w(x, t) < 0$. Since $w \ge 0$ at $t = 0$, we have $t_0 > 0$ and hence the function $w(\cdot, t_0) \in C^0(\overline{\Omega}; (0, \infty]) \cap C^2(\Omega\backslash\{0\})$ attains its negative minimum at x_0. From the inequality $w \ge 0$ on $\partial\Omega$, we know that $x_0 \in \Omega$, and since $w(0, t_0) = +\infty$, we have $x_0 \ne 0$. It follows that $\Delta u^m(x_0, t_0) \ge 0$, because $u^m(\cdot, t_0) = (w(\cdot, t_0) + c - \delta t_0)^m$ attains a minimum at x_0. Thus, $0 \ge w_t = u_t + \delta = \frac{1}{m}\Delta u^m + \delta \ge \delta$ at (x_0, t_0). This contradiction shows that $w \ge 0$ and thus yields the claim upon taking $\delta \searrow 0$. $\square$

The next lemma shows that a singular classical solution cannot attain values below the proper solution u_p defined through (1.2) and (1.1):

LEMMA 4. *If u is a singular classical solution of* (0.1) *in* $\Omega \times (0, T)$ *for some* $T \in (0, \infty]$*, then* $u \geq u_p$ *in* $\Omega \times (0, T)$.

Proof. To see that $u \geq u_\varepsilon$ for all $\varepsilon \in \big(0, \frac{1}{a}\big)$, we pick a nonnegative $\chi \in C^\infty(\mathbb{R})$ such that $\chi \equiv 0$ in $(-\infty, 0)$, $\chi \equiv 1$ in $(1, \infty)$ and $0 \leq \chi' \leq 2$ on $\mathbb{R}$, and let $\chi_\delta(s) := \chi(\frac{s}{\delta})$ for $s \in \mathbb{R}$ and $\delta \in (0, 1)$. Now if $(x, t) \in \Omega \times (0, T)$ is such that $u(x, t) < u_\varepsilon(x, t)$, then u is smooth at (x, t) and hence $(u_\varepsilon - u)_t = \frac{1}{m}\Delta(u_\varepsilon^m - u^m)$ holds at this point. Therefore, the identity $(u_\varepsilon - u)_t \cdot \chi_\delta(u_\varepsilon^m - u^m) = \frac{1}{m}\Delta(u_\varepsilon^m - u^m) \cdot \chi_\delta(u_\varepsilon^m - u^m)$ is valid in $\Omega \times (0, T)$, and integrating this yields

$$\begin{aligned} -\frac{1}{m}\int_0^t \int_\Omega |\nabla \left(u_\varepsilon^m - u^m\right)|^2 \cdot \chi_\delta' \left(u_\varepsilon^m - u^m\right) &= \int_0^t \int_\Omega (u_\varepsilon - u)_t \cdot \chi_\delta \left(u_\varepsilon^m - u^m\right) \\ &= \int_\Omega (u_\varepsilon - u) \cdot \chi_\delta \left(u_\varepsilon^m - u^m\right)\Big|_0^t \\ &\quad - \int_0^t \int_\Omega (u_\varepsilon - u) \cdot \chi_\delta' \left(u_\varepsilon^m - u^m\right) \\ &\quad \times \left(\left(u_\varepsilon^m\right)_t - \left(u^m\right)_t\right) \end{aligned} \tag{1.4}$$

for $t \in (0, T)$. Since $u \geq c$ in $\Omega \times (0, T)$ for some $c > 0$ by Lemma 3, it follows from parabolic regularity theory that both $(u^m)_t$ and $(u_\varepsilon^m)_t$ are bounded in the set $\{u < u_\varepsilon\}$ by a constant $C(\varepsilon) > 0$, so that

$$\begin{aligned} \left|(u_\varepsilon - u) \cdot \chi_\delta'(u_\varepsilon^m - u^m) \cdot \left((u_\varepsilon^m)_t - (u^m)_t\right)\right| &\leq (u_\varepsilon - u) \cdot \frac{2}{\delta} \cdot \chi_{\{u_\varepsilon^m - u^m \leq \delta\}} \cdot 2C(\varepsilon) \\ &\leq \frac{1}{m} u_\varepsilon^{1-m} \cdot 2 \cdot 2C(\varepsilon). \end{aligned}$$

As each u_ε is bounded, we conclude by the dominated convergence theorem that the last integral in (1.4) vanishes in the limit $\delta \searrow 0$. Consequently,

$$\int_\Omega (u_\varepsilon - u)_+(\cdot, t) \leq \int_\Omega (u_\varepsilon - u)_+(\cdot, 0) = 0$$

for all $t \in (0, T)$, which proves the lemma. □

The next statement on local in-time boundedness of certain generalized moments of classical solutions is independent of the size of γ_1 and γ_2 in (0.3).

LEMMA 5. *Assume that* $u_0 \in C^2(\overline{\Omega}\backslash\{0\})$ *satisfies* (0.3) *with positive constants* $\underline{a}, \overline{a}$ *and* γ_2 *and that u is a classical solution of* (0.1) *in* $(\Omega\backslash\{0\}) \times (0, T)$ *for some* $T \in (0, \infty]$*. Then, for all* $\alpha > \max\{2, \gamma_2 - n\}$ *and each* $T' \in (0, T)$*, one can pick* $C(\alpha, T') > 0$ *with the property that*

$$\int_\Omega |x|^\alpha u(x, t)dx \leq C(\alpha, T') \quad \textit{for all } t \in (0, T'). \tag{1.5}$$

Proof. We fix any $\alpha > \max\{2, \gamma_2 - n\}$ and let

$$\varphi_\delta(x) := (|x| - \delta)_+^\alpha, \quad x \in \overline{\Omega},$$

for $\delta \in (0,1)$. Then, $\alpha > 2$ guarantees that $\varphi_\delta \in C^2(\overline{\Omega})$ and

$$\begin{aligned}\Delta\varphi_\delta &= \alpha(\alpha-1)(|x|-\delta)_+^{\alpha-2} + \frac{(n-1)\alpha}{|x|}\cdot(|x|-\delta)_+^{\alpha-1}\\ &\le \alpha(n+\alpha-2)(|x|-\delta)_+^{\alpha-2}.\end{aligned} \tag{1.6}$$

Since u is a classical solution at each $x \in \Omega$ with $|x| > \delta$, we may multiply $u_t = \frac{1}{m}\Delta u^m$ by φ_δ and integrate over Ω to obtain from Green's formula

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi_\delta(x)u(x,t)\mathrm{d}x &= \frac{1}{m}\int_\Omega \varphi_\delta\Delta u^m\\ &= \frac{1}{m}\int_\Omega \Delta\varphi_\delta\cdot u^m + \frac{1}{m}\int_{\partial\Omega}\varphi_\delta\cdot\frac{\partial u^m}{\partial\nu}\\ &\quad -\frac{1}{m}\int_{\partial\Omega}\frac{\partial\varphi_\delta}{\partial\nu}\cdot u^m\end{aligned} \tag{1.7}$$

for $t \in (0,T)$. Since $u = a$ on $\partial\Omega$, it is obvious that

$$\left|\frac{1}{m}\int_{\partial\Omega}\frac{\partial\varphi_\delta}{\partial\nu}\cdot u^m\right| \le C_1(\alpha)$$

with $C_1(\alpha)$ independent of δ and T. Next, as a classical solution for $t \in (0,T)$, u is bounded from above and below in any fixed neighbourhood $U \subset \Omega$ of $\partial\Omega$ for $t \in (0,T')$, which by parabolic regularity theory [12] entails that $\frac{\partial u^m}{\partial\nu}$ is bounded on $\partial\Omega \times (0,T')$ and hence

$$\left|\frac{1}{m}\int_{\partial\Omega}\varphi_\delta\cdot\frac{\partial u^m}{\partial\nu}\right| \le C_2(\alpha,T')$$

holds for all $t \in (0,T')$, where $C_2(\alpha,T')$ is independent of $\delta \in (0,1)$. Altogether, using (1.6), we find from (1.7) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|x|-\delta)_+^\alpha u(x,t)\mathrm{d}x \le C_1(\alpha) + C_2(\alpha,T') + \frac{\alpha(n+\alpha-2)}{m}\int_\Omega(|x|-\delta)_+^{\alpha-2}u^m$$

for $t \in (0,T')$. By Young's inequality, this yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|x|-\delta)_+^\alpha u(x,t)\mathrm{d}x &\le C_1(\alpha) + C_2(\alpha,T') + \alpha(n+\alpha-2)\int_\Omega(|x|-\delta)_+^\alpha u\\ &\quad + \frac{(1-m)\alpha(n+\alpha-2)}{m}\int_\Omega(|x|-\delta)_+^{\alpha-\frac{2}{1-m}}\end{aligned}$$

for such t. Since $\alpha - \frac{2}{1-m} > -\frac{2}{1-m} > -n$ due to the fact that $m < m_c$, we conclude upon integrating this differential inequality and using (0.3) that

$$\begin{aligned}\int_\Omega(|x|-\delta)_+^\alpha u(x,t)\mathrm{d}x &\le \left(\int_\Omega(|x|-\delta)_+^\alpha u_0(x)\mathrm{d}x\right)\cdot e^{\beta t}\\ &\le \overline{a}\left(\int_\Omega(|x|-\delta)_+^\alpha|x|^{-\gamma_2}\mathrm{d}x\right)\cdot e^{\beta t}\end{aligned}$$

for all $t \in (0, T')$ and some $\beta = \beta(\alpha, T') > 0$ independent of $\delta \in (0, 1)$. Recalling that $\alpha - \gamma_2 > -n$, we may let $\delta \searrow 0$ to arrive at (1.5). $\square$

Building on the latter result, we can proceed to show uniqueness of singular classical solutions when $\gamma_1 > \frac{2}{1-m}$.

THEOREM 6. *Assume that* (0.3) *is valid with some* $0 < \underline{a} \le \overline{a}$ *and* $\frac{2}{1-m} < \gamma_1 \le \gamma_2 < \infty$. *Let* $T \in (0, \infty]$ *and* u *be a singular classical solution of* (0.1) *in* $\Omega \times (0, T)$. *Then,* $u \equiv u_p$ *in* $\Omega \times (0, T)$.

Proof. We pick $\alpha > \max\{2, \gamma_2 - n\}$ and multiply the equation $(u - u_p)_t = \frac{1}{m}\Delta(u^m - u_p^m)$ by $\varphi_\delta(x) = (|x| - \delta)_+^\alpha$. After an integration, this results in

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi_\delta(x)\left(u(x,t) - u_p(x,t)\right)\mathrm{d}x &= \frac{1}{m}\int_\Omega \varphi_\delta \Delta\left(u^m - u_p^m\right)\\ &= \frac{1}{m}\int_\Omega \Delta\varphi_\delta \cdot \left(u^m - u_p^m\right)\\ &\quad + \frac{1}{m}\int_{\partial\Omega} \varphi_\delta \cdot \frac{\partial\left(u^m - u_p^m\right)}{\partial\nu}\end{aligned}$$

for $t \in (0, T)$, because $u = u_p$ on $\partial\Omega$. Since we already know from Lemma 4 that

$$u \ge u_p \text{ in } \Omega \times (0, T), \tag{1.8}$$

we necessarily have $\frac{\partial(u^m - u_p^m)}{\partial\nu} \le 0$ on $\partial\Omega$. Recalling (1.6), we thus find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|x| - \delta)_+^\alpha (u(x,t) - u_p(x,t))\mathrm{d}x\\ &\quad \le \frac{\alpha(n + \alpha - 2)}{m}\int_\Omega (|x| - \delta)_+^{\alpha-2}(u^m - u_p^m)\end{aligned} \tag{1.9}$$

for $t \in (0, T)$. Now by the mean value theorem and (1.8),

$$u^m - u_p^m \le m u_p^{m-1}(u - u_p) \text{ in } \Omega \times (0, T).$$

Here, Lemma 2 says that (0.3) ensures the lower estimate $u_p \ge c_0(T)|x|^{-\gamma_1}$ with some $c_0(T) > 0$ and hence

$$\begin{aligned}u^m - u_p^m &\le m c_0^{m-1}(T)|x|^{(1-m)\gamma_1}(u - u_p)\\ &\le m c_0^{m-1}(T) \cdot R^{(1-m)\gamma_1 - 2} \cdot |x|^2(u - u_p) \quad \text{in } \Omega \times (0, T),\end{aligned}$$

where $R > 0$ is large such that $\Omega \subset B_R(0)$. Inserted into (1.9), upon another integration this gives

$$\int_\Omega (|x| - \delta)_+^\alpha (u(x,t) - u_p(x,t))\mathrm{d}x \le C_1 \int_0^t \int_\Omega (|x| - \delta)_+^{\alpha-2}|x|^2(u - u_p) \tag{1.10}$$

for all $t \in (0, T)$ with $C_1 := \alpha(n + \alpha - 2)c_0^{m-1}(T)R^{(1-m)\gamma_1 - 2}$, where we have used that both u and u_p coincide with u_0 initially. Again using (1.8), we now invoke the monotone convergence theorem in taking $\delta \searrow 0$ on both sides of (1.10) to achieve

$$I(t) := \int_\Omega |x|^\alpha (u(x,t) - u_p(x,t))\mathrm{d}x \leq C_2 \int_0^t \int_\Omega |x|^\alpha (u - u_p)$$

for $t \in (0, T)$. Since the nonnegative function I belongs to $L^\infty((0, T'))$ for all $T' \in (0, T)$ by Lemma 5, Gronwall's lemma asserts that $I \equiv 0$ in $(0, T)$, which entails the desired identity. □

Let us state an immediate consequence of the asserted uniqueness property.

COROLLARY 7. *Let* $\underline{u}_0$ *and* $\overline{u}_0$ *belong to* $C^2(\overline{\Omega}\backslash\{0\}$ *and fulfil* $\underline{u}_0|_{\partial\Omega} = \overline{u}_0|_{pO} = a$ *and*

$$\underline{a}|x|^{-\gamma_1} \leq \underline{u}_0(x) \leq \overline{u}_0(x) \leq \overline{a}|x|^{-\gamma_2} \quad \textit{for all } x \in \overline{\Omega}\backslash\{0\} \tag{1.11}$$

with positive constants $\underline{a}$ *and* $\overline{a}$ *and* $\gamma_2 \geq \gamma_1 > \frac{2}{1-m}$. *Then,* (0.1) *possesses uniquely determined global singular classical solutions* $\underline{u}$ *and* $\overline{u}$ *with initial data* $\underline{u}_0$ *and* $\overline{u}_0$, *respectively, that satisfy*

$$\underline{u}(x,t) \leq \overline{u}(x,t) \quad \textit{for all } x \in \overline{\Omega}\backslash\{0\} \quad \textit{and } t \geq 0.$$

Proof. Since $\underline{u}_{0\varepsilon} \leq \overline{u}_{0\varepsilon}$ for all $\varepsilon \in \left(0, \frac{1}{a}\right)$, applying the comparison principle to the solutions of (1.1), we find that the corresponding proper solutions $\underline{u}_p$ and $\overline{u}_p$ satisfy $\underline{u}_p \leq \overline{u}_p$. But Lemma 6 guarantees that under the assumptions (1.11) no singular solutions other than the proper solution exist. □

1.2. Continuous dependence on the initial data

We can now derive the following statement on continuous dependence of our solutions on u_0 with respect to convergence in $C^2_{\mathrm{loc}}(\overline{\Omega}\backslash\{0\})$.

LEMMA 8. *Let* $u_0 \in C^2(\overline{\Omega}\backslash\{0\})$ *satisfy* (0.3) *with positive* $\overline{a} \geq \underline{a} > 0$ *and* $\gamma_2 \geq \gamma_1 > \frac{2}{1-m}$. *Suppose that* $(u_{0,l})_{l\in\mathbb{N}} \subset C^2(\overline{\Omega}\backslash\{0\})$ *is a sequence of functions such that for all* $l \in \mathbb{N}$, $u_{0,l}$ *satisfies* (0.3) *with the same constants* $\underline{a}, \overline{a}, \gamma_1$ *and* γ_2, *and that* $u_{0,l} = a$ *on* $\partial\Omega$. *Moreover, assume that* $u_{0,l}(x) \to u_0(x)$ *in* $C^2_{\mathrm{loc}}(\overline{\Omega}\backslash\{0\})$ *as* $l \to \infty$, *and let* u_l *and* u *denote the singular classical solutions of Problem* (0.1) *with initial data* $u_{0,l}$ *and* u_0, *respectively. Then,*

(i) *If* $u_{0,l+1} \leq u_{0,l}$ *in* Ω *for all* $l \in \mathbb{N}$, *then for all* $\delta > 0$ *and* $T > 0$ *and each compact set* $K \subset \Omega\backslash\{0\}$ *there exists* $l_0 = l_0(\delta, T, K) \in \mathbb{N}$ *such that*

$$u_l(x,t) \leq u(x,t) + \delta \quad \textit{for all } x \in K \quad \textit{and } t \in [0, T] \tag{1.12}$$

holds whenever $l \geq l_0$.

(ii) *If* $u_{0,l+1} \geq u_{0,l}$ *in* Ω *for all* $l \in \mathbb{N}$, *then for all* $\delta > 0$ *and* $T > 0$ *and any compact* $K \subset \Omega \backslash \{0\}$ *one can find* $l_0 = l_0(\delta, T, K) \in \mathbb{N}$ *such that for all* $l \geq l_0$,

$$u_l(x,t) \geq u(x,t) - \delta \quad \textit{for all } x \in K \textit{ and } t \in [0,T]. \tag{1.13}$$

Proof. (i) Since $\underline{a}|x|^{-\gamma_1} \leq u_0(x) \leq u_{0,l}(x) \leq \overline{a}|x|^{-\gamma_2}$ in Ω, it follows from Lemma 2 and Corollary 7 that both u and u_l exist globally in time and, given $T \in (0,\infty)$, satisfy

$$c_0(T)|x|^{-\gamma_1} \leq u(x,t) \leq u_l(x,t) \leq c_1(T)|x|^{-\gamma_2}$$
$$\text{for all} \quad x \in \overline{\Omega} \backslash \{0\} \quad \text{and } t \in [0,T] \tag{1.14}$$

with positive constants $c_0(T)$ and $c_1(T)$. Moreover, $u_{l+1} \leq u_l$, hence

$$u_l \searrow \hat{u} \quad \text{as } l \to \infty$$

holds in $(\overline{\Omega} \backslash \{0\}) \times ([0,\infty)$ with some limit function $\hat{u}$ fulfilling $\hat{u} \geq u$. Due to (1.14), parabolic regularity theory [12] can be applied to assert that $(u_l)_{l \in \mathbb{N}}$ is relatively compact in both $C^0_{\mathrm{loc}}((\overline{\Omega} \backslash \{0\}) \times [0,\infty))$ and $C^{2,1}((\Omega \backslash \{0\}) \times (0,\infty))$ and that accordingly the convergence $u_l \to \hat{u}$ actually takes place in these spaces. Therefore, it is clear that

$$\hat{u}_t = \nabla \cdot \left(\hat{u}^{m-1} \nabla \hat{u}\right) \quad \text{in } (\Omega \backslash \{0\}) \times (0,\infty),$$

that $\hat{u}(x,0) = u_0(x)$ holds for all $x \in \overline{\Omega} \backslash \{0\}$, and that $\hat{u}(x,t) = a$ is valid for all $x \in \partial\Omega$ and $t > 0$. Moreover, (1.14) guarantees that $\hat{u}(x,t) \geq c_0(T)|x|^{-\gamma_1}$ in $(\overline{\Omega} \backslash \{0\}) \times [0,T]$, which shows that $\hat{u} \in C^0(\overline{\Omega} \times [0,T]; (0,+\infty])$ for all $t > 0$ with $\hat{u}(0,t) = +\infty$ for each $t \geq 0$. Thus, $\hat{u}$ is a singular classical solution of (0.1) in the sense of Definition 1 and hence must coincide with u. Since this implies $u_l \to u$ in $C^0(K \times [0,T])$ as $l \to \infty$, the conclusion (1.12) is an immediate consequence.

(ii) This part can be proved similarly. □

1.3. Stabilizing and growing up solutions

For a proof of the following lemma on grow up of solutions emanating from sufficiently strongly singular initial data, we refer to [14, Theorem 3.13].

LEMMA 9. *Assume that* u_0 *satisfies* (0.3) *with* $\overline{a} \geq \underline{a} > 0$ *and* $\gamma_2 \geq \gamma_1 > \frac{n-2}{m}$. *Then the singular classical solution* u *of* (0.1) *satisfies*

$$u(\cdot,t) \to \infty \textit{ locally uniformly in } \Omega \textit{ as } t \to \infty$$

in the sense that for each compact $K \subset \Omega$, $\inf_{x \in K} u(x,t) \to \infty$ *as* $t \to \infty$.

We next consider initial data with a singularity that is strong enough to persist, but weak enough to be smoothed out in the limit $t \to \infty$. As seen in [14, Lemma 3.9, Theorem 3.11], in the framework of initial data satisfying (0.3) this is possible only when $m < m_c$ and $\frac{2}{1-m} < \gamma_1 \leq \gamma_2 < \frac{n-2}{m}$. In a slightly smaller range of γ_1, γ_2—which is yet nonempty whenever $m < m_c$—we can show that the corresponding solution in fact stabilizes towards a constant as $t \to \infty$.

LEMMA 10. *Let* u_0 *fulfil* (0.3) *with* $0 < \underline{a} \le \overline{a}$ *and* $\frac{2}{1-m} < \gamma_1 \le \gamma_2 < n$*, and assume that* $u_0 \ge a$ *in* Ω *as well as* $u_0|_{\partial\Omega} = a$*. Then there exists a sequence of times* $t_k \to \infty$ *such that the singular classical solution* u *of* (0.1) *satisfies*

$$u(\cdot, t_k) \to a \text{ locally uniformly in } \overline{\Omega}\backslash\{0\} \tag{1.15}$$

as $k \to \infty$.

Proof. Since u is unique by Lemma 6, it is sufficient to prove (1.15) for the proper solution $u = u_p = \lim_{\varepsilon\searrow 0} u_\varepsilon$ as defined through (1.1). Let us fix $\alpha > 1$ such that $\alpha\gamma_2 < n$ and multiply (1.1) by $u_\varepsilon^{\alpha-1}$. Using that $u_0 \ge a$ implies $u_\varepsilon \ge a$ by comparison, we obtain that $\frac{\partial u_\varepsilon}{\partial\nu} \le 0$ on $\partial\Omega$, and thus we find after integrating by parts that

$$\begin{aligned}\frac{1}{\alpha}\int_\Omega u_\varepsilon^\alpha(x,t)\mathrm{d}x + (\alpha-1)\int_0^t\int_\Omega u_\varepsilon^{m+\alpha-3}|\nabla u_\varepsilon|^2 \\ = \frac{1}{\alpha}\int_\Omega u_{0\varepsilon}^\alpha(x)\mathrm{d}x + \int_{\partial\Omega} u_\varepsilon^{m+\alpha-2}\frac{\partial u_\varepsilon}{\partial\nu} \le \frac{1}{\alpha}\int_\Omega u_0^\alpha(x)\mathrm{d}x\end{aligned}$$

for all $t > 0$. Since $\alpha\gamma_2 < n$, the right-hand side is bounded from above, whence we may take $\varepsilon\searrow 0$ and then $t\to\infty$ to obtain, using Fatou's lemma, that $\int_0^\infty\int_\Omega u^{m+\alpha-3}|\nabla u|^2$ is finite. Therefore, along some sequence $t_k \to \infty$, we have $u^{\frac{m+\alpha-1}{2}}(\cdot, t_k) \to c$ in $W^{1,2}(\Omega)$ with some $c > 0$. Since Lemma 2 in combination with standard parabolic regularity theory [12] implies that $(u(\cdot,t))_{t>1}$ is relatively compact in $C^0_{\mathrm{loc}}(\overline{\Omega}\backslash\{0\})$ with $u(\cdot,t)|_{\partial\Omega} \equiv a$, we must have $c = a^{\frac{m+\alpha-1}{2}}$, and hence (1.15) follows. □

2. Construction of oscillating solutions

Proof of Theorem 1. Let us fix constants γ_1 and γ_2 satisfying the conditions (0.6), which is possible since $m < m_c$. Our plan is to construct u_0 as the limit of a recursively defined sequence of data $u_{0,j}$ that lead to solutions u_j which along some sequence $t_j \to \infty$ approach $+\infty$ if j is odd, and a if j is even.

As a preparation, we choose a nondecreasing cut-off function $\chi \in C^\infty(\mathbb{R})$ such that $\chi \equiv 0$ in $(-\infty, \frac{1}{2})$ and $\chi \equiv 1$ in $(1,\infty)$. For $\delta > 0$ and $x \in \overline{\Omega}$, we shall abbreviate $\chi_\delta(x) := \chi(|x|/\delta)$. We then put $\psi = 1 - \chi$ and $\psi_\delta = 1 - \chi_\delta$. Then, $0 \le \psi_\delta \le 1$ and $\psi_\delta = 1$ for $|x| \le \delta/2$, $\psi_\delta = 0$ for $|x| \ge \delta$. Moreover, we let $\Omega_j := \left\{x \in \Omega \mid |x| > \frac{1}{j} \text{ and } \operatorname{dist}(x, \partial\Omega) > \frac{1}{j}\right\}$ for $j \in \mathbb{N}$.

STEP 1 Construction of $u_{0,0}$ and $u_{0,1}$.

For convenience, let us set $u_{0,0}(x) := a$ for all $x \in \overline{\Omega}$, and pick any $\varepsilon_1 > 0$. Then there exists $\delta_1 > 0$ such that $\varepsilon_1|x|^{-\gamma_2} \ge u_{0,0}(x)$ for all $x \in B_{\delta_1}(0)$, which implies that the combination

$$u_{0,1}(x) := (1 - \psi_{\delta_1}(x))\, u_{0,0}(x) + \varepsilon_1\psi_{\delta_1}(x)|x|^{-\gamma_2}, \quad x \in \overline{\Omega}\backslash\{0\},$$

belongs to $C^\infty(\overline{\Omega}\backslash\{0\})$ and satisfies $u_{0,1} \ge a$ in Ω as well as $u_{0,1}|_{\partial\Omega} = a$.

STEP 2 Iterative construction of $u_{0,2i}$ and $u_{0,2i+1}$ for $i \ge 1$.

At each step, we perform a modification of the previous initial function in smaller neighbourhood of the origin, oscillating between the two desired singular rates, $|x|^{-\gamma_1}$ and $|x|^{-\gamma_2}$, depending on even or odd subindex. A delicate choice of radii and constants is needed for the solutions to behave in an oscillating way in time as expected.

Suppose that for some $i \ge 1$, we have already found numbers $\varepsilon_1, \dots, \varepsilon_{2i-1}$, $\delta_1, \dots, \delta_{2i-1}$, $t_2 > 2, \dots, t_{2i-1} > 2i-1$ and functions $u_{0,0}, \dots, u_{0,2i-1} \in C^\infty(\overline{\Omega}\backslash\{0\})$ with the properties $u_{0,0} \equiv a$,

$$\begin{aligned} &\left\{ \begin{array}{l} u_{0,j}(x) = (1-\psi_{\delta_j}(x))\, u_{0,j-1}(x) + \varepsilon_j \psi_{\delta_j}(x)|x|^{-\gamma_2} \\ \text{and } u_{0,j-1}(x) \le \varepsilon_j |x|^{-\gamma_2} \quad \text{for } x \in B_{\delta_j}(x)\backslash\{0\} \end{array} \right\} \\ &\qquad\qquad\qquad\qquad \text{if } j \in \{1, \dots, 2i-1\} \text{ is odd}, \\ &\left\{ \begin{array}{l} u_{0,j}(x) = (1-\psi_{\delta_j}(x))\, u_{0,j-1}(x) + \frac{1}{\varepsilon_j} \psi_{\delta_j}(x)|x|^{-\gamma_1} \\ \text{and } u_{0,j-1}(x) \ge \frac{1}{\varepsilon_j} |x|^{-\gamma_1} \ge a \quad \text{for } x \in B_{\delta_j}(x)\backslash\{0\} \end{array} \right\} \\ &\qquad\qquad\qquad\qquad \text{if } j \in \{2, \dots, 2i-2\} \text{ is even}, \end{aligned} \tag{2.1}$$

where the constants ε_j, δ_j satisfy

$$0 < \varepsilon_j < \frac{\varepsilon_{j-1}}{2} \quad \text{if } j \in \{2, \dots, 2i-1\}, \tag{2.2}$$

$$0 < \delta_j < \frac{\delta_{j-1}}{\sqrt{2}} \quad \text{if } j \in \{2, \dots, 2i-1\}, \tag{2.3}$$

Let u_j be the singular classical solutions of (0.1) with $u_j|_{t=0} = u_{0,j}$. We also assume that at the selected times $t_2, \dots, t_{2i-1}$, they exhibit the following oscillatory behaviour:

$$\begin{aligned} &u_j(x, t_j) \ge j \quad \text{for all } x \in \Omega_j \quad \text{if } j \in \{2, \dots, 2i-2\} \text{ is even}, \\ &u_j(x, t_j) \le a + \frac{1}{j} \quad \text{for all } x \in \Omega_j \quad \text{if } j \in \{3, \dots, 2i-1\} \text{ is odd}. \end{aligned} \tag{2.4}$$

In order to fulfil the induction step, we now have to construct $u_{0,2i}$ and $u_{0,2i+1}$ with the same properties and prove that the solutions have the same oscillating behaviour at suitable times t_{2i}, t_{2i+1}. Let us examine the details.

(2A) To construct $u_{0,2i}$, we observe that due to the first identity in (2.1) we have

$$u_{0,2i-1}(x) = \varepsilon_{2i-1}|x|^{-\gamma_2} \quad \text{for } 0 < |x| < \frac{\delta_{2i-1}}{2}. \tag{2.5}$$

Since $\gamma_2 > \frac{n-2}{m}$, Lemma 9 implies that $u_{2i-1}(\cdot, t) \to \infty$ locally uniformly in Ω as $t \to \infty$. Thus, there exists some large $t_{2i} > 2i$, $t_{2i} > t_{2i-1}$, such that

$$u_{2i-1}(x, t_{2i}) \ge 2i + 1 \quad \text{for all } x \in \Omega_{2i}. \tag{2.6}$$

Now again due to (2.5) and the fact that $\gamma_1 < \gamma_2$, for each $\varepsilon > 0$, we can find $\delta(\varepsilon) > 0$ such that $u_{0,2i-1}(x) \ge \frac{1}{\varepsilon}|x|^{-\gamma_1} \ge a$ for all $x \in B_{\delta(\varepsilon)}(0)\backslash\{0\}$. Accordingly, if we let $\varepsilon^{(l)} := \frac{1}{l}$ for $l \in \mathbb{N}$, then we can pick a decreasing sequence $(\delta^{(l)})_{l\in\mathbb{N}} \subset (0,\infty)$ such that $\delta^{(l)}\searrow 0$ as $l \to \infty$ and $u_{0,2i-1}(x) \ge \frac{1}{\varepsilon^{(l)}}|x|^{-\gamma_1} \ge a$ for all $x \in B_{\delta^{(l)}}(0)\backslash\{0\}$—in fact, one may employ

$$\delta^{(l)} := \min\left\{\frac{\delta_{2i-1}}{2},\ \left(\frac{\varepsilon_{2i-1}}{l}\right)^{\frac{1}{\gamma_2-\gamma_1}},\ (\varepsilon_{2i-1}a)^{-\frac{1}{\gamma_1}}\right\}$$

for this purpose. This implies that

$$u^{(l)}_{0,2i}(x) := u_{0,2i-1}(x) - \psi_{\delta^{(l)}}\left(u_{0,2i-1}(x) - \frac{1}{\varepsilon^{(l)}}|x|^{-\gamma_1}\right), \quad x \in \overline{\Omega}\backslash\{0\},$$

defines an increasing sequence $(u^{(l)}_{0,2i})_{l\in\mathbb{N}} \subset C^\infty(\overline{\Omega}\backslash\{0\})$ of initial data $u^{(l)}_{0,2i} \le u_{0,2i-1}$ which evidently converge to $u_{0,2i-1}$ in $C^\infty_{\mathrm{loc}}(\overline{\Omega}\backslash\{0\})$ as $l \to \infty$, because for each compact $K \subset \overline{\Omega}\backslash\{0\}$ we have $u^{(l)}_{0,2i} \equiv u_{0,2i-1}$ in K for all sufficiently large l. Therefore, we can use the continuous dependence result of Lemma 8 (ii) to make sure that there exists $l_0 \in \mathbb{N}$ such that if $l \ge l_0$ then the singular classical solution $u^{(l)}_{2i}$ of (0.1) with $u^{(l)}_{2i}|_{t=0} = u^{(l)}_{0,2i}$ satisfies $u^{(l)}_{2i} \ge u_{2i-1} - 1$ in $\overline{\Omega}_{2i} \times [0, t_{2i}]$. Choosing l appropriately large, we thus infer that there exist $\varepsilon_{2i} \in (0, \frac{\varepsilon_{2i-1}}{2})$ and $\delta_{2i} \in (0, \frac{\delta_{2i-1}}{\sqrt{2}})$ such that

$$u_{0,2i-1}(x) \ge \frac{1}{\varepsilon_{2i}}|x|^{-\gamma_1} \ge a \text{ for all } x \in B_{\delta_{2i}}(0)\backslash\{0\}, \tag{2.7}$$

and such that the singular classical solution u_{2i} of (0.1) emanating from

$$u_{0,2i}(x) := u_{0,2i-1}(x) - \psi_{\delta_{2i}}(x)\left(u_{0,2i-1}(x) - \frac{1}{\varepsilon_{2i}}|x|^{-\gamma_1}\right), \quad x \in \overline{\Omega}\backslash\{0\}, \tag{2.8}$$

satisfies $u_{2i}(\cdot, t_{2i}) \ge u_{2i-1}(\cdot, t_{2i}) - 1$ in Ω_{2i}; hence, by (2.6),

$$u_{2i}(x, t_{2i}) \ge 2i \quad \text{for all } x \in \Omega_{2i}. \tag{2.9}$$

Observe that by (2.8), (2.9) and our choice of ε_{2i} and δ_{2i}, the requirements (2.1)–(2.4) are now fulfilled also up to $j = 2i$.

(2B) Pursuing the same basic idea but referring to Lemmas 10 and 8 (i) rather than to Lemmas 9 and 8 (ii), we proceed to define $u_{0,2i-1}$ as follows: According to (2.8),

$$u_{0,2i}(x) = \frac{1}{\varepsilon_{2i}}|x|^{-\gamma_1} \quad \text{for } 0 < |x| < \frac{\delta_{2i}}{2}, \tag{2.10}$$

so that since $\gamma_1 < n$, Lemma 10 says that $u_{2i}\left(\cdot, t^{(k)}\right) \to a$ in $L^\infty_{\mathrm{loc}}(\overline{\Omega}\backslash\{0\})$ along some sequence $t^{(k)} \to \infty$, and therefore

$$u_{2i}(x, t_{2i-1}) \le a + \frac{1}{2(2i+1)} \quad \text{for all } x \in \Omega_{2i+1} \tag{2.11}$$

is valid with some large $t_{2i+1} > 2i + 1$.

Arguing as above, writing $\varepsilon^{(l)} := \frac{1}{l}$ for $l \in \mathbb{N}$ we find $(\hat{\delta}^{(l)})_{l\in\mathbb{N}} \subset (0,\infty)$ satisfying $\hat{\delta}^{(l)} \searrow o$ as $l \to \infty$ and

$$u_{0,2i}(x) \le \varepsilon^{(l)}|x|^{-\gamma_2} \quad \text{for all } x \in B_{\hat{\delta}^{(l)}}(0)\backslash\{0\}. \tag{2.12}$$

Consequently, $u^{(l)}_{0,2i+1} \in C^\infty(\overline{\Omega}\backslash\{0\})$, as defined by

$$u^{(l)}_{0,2i+1}(x) := u_{0,2i}(x) + \psi_{\hat{\delta}^{(l)}}(x)\left(\varepsilon^{(l)}|x|^{-\gamma_2} - u_{0,2i}(x)\right), \quad x \in \overline{\Omega}\backslash\{0\},$$

decreases to $u_{0,2i}$ as $l \to \infty$, this convergence also taking place in $C^\infty_{\text{loc}}(\overline{\Omega}\backslash\{0\})$. Invoking Lemma 8 (i), we obtain that if l is large, then the singular classical solution $u_{2i+1} := u^{(l)}_{2i+1}$ of (0.1) evolving from $u_{0,2i+1} := u^{(l)}_{0,2i+1}$, that is, from

$$u_{0,2i+1}(x) := u_{0,2i}(x) + \left(1 - \chi_{\delta_{2i+1}}(x)\right)\left(\varepsilon_{2i+1}|x|^{-\gamma_2} - u_{0,2i}(x)\right), \quad x \in \overline{\Omega}\backslash\{0\},$$

with $\varepsilon_{2i+1} := \varepsilon^{(l)} \in \left(0, \frac{\varepsilon_{2i}}{2}\right)$ and $\delta_{2i+1} := \hat{\delta}^{(l)} \in \left(0, \frac{\delta_{2i}}{\sqrt{2}}\right)$, satisfies $u_{2i+1}(\cdot, t_{2i+1}) \le u_{2i}(\cdot, t_{2i+1}) + \frac{1}{2(2i+1)}$ in Ω_{2i+1}. Hence, (2.11) implies

$$u_{2i+1}(x, t_{2i+1}) \le a + \frac{1}{2i+1} \quad \text{for all } x \in \Omega_{2i+1},$$

and recalling (2.12) we see that (2.1)–(2.4) become valid even up to $j = 2i + 1$.

Step 3 Construction and properties of u_0.

Since $\delta_j \searrow 0$ y (2.3), it follows from (2.1) that in each compact subset of $\bar{\Omega}\backslash\{0\}$, $u_{0,j} \equiv u_{0,j-1}$ holds for all sufficiently large $j \in \mathbb{N}$. Therefore, trivially, $u_{0,j}$ converges to some limit function u_0 in $C^\infty_{\text{loc}}(\overline{\Omega}\backslash\{0\})$. In order to gain more information about u_0, we claim that $(u_{0,j})_{j\in\mathbb{N}}$, besides

$$u_{0,0} \le u_{0,1}, \quad u_{0,1} \ge u_{0,2}, \quad u_{0,2} \le u_{0,3}, \quad u_{0,3} \ge u_{0,4}, \ldots, \tag{2.13}$$

enjoys the ordering properties

$$u_{0,0} \le u_{0,2} \le u_{0,4} \le \cdots \quad \text{in } \overline{\Omega}\backslash\{0\} \tag{2.14}$$

and

$$u_{0,1} \ge u_{0,3} \ge u_{0,5} \ge \cdots \quad \text{in } \overline{\Omega}\backslash\{0\}. \tag{2.15}$$

Indeed, whereas (2.13) is obvious from (2.1), to see (2.14) we let $j \in \mathbb{N}$ be an even nonnegative integer and suppose that $x \in \overline{\Omega}\backslash\{0\}$. In the case $|x| \ge \delta_{j+2}$, we know that $\chi_{\delta_{j+2}}(x) = 1$ and hence, by (2.1) and (2.13),

$$u_{0,j+2}(x) = u_{0,j+1}(x) \le u_{0,j}(x)$$

for such x.

If $|x| < \delta_{j+2}$, however, (2.1) says that in the case $j = 0$ we have $u_{0,j+2}(x) \geq a = u_{0,0}(x)$ as desired, while if $j \geq 2$ then $u_{0,j+1}(x) \geq \frac{1}{\varepsilon_{j+2}}|x|^{-\gamma_1}$ and therefore

$$\begin{aligned} u_{0,j+2}(x) &= \chi_{\delta_{j+2}}(x) \cdot u_{0,j+1}(x) + \big(1 - \chi_{\delta_{j+2}}(x)\big) \cdot \frac{1}{\varepsilon_{j+2}}|x|^{-\gamma_1} \\ &\geq \chi_{\delta_{j+2}}(x) \cdot \frac{1}{\varepsilon_{j+2}}|x|^{-\gamma_1} + \big(1 - \chi_{\delta_{j+2}}(x)\big) \cdot \frac{1}{\varepsilon_{j+2}}|x|^{-\gamma_1} \\ &= \frac{1}{\varepsilon_{j+2}}|x|^{-\gamma_1}. \end{aligned} \tag{2.16}$$

Since $|x| < \delta_{j+2}$ entails $|x| < \frac{\delta_j}{2}$ by (2.3), on the other hand we have $\chi_{\delta_j}(x) = 0$ and thus

$$u_{0,j}(x) = \frac{1}{\varepsilon_j}|x|^{-\gamma_1} \tag{2.17}$$

for these x. As $\varepsilon_{j+2} < \varepsilon_j$ by (2.2), (2.16) and (2.17) complete the proof of (2.14), and the inequalities in (2.15) can be seen quite similarly.

STEP 4 Conclusion.

Now, (2.14) and (2.15) in particular imply that $u_{0,2} \leq u_0 \leq u_{0,1}$, from which it follows that (0.3) is true for u_0 with some positive constants $\underline{a}$ and $\overline{a}$. Accordingly, (0.1) possesses a unique singular classical solution u with initial data u_0. To see that this solution exhibits the claimed oscillatory behaviour, we observe that (2.14) also entails that for all even j, $u_{0,j} \leq u_0$ and thus, by Corollary 7, that $u_j \leq u$. In view of (2.4), this implies that

$$u(x, t_j) \geq j \quad \text{for all } x \in \Omega_j \quad \text{if } j \in \mathbb{N} \text{ is even,} \tag{2.18}$$

so that $u(\cdot, t_{2i}) \to \infty$ locally uniformly in $\Omega\backslash\{0\}$ as $i \to \infty$. Similarly, (2.15) yields

$$u(x, t_j) \leq a + \frac{1}{j} \quad \text{for all } x \in \Omega_j \quad \text{if } j \in \mathbb{N} \text{ is odd.} \tag{2.19}$$

Since (2.14) also implies that $u_0 \geq u_{0,0} \equiv a$, one more comparison argument shows that $u \geq a$ in $\Omega \times (0, \infty)$, whence (2.19) entails that $u(\cdot, t_{2i+1}) \to a$ in $L^\infty_{\mathrm{loc}}(\Omega\backslash\{0\})$ as $i \to \infty$ and thereby completes the proof. □

3. Extensions

There are a number of variants and improvements that can be made by using the previous ideas. One of them is to have solutions that exhibit three types of behaviour at sequences of time, namely diverging, stabilizing to the nonsingular steady state and finally stabilizing to the singular steady state. This is a more complex variant of the previous result, where only the two first forms are represented.

Another possibility is to refine the analysis and distinguish different time sequences where the solution diverges as time grows to infinity with different rates.

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Appendix: absence of oscillations in the case $m \in (m_c, 1]$

Let us finally demonstrate that oscillations in the style detected above do not occur in the problem (0.1) when $m \in (m_c, 1]$. As in the previous part of the paper, we restrict our considerations to the proper solution of (0.1), that is, we let u_p denote the limit of solutions u_ε of (1.1) as $\varepsilon \searrow 0$, and, particularly when $m = 1$, ignore the question whether or not u_p satisfies (0.1) in a reasonable sense.

THEOREM 11. *Let $m \in (m_c, 1]$, and assume that $u_0 \in C^0(\overline{\Omega})$ is continuous with values in $(0, +\infty]$ and such that $u_0|_{\partial\Omega} = a$. Then as $t \to \infty$, the proper solution u_p of* (0.1) *satisfies*

$$u_p(\cdot, t) \to \begin{cases} a & \text{uniformly in } \Omega \quad \text{if } \int_\Omega u_0 < \infty, \\ +\infty & \text{locally uniformly in } \Omega \quad \text{if } \int_\Omega u_0 = \infty. \end{cases} \tag{4.20}$$

Proof. (i) Let us first consider the case when $\int_\Omega u_0 < \infty$. Then, well-known smoothing results [9] state that (0.1) has a unique weak solution which clearly coincides with u_p and is smooth and bounded in $\Omega \times (\tau, \infty)$ for any $\tau > 0$. Hence, a standard energy argument can be applied to show that $u_p(\cdot, t)$ approaches the unique steady state $u_\infty \equiv a$ of (0.1).

(ii) On the other hand, if $\int_\Omega u_0 = +\infty$, then in the case $m = 1$ it can easily be seen upon representing u_ε via a convolution involving Green's function of the heat semigroup on Ω that since $\int_\Omega u_{0\varepsilon} \nearrow +\infty$, in view of the monotone convergence theorem we must have $u_\varepsilon(x, t) \nearrow +\infty$ for all $(x, t) \in \Omega \times (0, \infty)$ as $\varepsilon \searrow 0$. Thus, in this situation, we actually have immediate global blow-up, that is, $u_p(x, t) \equiv +\infty$ in $\Omega \times (0, \infty)$, so that (4.20) is trivially satisfied.

(ii-b) We are thus left with the case $\int_\Omega u_0 = \infty$ when $m \in (m_c, 1)$, in which we can proceed as follows: Let us fix $\underline{u}_0 \in C^0(\mathbb{R}^n; (0, +\infty])$ by letting $\underline{u}_0(x) := \frac{1}{8} u_0(x)$ if $x \in \overline{\Omega}$ and $\underline{u}_0(x) := \frac{a}{8}$ else. Then, for some $x_0 \in \Omega$, we necessarily have $\int_{B_r(0)} \underline{u}_0(x)dx = +\infty$ for all $r > 0$. Accordingly, Theorem 2.2 and Lemma 2.1 in [5] show that the corresponding proper solution $\underline{u}$ of $\underline{u}_t = \nabla \cdot (\underline{u}^{m-1} \nabla \underline{u})$ in $\mathbb{R}^n$ with initial data $\underline{u}_0$ satisfies

$$\underline{u}(x,t) \geq c_1 \left(\frac{t}{|x-x_0|^2} \right)^{\frac{1}{1-m}} \quad \text{for all } x \in \mathbb{R}^n \text{ and } t > 0 \tag{4.21}$$

with some $c_1 > 0$. Since moreover $\underline{u}_0(x) \leq \frac{a}{4}$ holds for all $x \in \mathbb{R}^n \backslash K$ with a compact subset K of Ω, Theorem 2.2 along with Definition 1 in [5] entails that $\underline{u}$ is continuous in $(\mathbb{R}^n \backslash K) \times [0, \infty)$ and thus, in particular, there exists $\tau > 0$ such that $\underline{u}(x,t) \leq \frac{a}{2}$ holds for all $x \in \partial\Omega$ and each $t \in (0, \tau)$. By comparison, we therefore have $\underline{u}(x,t) \leq u_p(x,t)$ for all $x \in \Omega$ and $t \in (0,\tau)$, so that (4.21) entails that

$$u_p(x,\tau) \geq c_2 |x-x_0|^{\frac{2}{1-m}} \quad \text{for all } x \in \Omega$$

is valid with a positive constant c_2. But now Theorem 3.13 in [14] becomes applicable to ensure that $u_p(\cdot,t) \to +\infty$ uniformly with respect to $x \in \Omega$. □

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