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Complexity of asymptotic behavior of the porous medium equation in $\mathbb{R}^N$

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Abstract. In this paper, we consider the complexity of large time behavior of solutions to the porous medium equation $u_t - \Delta u^m = 0$ in $\mathbb{R}^N$ with $m > 1$. We first show that for any given $0 < \mu < \frac{2N}{N(m-1)+2}$ and $\beta > \frac{2-\mu(m-1)}{4}$, the ω-limit set of $t^{\frac{\mu}{2}} u(t^\beta \cdot, t)$ includes all of the nonnegative functions f in the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ with $f(0) = 0$. Furthermore, we prove that, for a given countable subset E of the interval $\left(0, \frac{2N}{(N(m-1)+2)(2+\mu(m-1))}\right)$, there exists an initial value $u_0(x)$ such that for all μ and β satisfying $0 < \mu < \frac{2N}{N(m-1)+2}$, $\beta > \frac{2-\mu(m-1)}{4}$ and $\frac{\mu}{2\beta} \in E$, the ω-limit set of $t^{\frac{\mu}{2}} u(t^\beta \cdot, t)$ is equal to $C_0^+(\mathbb{R}^N) \equiv \{f \in C_0(\mathbb{R}^N);\ f(x) \geq 0,\ f(0) = 0\}$.

1. Introduction

In this paper, we consider the complexity of asymptotic behavior of solutions of the Cauchy problem for the porous medium equation

$$\frac{\partial u}{\partial t} - \Delta u^m = 0, \qquad \text{in} \quad \mathbb{R}^N \times (0, \infty), \tag{1.1}$$

$$u(x, t) = u_0(x), \qquad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

where $m > 1$ is a physical constant.

The asymptotic behavior of solutions of (1.1) has attracted a lot of attentions of mathematicians for a long time, and many interesting results have been obtained (see [2,6–14] and the references therein). In 2003, Vázquez [11] investigated the convergence of solutions to source-type solutions and proved that

$$\lim_{t\to\infty} \|u(x, t) - U_M(x, t)\|_{L^1(\mathbb{R}^N)} = 0$$

and

$$\lim_{t\to\infty} t^{\frac{N}{N(m-1)+2}} \|u(x, t) - U_M(x, t)\|_{L^\infty(\mathbb{R}^N)} = 0,$$

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where $U_M(x,t)$ is the source-type solution with the same mass as that of $u_0(x)$, provided that the initial value satisfies

$$u_0(x) \in L^1(\mathbb{R}^N) \quad \text{and} \quad u_0(x) \geq 0.$$

Such results reflect one of the important features to the asymptotic behavior of solutions, see also [2,7,8] and [10]. However, it has been shown that there are much more fruitful characteristics of asymptotic behavior of solutions. In other words, the complexity of asymptotic behavior happens for solutions of this kind of equations. For example, Vázquez and Zuazua [14] revealed that if $u(x,t)$ is a solution of (1.1)–(1.2), then the set of accumulation points of the spatially rescaled version of the solution

$$u(\sqrt{t}\cdot, t)$$

can contain a family of functions $\{S(1)\phi_j(x)\}_{j\geq 1}$ in the weak-star topology of $L^\infty(\mathbb{R}^N)$. This complexity reflects the intricate structure of the equation and that there might not exist a simple model for the asymptotic attractor. A more general rescaled version of the solution

$$t^{\frac{\mu}{2}} u(t^\beta \cdot, t) \tag{1.3}$$

can present more complexity of asymptotic behavior in the linear case of the Eq. (1.1), i.e., $m = 1$, and this has recently been discovered by Cazenave, Dickstein and Weissler in their recent works, see [3–5].

In this paper, we are concerned with the complexity of asymptotic behavior of solutions for the general case, i.e., $m > 1$, of the Eq. (1.1). Precisely speaking, we are interested in the set of accumulation points of (1.3) in $L^\infty(\mathbb{R}^N)$. We are also interested in the set of accumulation points of the space-time dilations of solutions

$$\lambda^\mu u(\lambda^{2\beta} x, \lambda^2 t) \tag{1.4}$$

in $L^\infty([\epsilon, T); L^\infty(\mathbb{R}^N))$ arguments. It is worthy of noticing that special form of (1.4) has been used to the discussion of asymptotic behavior of solutions. For example, Vázquez [11] took

$$\mu = \frac{2N}{N(m-1)+2}, \qquad \beta = \frac{1}{N(m-1)+2}$$

in (1.4) to explore the asymptotic convergence and Zuazua and Vázquez [14] chose

$$\mu = 0, \qquad \beta = \frac{1}{2}$$

in (1.4) to investigate the complexity of asymptotic behavior. In our arguments, we will take all

$$0 < \mu < \frac{2N}{N(m-1)+2} \tag{1.5}$$

and

$$\beta > \frac{2-\mu(m-1)}{4} > \frac{1}{N(m-1)+2} \tag{1.6}$$

such that $\frac{\mu}{2\beta}$ is in a countable subset of $\left(0, \frac{2N}{(N(m-1)+2)(2+\mu(m-1))}\right)$ in (1.4) to study the complexity of asymptotic behavior. The main difficulties of this paper come from the following three aspects: Obviously, the first one is the nonlinearity of the porous medium equation comparing to the works of Cazenave et al. (see [3–5]), where the heat equation was considered; The second one is the more general version of rescaling for the solutions. Vázquez [11] and Zuazua et al. [14] only considered the rescaling for fixed μ and fixed β. But, in our arguments, μ and β can have infinite numbers of values in some intervals; Lastly, our working space is more strong than that in the previous literatures. For example, Vázquez and Zuazua [14] consider the complexity of asymptotic behavior of the porous medium equation in the weak-star topology of $L^\infty(\mathbb{R}^N)$, while in this paper, we study the complexity of asymptotic behavior of porous medium equation in $L^\infty(\mathbb{R}^N)$. To overcome these difficulties, we first make use of the commutative relations between the semigroup operators and the dilation operators, together with the property of finite propagation of the porous medium equation.

This paper is organized as follows. In Sect. 2, we first give some definitions and demonstrate the commutative relations between the semigroup operators and the dilation operators. Then, we state the main results of this paper. In Sect. 3, we give the proof of Theorem 2.8. This proof mainly depends on the finite propagation of the porous medium equation which is very similar to the behavior of the characteristic lines of the wave equation. Finally, in Sect. 4, we give the proofs of Theorem 2.10 and Theorem 2.12 and show that the asymptotic behavior of the porous medium equation is really complex.

2. The preliminaries and main results

In this section, before the proofs of our main theorems, we first give some preliminaries. We will introduce the definition of solutions and show some important properties of these solutions which will be used in the proofs of the main results of this paper and study the relations between the semigroup operators and the dilation operators which plays key role in the proofs of our theorems. In the end of this section, we present our main results of this paper.

The Cauchy problem (1.1)–(1.2) does not admit classical solutions for general data u_0 in $L^1_{loc}(\mathbb{R}^N)$ or even in a smaller class, for example the set of smooth nonnegative and rapidly decaying initial data. This is due to the fact that the equation is parabolic only for $u > 0$, but degenerate when $u = 0$. Therefore, we need to introduce a concept of generalized solutions and make sure that the problem is well posed in that class. In order to prepare for this definition, we need some concepts. For $f \in L^1_{loc}(\mathbb{R}^N)$ and

$r > 0$, let

$$|f|_r = \sup_{R \geq r} R^{-\frac{N(m-1)+2}{m-1}} \int_{|x| \leq R} f(x) \mathrm{d}x,$$

then define the space $X = X(\mathbb{R}^N)$ as

$$X \equiv \{f \in L^1_{loc}(\mathbb{R}^N); |f|_r < \infty\},$$

and equip this space with the norm $\| \cdot \|_1$. Hence, it is a Banach space. The space $X_0 = X_0(\mathbb{R}^N)$ is defined by

$$X_0 \equiv \{f \in X; \lim_{r \to \infty} |f|_r = 0\}.$$

Note that $L^1(\mathbb{R}^N) \subset X_0 \subset X \subset L^1_{loc}(\mathbb{R}^N)$ with continuous inclusions. Similarly, $C_0(\mathbb{R}^N) \subset X_0$ with continuous inclusions. At present, we introduce the *weighted spaces*

$$L^1_{\rho_\alpha}(\mathbb{R}^N) = \{f \in L^1_{loc}(\mathbb{R}^N); \int_{\mathbb{R}^N} f \rho_\alpha \mathrm{d}x < \infty\},$$

where $\rho_\alpha(x) = (1+|x|^2)^\alpha$. These weighted spaces have first appeared in [1] to observe the existence and uniqueness of solutions of (1.1)–(1.2) for the general class of initial data $u_0 \in X$. We will follow their methods to obtain the solutions which we want to study. For any $u_0 \in X_0$, we perform a standard regularization of the data into

$$u_{0n}(x) = \begin{cases} \min(n, \max(-n, u_0(x))) & \text{if} \quad |x| \leq n, \\ 0 & \text{if} \quad |x| > n. \end{cases}$$

Hence, there exists a unique strong solution u_n to the Cauchy problem of the porous medium equation with initial data u_{0n}. We say the limit

$$u(x,t) = \lim_{n \to \infty} u_n(x,t) \quad \text{in} \quad L^1_{\rho_\alpha}(\mathbb{R}^N),$$

for some $\alpha > \frac{N(m-1)+2}{2(m-1)}$, is a *solution* of (1.1)–(1.2). The existence and uniqueness of these solutions of (1.1)–(1.2) for the initial value $u_0 \in X_0$ is shown by Bénilan [1] and Vázquez [15]. Then, we list some important properties of these solutions for the use in our proofs of the main results in this paper.

PROPOSITION 2.1. [15] *The porous medium equation generates a bounded continuous semigroup in the space X_0 given by $S(t)$:*

$$u_0 \to u(x,t), \tag{2.1}$$

that is, $S(t)u_0 \in C([0,\infty); X_0)$.

PROPOSITION 2.2. [16] *When* $0 \le u_0 \in L^1(\mathbb{R}^N)$ *the solution* $u(x,t)$ *is the strong solution and satisfies the* L^1–L^∞ *smoothing effect: for every* $t > 0$,

$$u(x,t) \le C \|u_0\|_{L^1}^{\frac{1}{N(m-1)+2}} t^{-\frac{N}{N(m-1)+2}},$$

where C *is a constant dependent on* m *and* N. *Moreover, if* $u_0 \in C^\alpha(\mathbb{R}^N)$ *for some* $0 < \alpha < 1$, *we can get that* $u(x,t) \in C^\alpha([0,\infty) \times \mathbb{R}^N)$.

PROPOSITION 2.3. [1] *The maximum principle holds. That is, if* $u_0 \in L^\infty(\mathbb{R}^N)$, *then for every* $t \ge 0$,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \|u_0\|_{L^\infty(\mathbb{R}^N)}.$$

PROPOSITION 2.4. (Finite propagation property) [12] *If the initial value* $u_0 \ge 0$ *is compactly supported, so are the functions* $u(\cdot,t)$ *for every* $t > 0$. *Under these conditions, there exists a free boundary or interface which separates the regions* $\{(x,t) \in Q; u(x,t) > 0\}$ *and* $\{(x,t) \in Q; u(x,t) = 0\}$. *This interface is usually an* N*-dimensional hypersurface in* $\mathbb{R}^{N+1}$.

PROPOSITION 2.5. [15] *The comparison principle holds: if* $u_0, v_0 \in X_0$ *and* $u_0 \le v_0$, *then* $u(t) \le v(t)$ *for every* $t \in (0,\infty)$.

Next we will give the definitions of scalings and prove the commutative relations between the semigroup operators and the dilation operators. Applying a rescaling is an efficient way in the analysis of the finer asymptotic behavior of global solutions which often decay to 0 as $t \to \infty$. Assuming $\lambda, \mu, \beta > 0$, $u_0 \in L^1(\mathbb{R}^N)$, we introduce the *space-time dilation* $\Gamma_\lambda^{\mu,\beta}$

$$\Gamma_\lambda^{\mu,\beta}[u_0] \equiv D_\lambda^{\mu,\beta}[S(\lambda^2 t)u_0], \tag{2.2}$$

where the *dilation* $D_\lambda^{\mu,\beta}$ is given by

$$D_\lambda^{\mu,\beta} w(x) \equiv \lambda^\mu w(\lambda^{2\beta} x)$$

and $S(t)$ is the semigroup defined in (2.1). Letting $\lambda, \mu, \beta > 0$, $u_0 \in L^1(\mathbb{R}^N)$, we call the scaling

$$D_{\sqrt{t}}^{\mu,\beta}[S(t)u_0](x) \equiv t^{\frac{\mu}{2}} u(t^\beta x, t) \tag{2.3}$$

a *spatially dilation*. Now we adopt the following notation to denote a subset of the space $C_0(\mathbb{R}^N)$:

$$C_0^+(\mathbb{R}^N) \equiv \{f \in C_0(\mathbb{R}^N);\ f \ge 0 \quad \text{and} \quad f(0) = 0\}.$$

The Γ-limit set and the Ω-limit set will be our main objects of study in this paper. The function set

$$\gamma^{\mu,\beta}(u_0) \equiv \Big\{h \in C((0,\infty); C_0^+(\mathbb{R}^N)); \exists \lambda_n \to \infty\ s.t.\ \Gamma_\lambda^{\mu,\beta}[u_0] \xrightarrow{n\to\infty} h$$
$$\text{in } L^\infty([\epsilon,\infty); L^\infty(\mathbb{R}^N)) \text{ for all } 0 < \epsilon < T < \infty\Big\}$$

is called Γ-*limit set* and the corresponding Ω-*limit set* is given by

$$\omega^{\mu,\beta}(u_0) \equiv \left\{ f \in C_0^+(\mathbb{R}^N);\ \exists t_n \to \infty\ s.t.\ D^{\mu,\beta}_{\sqrt{t_n}}[S(t_n)u_0] \xrightarrow{n\to\infty} f \text{ in } L^\infty(\mathbb{R}^N) \right\}.$$

REMARK 2.6. This two limit sets first appeared in [4] to study the asymptotic behavior of solutions of the heat equation. Therefore, our study extends the well-known theory of the classical heat equation to a nonlinear situation, which needs a whole set of new tools.

REMARK 2.7. Letting $\mu = 0$ and $\beta = 1/2$ in (2.2) and (2.3), we have

$$\Gamma^{\mu,\beta}_\lambda[u_0] = u(\lambda x, \lambda^2 t)$$

and

$$D^{\mu,\beta}_{\sqrt{t}}[S(t)u_0] = u(\sqrt{t}x, t)$$

which have been used in [14] to investigate the complexity of asymptotic behavior of the porous medium equation. So our study can be seen as some extending to the work of [14].

Then, we study the relations between the semigroup operators $S(t)$ and the dilation operators $D^{\mu,\beta}_\lambda$. Suppose $u(x,t)$ is a solution of the problem (1.1)–(1.2) with initial value $u_0 \in X_0^+$, i.e., $u_0 \in X_0$ and $u_0 \geq 0$, and let

$$v(x,t) = (\Gamma^{\mu,\beta}_\lambda[u_0])(x) = \lambda^\mu (S(\lambda^2 t)u_0)(\lambda^{2\beta}x) = \lambda^\mu u(\lambda^{2\beta}x, \lambda^2 t), \tag{2.4}$$

where $\lambda \geq 1$, μ satisfies (1.5) and β satisfies (1.6). Combining (1.1) and (2.4), we have

$$\frac{\partial v}{\partial t} = \lambda^{2-4\beta-\mu(m-1)} \Delta v^m.$$

Letting

$$w(x,t) = v(x, \lambda^{-2+4\beta+\mu(m-1)}t), \tag{2.5}$$

we obtain that $w(x,t)$ is the solution of the following Cauchy problem

$$\begin{cases} \frac{\partial w}{\partial t} = \Delta w^m, & \text{in} \quad \mathbb{R}^N \times (0,\infty), \\ w(x,0) = \lambda^\mu u_0(\lambda^{2\beta}x) = D^{\mu,\beta}_\lambda u_0, & \text{in} \quad \mathbb{R}^N. \end{cases}$$

Notice that for any $r > 0$ and $\lambda \geq 1$,

$$|w(\cdot,0)|_r = \lambda^\mu \sup_{R\geq r} R^{-\frac{N(m-1)+2}{m-1}} \int_{|x|\leq R} |u_0(\lambda^{2\beta}x)| \mathrm{d}x \leq \lambda^{\mu+\frac{4\beta}{m-1}} |u_0|_{\lambda^{2\beta}r}.$$

We thus get that $w(x,0) \in X_0^+$. Therefore,

$$w(x,t) = S(t)[D^{\mu,\beta}_\lambda u_0](x),$$

and this implies, via (2.4) and (2.5), that

$$(\Gamma_{\lambda}^{\mu,\beta}[u_0])(x) = v(x,t) = w(x, \lambda^{2-4\beta-\mu(m-1)}t)$$
$$= S(\lambda^{2-4\beta-\mu(m-1)}t)[D_{\lambda}^{\mu,\beta}u_0](x).$$

Consequently, we get the following commutative relations between the semigroup operators $S(t)$ and the dilation operators $D_{\lambda}^{\mu,\beta}$

$$(\Gamma_{\lambda}^{\mu,\beta}[u_0])(x) = D_{\lambda}^{\mu,\beta}[S(\lambda^2 t)u_0)] = S(\lambda^{2-4\beta-\mu(m-1)}t)[D_{\lambda}^{\mu,\beta}u_0]. \qquad (2.6)$$

In particular,

$$D_{\sqrt{t}}^{\mu,\beta}[S(t)u_0)] = S(t^{1-2\beta-\frac{\mu(m-1)}{2}})[D_{\sqrt{t}}^{\mu,\beta}u_0].$$

In the last, we will present our main results of this paper. We first show that the element of Γ-limit set and the Ω-limit set is very free.

THEOREM 2.8. *Assume* $f \in \mathscr{S}^+(\mathbb{R}^N) = \{f \in \mathscr{S}(\mathbb{R}^N);\ f(x) \geq 0$ *and* $f(0) = 0\}$, *where* $\mathscr{S}(\mathbb{R}^N)$ *is the Schwartz space. Then, for any given*

$$0 < \mu < \frac{2N}{N(m-1)+2} \qquad (2.7)$$

and

$$\beta > \frac{2-\mu(m-1)}{4}, \qquad (2.8)$$

there exists an initial value $u_0 \in C_0^+(\mathbb{R}^N)$ *satisfying the following properties.*

(i) *There exists a sequence* $t_n \to \infty$ *such that*

$$D_{\sqrt{t_n}}^{\mu,\beta}[S(t_n)u_0] \xrightarrow{n\to\infty} f \quad in \;\; L^{\infty}(\mathbb{R}^N).$$

In other words, $f \in \omega^{\mu,\beta}(u_0)$;

(ii) *There exists a sequence* $\lambda_n \to \infty$ *such that for all* $0 < \epsilon < T < \infty$,

$$\Gamma_{\lambda_n}^{\mu,\beta}[u_0] \xrightarrow{\lambda_n\to\infty} h \quad in\; L^{\infty}([\epsilon, T);\; L^{\infty}(\mathbb{R}^N))$$

where $h(t) \equiv f$. *That is,* $h \in \gamma^{\mu,\beta}(u_0)$.

REMARK 2.9. The fact that if $u_0 \in L^1(\mathbb{R}^N)$ and $U_M(x,t)$ is the corresponding source-type solution, then $\lim_{t\to\infty} t^{\frac{N}{N(m-1)+2}} \|[S(t)u_0](x) - U_M(x,t)\|_{L^\infty} = 0$ is well known. Generally, we get in this theorem that the asymptotic limit of spatially dilation of the solutions can be very free, so the asymptotic behavior of these solutions is indeed complex.

In the following theorem, we show how complex the asymptotic behavior of solutions for the porous medium equation take place.

THEOREM 2.10. *Let E be a countable subset of* $\left(0, \frac{2N}{(N(m-1)+2)(2+\mu(m-1))}\right)$. *Assume F is a countable subset of $\mathscr{S}^+(\mathbb{R}^N)$. Then, for all $\phi \in F$ and all μ, β satisfy (2.7), (2.8) and*

$$\sigma = \frac{\mu}{2\beta} \in E, \tag{2.9}$$

there exists an initial value $u_0 \in C_0^+(\mathbb{R}^N)$ satisfying the following properties.

(i) *There exists a sequence $t_n \to \infty$ such that*

$$D^{\mu,\beta}_{\sqrt{t_n}}[S(t_n)u_0] \xrightarrow{n\to\infty} \phi \quad in\ L^\infty(\mathbb{R}^N).$$

In other words, for all $\phi \in F$ and all $\mu\beta$ satisfying (2.7), (2.8) and (2.9),

$$\phi \in \omega^{\mu,\beta}(u_0).$$

(ii) *There exists a sequence $\lambda_n \to \infty$ such that for all $0 < \epsilon < T < \infty$,*

$$\Gamma^{\mu,\beta}_{\lambda_n}[u_0] \xrightarrow{\lambda_n\to\infty} \Phi \quad in\ L^\infty([\epsilon, T);\ L^\infty(\mathbb{R}^N))$$

where $\Phi(t) \equiv \phi$. In other words, for all $\phi \in F$ and all $\mu\beta$ satisfying (2.7), (2.8) and (2.9),

$$\Phi \in \gamma^{\mu,\beta}(u_0).$$

REMARK 2.11. This theorem clearly shows that the asymptotic behavior of the solutions for the heat equation can occur in the porous medium equation. That is, we generalize the work of [4]. Obviously, the proof of this theorem is more difficult for the nonlinearity of the porous medium equation.

It is a relatively simple step from Theorem 2.10 to the next theorem. This theorem clearly give the structures of the Γ-limit set and the Ω-limit set.

THEOREM 2.12. *There exists an initial value $u_0 \in C_0^+(\mathbb{R}^N)$ such that:*

(i) *for all μ and β satisfying (2.7), (2.8) and (2.9), we have*

$$\omega^{\mu,\beta}(u_0) = C_0^+(\mathbb{R}^N);$$

(ii) *for all μ and β satisfying (2.7), (2.8) and (2.9), we have*

$$\gamma^{\mu,\beta}(u_0) = \Omega,$$

where

$$\Omega = \left\{h \in C((0,\infty); C_0^+(\mathbb{R}^N));\ h(t) = h(1)\right\}.$$

3. Proof of Theorem 2.8

In this section, we give the proof of Theorem 2.8. For convenience, let

$$\eta = \frac{1}{N(m-1)+2}$$

and

$$\beta' = 4\beta + \mu(m-1) - 2.$$

Obviously, (2.7) and (2.8) imply that $\beta' > 0$. The proof of Theorem 2.8 is based on the following lemmas. The first lemma is the Hole-filling lemma which has been proved in [12].

LEMMA 3.1. (Hole-filling lemma) *Let $u \geq 0$ be a bounded local solution of the porous medium equation posed in $\mathbb{R}^N \times (0, \infty)$ and let us assume that it takes (continuously or in the sense of $L^1(\mathbb{R}^N)$ convergence) initial data given by a bounded function, $u_0 \in L^\infty(\mathbb{R}^N)$, and let us also assume that u_0 vanishes a.e. in a ball $B_R(x_0)$. Then, there is a time $T = T(\|u\|_{L^\infty})$ such that for every $0 < t < T$, the solution $u(t)$ vanishes at least in a smaller ball $B_{R(t)}(x_0)$ with $0 < R(t) \leq R$. The function $R(t)$ is monotone nonincreasing. Moreover, we have the bounds*

$$T \geq CR^2 \|u\|_{L^\infty}^{\frac{m-1}{2}}$$

and

$$R(t) \geq R - C\|u\|_{L^\infty}^{\frac{m-1}{2}} t^{\frac{1}{2}},$$

where C depend on m, N.

LEMMA 3.2. *Let $u(t) \geq 0$ be a solution of (1.1)–(1.2).*

(i) *Assume the nonnegative initial value $u_0 \in L^\infty(\mathbb{R}^N)$ is continuous with*

$$\text{supp} u_0 \subset \{x \in \mathbb{R}^N; |x| \leq R\}.$$

Then, for any $0 < t < \infty$,

$$\text{supp} u(t) \subset \{x \in \mathbb{R}^N; |x| \leq R_1(t)\}.$$

Here, $R_1(t)$ is an increase function and satisfies the following estimates

$$R \leq R_1(t) \leq 4R + C\|u_0\|_{L^\infty}^{\frac{m-1}{2}} t^\eta,$$

where the constant C depends on m and N.

(ii) *Assume the nonnegative initial value* $u_0 \in L^1(\mathbb{R}^N)$ *is continuous with*

$$\mathrm{supp} u_0 \subset \{x \in \mathbb{R}^N; |x| \le R\}.$$

Then, for any $0 < t < \infty$,

$$\mathrm{supp} u(t) \subset \{x \in \mathbb{R}^N; |x| \le R_2(t)\},$$

where $R_2(t) \ge R$, *the function* $R_2(t)$ *is increasing. Moreover, we have*

$$R_2(t) \le R + C\|u_0\|_{L^1}^{(m-1)\eta} t^{\eta},$$

where the constant C *depends on* m *and* N.

Proof. We first prove Property (i) of this Lemma. We use a comparison argument by considering the supersolution $V(x,t)$ formed by a source-type solution $U_M(x,t)$, namely,

$$V(x,t) = U_M(x, t+\tau) = (t+\tau)^{-\frac{N}{N(m-1)+2}} (M - k|x|^2 (t+\tau)^{-2\eta})_+^{\frac{1}{m-1}}, \quad (3.1)$$

where $k = \frac{\eta(m-1)}{2m}$, $M = \frac{4}{3}\|u_0\|_{L^\infty}^{m-1}$ and $\tau = (3kR^2\|u_0\|_{L^\infty}^{m-1})^{-\frac{1}{2\eta}}$. From this, it is easy to verify that

$$\mathrm{supp}(V(0)) \subset \{x \in \mathbb{R}^N; |x| \le 2R\}$$

and

$$V(x,0) \ge u_0(x), \qquad \text{for all } x \in \mathbb{R}^N.$$

Therefore, by comparison principle, we have

$$V(x,t) \ge u(x,t), \qquad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}^+.$$

Note also, from (3.1), that

$$\begin{aligned} \mathrm{supp}(V(t)) &\subset \{x \in \mathbb{R}^N; |x| \le (Mk^{-1})^{\frac{1}{2}} (t+\tau)^{\eta}\} \\ &\subset \{x \in \mathbb{R}^N; |x| \le 2^{\eta+1} R + 2^{\eta+1} k^{-1} \|u_0\|_{L^\infty}^{\frac{m-1}{2}} t^{\eta}\}, \end{aligned}$$

hence

$$\mathrm{supp}(u(t)) \subset \{x \in \mathbb{R}^N; |x| \le 4R + C\|u_0\|_{L^\infty}^{\frac{m-1}{2}} t^{\eta}\},$$

where $C = 2^{\eta+1}k^{-1}$. So we complete the proof of Property (i) of this Lemma.

To prove Property (ii) of Lemma 3.2, we begin by showing that for $u_0 \in L^\infty(\mathbb{R}^N)$ and $t \ge 0$,

$$\mathrm{supp}(u(t)) \subset \left\{x \in \mathbb{R}^N; |x| \le R + 2C_2\|u_0\|_{L^\infty}^{\frac{m-1}{2}} t^{1/2}\right\}. \quad (3.2)$$

In fact, if $t = 0$, then the proof is trivial. If $t > 0$, then for all $|x_0| > R + 2C_2\|u_0\|_{L^\infty}^{\frac{m-1}{2}} t^{\frac{1}{2}}$,

$$B_{R_1}(x_0) \bigcap \operatorname{supp}(u_0) = \emptyset,$$

where $B_{R_1}(x_0)$ is a ball centered at x_0 with radius $R_1 = 2C_2\|u_0\|_{L^\infty}^{\frac{m-1}{2}} t^{\frac{1}{2}}$. Using this, we can deduce from Lemma 3.1 that for any $0 \le s \le t$ and $x \in B_{\frac{1}{3}R_1}(x_0)$,

$$u(x, s) = 0.$$

In particular,

$$u(x_0, s) = 0, \qquad \text{for all } 0 \le s \le t.$$

Therefore, we complete the proof of (3.2). Now we use the symbol $R(t)$ to denote the radius of the smallest ball which contains the support of $u(x, t)$. For any $0 \le t < \infty$, we select a sequence of times $t_k = 2^{-k}t \to 0$ and estimate the increase of the support in those times, i.e., we consider the evolution in the different time intervals $I_k = [t_k, t_{k-1}]$. From the L^1–L^∞ smoothing effect (Proposition 2), at each initial time $t = t_k$ of the evolution, we have

$$K_k = \|u(t_k)\|_{L^\infty(\mathbb{R}^N)} \le C(m, d)\|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{2}{N(m-1)+2}} t_k^{-\frac{N}{N(m-1)+2}}.$$

Therefore, using (3.2), we get

$$\begin{aligned} R(t_{k-1}) &\le R(t_k) + CK_k^{\frac{m-1}{2}}(t_{k-1} - t_k)^{\frac{1}{2}} \\ &\le R(t_k) + 2^{-\frac{k}{N(m-1)+2}} C\|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{m-1}{N(m-1)+2}} t^{\frac{1}{N(m-1)+2}}. \end{aligned}$$

Iterating, we thus have

$$R(t) \le R(0) + C\|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{m-1}{N(m-1)+2}} t^{\frac{1}{N(m-1)+2}} = R + C\|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{m-1}{N(m-1)+2}} t^{\frac{1}{N(m-1)+2}}.$$

where the constant C depends on m and N. In other words,

$$\operatorname{supp}(u(t)) \subset \left\{x \in \mathbb{R}^N; |x| \le R + C\|u_0\|_{L^1}^{(m-1)\eta} t^\eta\right\},$$

From this, we find that Property (ii) of this lemma is true. So we complete the proof of Lemma 3.2. □

The following lemma is very important in the proof of our main results.

LEMMA 3.3. *For all* $j \ge 1$, *let*

$$\phi_j \in \mathscr{S}^+(\mathbb{R}^N),$$

and

$$\chi_j(x) = \chi'_j(x/\lambda_j^{2\beta}).$$

Here, $\chi'_j(x)$ *is the cut-off function on* $A_j = \{2^{-j} < |x| < 2^j\}$ *relative to* $A'_j = \{2^{-(j-1)} < |x| < 2^{j-1}\}$*, that is,*

$$\chi'_j(x) \in C_0^\infty(A_j),$$
$$0 \le \chi'_j(x) \le 1 \quad \text{if} \quad x \in A_j$$

and

$$\chi'_j(x) = 1 \quad \text{if} \quad x \in A'_j.$$

Suppose

$$v_n^1(x) = \lambda_n^\mu \chi_{n-1}(\lambda_n^{2\beta} x)\lambda_{n-1}{}^{-\mu}\phi_{n-1}((\lambda_n/\lambda_{n-1})^{2\beta} x), \tag{3.3}$$
$$v_n(x) = \chi_n(\lambda_n^{2\beta} x)\phi_n(x), \tag{3.4}$$
$$v_{n-1}(x) = \chi_{n-1}(\lambda_{n-1}^{2\beta} x)\phi_{n-1}(x) \tag{3.5}$$

and

$$v_n^2(x) = \lambda_{n-1}^\mu \chi_n(\lambda_{n-1}^{2\beta} x)\lambda_n{}^{-\mu}\phi_n((\lambda_{n-1}/\lambda_n)^{2\beta} x), \tag{3.6}$$

where μ *and* β *satisfy* (2.7) *and* (2.8)*. Then, for* $0 \le t \le n$ *and any fixed* $\lambda_{n-1} > 1$*, there exists a* $\lambda'_n \ge \lambda_{n-1}$ *such that for* $\lambda_n \ge \lambda'_n$*,*

$$\operatorname{supp}(S(\lambda_n^{-\beta'} t)(v_n)) \bigcap \operatorname{supp}(S(\lambda_n^{-\beta'} t)(v_n^1)) = \emptyset \tag{3.7}$$

and

$$\operatorname{supp}(S(\lambda_{n-1}^{-\beta'} t)(v_{n-1})) \bigcap \operatorname{supp}(S(\lambda_{n-1}^{-\beta'} t)(v_n^2)) = \emptyset. \tag{3.8}$$

Proof. Clearly, the hypotheses (3.3)–(3.6) imply

$$\begin{aligned}
&\operatorname{supp}(v_n^1) \subset \{x \in \mathbb{R}^N;\, 2^{-n+1}(\lambda_{n-1}/\lambda_n)^{2\beta} \le |x| \le 2^{n-1}(\lambda_{n-1}/\lambda_n)^{2\beta}\},\\
&\operatorname{supp}(v_n) \subset \{x \in \mathbb{R}^N;\, 2^{-n} \le |x| \le 2^n\},\\
&\operatorname{supp}(v_{n-1}) \subset \{x \in \mathbb{R}^N;\, 2^{-n+1} \le |x| \le 2^{n-1}\},\\
&\operatorname{supp}(v_n^2) \subset \{x \in \mathbb{R}^N;\, 2^{-n}(\lambda_n/\lambda_{n-1})^{2\beta} \le |x| \le 2^n(\lambda_n/\lambda_{n-1})^{2\beta}\},\\
&\|v_n^1\|_{L^1} \le \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^{\mu-2N\beta} \|\phi_{n-1}\|_{L^1} \le \|\phi_{n-1}\|_{L^1}
\end{aligned}$$

and

$$\|v_n^2\|_{L^\infty} \le \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{\mu} \|\phi_n\|_{L^\infty} \le \|\phi_n\|_{L^\infty}.$$

Here, we used the facts that $\lambda_n \geq \lambda_{n-1}$ and $\mu - 2N\beta \leq 0$. So, Lemma 3.1 and Lemma 3.2 imply that for $0 \leq t \leq n$,

$$\begin{aligned} &\mathrm{supp}(S(\lambda_n^{-\beta'} t)v_n^1) \subset \{x \in \mathbb{R}^N; |x| \leq 2^{n+1}(\lambda_{n-1}/\lambda_n)^{2\beta} \\ &\quad + C\|\phi_{n-1}\|_{L^1}^{(m-1)\eta}(\lambda_n^{-\beta'} n)^\eta\}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\mathrm{supp}(S(\lambda_n^{-\beta'} t)v_n) \subset \{x \in \mathbb{R}^N; 2^{-n} - C\|\phi_n\|_{L^\infty}^{\frac{m-1}{2}}(\lambda_n^{-\beta'} n)^{\frac{1}{2}} \\ &\quad \leq |x| \leq 2^{n+2} + C\|\phi_n\|_{L^\infty}^{\frac{m-1}{2}}(\lambda_n^{-\beta'} n)^\eta\}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\mathrm{supp}(S(\lambda_{n-1}^{-\beta'} t)v_{n-1}) \subset \{x \in \mathbb{R}^N; 2^{-n+1} - C\|\phi_{n-1}\|_{L^\infty}^{\frac{m-1}{2}}(\lambda_{n-1}^{-\beta'} n)^{\frac{1}{2}} \\ &\quad \leq |x| \leq 2^{n+1} + C\|\phi_{n-1}\|_{L^1}^{(m-1)\eta}(\lambda_{n-1}^{-\beta'} n)^\eta\} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\mathrm{supp}(S(\lambda_{n-1}^{-\beta'} t)v_n^2) \subset \{x \in \mathbb{R}^N; 2^{-n}(\lambda_n/\lambda_{n-1})^{2\beta} - C\|\phi_n\|_{L^\infty}^{\frac{m-1}{2}}(\lambda_{n-1}^{-\beta'} n)^{\frac{1}{2}} \\ &\quad \leq |x| \leq 2^{n+2}(\lambda_n/\lambda_{n-1})^{2\beta} + C\|\phi_n\|_{L^\infty}^{\frac{m-1}{2}}(\lambda_{n-1}^{-\beta'} n)^\eta\}, \end{aligned} \tag{3.12}$$

where C only depends on m and N. We now assume that

$$\lambda_n' = 2^{\frac{2n+2}{2\beta}}\lambda_{n-1} + CC_n 2^{\frac{n+3}{\eta\beta'}} n^{\frac{1}{\beta'}}, \tag{3.13}$$

where $C_n = \max\,(\|\phi_n\|_{L^\infty}^{\frac{m-1}{4\beta'}}, \|\phi_{n-1}\|_{L^1}^{\frac{m-1}{\beta}})$ and the constant C takes the same value as above. Hence, if $\lambda_n \geq \lambda_n'$, then

$$2^{n+1}\,(\lambda_{n-1}/\lambda_n)^{2\beta} + C(\lambda_n^{-\beta'} n)^\eta \|\phi_{n-1}\|_{L^1}^{(m-1)\eta} \leq 2^{-n} - C\|\phi_n\|_{L^\infty}^{\frac{m-1}{2}}(\lambda_n^{-\beta'} n)^{\frac{1}{2}}$$

and

$$2^{n+1} + C\|\phi_{n-1}\|_{L^1}^{(m-1)\eta}(\lambda_{n-1}^{-\beta'} n)^\eta \leq 2^{-n}(\lambda_n/\lambda_{n-1})^{2\beta} - C\|\phi_n\|_{L^\infty}^{\frac{m-1}{2}}(\lambda_{n-1}^{-\beta'} n)^{\frac{1}{2}}.$$

We thus obtain from (3.9)–(3.13) that (3.7) and (3.8) are true. The proof of this Lemma is complete. □

Now, we are in the position to give the proof of Theorem 2.8.

Proof of Theorem 2.8. For any given $f \in \mathscr{S}^+(\mathbb{R}^N)$, let

$$\ell = \max\left(\|f\|_{L^1(\mathbb{R}^N)}, \|f\|_{L^\infty(\mathbb{R}^N)}\right)$$

and

$$\sigma = \frac{\mu}{2\beta} \in \left(0, \frac{4N}{(N(m-1)+2)(2+\mu(m-1))}\right).$$

The initial value u_0 is defined as follows:

$$u_0(x) = \sum_{j=1}^{\infty} \chi_j(x)\lambda_j{}^{-\mu} f\left(x/\lambda_j^{2\beta}\right), \tag{3.14}$$

where $\chi_j(x) = \chi_j'(x/\lambda_j^{2\beta})$, $\chi_j'(x)$ is the cut-off function on set $A_j = \{2^{-j} < |x| < 2^j\}$ relative to $A_j' = \{2^{-(j-1)} < |x| < 2^{j-1}\}$. The sequence $\{\lambda_j\}_{j\geq 1}$ will be determined by the following method. Firstly, suppose $\lambda_0 \geq 1$ is large enough to satisfy

$$\exp x \geq (2x)^\tau, \quad \text{for all } x \geq \lambda_0 \geq 1, \tag{3.15}$$

where $\tau = \frac{2N}{N(m-1)+2}$. Then, let

$$\begin{cases} \lambda_1 = \lambda_0, \\ \lambda_j = \max\left((\lambda_j'), \exp\frac{\lambda_{j-1}}{\mu}\right), & j > 1. \end{cases} \tag{3.16}$$

where λ_j' take the same values as description in Lemma 3.3. We obtain from (3.15) and (3.16) that

$$\lambda_{j+1} \geq \exp\frac{\lambda_j}{\mu} > \exp\frac{\lambda_j}{\tau} \geq 2\lambda_j.$$

Iterating, we find that

$$\lambda_j \geq 2^{j-1}, \qquad \text{for all } j \geq 1.$$

Note also that

$$\|\chi_j(\cdot) f(\cdot/\lambda_j^{2\beta})\|_{L^\infty} \leq \|f\|_{L^\infty}.$$

Therefore, the sequence of (3.14) is convergent in $L^\infty(\mathbb{R}^N)$. So,

$$u_0 \in C_0(\mathbb{R}^N).$$

In light of $f \in \mathscr{S}^+(\mathbb{R}^N)$ and (3.14), we have

$$u_0 \geq 0$$

and

$$u_0(0) = 0.$$

Hence,

$$u_0 \in C_0^+(\mathbb{R}^N) \subset X_0^+.$$

So $S(t)u_0$ is well defined.

As a result of (2.6), we see that for $0 < t < T < \infty$,

$$[\Gamma^{\mu,\beta}_{\lambda_n} u_0](x,t) = S(t\lambda_n^{-\beta'})(u_n + v_n + w_n),$$

where

$$u_n = \lambda_n^{\mu} \sum_{j=1}^{n-1} \chi_j(x\lambda_n^{2\beta})\lambda_j{}^{-\mu} f(x\lambda_n^{2\beta}/\lambda_j^{2\beta}), \tag{3.17}$$

$$v_n = \chi_n(x\lambda_n^{2\beta}) f(x) \tag{3.18}$$

and

$$w_n = \lambda_n^{\mu} \sum_{j=n+1}^{\infty} \chi_j(x\lambda_n^{2\beta})\lambda_j{}^{-\mu} f(x\lambda_n^{2\beta}/\lambda_j^{2\beta}). \tag{3.19}$$

In light of (3.17)–(3.19), we have

$$\mathrm{supp}(u_n) = \left\{x \in \mathbb{R}^N; |x| \le 2^{n-1}(\lambda_{n-1}/\lambda_n)^{2\beta}\right\},$$
$$\mathrm{supp}(v_n) = \left\{x \in \mathbb{R}^N; 2^{-n} \le |x| \le 2^n\right\}$$

and

$$\mathrm{supp}(w_n) = \left\{x \in \mathbb{R}^N; |x| \ge 2^{-(n+1)}(\lambda_{n+1}/\lambda_n)^{2\beta}\right\}.$$

Here, we have used the fact that for all $j \ge 2$,

$$2^{-(n+1)}(\lambda_{n+1}/\lambda_n)^{2\beta} \le 2^{-(n+j)}(\lambda_{n+j}/\lambda_n)^{2\beta}.$$

Note also that

$$\|u_n\|_{L^1} \le \sum_{j=1}^{n-1} \left(\frac{\lambda_n}{\lambda_j}\right)^{\mu-2\beta N} \|\chi_j' f\|_{L^1} \le \|f\|_{L^1} \le \ell,$$

and

$$\|u_n\|_{L^\infty} \le \| \sum_{j=n+1}^{\infty} \left(\frac{\lambda_n}{\lambda_j}\right)^{-\mu} \chi_j f\|_{L^\infty} \le \|f\|_{L^\infty} \le \ell.$$

We thus get from the results of Lemma 3.1 and Lemma 3.2 that for $0 < T < \infty$, if $0 \le t < T \le n$, then

$$\begin{aligned}\mathrm{supp} S(\lambda_n^{-\beta'} t)(u_n) = \{x \in \mathbb{R}^N; |x| \le 2^{n+1}(\lambda_{n-1}/\lambda_n)^{2\beta},\\ +Cn^{\eta}\ell^{(m-1)\eta}\lambda_n^{-\beta'\eta}\}\end{aligned} \tag{3.20}$$

$$\begin{aligned}\mathrm{supp} S(\lambda_n^{-\beta'} t)(v_n) = \{x \in \mathbb{R}^N; 2^{-n} - Cn^{\frac{1}{2}}\ell^{\frac{m-1}{2}}\lambda_n^{-\frac{\beta'}{2}} \le |x| \le 2^n\\ +Cn^{\eta}\ell^{\frac{m-1}{2}}\lambda_n^{-\beta'\eta}\}\end{aligned} \tag{3.21}$$

and

$$\mathrm{supp}S(\lambda_n^{-\beta'}t)(w_n)=\left\{x\in\mathbb{R}^N;\,|x|\geq 2^{-(n+1)}(\lambda_{n+1}/\lambda_n)^{2\beta}-Cn^{\frac{1}{2}}\ell^{\frac{m-1}{2}}\lambda_n^{-\frac{\beta'}{2}}\right\}. \tag{3.22}$$

From (3.13), (3.16) and Lemma 3.3, we have

$$\begin{aligned}&2^{n+1}(\lambda_{n-1}/\lambda_n)^{2\beta}+Cn^{\eta}\ell^{(m-1)\eta}\lambda_n^{-\beta'\eta}\\&\quad\leq 2^{-n}-Cn^{\frac{1}{2}}\ell^{\frac{m-1}{2}}\lambda_n^{-\frac{\beta'}{2}}\\&\quad\leq 2^{n}+Cn^{\eta}\ell^{\frac{m-1}{2}}\lambda_n^{-\beta'\eta}\\&\quad\leq 2^{-(n+1)}(\lambda_{n+1}/\lambda_n)^{2\beta}-Cn^{\frac{1}{2}}\ell^{\frac{m-1}{2}}\lambda_n^{-\frac{\beta'}{2}},\end{aligned}$$

if we take $\phi_j=f$ for all $j\geq 1$ in (3.13). Combining this with (3.20)–(3.22), we obtain that for $0<T<\infty$, if $0\leq t<T\leq n$, then

$$\mathrm{supp}S(\lambda_n^{-\beta'}t)(u_n)\bigcap\mathrm{supp}S(\lambda_n^{-\beta'}t)(v_n)=\emptyset,$$
$$\mathrm{supp}S(\lambda_n^{-\beta'}t)(u_n)\bigcap\mathrm{supp}S(\lambda_n^{-\beta'}t)(w_n)=\emptyset$$

and

$$\mathrm{supp}S(\lambda_n^{-\beta'}t)(v_n)\bigcap\mathrm{supp}S(\lambda_n^{-\beta'}t)(w_n)=\emptyset.$$

Therefore, for $0<T<\infty$ and $x\in\mathbb{R}^n$, if $0\leq t<T\leq n$, then

$$S(t\lambda_n^{-\beta'})(u_n+v_n+w_n)=S(t\lambda_n^{-\beta'})(u_n)+S(t\lambda_n^{-\beta'})(v_n)+S(t\lambda_n^{-\beta'})(w_n), \tag{3.23}$$

i.e., superposition holds as long as the supports are disjoint. Using (3.17), we have

$$\|u_n\|_{L^1}\leq\lambda_n^{\mu}\sum_{j=1}^{n-1}\|f(\cdot\lambda_n^{2\beta}/a_j)\|_{L^1}=\ell\lambda_n^{\mu-2N\beta}\sum_{j=1}^{n-1}\lambda_j^{2N\beta}\leq n\ell\lambda_n^{\mu-2N\beta}\lambda_{n-1}^{2N\beta}.$$

Consequently, the L^1–L^∞ smoothing estimates (Proposition 2.2) indicates

$$\begin{aligned}&\|S(\lambda_n^{-\beta'}t)u_n(x)\|_{L^\infty}\\&\quad\leq C(\lambda_n^{-\beta'}t)^{-\frac{N}{N(m-1)+2}}\|u_n(x)\|_{L^1}^{\frac{2}{N(m-1)+2}}\\&\quad\leq C\lambda_{n-1}^{\frac{N\eta}{\beta}}t^{-N\eta}(n\ell)^{2\eta}\lambda_n^{(2\mu+N\mu(m-1)-2N)\eta}.\end{aligned} \tag{3.24}$$

Clearly, (3.19) implies

$$\|w_n\|_{L^\infty}\leq\ell\lambda_n^{\mu}\sum_{j=n+1}^{\infty}\lambda_j{}^{-\mu}=\ell\lambda_n^{\mu}\sum_{j=n}^{\infty}\lambda_{j+1}{}^{-\mu}.$$

Applying Proposition 2.3 (i.e., the maximum principle) to $S(t\lambda_n^{-\beta'})w_n$, we thus have

$$\|S(t\lambda_n^{-\beta'})w_n\|_{L^\infty} \le C\lambda_n^\mu \sum_{j=n}^{\infty} \lambda_{j+1}{}^{-\mu} \tag{3.25}$$

We will first prove Property (ii) of Theorem 2.8. It is clear from (2.7) that

$$2N/\mu - 2 - N(m-1) > 0.$$

As an immediate consequence of (3.16) and (3.24), we have

$$\|S(t\lambda_n^{-\beta'})u_n\|_{L^\infty} \le Cnt^{N\eta}\lambda_{n-1}^{\frac{N\eta}{\beta}} \exp(-\lambda_{n-1}(2N/\mu - 2 - N(m-1))\eta).$$

From (3.16), we get that for any fixed $\epsilon > 0$,

$$\|S(t\lambda_n^{-\beta'})u_n\|_{L^\infty} \xrightarrow{n\to\infty} 0 \tag{3.26}$$

uniformly for $t \ge \epsilon$. Moveover, (3.16), (3.25) and $\mu > 0$ imply

$$\|S(t\lambda_n^{-\beta'})w_n\|_{L^\infty} \le C\lambda_n^\mu \sum_{j=n+1}^{\infty} \lambda_j{}^{-\mu} \le C \sum_{j=n+1}^{\infty} \lambda_j^{-\mu}\lambda_n{}^\mu \xrightarrow{n\to\infty} 0. \tag{3.27}$$

At present, we want to verify the claim that for $0 \le t < T < \infty$,

$$\|S(\lambda_n^{-\beta'}t)v_n - f\|_{L^\infty(\mathbb{R}^N)} \xrightarrow{n\to\infty} 0. \tag{3.28}$$

In fact, if $0 < t < \infty$, then

$$\begin{aligned}\|S(\lambda_n^{-\beta'}t)v_n - f\|_{L^\infty(\mathbb{R}^N)} &\le \|S(\lambda_n^{-\beta'}t)v_n - f\|_{L^\infty(A_{n_0})} \\ &+ \|S(\lambda_n^{-\beta'}t)v_n\|_{L^\infty(\mathbb{R}^N\setminus A_{n_0})} + \|f\|_{L^\infty(\mathbb{R}^N\setminus A_{n_0})}.\end{aligned} \tag{3.29}$$

The results of Lemma 3.1 and Lemma 3.2 imply that for $0 < T < \infty$, if $0 \le t < T \le n_0 \le n$, then

$$\begin{aligned}&\operatorname{supp}(S(\lambda_n^{-\beta'}t)[\chi'_{n_0-4}v_n]) \subset \Big\{x \in \mathbb{R}^N; \\ &2^{-n_0+4} - C\|v_n\|_{L^1}^{\frac{m-1}{2}} \lambda_n^{-\frac{\beta'}{2}} t^{1/2} \le |x| \le 2^{n_0-2} + C\|v_n\|_{L^1}^{(m-1)\eta} \lambda_n^{-\beta'\eta} t^\eta\Big\}.\end{aligned}$$

In view of the selection of λ_n and (3.15), we obtain that for $0 < T < \infty$, if $0 \le t < T \le n_0 \le n$, then

$$2^{-n_0+4} - C\|v_n\|_{L^1}^{\frac{m-1}{2}} \lambda_n^{-\frac{\beta'}{2}} t^{1/2} \ge 2^{-n_0+4} - 2^{-n_0} > 2^{-n_0+1}$$

and

$$2^{n_0-4} + C\|v_n\|_{L^1}^{(m-1)\eta} \lambda_n^{-\beta'\eta} t^\eta \le 2^{n_0-4} + 2^{-n_0} < 2^{n_0-1}.$$

Hence, we obtain that for all $0 < T < \infty$, if $0 \leq t < T \leq n_0 \leq n$, then

$$\mathbb{R}^N \setminus A_{n_0} \bigcap \operatorname{supp}(S(\lambda_n^{-\beta'} t)[\chi'_{n_0-4} v_n]) = \emptyset. \tag{3.30}$$

In other words, for all $0 < T < \infty$, if $0 \leq t < T \leq n_0 \leq n$, then the value of $S(\lambda_n^{-\beta'} t)[v_n]$ in $\mathbb{R}^N \setminus A_{n_0}$ is independent of the initial value v_n in A_{n_0-4}. Therefore, for any $\epsilon > 0$, there exists a n_1 such that for all $0 < T < \infty$, if $0 \leq t < T \leq n_0 \leq n_1 \leq n$, then

$$\begin{aligned}
&\|S(\lambda_n^{-\beta'} t) v_n\|_{L^\infty(\mathbb{R}^N \setminus A_{n_0})} \\
&\quad = \|S(\lambda_n^{-\beta'} t)(1 - \chi'_{n_0-4}) v_n\|_{L^\infty(\mathbb{R}^N \setminus A_{n_0})} \\
&\quad \leq \|(1 - \chi'_{n_0-4}) v_n\|_{L^\infty} \leq \|(1 - \chi'_{n_0-4}) f\|_{L^\infty} \\
&\quad = \|f\|_{L^\infty(|x| \geq 2^{n_0-4})} + \|f\|_{L^\infty(|x| \leq 2^{-n_0+4})} \\
&\quad \leq \|f\|_{L^\infty(|x| \geq 2^{n_1-4})} + \|f\|_{L^\infty(|x| \leq 2^{-n_1+4})} \\
&\quad < \frac{\epsilon}{3}.
\end{aligned} \tag{3.31}$$

Here, we have used the fact that $f \in \mathscr{S}^+(\mathbb{R}^N)$. On the other hand, by a similar arguments to the proof of (3.30), we can get that for all $0 < T < \infty$, if $0 \leq t < T \leq n_0 < n_0 + 6 \leq n$, then the value of $S(\lambda_n^{-\beta'} t) v_n$ in A_{n_0} is only dependent on the initial value of v_n in A_{n_0+4}. Indeed, from Lemma 3.1 and Lemma 3.2, we obtain that for all $0 < T < \infty$, if $0 \leq t < T \leq n_0 \leq n$, then

$$\operatorname{supp}(S(\lambda_n^{-\beta'} t)[\chi'_{R_1} v_n]) \subset \left\{x;\, |x| \leq 2^{-n_0-4} + C\|v_n\|_{L^\infty}^{\frac{m-1}{2}} \lambda_n^{-\eta\beta'} t^\eta\right\} \tag{3.32}$$

and

$$\operatorname{supp}(S(\lambda_n^{-\beta'} t)[(1 - \chi'_{R_2}) v_n]) \subset \left\{x;\, |x| \geq 2^{n_0+4} - C\|v_n\|_{L^1}^{(m-1)\eta} \lambda_n^{-\frac{\beta'}{2}} t^{\frac{1}{2}}\right\}, \tag{3.33}$$

where χ'_{R_1} is the characteristic function of $B_{R_1} = \{x;\, |x| \leq 2^{-n_0-4}\}$ and χ'_{R_2} is the characteristic function of $B_{R_2} = \{x;\, |x| \leq 2^{n_0+4}\}$. From (3.13), (3.32) and (3.33), we thus obtain that for all $0 < T < \infty$, if $0 \leq t < T \leq n_0 \leq n$, then

$$\operatorname{supp}(S(\lambda_n^{-\beta'} t)[\chi'_{R_1} v_n]) \bigcap A_{n_0} = \emptyset \tag{3.34}$$

and

$$\operatorname{supp}(S(\lambda_n^{-\beta'} t)[(1 - \chi'_{R_2}) v_n]) \bigcap A_{n_0} = \emptyset. \tag{3.35}$$

The expression of (3.18) clearly means that

$$v_n = f \quad \text{for all } x \in A_n = \{2^{-j+1} < |x| < 2^{j-1}\}. \tag{3.36}$$

So, if $n \geq n_0 + 6$, then

$$v_n = f \quad \text{for all } x \in A_{n_0+4}.$$

This implies, via (3.34) and (3.35), that for all $0 < T < \infty$, if $0 \leq t < T \leq n_0 \leq n_0 + 6 \leq n$, then

$$\|S(\lambda_n^{-\beta'} t)v_n - S(\lambda_n^{-\beta'} t) f\|_{L^\infty(A_{n_0})} = 0. \tag{3.37}$$

Since $f \in \mathscr{S}^+(\mathbb{R}^N)$, a consequence of Proposition 2.2 is

$$S(t)f \in C([0,\infty) \times \mathbb{R}^N).$$

Therefore, for any $\epsilon > 0$ and A_{n_0}, there exists a $n_2 \geq n_0$ such that for all $0 < T < \infty$, if $0 \leq t < T \leq n_0 \leq n_2 \leq n$, then

$$\|S(\lambda_n^{-\beta'} t) f - f\|_{L^\infty(A_{n_0})} < \frac{\epsilon}{3}. \tag{3.38}$$

The facts that A_{n_0} is a relative compact set in $\mathbb{R}^N$ and $\beta' > 0$ have been used in (3.38). The estimate (3.37) thus tells us that for all $n \geq \max(n_0 + 6, n_2)$, if $0 \leq t < T \leq n_0$, then

$$\begin{aligned}
&\|S(\lambda_n^{-\beta'} t)v_n - f\|_{L^\infty(A_{n_0})} \\
&\quad \leq \|S(\lambda_n^{-\beta'} t)v_n - S(\lambda_n^{-\beta'} t) f\|_{L^\infty(A_{n_0})} + \|S(\lambda_n^{-\beta'} t) f - f\|_{L^\infty(A_{n_0})} \\
&\quad < \frac{\epsilon}{3}.
\end{aligned} \tag{3.39}$$

If $n_0 \geq n_1$, we obtain from (3.31) that

$$\|f\|_{L^\infty(\mathbb{R}^N \setminus A_{n_0})} < \frac{\epsilon}{3}.$$

Combining this with (3.29), (3.31), (3.38) and (3.39), we get that for any $\epsilon > 0$ and all $0 < T < \infty$, if $0 \leq t < T \leq n_0$ and $n \geq \max(n_0 + 6, n_1, n_2)$, then

$$\begin{aligned}
&\|S(\lambda_n^{-\beta'} t)v_n - f\|_{L^\infty(\mathbb{R}^N)} \\
&\quad \leq \|S(\lambda_n^{-\beta'} t)v_n - f\|_{L^\infty(A_{n_0})} + \|S(\lambda_n^{-\beta'} t)v_n\|_{L^\infty(\mathbb{R}^N \setminus A_{n_0})} + \|f\|_{L^\infty(\mathbb{R}^N \setminus A_{n_0})} \\
&\quad < \epsilon.
\end{aligned}$$

Hence, the claim of (3.28) is proved. For all $0 < T < \infty$, (3.28) clearly indicates that

$$S(t\lambda_n^{-\beta'})v_n \xrightarrow{n\to\infty} f \quad \text{in} \quad L^\infty(\mathbb{R}^N) \tag{3.40}$$

uniformly for $t \in [0, T)$. So, from (2.6), (3.23), (3.26), (3.27) and (3.40), we obtain that for all $0 < \epsilon < T < \infty$,

$$\Gamma_{\lambda_n}^{\mu,\beta}[u_0] = S(\lambda_n^{-\beta'} t)(u_n + v_n + w_n) \xrightarrow{n\to\infty} h(t) \quad \text{in} \quad L^\infty\left(\mathbb{R}^N\right)$$

uniformly for $t \in [\epsilon, T)$. In other words,

$$\Gamma_{\lambda_n}^{\mu,\beta}[u_0] \xrightarrow{n\to\infty} h, \quad \text{in} \quad L^\infty([\epsilon, T); L^\infty\left(\mathbb{R}^N)\right)$$

for all $0 < \epsilon < T < \infty$. So we complete the proof of Property (ii) of Theorem 2.8. Property (i) of Theorem 2.8 follows from Property (ii) by setting $t = 1$ and $t_n = \lambda_n^2$. The proof of this theorem is complete. □

4. Proofs of Theorem 2.10 and Theorem 2.12

In this section, we will give the proofs of Theorem 2.10 and Theorem 2.12. We first prove the following lemma which will be used in the proof of Theorem 2.10.

LEMMA 4.1. *Let E be a countable subset of $\left(0, \frac{2N}{(N(m-1)+2)(2+\mu(m-1))}\right)$ and F be a countable subset of $\mathscr{S}^+(\mathbb{R}^N)$. Then, there exist a constant $M > 0$, a sequence $\{\sigma_j\}_{j\geq 1}$ and a sequence $\{\phi_j\}_{j\geq 1}$ such that :*

(i) *every element of $E \times F$ occurs infinitely often in the sequence $\{(\sigma_j, \phi_j)\}_{j\geq 1}$;*

(ii) *the sequence $\{\phi_j\}_{j\geq 1}$ satisfies*

$$\max(\|\phi_j\|_{L^1}, \|\phi_j\|_{L^\infty}) \leq \max(j, M) \quad \textit{for all } j \geq 1. \tag{4.1}$$

Proof. For any given $\phi \in F$, let

$$M = \max\left(\|\phi\|_{L^1}, \|\phi\|_{L^\infty}\right).$$

Since the set $E \times F$ is countable, it is easy to see that there exists a sequence $\{\sigma_j\}_{j\geq 1} \subset E$ and a sequence $\{\widetilde{\phi_j}\}_{j\geq 1} \subset F$ such that every element $(\delta, \varphi) \in E \times F$ occurs infinitely often in the sequence $\{(\sigma_j, \widetilde{\phi_j})\}$. Then, let

$$\phi_j = \begin{cases} \widetilde{\phi_j}, & \text{if } \max\left(\|\widetilde{\phi_j}\|_{L^1}, \|\widetilde{\phi_j}\|_{L^\infty}\right) \leq j, \\ \phi, & \text{if } \max\left(\|\widetilde{\phi_j}\|_{L^1}, \|\widetilde{\phi_j}\|_{L^\infty}\right) > j. \end{cases} \tag{4.2}$$

By $\phi \in E$, we have $\phi_j \in E$ for all $j \geq 1$. Next from (4.2), we get that $\{\phi_j\}$ satisfies (4.1) for all $j \geq 1$. Thus, Property (ii) of Lemma 4.1 is satisfied. Then, for any $(\delta, \varphi) \in E \times F$, we let $j_k \to \infty$ to satisfy $(\sigma_{j_k}, \widetilde{\phi_{j_k}}) = (\delta, \varphi)$ for all $j_k \geq 1$. We deduce from $j_k \to \infty$ that for all sufficiently large k, $\phi_{j_k} = \widetilde{\phi_{j_k}}$. So, Property (i) of Lemma 4.1 is satisfied. The proof of this lemma is complete. □

Now, we give the proof of Theorem 2.10.

Proof of Theorem 2.10. For E and F, let $\{(\sigma_j,\ \phi_j)\}_{j\geq 1} \subset E \times F$ be as that in Lemma 4.1. Then, the initial value u_0 is defined as follows:

$$u_0(x) = \sum_{j=1}^{\infty} \chi_j(x) a_j{}^{-\sigma_j} \phi_j(x/a_j) = \sum_{j=1}^{\infty} \chi_j(x) a_j^{-\sigma_j} \phi_j(x/a_j), \tag{4.3}$$

where $\chi_j(x) = \chi_j'(x/a_j)$, $\chi_j'(x)$ is the cut-off function on set $A_j = \{2^{-j} < |x| < 2^j\}$ relative to $A_j' = \{2^{-(j-1)} < |x| < 2^{j-1}\}$. The sequence $\{a_j\}_{j\geq 1}$ will be determined by the following method. Firstly, suppose $a_0 \geq 1$ is large enough to satisfy

$$\exp x \geq (2x)^\tau, \qquad \text{for all } x \geq a_0 \geq 1,$$

where $\tau = \frac{2N}{(N(m-1)+2)(2+\mu(m-1))}$. If let

$$\begin{cases} a_1 = a_0, \\ a_j = \max\left((\lambda_j')^{2\beta}, \exp \frac{a_{j-1}}{\sigma_j}\right), \qquad j > 1, \end{cases} \tag{4.4}$$

where λ'_j take the same values as that in Lemma 3.3, we thus have

$$a_j \geq \exp \frac{a_{j-1}}{\sigma_j} > \exp \frac{a_{j-1}}{\tau} \geq 2a_{j-1}.$$

Iterating, we see that

$$a_j \geq 2^{j-1}, \qquad \text{for all } j \geq 1,$$

which implies, via (4.1) and (4.3), that

$$\|u_0\|_{L^\infty} \leq \sum_{j=1}^{\infty} \|\chi_j(x) a_j{}^{-\sigma_j} \phi_j(x/a_j)\|_{L^\infty} \leq \sum_{j=1}^{\infty} \max(M, j) a_j{}^{-\sigma_j} < \infty.$$

Therefore, the sequence of (4.3) is convergent in $C_0(\mathbb{R}^N)$. So

$$u_0 \in C_0(\mathbb{R}^N).$$

Note also, from (4.3), that

$$u_0 \geq 0$$

and

$$u_0(0) = 0.$$

We thus have

$$u_0 \in C_0^+(\mathbb{R}^N) \subset X_0^+.$$

So $S(t)u_0$ is well defined for all $0 \leq t < \infty$. The Property (ii) of Theorem 2.10 will be considered first. For $0 \leq t < \infty$, (2.6) clearly indicates

$$\Gamma_{\lambda_n}^{\mu,\beta}[u_0] = S(\lambda_n^{-\beta'} t)[D_{\lambda_n}^{\mu,\beta} u_0].$$

For any $(\sigma, \phi) \in E \times F$, Lemma 4.1 implies that there exists a sequence of integers $\{n_k\}_{k \geq 1}$ going to infinity such that

$$\sigma_{n_k} = \sigma, \qquad \phi_{n_k} = \phi \tag{4.5}$$

for all $k \geq 1$. Hence,

$$\Gamma_{\lambda_{n_k}}^{\mu,\beta}[u_0] = S(\lambda_{n_k}^{-\beta'} t)(u_{n_k} + v_{n_k} + w_{n_k}), \tag{4.6}$$

where

$$u_{n_k} = \lambda_{n_k}^{\mu} \sum_{j=1}^{n_k - 1} \chi_j(\lambda_{n_k}^{2\beta} x) a_j{}^{-\sigma_j} \phi_j(\lambda_{n_k}^{2\beta}/a_j x), \tag{4.7}$$

$$v_{n_k} = \lambda_{n_k}^{\mu} \chi_{n_k}(\lambda_{n_k}^{2\beta} x) a_{n_k}{}^{-\sigma_{n_k}} \phi_j(\lambda_{n_k}^{2\beta}/a_{n_k} x) \tag{4.8}$$

and

$$w_{n_k} = \lambda_{n_k}^{\mu} \sum_{j=n_k+1}^{\infty} \chi_j(\lambda_{n_k}^{2\beta} x) a_j{}^{-\sigma_j} \phi_j(\lambda_{n_k}^{2\beta}/a_j x). \tag{4.9}$$

Let

$$\lambda_{n_k} = a_{n_k}^{1/2\beta} \tag{4.10}$$

and

$$n_k \geq M,$$

where M is given in Lemma 4.1. First note, from (2.7), that

$$\mu - 2N\beta < 0.$$

We thus get from (4.4) that there is a $n_{k_0} > 1$ such that for all $n_k \geq n_{k_0}$,

$$\begin{aligned} &\|u_{n_k}\|_{L^1} \leq \sum_{j=1}^{n_k} \lambda_{n_k}^{\mu-2N\beta} a_j^{N-\sigma} \|\phi_j\|_{L^1} \leq \min(n_k^2, n_k^2 \lambda_{n_k}^{\mu-2N\beta} a_{n_k}^N), \\ &\|v_{n_k}\|_{L^\infty} \leq n_k \end{aligned} \tag{4.11}$$

and

$$\|w_{n_k}\|_{L^\infty} \leq \sum_{j=n_k+1}^{\infty} \lambda_{n_k}^{\mu} a_j^{-\sigma} \|\phi_j\|_{L^\infty} \leq \min(1, \sum_{j=n_k+1}^{\infty} \lambda_{n_k}^{\mu} a_j^{-\sigma}(n_k+1)). \tag{4.12}$$

In light of (4.7), (4.8) and (4.9), we have

$$\operatorname{supp}(u_{n_k}) = \left\{x \in \mathbb{R}^N; |x| \leq 2^{n_k-1}(\lambda_{n_k-1}/\lambda_{n_k})^{2\beta}\right\}, \tag{4.13}$$

$$\operatorname{supp}(v_{n_k}) = \left\{x \in \mathbb{R}^N; 2^{-n_k} \leq |x| \leq 2^{n_k}\right\} \tag{4.14}$$

and

$$\operatorname{supp}(w_{n_k}) = \left\{x \in \mathbb{R}^N; |x| \geq 2^{-(n_k+1)}(\lambda_{n_k+1}/\lambda_{n_k})^{2\beta}\right\}. \tag{4.15}$$

Here, we have used the fact that

$$2^{-(n_k+1)}(\lambda_{n_k+1}/\lambda_{n_k})^{2\beta} \leq 2^{-(n_k+j)}(\lambda_{n_k+j}/\lambda_{n_k})^{2\beta}$$

for all $j \geq 2$. Combining (4.11)–(4.15) with Lemma 3.1 and Lemma 3.2, we find that for $0 < T < \infty$, if $0 \leq t < T \leq n_k$, then

$$\operatorname{supp} S(\lambda_{n_k}^{-\beta'} t)(u_{n_k}) = \{x \in \mathbb{R}^N; |x| \leq 2^{n_k+1}(\lambda_{n_k-1}/\lambda_{n_k})^{2\beta} + C n_k{}^{\eta+2(m-1)\eta} \lambda_{n_k}^{-\beta'\eta}\},$$

$$\mathrm{supp} S(\lambda_{n_k}^{-\beta'} t)(v_{n_k}) = \Big\{ x \in \mathbb{R}^N;$$
$$2^{-n_k} - C{n_k}^{\frac{m-1}{2}} \lambda_{n_k}^{-\frac{\beta'}{2}} \le |x| \le 2^{n_k+2} + C{n_k}^{\eta+\frac{m-1}{2}} \lambda_{n_k}^{-\beta'\eta} \Big\}$$

and

$$\mathrm{supp} S(\lambda_{n_k}^{-\beta'} t)(w_{n_k}) = \left\{ x \in \mathbb{R}^N; |x| \ge 2^{-(n_k+1)} (\lambda_{n_k+1}/\lambda_{n_k})^{2\beta} - C{n_k}^{\frac{1}{2}} \lambda_{n_k}^{-\frac{\beta'}{2}} \right\}.$$

Letting $C_{n_k} = n_k^2$ in (3.13), we obtain from Lemma 3.3 that

$$\begin{aligned} &2^{n_k+1} (\lambda_{n_k-1}/\lambda_{n_k})^{2\beta} + C{n_k}^{\eta+(m-1)\eta} \lambda_{n_k}^{-\beta'\eta} \\ &\le 2^{-n_k} - C{n_k}^{\frac{m-1}{2}} \lambda_n^{-\frac{\beta'}{2}} \\ &\le 2^{n_k+2} + C{n_k}^{\eta+\frac{m-1}{2}} \lambda_n^{-\beta'\eta} \\ &\le 2^{-(n_k+1)} (\lambda_{n_k+1}/\lambda_{n_k})^{2\beta} - C{n_k}^{\frac{1}{2}} \lambda_{n_k}^{-\frac{\beta'}{2}}. \end{aligned}$$

Hence,

$$\mathrm{supp} S(\lambda_{n_k}^{-\beta'} t)(u_{n_k}) \bigcap \mathrm{supp} S(\lambda_{n_k}^{-\beta'} t)(v_{n_k}) = \emptyset,$$
$$\mathrm{supp} S(\lambda_{n_k}^{-\beta'} t)(u_{n_k}) \bigcap \mathrm{supp} S(\lambda_{n_k}^{-\beta'} t)(w_{n_k}) = \emptyset$$

and

$$\mathrm{supp} S(\lambda_{n_k}^{-\beta'} t)(v_{n_k}) \bigcap \mathrm{supp} S(\lambda_{n_k}^{-\beta'} t)(w_{n_k}) = \emptyset.$$

So, for $x \in \mathbb{R}^N$ and $0 < T < \infty$, if $0 \le t < T \le n_k$, then

$$\begin{aligned} &S(t\lambda_{n_k}^{-\beta'})(u_{n_k} + v_{n_k} + w_{n_k}) \\ &\quad = S(t\lambda_{n_k}^{-\beta'})(u_{n_k}) + S(t\lambda_{n_k}^{-\beta'})(v_{n_k}) + S(t\lambda_{n_k}^{-\beta'})(w_{n_k}). \end{aligned} \tag{4.16}$$

The maximum principle implies, via (4.12), that

$$\|S(t\lambda_{n_k}^{-\beta'}) w_{n_k}\|_{L^\infty} \le \|w_{n_k}\|_{L^\infty} \le \lambda_{n_k}^{\mu} \sum_{j=n_k}^{\infty} (j+1) {a_{j+1}}^{-\sigma}.$$

Hence,

$$\begin{aligned} \|S(t\lambda_{n_k}^{-\beta'}) w_{n_k}\|_{L^\infty} &\le a_{n_k}^{\sigma} \sum_{j=n_k}^{\infty} (j+1) {a_{j+1}}^{-\sigma} \\ &\le \sum_{j=n_k}^{\infty} (j+1) a_{n_k}^{\sigma} {a_{j+1}}^{-\sigma} \xrightarrow{n\to\infty} 0. \end{aligned} \tag{4.17}$$

Here, we have used (2.9), (4.4) and (4.10). It follows from (4.10) and (4.11) that

$$\begin{aligned}&\|S(t\lambda_{n_k}^{-\beta'})u_{n_k}\|_{L^\infty}\\&\leq Ca_{n_k-1}^{\frac{2N}{N(m-1)+2}}t^{-\frac{N}{N(m-1)+2}}a_{n_k}^{-\frac{2N-2\mu-\mu N(m-1)}{2N\beta(m-1)+4\beta}}n_k^{\frac{2}{N(m-1)+2}}\\&\leq Cn_k{}^2a_{n_k-1}^{\frac{2N}{N(m-1)+2}}t^{-\frac{N}{N(m-1)+2}}a_{n_k}^{-\frac{\eta}{2\beta}(\frac{2N}{\mu}-2-N(m-1))}.\end{aligned} \tag{4.18}$$

Obviously, (2.7) indicates

$$\frac{2N}{\mu}-2-N(m-1)<0.$$

Therefore, by (4.4) and (4.18), we get that for any fixed $\epsilon>0$,

$$\|S(t\lambda_{n_k}^{-\beta'})u_{n_k}\|_{L^\infty}\xrightarrow{n_k\to\infty}0 \tag{4.19}$$

uniformly for $t\geq\epsilon$. From (2.9), (4.5), (4.8) and (4.11), we can see that for all $x\in A_{n_k}=\{2^{-n_k+1}<|x|<2^{n_k-1}\}$,

$$v_{n_k}=a_{n_k}^{\frac{\mu}{2\beta}-\sigma_{n_k}}\phi_{n_k}=a_{n_k}^{\sigma-\sigma_{n_k}}\phi_{n_k}=\phi. \tag{4.20}$$

Next, we want to show that for $0<T<\infty$, if $0\leq t<T$, then

$$\|S(\lambda_{n_k}^{-\beta'}t)v_{n_k}-\phi\|_{L^\infty(\mathbb{R}^N)}\xrightarrow{n_k\to\infty}0. \tag{4.21}$$

In fact,

$$\begin{aligned}\|S(\lambda_{n_k}^{-\beta'}t)v_{n_k}-\phi\|_{L^\infty(\mathbb{R}^N)}&\leq\|S(\lambda_{n_k}^{-\beta'}t)v_{n_k}-\phi\|_{L^\infty(A_{n_{k_0}})}\\&+\|S(\lambda_{n_k}^{-\beta'}t)v_{n_k}\|_{L^\infty(\mathbb{R}^N\setminus A_{n_{k_0}})}+\|\phi\|_{L^\infty(\mathbb{R}^N\setminus A_{n_{k_0}})}.\end{aligned} \tag{4.22}$$

Clearly, Lemma 3.1 and Lemma 3.2 imply that for $0<T<\infty$, if $0\leq t<T$ and $n_k\geq n_{k_0}$, then

$$\begin{aligned}&\mathrm{supp}(S(\lambda_{n_k}^{-\beta'}t)[\chi'_{n_{k_0}-4}v_{n_k}])\subset\{x\in\mathbb{R}^N;\\&2^{-n_{k_0}+4}-C\|v_{n_k}\|_{L^\infty}^{\frac{m-1}{2}}\lambda_{n_k}^{-\frac{\beta'}{2}}t^{1/2}\leq|x|\leq2^{n_{k_0}-2}+C\|v_{n_k}\|_{L^\infty}^{\frac{m-1}{2}}\lambda_{n_k}^{-\beta'\eta}t^\eta\}.\end{aligned}$$

Combining this with (3.13) and (4.4) yields that for $0<T<\infty$, if $n_k\geq n_{k_0}\geq T>t\geq0$, then

$$\mathbb{R}^N\setminus A_{n_{k_0}}\bigcap\mathrm{supp}(S(\lambda_{n_k}^{-\beta'}t)[\chi'_{n_{k_0}-4}v_{n_k}])=\emptyset. \tag{4.23}$$

In other words, for $0<T<\infty$, if $n_k\geq n_{k_0}\geq T>t\geq0$, then the value of $S(\lambda_{n_k}^{-\beta'}t)[v_{n_k}]$ in $\mathbb{R}^N\setminus A_{n_{k_0}}$ is independent of the initial value v_{n_k} in $A_{n_{k_0}-4}$. So, for any $\epsilon > 0$, there exists a n_{k_1} such that for $0 < T < \infty$, if $n_k \geq n_{k_1} \geq n_{k_0} \geq T > t \geq 0$, then

$$\begin{aligned}
&\|S(\lambda_{n_k}^{-\beta'} t) v_{n_k}\|_{L^\infty(\mathbb{R}^N \setminus A_{n_{k_0}})} \\
&\quad \leq \|(1 - \chi'_{n_{k_0}-4}) v_{n_k}\|_{L^\infty} \leq \|(1 - \chi'_{n_{k_0}-4}) \phi\|_{L^\infty} \\
&\quad = \|\phi\|_{L^\infty(|x| \geq 2^{n_{k_0}-4})} + \|\phi\|_{L^\infty(|x| \leq 2^{-n_{k_0}+4})} \\
&\quad \leq \|\phi\|_{L^\infty(|x| \geq 2^{n_{k_1}-4})} + \|\phi\|_{L^\infty(|x| \leq 2^{-n_{k_1}+4})} \\
&\quad < \frac{\epsilon}{3}.
\end{aligned} \tag{4.24}$$

Here, we have used the fact that $\phi \in \mathscr{S}^+(\mathbb{R}^N)$.

By a similar analysis works to (4.23), we can get that for $0 < T < \infty$, if $n_k \geq n_{k_0} \geq T > t \geq 0$, then the value of $S(\lambda_{n_k}^{-\beta'} t) v_{n_k}$ in $A_{n_{k_0}}$ is only dependent on the initial value of v_{n_k} in $A_{n_{k_0}+4}$.

Indeed, we obtain from Lemma 3.1 and Lemma 3.2 that for $0 < T < \infty$, if $n_k \geq n_{k_0} \geq T > t \geq 0$, then

$$\operatorname{supp}(S(\lambda_{n_k}^{-\beta'} t)[\chi'_{R_1} v_{n_k}]) \subset \{x; |x| \leq 2^{-n_{k_0}-2} + C\|v_{n_k}\|_{L^\infty}^{\frac{m-1}{2}} \lambda_{n_k}^{-\eta\beta} t^\eta\} \tag{4.25}$$

and

$$\operatorname{supp}(S(\lambda_{n_k}^{-\beta'} t)[(1 - \chi'_{R_2}) v_{n_k}]) \subset \{x; |x| \geq 2^{n_{k_0}+4} - C\|v_{n_k}\|_{L^\infty}^{\frac{m-1}{2}} \lambda_{n_k}^{-\frac{\beta}{2}} t^{\frac{1}{2}}\}, \tag{4.26}$$

where χ'_{R_1} is the characteristic function of $B_{R_1} = \{x; |x| \leq 2^{-n_{k_0}-4}\}$ and χ'_{R_2} is the characteristic function of $B_{R_2} = \{x; |x| \leq 2^{n_{k_0}+4}\}$. These imply, via (3.13), (4.25) and (4.26), that for $0 < T < \infty$, if $n_k \geq n_{k_0} \geq T > t \geq 0$, then

$$\operatorname{supp}(S(\lambda_{n_k}^{-\beta'} t)[\chi'_{R_1} v_{n_k}]) \bigcap A_{n_{k_0}} = \emptyset \tag{4.27}$$

and

$$\operatorname{supp}(S(\lambda_{n_k}^{-\beta'} t)[(1 - \chi'_{R_2}) v_{n_k}]) \bigcap A_{n_{k_0}} = \emptyset. \tag{4.28}$$

A simple result of (4.20) is that if $n_k \geq n_{k_0} + 6$, then

$$v_{n_k} = \phi, \qquad \text{for all } x \in A_{n_{k_0}+4}.$$

This implies, via (4.27) and (4.28), that for $0 < T < \infty$,

$$\|S(\lambda_{n_k}^{-\beta'} t) v_{n_k} - S(\lambda_{n_k}^{-\beta'} t)\phi\|_{L^\infty(A_{n_{k_0}})} = 0 \tag{4.29}$$

provided $0 \leq t < T \leq n_{k_0} < n_{k_0} + 6 \leq n_k$. Since $\phi \in \mathscr{S}^+(\mathbb{R})$, by Proposition 2.2, we have

$$S(t)\phi \in C([0, \infty) \times \mathbb{R}^N). \tag{4.30}$$

For any $\epsilon > 0$ and $A_{n_{k_0}}$, we deduce from $\beta' > 0$ and (4.30) that there exists a $n_{k_2} \geq n_{k_0}$ such that for $0 < T < \infty$, if $n_k \geq n_{k_2} \geq n_{k_0} \geq T > t \geq 0$, then

$$\|S(\lambda_{n_k}^{-\beta'} t)\phi - \phi\|_{L^\infty(A_{n_{k_0}})} < \frac{\epsilon}{3}. \tag{4.31}$$

Here, we used the fact that $A_{n_{k_0}}$ is a relative compact set in $\mathbb{R}^N$. So, from (4.29), (4.30) and (4.31), we find that for $0 < T < \infty$, if $0 \leq t < T \leq n_{k_0}$ and $n_k \geq \max(n_{k_0+6}, n_{k_1}, n_{k_2})$, then

$$\begin{aligned}
&\|S(\lambda_{n_k}^{-\beta'} t)v_{n_k} - \phi\|_{L^\infty(A_{n_{k_0}})} \\
&\leq \|S(\lambda_{n_k}^{-\beta'} t)v_{n_k} - S(\lambda_{n_k}^{-\beta'} t)\phi\|_{L^\infty(A_{n_{k_0}})} + \|S(\lambda_{n_k}^{-\beta'} t)\phi - \phi\|_{L^\infty(A_{n_{k_0}})} \\
&< \frac{\epsilon}{3}.
\end{aligned} \tag{4.32}$$

Obviously, (4.24) clearly indicates that for $n_{k_0} \geq n_{k_1}$,

$$\|\phi\|_{L^\infty(\mathbb{R}^N \setminus A_{k_0})} < \frac{\epsilon}{3}. \tag{4.33}$$

For any $\epsilon > 0$ and $0 < T < \infty$, we obtain from (4.22), (4.24), (4.32) and (4.33) that if $0 \leq t < T \leq n_{k_0}$ and $n_k \geq \max(n_{k_0} + 6, n_{k_1}, n_{k_2})$, then

$$\begin{aligned}
&\|S(\lambda_{n_k}^{-\beta'} t)v_{n_k} - \phi\|_{L^\infty(\mathbb{R}^N)} \\
&\leq \|S(\lambda_{n_k}^{-\beta'} t)v_{n_k} - \phi\|_{L^\infty(A_{n_{k_0}})} + \|S(\lambda_{n_k}^{-\beta'} t)v_{n_k}\|_{L^\infty(\mathbb{R}^N \setminus A_{n_{k_0}})} + \|\phi\|_{L^\infty(\mathbb{R}^N \setminus A_{n_{k_0}})} \\
&< \epsilon.
\end{aligned}$$

So, the claim of (4.21) is verified. The other meaning of (4.21) is that for $0 < T < \infty$,

$$\lim_{n_k \to \infty} \|S(t\lambda_{n_k}^{-\beta'})v_{n_k} - \Phi(t)\|_{L^\infty(\mathbb{R}^N)} = 0 \tag{4.34}$$

uniformly for $t \in [0, T)$. So, by (4.6), (4.16), (4.17), (4.19), (4.21) and (4.34), we obtain that for all $0 < \epsilon < T < \infty$,

$$\Gamma_{\lambda_{n_k}}^{\mu,\beta}[u_0] = S(\lambda_{n_k}^{-\beta'} t)[u_{n_k} + v_{n_k} + w_{n_k}] \xrightarrow{n_k \to \infty} \Phi(t) \quad \text{in} \quad L^\infty(\mathbb{R}^N)$$

uniformly for any $t \in [\epsilon, T)$. That is, for all $0 < \epsilon < T < \infty$,

$$\Gamma_{\lambda_{n_k}}^{\mu,\beta}[u_0] \xrightarrow{n_k \to \infty} \Phi \quad \text{in} \quad L^\infty([\epsilon, T); L^\infty(\mathbb{R}^N)).$$

So, we complete the proof of Property (ii). Property (i) follows from Property (ii) by setting $t = 1$ and $t_{n_k} = \lambda_{n_k}^2$. The proof of this theorem is complete. □

From Theorem 2.10, we can give the proof of Theorem 2.12 as follows.

Proof of Theorem 2.12. Let $F \subset \mathscr{S}^+(\mathbb{R}^N)$ be a countable dense subset of $C_0^+(\mathbb{R}^N)$, and consider u_0 given by Theorem 2.10 applied with this set F. Since F is dense in $C_0^+(\mathbb{R}^N)$ and $\omega^{\mu,\beta}(u_0)$ is clearly closed in $C_0^+(\mathbb{R}^N)$, Property (i) follows from Property (i) of Theorem 2.10. Note that $\gamma^{\mu,\beta}(u_0)$ is a closed subset of $C((0,\infty); C_0^+(\mathbb{R}^N))$ for the topology of uniform convergence on $[\epsilon, T) \times \mathbb{R}^N$ for all $0 < \epsilon < T < \infty$. Since F is dense in $C_0^+(\mathbb{R}^N)$, Property (ii) follows from Property (ii) of Theorem 2.10. So we complete the proof of this theorem. □

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