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Doubly nonlinear evolution equations with non-monotone perturbations in reflexive Banach spaces

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Dedicated to Professor Mitsuharu Ôtani on the occasion of his 60th birthday

Abstract. Let V and V^* be a real reflexive Banach space and its dual space, respectively. This paper is devoted to the abstract Cauchy problem for doubly nonlinear evolution equations governed by subdifferential operators with non-monotone perturbations of the form: $\partial_V \psi^t(u'(t)) + \partial_V \varphi(u(t)) + B(t, u(t)) \ni f(t)$ in $V^*, 0 < t < T, u(0) = u_0$, where $\partial_V \psi^t, \partial_V \varphi : V \to 2^{V^*}$ denote the subdifferential operators of proper, lower semicontinuous and convex functions $\psi^t, \varphi : V \to (-\infty, +\infty]$, respectively, for each $t \in [0, T]$, and $f : (0, T) \to V^*$ and $u_0 \in V$ are given data. Moreover, let B be a (possibly) multi-valued operator from $(0, T) \times V$ into V^*. We present sufficient conditions for the local (in time) existence of strong solutions to the Cauchy problem as well as for the global existence. Our framework can cover evolution equations whose solutions might blow up in finite time and whose unperturbed equations (i.e., $B \equiv 0$) might not be uniquely solved in a doubly nonlinear setting. Our proof relies on a couple of approximations for the equation and a fixed point argument with a multi-valued mapping. Moreover, the preceding abstract theory is applied to doubly nonlinear parabolic equations.

1. Introduction

Let V and V^* be a reflexive Banach space and its dual space, respectively, and let H be a Hilbert space whose dual space H^* is identified with itself such that

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

with continuous and densely defined canonical injections. Let $\partial_V \psi^t$ (for each $t \in [0, T]$) and $\partial_V \varphi : V \to 2^{V^*}$ stand for the subdifferential operators of proper, lower semicontinuous and convex functions ψ^t and φ, respectively, from V into $(-\infty, +\infty]$. Moreover, let B be a (possibly) multi-valued mapping from $(0, T) \times V$ into V^* such

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that $B(t,\cdot)$ might be non-monotone in $V \times V^*$ for each fixed t. We discuss the existence of local and global (in time) strong solutions to the following Cauchy problem for a doubly nonlinear evolution equation:

$$\text{(CP)} \begin{cases} \partial_V \psi^t(u'(t)) + \partial_V \varphi(u(t)) + B(t,u(t)) \ni f(t) \text{ in } V^*, \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where $u'(t) = du(t)/dt$, and $f : (0,T) \to V^*$ and $u_0 \in V$ are given.

Studies of evolution equations governed by subdifferential operators were initiated with the following simple case:

$$u'(t) + \partial_H \varphi(u(t)) \ni 0, \quad 0 < t < T \tag{1.1}$$

in a Hilbert space H (see, e.g., Brézis [20]), and various generalized forms of (1.1) have been studied by many authors to reinforce the applicability of theories of evolution equations to nonlinear PDEs. We particularly choose three directions of generalization among successful ones in applications to nonlinear PDEs.

Non-monotone perturbations: The development of perturbation theory for (1.1) is further extending the applicability of subdifferential approaches to nonlinear PDEs. Indeed, Navier–Stokes equation (see Ôtani-Yamada [43], Ôtani [41,42]), Allen-Cahn equation and Cahn–Hilliard equation (see Kenmochi et al. [34]) are reduced to the perturbation problem for (1.1) of the form:

$$u'(t) + \partial_H \varphi(u(t)) + B(t,u(t)) \ni f(t) \tag{1.2}$$

with a possibly non-monotone operator $B : (0,T) \times H \to H$ in a Hilbert space H. In [41], Mitsuharu Ôtani first established an abstract theory on the existence of local and global (in time) strong solutions to Cauchy problems for (1.2), and his framework can cover nonlinear PDEs whose solutions possibly blow up in finite time, e.g., degenerate parabolic equations with blow-up terms (see also [39,40]). Moreover, his abstract theory has been applied to various nonlinear parabolic equations and systems such as Navier–Stokes equation, heat-convection equation, magneto-micropolar fluid equation and various nonlinear parabolic equations and systems.

Doubly nonlinear evolution equation: Barbu [14], Arai [10], Senba [48] and Colli-Visintin [24] investigated sufficient conditions for the existence of strong solutions to Cauchy problems for doubly nonlinear evolution equations in the form

$$\partial_H \psi(u'(t)) + \partial_H \varphi(u(t)) \ni f(t) \text{ in } H, \quad 0 < t < T \tag{1.3}$$

with two subdifferential operators $\partial_H \psi$ and $\partial_H \varphi$, and their results were applied to doubly nonlinear parabolic equations such as

$$\alpha\,(u_t(x,t)) - \operatorname{div} \mathbf{a}(\nabla u(x,t)) \ni f(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $\alpha : \mathbb{R} \to \mathbb{R}$ and $\mathbf{a} : \mathbb{R}^N \to \mathbb{R}^N$ are maximal monotone graphs, and $f : \Omega \times (0, T) \to \mathbb{R}$ is a given function (see also [4,8,17,23,37], [45, Sect. 11], [11,38,47] and [46]).

Moreover, Grange-Mignot [30], Barbu [16] and Kenmochi-Pawlow [35] also studied other types of doubly nonlinear evolution equations such as

$$v'(t) + \partial_H \varphi(u(t)) \ni f(t), \ v(t) \in \partial_H \psi(u(t)) \text{ in } H, \quad 0 < t < T \tag{1.5}$$

(see also [9,12,25,36,49,53], [45, Sect. 11], [1–3,5]).

Banach space framework: It helps our analysis of nonlinear PDEs to choose a proper function space as a base space of each setting. Indeed, one can find advantages of frameworks which admit a flexible choice of function spaces particularly in studies on doubly nonlinear parabolic equations, e.g., (1.4) and the following

$$\frac{\partial}{\partial t}|u|^{p-2}u(x,t) - \Delta_m u(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T),$$

where $p, m \in (1, \infty)$ and Δ_m denotes the so-called m-Laplacian given by $\Delta_m u(x) = \mathrm{div}(|\nabla u(x)|^{m-2}\nabla u(x))$ (see Raviart [44], Tsutsumi [51]). However, evolution equations governed by subdifferential operators were originally studied only in Hilbert space settings. Hence, several authors (e.g., Brézis [19], Kenmochi [32], Barbu [16] and Colli [23]) made attempts to establish V-V^* frameworks that enable us to treat evolution equations in Banach spaces V and their dual spaces V^* (see also Akagi-Ôtani [6–8], Akagi [5], Aso et al's[12]).

In order to cover a broader range of nonlinear PDEs, particularly, doubly nonlinear versions of various PDEs, e.g., Allen-Cahn equations and Navier–Stokes equations, it would be necessary to study (CP) with as general assumptions as possible. However, there seems to be no contribution to (CP) with three options, double nonlinearity, non-monotone perturbations and Banach space framework. The purpose of the current paper is to present sufficient conditions for the local (in time) existence of strong solutions to (CP) as well as for the global existence. To do so, we overcome a couple of difficulties, e.g., the strong nonlinearity of the equation and the defect of useful properties of maximal monotone operators defined in Banach spaces (cf. maximal monotone operators in Hilbert spaces have fine properties such as the Lipschitz continuity of their resolvents and Yosida approximations).

It is particularly noteworthy that the following unperturbed problems corresponding to (CP) might not be uniquely solved.

$$\partial_V \psi^t(u'(t)) + \partial_V \varphi(u(t)) \ni f(t) \text{ in } V^*, \ 0 < t < T, \quad u(0) = u_0. \tag{1.6}$$

Indeed, a simple example of non-unique solutions was given in [23] even for the case where V is a Hilbert space and ψ^t is independent of t. Following a classical approach to perturbation problems, one employs mappings $\mathcal{S}_T : g \mapsto u$, which maps a function

$g : (0, T) \to V^*$ to the strong solution(s) u of (1.6) with f replaced by $f - g$ on $[0, T]$, and $\mathcal{F}_T : g \mapsto B(\cdot, u(\cdot))$ to obtain a strong solution $u_* := \mathcal{S}_T g_*$ of (CP) with a fixed point g_* of $\mathcal{F}_T$. However, since we cannot ensure the uniqueness of strong solutions of (1.6), $\mathcal{F}_T$ could be a multi-valued mapping. Fixed point theorems for multi-valued mappings have already been established; in particular, several authors extended Schauder-Tychonoff's fixed point theorem to multi-valued mappings (see, e.g., [22,27,29]). Here, we note that such fixed point theorems require the convexity of the set $\mathcal{F}_T g$ for every g; however, the convexity is not obvious in our case. In order to overcome such a difficulty, we introduce approximate problems for (CP) whose solutions can be constructed by the fixed point argument mentioned previously. More precisely, the unperturbed problem corresponding to our approximation has a unique solution, so the fixed point argument can work well for the approximate problems. Furthermore, our unperturbed problem with approximation could be a new example of doubly nonlinear problems with the uniqueness of solutions (cf. [23]). Thus, we can construct approximate solutions for (CP), and then, we derive the convergence of the approximate solutions to obtain a solution of (CP) (see Sect. 4 for more details).

We apply the preceding abstract theory to the initial-boundary value problems for doubly nonlinear parabolic equations of degenerate type such as

$$|u_t|^{p-2}u_t(x, t) - \Delta_m u(x, t) - |u|^{q-2}u(x, t) = f(x, t) \tag{1.7}$$

for $(x, t) \in \Omega \times (0, T)$, where Ω is a bounded domain in $\mathbb{R}^N$, $1 < m, p, q < \infty$ and $f : \Omega \times (0, T) \to \mathbb{R}$ is given. Such doubly nonlinear degenerate parabolic equations can be regarded as a special case of generalized Allen-Cahn equations due to Gurtin [31]. Indeed, the solution $u(x, t)$ of (1.7) corresponds to the order parameter at (x, t) generated by a generalized gradient system $\mathcal{A}(u'(t))u'(t) = -\mathcal{F}'(u(t))$ of the free energy

$$\mathcal{F}(u) := \frac{1}{m}\int_\Omega |\nabla u(x)|^m dx - \frac{1}{q}\int_\Omega |u(x)|^q dx - \int_\Omega f(x, t)u(x) dx$$

and the constitutive modulus $\mathcal{A}(u) := |u|^{p-2}$. Moreover, we also treat a semilinear equation with a nonlinear term involving the gradient of u, e.g.,

$$|u_t|^{p-2}u_t(x, t) - \Delta u(x, t) - |u|^{q_1-2}u(x, t) \pm |\nabla u(x, t)|^{q_2-1} = f(x, t) \tag{1.8}$$

with $1 < q_1, q_2 < \infty$. It is noteworthy that (1.8) can be no longer written as a generalized gradient system, because of the gradient nonlinearity.

This paper consists of seven sections. In Sect. 2, we summarize without proofs the relevant material on maximal monotone operators and subdifferential operators. Section 3 is devoted to our main results on the existence of local and global (in time) strong solutions of (CP). Proofs of the main results will be given in Sects. 4, 5 (for the local existence) and in Sect. 6 (for the global existence). Finally, in Sect. 7 we discuss applications of the preceding abstract theory to nonlinear PDEs.

2. Preliminaries

In this section, several standard facts on maximal monotone operators and subdifferential operators are given for later use.

Let E and E^* be a reflexive Banach space and its dual space with the norms $|\cdot|_E$ and $|\cdot|_{E^*}$, respectively, and the duality pairing $\langle\cdot,\cdot\rangle$. According to [13], every reflexive Banach space can be equivalently renormed (along with its dual) to be strictly convex. Throughout this paper, we denote by $D(A)$ the domain of each operator $A : E \to 2^{E^*}$, and moreover, we denote by A the graph of A, that is, $[u, \xi] \in A$ means $u \in D(A)$ and $\xi \in A(u)$.

An operator $A : E \to 2^{E^*}$ is said to be monotone if $\langle \xi_1 - \xi_2, u_1 - u_2\rangle \geq 0$ for all $[u_1, \xi_1], [u_2, \xi_2] \in A$, and the maximality of A is known to be equivalent to the condition that the range of $F_E + A$ coincides with E^*, where F_E denotes the duality mapping between E and E^*, provided that E and E^* are strictly convex (see, e.g., [15,21]). The following proposition is concerned with the closedness of maximal monotone operators in an appropriate topology (see [21]).

PROPOSITION 2.1. *Let E be a reflexive Banach space. Let $A : E \to 2^{E^*}$ be a maximal monotone operator and let $[u_n, \xi_n] \in A$ and $[u, \xi] \in E \times E^*$ be such that $u_n \to u$ weakly in E and $\xi_n \to \xi$ weakly in E^*. Moreover, suppose that*

$$\limsup_{n\to\infty} \langle \xi_n, u_n\rangle \leq \langle \xi, u\rangle.$$

Then, it follows that $[u, \xi] \in A$ and $\langle \xi_n, u_n\rangle \to \langle \xi, u\rangle$.

We denote by $\Phi(E)$ the set of all proper, lower semicontinuous and convex functions ϕ from E into $(-\infty, +\infty]$, where the "proper" means $\phi \not\equiv \infty$. For each $\phi \in \Phi(E)$, the *effective domain* $D(\phi)$ of ϕ is given as follows:

$$D(\phi) := \{u \in E; \phi(u) < \infty\},$$

and the *subdifferential operator* $\partial_E\phi : E \to 2^{E^*}; u \mapsto \partial_E\phi(u)$ of ϕ is defined by

$$\partial_E\phi(u) := \{\xi \in E^*; \phi(v) - \phi(u) \geq \langle \xi, v - u\rangle \text{ for all } v \in D(\phi)\}$$

with the domain $D(\partial_E\phi) := \{u \in D(\phi); \partial_E\phi(u) \neq \emptyset\}$. It is well known that every subdifferential operator is maximal monotone (see, e.g., [15,21]).

Now, let H be a Hilbert space whose dual space H^* is identified with itself and define the subdifferential operator $\partial_H\phi : H \to 2^H$ of $\phi \in \Phi(H)$ as follows:

$$\partial_H\phi(u) := \{\xi \in H; \phi(v) - \phi(u) \geq (\xi, v - u)_H \text{ for all } v \in D(\phi)\}$$

with the domain $D(\partial_H\phi) := \{u \in D(\phi); \partial_H\phi(u) \neq \emptyset\}$. Here, $(\cdot,\cdot)_H$ denotes the inner product of H. Then, since $\partial_H\phi$ becomes maximal monotone, for $\lambda > 0$, one can define the *resolvent* $J_\lambda : H \to D(\partial_H\phi)$ and the *Yosida approximation* $(\partial_H\phi)_\lambda : H \to H$ of $\partial_H\phi$ by

$$J_\lambda := (I + \lambda\partial_H\phi)^{-1}, \quad (\partial_H\phi)_\lambda := (I - J_\lambda)/\lambda,$$

where I stands for the identity mapping of H. Furthermore, for $\lambda > 0$ the *Moreau-Yosida regularization* $\phi_\lambda : H \to \mathbb{R}$ of $\phi \in \Phi(H)$ is given by

$$\phi_\lambda(u) := \inf_{v \in H} \left\{ \frac{1}{2\lambda}|u - v|_H^2 + \phi(v) \right\} \quad \text{for all } u \in H. \tag{2.1}$$

The following proposition provides fine properties of resolvents, Yosida approximations and Moreau-Yosida regularizations in H (see [18] for its proof).

PROPOSITION 2.2. *Let H be a Hilbert space and let $\phi \in \Phi(H)$. Then, ϕ_λ is a Fréchet differentiable convex function from H into $\mathbb{R}$. Moreover, the infimum in* (2.1) *is attained by $J_\lambda u$, where J_λ denotes the resolvent of $\partial_H \phi$, i.e.,*

$$\phi_\lambda(u) = \frac{1}{2\lambda}|u - J_\lambda u|_H^2 + \phi(J_\lambda u) = \frac{\lambda}{2}|(\partial_H \phi)_\lambda(u)|_H^2 + \phi(J_\lambda u).$$

Furthermore, the following (i)–(iii) *hold.*

(i) $\partial_H(\phi_\lambda) = (\partial_H \phi)_\lambda$, *where $\partial_H(\phi_\lambda)$ is the subdifferential* (*Fréchet derivative*) *of* ϕ_λ.

(ii) $\phi(J_\lambda u) \le \phi_\lambda(u) \le \phi(u)$ *for all $u \in H$ and $\lambda > 0$.*

(iii) $\phi_\lambda(u) \to \phi(u)$ *as $\lambda \to 0_+$ for all $u \in H$.*

Finally, we recall the chain rule for subdifferential operators in a Banach space setting, and it also plays important roles to deal with evolution problems (see [5,23,32]). Throughout this paper, for each $p \in (1, \infty)$, we denote by p' the Hölder conjugate of p, i.e., $p' := p/(p-1)$.

PROPOSITION 2.3. *Let E be a reflexive Banach space and let $p \in (1, \infty)$. Let $\phi \in \Phi(E)$ and let $u \in W^{1,p}(0, T; E)$ be such that $u(t) \in D(\partial_E \phi)$ for a.e. $t \in (0, T)$. Suppose that there exists $g \in L^{p'}(0, T; E^*)$ such that $g(t) \in \partial_E \phi(u(t))$ for a.e. $t \in (0, T)$. Then, the function $t \mapsto \phi(u(t))$ is absolutely continuous on* $[0, T]$. *Moreover, let $\mathcal{I} := \{t \in [0, T];\ u(t) \in D(\partial_E \phi),\ u$ and $\phi(u(\cdot))$ are differentiable at $t\}$. Then, $[0, T] \setminus \mathcal{I}$ is negligible, i.e., its Lebesgue measure is zero, and*

$$\frac{d}{dt}\phi(u(t)) = \langle h, u'(t) \rangle \quad \textit{for every} \quad h \in \partial_E \phi(u(t)) \quad \textit{and} \quad t \in \mathcal{I}.$$

3. Main results

Let V and V^* be a real reflexive Banach space and its dual space, let H be a real Hilbert space whose dual space H^* is identified with itself such that

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^* \tag{3.1}$$

with continuous and densely defined canonical injections. Here, we set

$$C_H := \sup_{u \in V \setminus \{0\}} \frac{|u|_H}{|u|_V} > 0.$$

Let $\psi^t, \varphi \in \Phi(V)$ and let $\partial_V \psi^t$ and $\partial_V \varphi$ be the subdifferential operators of ψ^t and φ, respectively, for every $t \in [0, T]$ with $T > 0$. Moreover, let B be a mapping from $(0, T) \times V$ into 2^{V^*}. We consider the following Cauchy problem.

$$\text{(CP)} \begin{cases} \partial_V \psi^t(u'(t)) + \partial_V \varphi(u(t)) + B(t, u(t)) \ni f(t) \ \text{ in } V^*, \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where $f : (0, T) \to V^*$ and $u_0 \in V$ are given data. Here and henceforth, we are concerned with strong solutions of (CP) defined as follows.

DEFINITION 3.1. For each $S \in (0, T]$, a function $u \in C([0, S]; V)$ is said to be a strong solution of (CP) on $[0, S]$, if the following conditions are satisfied:

(i) u is a V-valued absolutely continuous function on $[0, S]$;
(ii) $u(0) = u_0$;
(iii) $u(t) \in D(\partial_V \varphi)$, $u'(t) \in D(\partial_V \psi^t)$ for a.e. $t \in (0, S)$, and there exist sections $\eta(t) \in \partial_V \psi^t(u'(t))$, $\xi(t) \in \partial_V \varphi(u(t))$ and $g(t) \in B(t, u(t))$ such that

$$\eta(t) + \xi(t) + g(t) = f(t) \ \text{ in } V^* \quad \text{for a.e. } t \in (0, S); \tag{3.2}$$

(iv) the function $t \mapsto \varphi(u(t))$ is absolutely continuous on $[0, S]$.

Before describing our main results, let us introduce assumptions on ψ^t, φ and B for $p \in (1, \infty)$ and $T > 0$. We first give assumptions on the coercivity and the boundedness of $\partial_V \psi^t : V \to V^*$ as follows.

(A1) There exist constants $C_1 > 0$ and $C_2 \geq 0$ such that

$$C_1 |u|_V^p \leq \psi^t(u) + C_2 \quad \text{for all } t \in [0, T] \text{ and } u \in D(\psi^t).$$

(A2) There exist a constant $C_3 \geq 0$ and $m_1 \in L^1(0, T)$ such that

$$|\eta|_{V^*}^{p'} \leq C_3 \psi^t(u) + m_1(t) \quad \text{for a.e. } t \in (0, T) \text{ and all } [u, \eta] \in \partial_V \psi^t.$$

Here we give a proposition, which will be used later.

PROPOSITION 3.2. *Let $p \in (1, \infty)$ and suppose that* (A2) *is satisfied. In addition, we assume that there exists a function $w : [0, T] \to V$ such that*

$$\mu_0 := \sup_{t \in [0, T]} \left\{ |w(t)|_V + |\psi^t(w(t))| \right\} < +\infty. \tag{3.3}$$

Then, the following (A2)$'$ *holds true*:

(A2)$'$ *For all $\zeta \in (0, 1)$, there exists $N_\zeta \in L^1(0, T)$ depending only on ζ, p, C_3, m_1, μ_0 such that*

$$(1 - \zeta)\psi^t(u) \leq \langle \eta, u \rangle + N_\zeta(t) \quad \textit{for all } t \in [0, T] \quad \textit{and} \quad [u, \eta] \in \partial_V \psi^t.$$

In particular, if $m_1 \equiv 0$ and $\mu_0 = 0$, then $N_\zeta \equiv 0$ for any ζ; hence $\psi^t(u) \leq \langle \eta, u \rangle$.

Proof. Let $t \in [0,T]$ and $[u,\eta] \in \partial_V \psi^t$ be fixed. By the definition of subdifferentials, it then follows that

$$\psi^t(u) - \psi^t(w(t)) \le \langle \eta, u - w(t) \rangle \le \langle \eta, u \rangle + |\eta|_{V^*} |w(t)|_V$$

for each $t \in [0,T]$. By (A2) and Young's inequality, for any $\zeta \in (0,1)$, there exists a constant $C_\zeta \ge 0$ such that

$$\psi^t(u) \le \langle \eta, u \rangle + \zeta \psi^t(u) + \zeta \frac{m_1(t)}{C_3} + C_\zeta \sup_{t\in[0,T]} |w(t)|_V^p + \sup_{t\in[0,T]} \psi^t(w(t)).$$

Hence, setting $N_\zeta(t) := \zeta m_1(t)/C_3 + C_\zeta \mu_0^p + \mu_0$, we obtain (A2)$'$, and moreover, we also notice that $N_\zeta \equiv 0$ if $m_1 \equiv 0$ and $\mu_0 = 0$. □

REMARK 3.3. Mielke and Theil [37] studied the *rate-independent processes* generated by some energy formulation of doubly nonlinear evolution equations with dissipation functionals ψ homogeneous of degree 1, i.e., $\psi(\alpha u) = \alpha \psi(u)$ for $\alpha \ge 0$ and $u \in V$. Unfortunately, our framework cannot handle their setting, which corresponds to the case $p = 1$ in our assumptions, since this case is excluded.

We write $\{\psi^t\}_{t\in[0,T]} \in \Phi(V,[0,T];\alpha,\beta,\ell_0)$ for functions $\alpha, \beta : (0,T) \to \mathbb{R}$ and a non-decreasing function ℓ_0 on $[0,\infty)$ if the following (i) and (ii) are satisfied:

(i) $\psi^t \in \Phi(V)$ for all $t \in [0,T]$;

(ii) there exists a constant $\delta > 0$ such that for all $t_0 \in [0,T]$ and $v_0 \in D(\psi^{t_0})$, we can take a function $v : I_\delta(t_0) := [t_0-\delta, t_0+\delta] \cap [0,T] \to V$ satisfying

$$\begin{aligned} |v(t) - v_0|_V &\le |\alpha(t) - \alpha(t_0)| \ell_0(|\psi^{t_0}(v_0)| + |v_0|_V) \\ \psi^t(v(t)) &\le \psi^{t_0}(v_0) + |\beta(t) - \beta(t_0)| \ell_0(|\psi^{t_0}(v_0)| + |v_0|_V) \end{aligned}$$

for all $t \in I_\delta(t_0)$.

Particularly, (ii) ensures a smooth movement of the graph for ψ^t in t, and this type of assumption was well studied by several authors (see, e.g., [32,33]) to treat time-dependent subdifferential operators. Then, our third assumption reads,

(A3) There exist functions $\alpha, \beta \in W^{1,1}(0,T)$ and a non-decreasing function ℓ_0 on $[0,\infty)$ such that $\{\psi^t\}_{t\in[0,T]} \in \Phi(V,[0,T];\alpha,\beta,\ell_0)$.

REMARK 3.4. The assumption (A3) ensures that the function $t \mapsto \psi^t(u(t))$ is measurable in $(0,T)$ whenever $u \in L^1(0,T;V)$, and moreover, by (A3) one can always take a function $w : [0,T] \to V$ satisfying (3.3) (see [8] and [32]).

Suppose that (A3) is satisfied and define $\Psi : L^p(0,T;V) \to (-\infty,+\infty]$ by

$$\Psi(u) := \begin{cases} \displaystyle\int_0^T \psi^t(u(t))dt & \text{if } [t \mapsto \psi^t(u(t))] \in L^1(0,T), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, $\Psi \in \Phi(L^p(0,T;V))$. Moreover, by Proposition 1.1 of [32], we can assure that

$$\eta \in \partial_{L^p(0,T;V)}\Psi(u) \quad \text{if } u \in L^p(0,T;V),\ \eta \in L^{p'}(0,T;V^*), \\ \text{and } [u(t),\eta(t)] \in \partial_V \psi^t \text{ for a.e. } t \in (0,T). \tag{3.4}$$

As to φ, we employ the following compactness condition.

(Φ1) There exist a reflexive Banach space X and a non-decreasing function ℓ_1 in $\mathbb{R}$ such that X is compactly embedded in V and

$$|u|_X \le \ell_1(\varphi(u) + |u|_H) \quad \text{for all } u \in D(\partial_V \varphi).$$

We next introduce assumptions on the non-monotone operator B. Condition $(\mathrm{B1})_\varepsilon$ provides some growth condition for $B(t,\cdot) : V \to V^*$ with a constant $\varepsilon > 0$. Condition (B2) can be regarded as a condition on the compactness and the closedness for the operator $\mathcal{B} : u \mapsto B(\cdot, u(\cdot))$ in the sense of multi-valued operators. Moreover, to treat multi-valued operators $B : (0,T) \times V \to V^*$, we also impose (B3) so that the operator $\mathcal{B}$ will be well defined and convex-valued in a proper Bochner-Lebesgue space (see also Remark 3.5).

$(\mathrm{B1})_\varepsilon$ $D(\partial_V \varphi) \subset D(B(t,\cdot))$ for a.e. $t \in (0,T)$. There exist $m_2^\varepsilon \in L^1(0,T)$ and a non-decreasing function ℓ_2^ε on $[0,\infty)$ satisfying the following:

$$|g|_{V^*}^{p'} \le \varepsilon |\xi|_{V^*}^\sigma + |m_2^\varepsilon(t)| \ell_2^\varepsilon(|\varphi(u)| + |u|_V), \quad \sigma := \min\{2, p'\}$$

for a.e. $t \in (0,T)$ and all $u \in D(\partial_V \varphi)$, $g \in B(t,u)$ and $\xi \in \partial_V \varphi(u)$.

(B2) Let $S \in (0,T]$ and let $\{u_n\}$ and $\{\xi_n\}$ be sequences in $C([0,S];V)$ and $L^\sigma(0,S;V^*)$ with $\sigma := \min\{2,p'\}$, respectively, such that

$$u_n \to u \text{ strongly in } C([0,S];V), \quad [u_n(t), \xi_n(t)] \in \partial_V \varphi \text{ for a.e. } t \in (0,S),$$
$$\sup_{t \in [0,S]} |\varphi(u_n(t))| + \int_0^S |u'_n(t)|_H^p dt + \int_0^S |\xi_n(t)|_{V^*}^\sigma dt$$

is bounded for all $n \in \mathbb{N}$.

Moreover, let $\{g_n\}$ be a sequence in $L^{p'}(0,S;V^*)$ such that

$$g_n(t) \in B(t, u_n(t)) \text{ for a.e. } t \in (0,S), \quad g_n \to g \text{ weakly in } L^{p'}(0,S;V^*).$$

Then, $\{g_n\}$ is precompact in $L^{p'}(0,S;V^*)$ and $g(t) \in B(t,u(t))$ for a.e. $t \in (0,S)$.

(B3) Let $S \in (0,T]$ and let $u \in C([0,S];V) \cap W^{1,p}(0,S;H)$ be such that $\varphi(u(\cdot)) \in L^\infty(0,S)$. Suppose that there exists $\xi \in L^{p'}(0,S;V^*)$ such that $\xi(t) \in \partial_V \varphi(u(t))$ for a.e. $t \in (0,S)$. Then, there exists a V^*-valued strongly measurable function g such that $g(t) \in B(t,u(t))$ for a.e. $t \in (0,S)$. Moreover, the set $B(t,u)$ is convex for all $t \in (0,T)$ and $u \in D(B(t,\cdot))$.

Here, we give a couple of remarks on $(\mathrm{B1})_\varepsilon$–(B3).

REMARK 3.5. (i) Let us show a couple of simpler (but more restrictive) alternatives to (B2).

$(B2)_1$ $B(t, u) = B(u)$ is single-valued and locally uniformly continuous from V into V^*.
If $(B2)_1$ is assumed, then for any sequence $u_n \to u$ strongly in $C([0, S]; V)$, it follows that $g_n(t) := B(u_n(t)) \to B(u(t))$ strongly in V^* uniformly on $[0, T]$. Hence (B2) follows. However, this condition could be somewhat restrictive in applications to PDEs.

$(B2)_2$ For each $S \in (0, T)$, the operator $\mathcal{B} : u \mapsto B(\cdot, u(\cdot))$ is single-valued, continuous and compact from $L^\infty(0, S; X) \cap W^{1,p}(0, S; H)$ into $L^{p'}(0, S; V^*)$.
Let (u_n) be a sequence in the assumption of (B2). Then, by (Φ1), we find that (u_n) is bounded in $L^\infty(0, S; X) \cap W^{1,p}(0, S; H)$. Hence by $(B2)_2$, up to a subsequence, we have $g_n := B(\cdot, u_n(\cdot)) \to B(\cdot, u)$ strongly in $L^{p'}(0, S; V^*)$.

(ii) Condition (B3) is not necessary to be assumed under (Φ1) and $(B1)_\varepsilon$ if X is separable and B is single-valued and $\mathfrak{M}(0, T) \times \mathfrak{B}(X)$-measurable, where $\mathfrak{M}(0, T)$ is the σ-algebra of Lebesgue measurable sets on $(0, T)$ and $\mathfrak{B}(X)$ is the Borel tribe generated by X. Indeed, the function $t \mapsto B(t, u(t))$ is $\mathfrak{M}(0, T)$-measurable for any $\mathfrak{M}(0, T)$-measurable function $u : (0, T) \to X$. Hence by $(B1)_\varepsilon$, we deduce that $\mathcal{B}(u)$ belongs to $L^{p'}(0, T; V^*)$, provided that u satisfies all assumptions in (B3). Moreover, $B(t, u)$ is always convex.

(iii) Suppose that both $(B1)_\varepsilon$ and (Φ1) are satisfied. Then we get, by (Φ1),

$$|u|_V \le C|u|_X \le C\ell_1(|\varphi(u)| + |u|_H).$$

Hence, we can derive the following $(B1)'_\varepsilon$ from $(B1)_\varepsilon$ by putting $\ell_3^\varepsilon(x) := \ell_2^\varepsilon(x + C\ell_1(x))$.

$(B1)'_\varepsilon$ $D(\partial_V \varphi) \subset D(B(t, \cdot))$ for a.e. $t \in (0, T)$. There exist $m_2^\varepsilon \in L^1(0, T)$ and a non-decreasing function ℓ_3^ε on $[0, \infty)$ satisfying the following:

$$|g|_{V^*}^{p'} \le \varepsilon|\xi|_{V^*}^\sigma + |m_2^\varepsilon(t)|\ell_3^\varepsilon(|\varphi(u)| + |u|_H), \quad \sigma := \min\{2, p'\}$$

for a.e. $t \in (0, T)$ and all $u \in D(\partial_V \varphi)$, $g \in B(t, u)$ and $\xi \in \partial_V \varphi(u)$.

Hence, we use $(B1)'_\varepsilon$ instead of $(B1)_\varepsilon$ to prove main results stated below.

Now, our result on the local (in time) existence is stated as follows:

THEOREM 3.6. (Local existence) *Let $p \in (1, \infty)$ and $T > 0$ be given. Suppose that* (A1)–(A3), (Φ1), $(B1)_\varepsilon$–(B3) *are all satisfied with a sufficiently small $\varepsilon > 0$ (the smallness of ε is determined only from p, C_1, C_3 and C_H). Then, for all $f \in L^{p'}(0, T; V^*)$ and $u_0 \in D(\varphi)$, there exists $T_* = T_*(\varphi(u_0) + |u_0|_H + \|f\|_{L^{p'}(0,T;V^*)}) \in (0, T]$ such that* (CP) *admits at least one strong solution $u \in W^{1,p}(0, T_*; V)$ on $[0, T_*]$ satisfying*

$$\eta, \xi, g \in L^{p'}(0, T_*; V^*), \quad \varphi(u(\cdot)) \in W^{1,1}(0, T_*),$$

where $\eta(t), \xi(t)$ and $g(t)$ denote the sections of $\partial_V \psi^t(u'(t)), \partial_V \varphi(u(t))$ and $B(t, u(t))$, respectively, as in (3.2) *for a.e. $t \in (0, T_*)$.*

A proof of Theorem 3.6 will be given in Sects. 4 and 5; its outline will be also shown at the beginning of Sect. 4.

As for the global (in time) existence, we have:

THEOREM 3.7. (Global existence) *Let* $p \in (1, \infty)$ *and* $T > 0$ *be fixed. Suppose that* (A1)–(A3), (Φ1), (B2), (B3) *and the following* $(\mathrm{B4})_\varepsilon$ *are satisfied with a sufficiently small* $\varepsilon > 0$ (*the smallness of* ε *is determined only from* p, C_1, C_3 *and* C_H).

$(\mathrm{B4})_\varepsilon$ $D(\partial_V\varphi) \subset D(B(t,\cdot))$ *for a.e.* $t \in (0,T)$. *There exists* $m_3^\varepsilon \in L^1(0,T)$ *satisfying the following*:

$$|g|_{V^*}^{p'} \leq \varepsilon|\xi|_{V^*}^{\sigma} + |m_3^\varepsilon(t)| \left\{|\varphi(u)| + |u|_V^p + 1\right\}, \quad \sigma := \min\{2, p'\}$$

for a.e. $t \in (0,T)$ *and all* $u \in D(\partial_V\varphi)$, $g \in B(t,u)$ *and* $\xi \in \partial_V\varphi(u)$.

Then, for all $f \in L^{p'}(0,T;V^*)$ *and* $u_0 \in D(\varphi)$, *there exists a strong solution* $u \in W^{1,p}(0,T;V)$ *of* (CP) *on* $[0,T]$ *such that*

$$\eta, \xi, g \in L^{p'}(0,T;V^*), \quad \varphi(u(\cdot)) \in W^{1,1}(0,T), \tag{3.5}$$

where $\eta(t), \xi(t)$ *and* $g(t)$ *denote the sections of* $\partial_V\psi^t(u'(t)), \partial_V\varphi(u(t))$ *and* $B(t,u(t))$, *respectively, as in* (3.2) *for a.e.* $t \in (0,T)$.

Furthermore, the global existence is assured for small data u_0 and f in a proper sense by employing the following (B5) and $(\mathrm{B6})_\varepsilon$ instead of $(\mathrm{B4})_\varepsilon$.

(B5) There exist a positive constant C_4 and non-decreasing functions ℓ_i $(i = 4, 5)$ on $[0,\infty)$ such that $\lim_{s\to+0} \ell_i(s) = 0$ and

$$C_4\varphi(u) \leq \langle \xi + g, u\rangle + \ell_4(\varphi(u))\varphi(u), \tag{3.6}$$

$$|u|_V^p \leq \ell_5(\varphi(u))\varphi(u), \tag{3.7}$$

for a.e. $t \in (0,T)$ and all $u \in D(\partial_V\varphi)$, $\xi \in \partial_V\varphi(u)$, $g \in B(t,u)$.

$(\mathrm{B6})_\varepsilon$ There exists a non-decreasing function ℓ_6^ε on $[0,\infty)$ such that $\lim_{s\to+0} \ell_6^\varepsilon(s) = 0$ and

$$|g|_{V^*}^{p'} \leq \varepsilon|\xi|_{V^*}^{p'} + \ell_6^\varepsilon(\varphi(u))\varphi(u) \tag{3.8}$$

for a.e. $t \in (0,T)$ and all $u \in D(\partial_V\varphi) \cap D(B(t,\cdot))$, $\xi \in \partial_V\varphi(u)$, $g \in B(t,u)$.

THEOREM 3.8. (Global existence for small data) *Let* $p \in (1,\infty)$ *and* $T > 0$ *be fixed. Suppose that* $\psi^t(0) \equiv 0$, (A1)–(A3), (Φ1), $(\mathrm{B1})_\varepsilon$–(B3) *and* (B5), $(\mathrm{B6})_\varepsilon$ *are all satisfied with* $C_2 = 0, m_1 \equiv 0$ *and a sufficiently small* $\varepsilon > 0$ (*the smallness of* ε *is determined only from* p, C_1, C_3 *and* C_H). *Then, there exists* $\delta > 0$ *independent of* T *such that for all* $f \in L^{p'}(0,T;V^*)$ *and* $u_0 \in D(\varphi)$ *satisfying* $\|f\|_\star + \varphi(u_0) < \delta$, *where* $\|f\|_\star$ *is given by*

$$\|f\|_\star := \begin{cases} \displaystyle\sup_{t\in[1,T]} \int_{t-1}^{t} |f(\tau)|_{V^*}^{p'} d\tau & \text{if } 1 \leq T, \\ \displaystyle\int_0^T |f(\tau)|_{V^*}^{p'} d\tau & \text{if } 0 < T < 1, \end{cases} \tag{3.9}$$

the Cauchy problem (CP) *admits a strong solution* $u \in W^{1,p}(0, T; V^*)$ *on* $[0, T]$ *and* (3.5) *holds true.*

REMARK 3.9. We can assume that $\psi^t \geq 0$ and $\varphi \geq 0$ without any loss of generality in our proofs of the main results. Indeed, putting $\hat{\psi}^t := \psi^t + C_2$ and using (A1), we find that $\hat{\psi}^t \geq 0$, $D(\hat{\psi}^t) = D(\psi^t)$, $D(\partial_V \hat{\psi}^t) = D(\partial_V \psi^t)$ and $\partial_V \hat{\psi}^t = \partial_V \psi^t$. As for φ, from the fact that $\varphi \in \Phi(V)$ and (Φ1), the extension by infinity $\tilde{\varphi}$ of φ onto H (see also (4.1) below) belongs to $\Phi(H)$. Hence, there exist $u^* \in H$ and $\mu \in \mathbb{R}$ such that $\tilde{\varphi}(u) \geq (u^*, u)_H + \mu$ for all $u \in H$ (see, e.g., Proposition 2.1 of [15, p. 51]). Thus, we have $\hat{\varphi}(u) := \varphi(u) - (u^*, u)_H - \mu \geq 0$ for all $u \in V$, and moreover, it holds that $D(\hat{\varphi}) = D(\varphi)$, $D(\partial_V \hat{\varphi}) = D(\partial_V \varphi)$ and $\partial_V \hat{\varphi} = \partial_V \varphi - u^*$. Therefore, the evolution equation of (CP) is equivalent to the following:

$$\partial_V \hat{\psi}^t(u'(t)) + \partial_V \hat{\varphi}(u(t)) + B(t, u(t)) \ni \hat{f}(t) := f(t) - u^*.$$

Moreover, (A1)–(A3), $(\mathrm{B1})_\varepsilon$–$(\mathrm{B4})_\varepsilon$ and (Φ1) are all satisfied with ψ^t and φ replaced by $\hat{\psi}^t$ and $\hat{\varphi}$, respectively. In particular, if (B5) is satisfied, it then follows from (3.7) that $\varphi \geq 0$ without any replacement of φ.

In the rest of this paper, we denote by C a non-negative constant, which does not depend on the elements of the corresponding space or set and may vary from line to line.

4. Approximate problems for (CP)

Our proof of Theorem 3.6 is divided into two steps. In the first step, we propose approximate problems for (CP) and construct their solutions by employing Kakutani-Fan's fixed point theorem for multi-valued mappings. To do so, we first define the extension of φ onto H as follows:

$$\tilde{\varphi}(u) := \begin{cases} \varphi(u) & \text{if } u \in V, \\ +\infty & \text{if } u \in H \setminus V. \end{cases} \tag{4.1}$$

Then, the assumption (Φ1) yields $\tilde{\varphi} \in \Phi(H)$. We now introduce approximate problems for (CP) as follows:

$$(\mathrm{CP})_\lambda \quad \begin{cases} \lambda u'(t) + \partial_V \psi^t(u'(t)) + \partial_H \tilde{\varphi}_\lambda(u(t)) + B(t, J_\lambda u(t)) \ni f(t) \ \text{ in } V^*, \\ \hfill 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where J_λ and $\partial_H \tilde{\varphi}_\lambda$ denote the resolvent and the Yosida approximation of $\partial_H \tilde{\varphi}$, respectively. Before discussing the existence of strong solutions for $(\mathrm{CP})_\lambda$, we first prove in Sect. 4.1 the existence and uniqueness of strong solutions for the following unperturbed problems with an arbitrary function $g \in L^{p'}(0, T; V^*)$:

$$(\mathrm{CP})_{\lambda,g} \quad \begin{cases} \lambda u'(t) + \partial_V \psi^t(u'(t)) + \partial_H \tilde{\varphi}_\lambda(u(t)) + g(t) \ni f(t) \ \text{ in } V^*, \quad 0 < t < T, \\ u(0) = u_0. \end{cases}$$

We next define the solution operator $\mathcal{S}_T : L^{p'}(0, T; V^*) \to W^{1,p}(0, T; V)$, which maps g into the unique strong solution u of $(\text{CP})_{\lambda,g}$ on $[0, T]$. In order to prove the existence of local (in time) strong solutions for $(\text{CP})_\lambda$, we find a fixed point of the mapping $\mathcal{F}_{T_0} : L^{p'}(0, T_0; V^*) \to 2^{L^{p'}(0,T_0;V^*)}$; $g \mapsto \mathcal{F}_{T_0} g$ given by

$$\mathcal{F}_{T_0} g := \left\{ h \in L^{p'}(0, T_0; V^*); h(t) \in B(t, J_\lambda(\mathcal{S}_{T_0} g)(t)) \text{ for a.e. } t \in (0, T_0) \right\}$$

for some $T_0 \in (0, T]$ independent of λ. Indeed, for every fixed point g_* of $\mathcal{F}_{T_0}$, the strong solution $u_* := \mathcal{S}_{T_0} g_*$ of $(\text{CP})_{\lambda,g_*}$ satisfies $B(t, J_\lambda u_*(t)) \ni g_*(t)$ for a.e. $t \in (0, T_0)$. Hence, u_* also becomes a strong solution of $(\text{CP})_\lambda$ on $[0, T_0]$. The detail of our proof for the existence of fixed points of $\mathcal{F}_{T_0}$ will be given in Sect. 4.2.

The second step is devoted to the limiting procedure of strong solutions u_λ for $(\text{CP})_\lambda$ as $\lambda \to +0$. To do so, we establish a priori estimates for u_λ (see Sect. 5).

REMARK 4.1. For the case where $V = V^* = H$ is a Hilbert space (see [11]), one can more easily prove the uniqueness of strong solutions for $(\text{CP})_{\lambda,g}$. Indeed, $(\text{CP})_{\lambda,g}$ can be rewritten into

$$u'(t) = (\lambda I + \partial_H \psi^t)^{-1} (f(t) - g(t) - \partial_H \tilde{\varphi}_\lambda(u(t))) \quad \text{in } H, \quad 0 < t < T,$$

and we observe that the mapping $u \mapsto (\lambda I + \partial_H \psi^t)^{-1} (f(t) - g(t) - \partial_H \tilde{\varphi}_\lambda(u))$ becomes Lipschitz continuous in H for every $t \in [0, T]$. Hence, the uniqueness of strong solutions follows immediately. However, for the case where V is not a Hilbert space, the mapping $(\lambda I + \partial_V \psi^t)^{-1} : V^* \to V$ is no longer Lipschitz continuous.

4.1. Unperturbed problem

In this subsection, the existence and uniqueness of strong solutions are proved for the unperturbed problems $(\text{CP})_{\lambda,g}$.

THEOREM 4.2. *Let $T > 0$ and $p \in (1, \infty)$ be fixed. Suppose that* (A1)–(A3) *and* (Φ1) *are satisfied. Then, for each $\lambda \in (0, \infty)$, $f, g \in L^{p'}(0, T; V^*)$ and $u_0 \in D(\varphi)$, the Cauchy problem* $(\text{CP})_{\lambda,g}$ *admits a unique strong solution $u \in W^{1,p}(0, T; V) \cap W^{1,2}(0, T; H)$ on $[0, T]$ such that*

$$J_\lambda u(\cdot) \in C([0, T]; V) \cap W^{1,p}(0, T; H),$$
$$\tilde{\varphi}_\lambda(u(\cdot)) \in W^{1,1}(0, T), \quad \eta \in L^{p'}(0, T; V^*),$$

where $\eta(t)$ denotes the section of $\partial_V \psi^t(u'(t))$ such that $\lambda u'(t) + \eta(t) + \partial_H \tilde{\varphi}_\lambda(u(t)) + g(t) = f(t)$ for a.e. $t \in (0, T)$.

Proof. We first prove the uniqueness part. Let u_1 and u_2 be strong solutions for $(\text{CP})_{\lambda,g}$ on $[0, T]$ and put $w := u_1 - u_2$. We then see that

$$\lambda w'(t) + \eta_1(t) - \eta_2(t) + \partial_H \tilde{\varphi}_\lambda(u_1(t)) - \partial_H \tilde{\varphi}_\lambda(u_2(t)) \ni 0,$$

where $\eta_i(t) := f(t) - g(t) - \partial_H \tilde{\varphi}_\lambda(u_i(t)) - \lambda u'_i(t) \in \partial_V \psi^t(u'_i(t))$ $(i = 1, 2)$. Multiplying this by $w'(t)$, we can deduce that

$$\begin{aligned}
&\lambda \left|w'(t)\right|_H^2 + \left\langle \eta_1(t) - \eta_2(t), w'(t)\right\rangle \\
&\quad = - \left(\partial_H \tilde{\varphi}_\lambda(u_1(t)) - \partial_H \tilde{\varphi}_\lambda(u_2(t)), w'(t)\right)_H \leq \frac{1}{\lambda} |w(t)|_H |w'(t)|_H.
\end{aligned}$$

Using the monotonicity of $\partial_V \psi^t$, we have

$$\lambda \left|w'(t)\right|_H^2 \leq \frac{1}{\lambda} |w(t)|_H |w'(t)|_H,$$

which implies

$$\frac{d}{dt} |w(t)|_H \leq \left|w'(t)\right|_H \leq \frac{1}{\lambda^2} |w(t)|_H.$$

Therefore, integrating this over $(0, t)$, we get

$$|w(t)|_H \leq |w(0)|_H + \frac{1}{\lambda^2} \int_0^t |w(\tau)|_H d\tau \quad \text{for all } t \in [0, T],$$

which together with Gronwall's inequality implies

$$|w(t)|_H \leq |w(0)|_H \exp\left(\frac{t}{\lambda^2}\right) \quad \text{for all } t \in [0, T].$$

Thus, the uniqueness of strong solutions follows, provided that $\lambda > 0$. □

REMARK 4.3. Several criteria have been provided for the uniqueness of solutions in [24] and [23] (see also [37]). However, $(\mathrm{CP})_{\lambda,g}$ could not be classified into their categories. We emphasize that $(\mathrm{CP})_{\lambda,g}$ is truly doubly nonlinear, i.e., both operators acting on $u(t)$ and $u'(t)$, respectively, are nonlinear and not self-adjoint, but its solution is unique.

As for the existence of strong solutions for $(\mathrm{CP})_{\lambda,g}$, we further introduce the following approximate problems in H:

$$(\mathrm{CP})^H_{\lambda,g_n} \quad \begin{cases} \lambda u_n'(t) + \partial_H \tilde{\psi}^t(u_n'(t)) + \partial_H \tilde{\varphi}_\lambda(u_n(t)) + g_n(t) \ni f_n(t) \ \text{ in } H, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < t < T, \\ u_n(0) = u_0, \end{cases}$$

where $\tilde{\psi}^t$ denotes the extension of ψ^t onto H defined as in (4.1), and $\{f_n\}$ and $\{g_n\}$ are approximate sequences in $C([0, T]; H)$ such that

$$f_n \to f \ \text{ and } \ g_n \to g \quad \text{strongly in } L^{p'}(0, T; V^*) \text{ as } n \to \infty.$$

Here, we remark that (A1) implies $\tilde{\psi}^t \in \Phi(H)$ for all $t \in [0, T]$, so that $\partial_H \tilde{\psi}^t$ becomes maximal monotone in H. Then, $(\mathrm{CP})^H_{\lambda,g_n}$ can be rewritten into

$$u_n'(t) = F_n(t, u_n(t)), \quad u_n(0) = u_0$$

with the mapping $F_n : [0, T] \times H \to H$ defined by

$$F_n : (t, u) \mapsto \left(\lambda I + \partial_H \tilde{\psi}^t\right)^{-1} (f_n(t) - g_n(t) - \partial_H \tilde{\varphi}_\lambda(u)).$$

Then, since $\partial_H \tilde{\varphi}_\lambda$ and $(\lambda I + \partial_H \tilde{\psi}^t)^{-1}$ are Lipschitz continuous in H, so is $F_n(t, \cdot)$ for all $t \in [0, T]$. By Lemma 2.9 of [8], we can deduce from (A3) that the function $t \mapsto F_n(t, u)$ is continuous in $[0, T]$ for all $u \in H$. Hence, the existence and uniqueness of strong solutions $u_n \in C^1([0, T]; H)$ for $(\mathrm{CP})^H_{\lambda, g_n}$ on $[0, T]$ are ensured by Cauchy-Lipschitz-Picard's existence theorem with obvious modifications (see, e.g., Corollary 1.1 of [20]). Furthermore, as in [8, p. 694], we can prove that u' is a V-valued weakly continuous function on $[0, T]$.

We next establish a priori estimates for u_n in the following lemmas.

LEMMA 4.4. *There exists a constant $M \geq 0$ such that for all $n \in \mathbb{N}$, all strong solutions u_n of $(\mathrm{CP})^H_{\lambda, g_n}$ on $[0, T]$ satisfy*

$$\begin{aligned} &\lambda \int_0^T |u_n'(t)|_H^2 dt + \int_0^T \psi^t(u_n'(t)) dt + \sup_{t \in [0,T]} \tilde{\varphi}_\lambda(u_n(t)) \\ &\quad \leq M \left\{ \varphi(u_0) + C_2 T + |N_{\frac{1}{2}}|_{L^1(0,T)} + \| f_n - g_n \|^{p'}_{L^{p'}(0,T;V^*)} \right\} \end{aligned} \tag{4.2}$$

with a constant $M = M(p, C_1)$ depending only on p and C_1.

Proof. Multiplying $(\mathrm{CP})^H_{\lambda, g_n}$ by $u_n'(t)$ and using Proposition 2.3, we get

$$\lambda |u_n'(t)|_H^2 + \langle \eta_n(t), u_n'(t) \rangle + \frac{d}{dt} \tilde{\varphi}_\lambda(u_n(t)) = \langle f_n(t) - g_n(t), u_n'(t) \rangle,$$

where $\eta_n(t) := f_n(t) - g_n(t) - \partial_H \tilde{\varphi}_\lambda(u_n(t)) - \lambda u_n'(t) \in \partial_H \tilde{\psi}^t(u_n'(t)) \subset \partial_V \psi^t(u_n'(t))$, for a.e. $t \in (0, T)$. Then, by virtue of $(\mathrm{A2})'$ with $\zeta = 1/2$, it follows that

$$\begin{aligned} &\lambda |u_n'(t)|_H^2 + \frac{1}{2} \psi^t(u_n'(t)) + \frac{d}{dt} \tilde{\varphi}_\lambda(u_n(t)) \\ &\quad \leq N_{\frac{1}{2}}(t) + c_0 \left(|f_n(t) - g_n(t)|^{p'}_{V^*} + C_2 \right) + \frac{1}{4} \psi^t(u_n'(t)) \end{aligned}$$

with a constant $c_0 = c_0(p, C_1)$ depending only on p and C_1. Thus,

$$\lambda |u_n'(t)|_H^2 + \frac{1}{4} \psi^t(u_n'(t)) + \frac{d}{dt} \tilde{\varphi}_\lambda(u_n(t)) \leq N_{\frac{1}{2}}(t) + c_0 \left(|f_n(t) - g_n(t)|^{p'}_{V^*} + C_2 \right).$$

Integrating this over $(0, t)$, we have

$$\begin{aligned} &\lambda \int_0^t |u_n'(\tau)|_H^2 d\tau + \frac{1}{4} \int_0^t \psi^\tau(u_n'(\tau)) d\tau + \tilde{\varphi}_\lambda(u_n(t)) \\ &\quad \leq \varphi(u_0) + |N_{\frac{1}{2}}|_{L^1(0,T)} + c_0 \left(\| f_n - g_n \|^{p'}_{L^{p'}(0,T;V^*)} + C_2 T \right) \end{aligned}$$

for all $t \in [0, T]$, since Proposition 2.2 gives $\tilde{\varphi}_\lambda(u_0) \leq \tilde{\varphi}(u_0) = \varphi(u_0)$. □

LEMMA 4.5. *There exist constants C and C_λ such that*

$$\sup_{t\in[0,T]} |u_n(t)|_V + \int_0^T |u_n'(t)|_V^p dt \leq C, \tag{4.3}$$

$$\sup_{t\in[0,T]} |J_\lambda u_n(t)|_H + \int_0^T \left|\frac{d}{dt} J_\lambda u_n(t)\right|_H^p dt \leq C, \tag{4.4}$$

$$\int_0^T |\eta_n(t)|_{V^*}^{p'} dt \leq C, \tag{4.5}$$

$$\sup_{t\in[0,T]} |\partial_H \tilde{\varphi}_\lambda(u_n(t))|_H \leq C_\lambda, \tag{4.6}$$

where C is independent of λ, but C_λ may not.

Proof. By (A1) and (4.2), we get $\int_0^T |u_n'(t)|_V^p dt \leq C$. Moreover, we note that

$$|u_n(t)|_V = |u_0|_V + \int_0^t \frac{d}{d\tau} |u_n(\tau)|_V d\tau \leq |u_0|_V + \int_0^t |u_n'(\tau)|_V d\tau,$$

which implies (4.3).

Since the resolvent J_λ is non-expansive in H, it follows that $|J_\lambda u_n(t+h) - J_\lambda u_n(t)|_H \leq |u_n(t+h) - u_n(t)|_H$ for all $t, t+h \in [0,T]$, which implies

$$\int_0^T \left|\frac{d}{dt} J_\lambda u_n(t)\right|_H^p dt \leq \int_0^T |u_n'(t)|_H^p dt \leq C.$$

Moreover, as in the proof of (4.3), we also derive that $\sup_{t\in[0,T]} |J_\lambda u_n(t)|_H \leq C$.

By virtue of the assumption (A2),

$$|\eta_n(t)|_{V^*}^{p'} \leq C_3 \psi^t(u_n'(t)) + m_1(t)$$

for a.e. $t \in (0,T)$. Thus, (4.2) also implies (4.5).

Moreover, since $\partial_H \tilde{\varphi}_\lambda$ is Lipschitz continuous in H, we can deduce that

$$|\partial_H \tilde{\varphi}_\lambda(u_n(t))|_H \leq C_\lambda \left(|u_n(t)|_H + 1\right),$$

which together with (4.3) yields (4.6). □

From these a priori estimates, we can derive the following convergences.

LEMMA 4.6. *There exist a subsequence $\{n'\}$ of $\{n\}$, $u \in W^{1,p}(0,T;V) \cap W^{1,2}(0,T;H)$ and $\eta \in L^{p'}(0,T;V^*)$ such that*

$$u_{n'} \to u \quad \text{weakly in } W^{1,p}(0,T;V) \cap W^{1,2}(0,T;H), \tag{4.7}$$

$$\eta_{n'} \to \eta \quad \text{weakly in } L^{p'}(0,T;V^*), \tag{4.8}$$

$$\partial_H \tilde{\varphi}_\lambda(u_{n'}(\cdot)) \to \partial_H \tilde{\varphi}_\lambda(u(\cdot)) \quad \text{weakly in } L^2(0,T;H), \tag{4.9}$$

$$J_\lambda u_{n'}(\cdot) \to J_\lambda u(\cdot) \quad \text{weakly in } W^{1,p}(0,T;H), \tag{4.10}$$

$$\text{strongly in } C([0,T];V). \tag{4.11}$$

Hence, we have $\tilde{\varphi}_\lambda(u(\cdot)) \in W^{1,1}(0,T)$ and $\lambda u' + \eta + \partial_H \tilde{\varphi}_\lambda(u(\cdot)) + g = f$.

Proof. By Lemmas 4.4 and 4.5, there exist $u \in W^{1,p}(0,T;V) \cap W^{1,2}(0,T;H)$ and $\eta \in L^{p'}(0,T;V^*)$ such that (4.7)–(4.8) hold, and moreover, there exist $\xi \in L^2(0,T;H)$ and $v \in W^{1,p}(0,T;H)$ such that

$$\partial_H \tilde{\varphi}_\lambda(u_{n'}(\cdot)) \to \xi \quad \text{weakly in } L^2(0,T;H), \tag{4.12}$$

$$J_\lambda u_{n'}(\cdot) \to v \quad \text{weakly in } W^{1,p}(0,T;H). \tag{4.13}$$

By Proposition 2.2 and Lemma 4.4, we see that

$$\varphi(J_\lambda u_n(t)) = \tilde{\varphi}(J_\lambda u_n(t)) \leq \tilde{\varphi}_\lambda(u_n(t)) \leq C\left\{\varphi(u_0) + 1 + \|f_n - g_n\|^{p'}_{L^{p'}(0,T;V^*)}\right\}$$

for each $t \in [0,T]$, which together with (4.4) and (Φ1) implies that $\{J_\lambda u_n(\cdot)\}$ is bounded in $L^\infty(0,T;X) \cap W^{1,p}(0,T;H)$. Therefore, since X is compactly embedded in V, and V is continuously embedded in H, Theorem 5 of [50] ensures that

$$J_\lambda u_{n'}(\cdot) \to v \quad \text{strongly in } C([0,T];V). \tag{4.14}$$

Hence, since $\partial_H \tilde{\varphi}_\lambda(u_n(t)) \in \partial_H \tilde{\varphi}(J_\lambda u_n(t))$, by Proposition 1.1 of [32] and Proposition 2.1 of Sect. 2, we can derive from (4.12) and (4.14) that $\xi(t) \in \partial_H \tilde{\varphi}(v(t))$ for a.e. $t \in (0,T)$.

Now, it remains to prove that $v(t) = J_\lambda u(t)$ and $\xi(t) = \partial_H \tilde{\varphi}_\lambda(u(t))$ for a.e. $t \in (0,T)$. To this end, from the definition of resolvents and Yosida approximations (see Sect. 2), we have $J_\lambda u_{n'}(t) + \lambda \partial_H \tilde{\varphi}_\lambda(u_{n'}(t)) = u_{n'}(t)$ for a.e. $t \in (0,T)$. Passing to the limit as $n' \to \infty$, we can deduce that $v(t) + \lambda\xi(t) = u(t)$ for a.e. $t \in (0,T)$, and therefore, since $\xi(t) \in \partial_H \tilde{\varphi}(v(t))$, we can deduce that $v(t) = J_\lambda u(t)$ and $\xi(t) = \partial_H \tilde{\varphi}_\lambda(u(t))$ for a.e. $t \in (0,T)$. Moreover, by Proposition 2.3, it follows that

$$\frac{d}{dt}\tilde{\varphi}_\lambda(u(\cdot)) = (\xi(\cdot), u'(\cdot))_H \in L^1(0,T),$$

which implies $\tilde{\varphi}_\lambda(u(\cdot)) \in W^{1,1}(0,T)$. □

We next verify that the limit u satisfies the initial condition $u(0) = u_0$, and moreover, the point wise convergence of $u_{n'}$ at each $t \in [0,T]$ is also derived in the following lemma.

LEMMA 4.7. *The limit u of $u_{n'}$ obtained in Lemma 4.6 by choosing the subsequence $\{n'\}$ of $\{n\}$ satisfies*

$$u(t) \to u_0 \quad \textit{strongly in } V \textit{ as } t \to +0.$$

Furthermore, for each $t \in [0,T]$, it follows that

$$u_{n'}(t) \to u(t) \quad \textit{weakly in } V. \tag{4.15}$$

Proof. By (4.3), for any $q \in (1, \infty)$, we can take a subsequence $\{n^q\}$ of $\{n\}$ such that $u_{n^q} \to u$ weakly in $L^q(0, t; V)$ for all $t \in (0, T)$. Hence we have, by (4.3),

$$\begin{aligned}\|u - u_0\|_{L^q(0,t;V)} &\leq \liminf_{n^q \to \infty} \|u_{n^q} - u_0\|_{L^q(0,t;V)} \\ &= \liminf_{n^q \to \infty} \left(\int_0^t \left| \int_0^\tau u'_{n^q}(s) ds \right|_V^q d\tau \right)^{1/q} \\ &\leq C \left(\frac{p'}{q + p'} \right)^{1/q} t^{1/q + 1/p'}\end{aligned}$$

for all $q \in (1, \infty)$. Therefore, passing to the limit as $q \to \infty$, since $u \in C([0, T]; V)$, we can deduce that, for each $t \in [0, T]$,

$$|u(t) - u_0|_V \leq \sup_{\tau \in [0,t]} |u(\tau) - u_0|_V = \lim_{q \to \infty} \|u - u_0\|_{L^q(0,t;V)} \leq C t^{1/p'},$$

which implies that $u(t) \to u_0$ strongly in V as $t \to +0$.

Moreover, since $u(0) = u_{n'}(0) = u_0$, we get, by (4.7),

$$\langle w, u_{n'}(t) - u(t) \rangle = \int_0^t \langle w, u'_{n'}(\tau) - u'(\tau) \rangle d\tau \to 0$$

for all $w \in V^*$ and $t \in [0, T]$. Thus, (4.15) holds. □

Finally, we prove that $\eta(t) \in \partial_V \psi^t(u'(t))$ for a.e. $t \in (0, T)$ to close our proof of Theorem 4.2. Multiply $\eta_n(t)$ by $u'_n(t)$ and integrate this over $(0, T)$. By Proposition 2.3, it then follows from $(\mathrm{CP})^H_{\lambda, g_n}$ that

$$\begin{aligned}&\int_0^T \langle \eta_n(t), u'_n(t) \rangle dt \\ &\quad = \int_0^T \langle f_n(t) - g_n(t), u'_n(t) \rangle dt - \tilde{\varphi}_\lambda(u_n(T)) + \tilde{\varphi}_\lambda(u_0) - \lambda \int_0^T |u'_n(t)|_H^2 dt.\end{aligned}$$

Hence, by Lemmas 4.6 and 4.7, we get

$$\begin{aligned}&\limsup_{n' \to \infty} \int_0^T \langle \eta_{n'}(t), u'_{n'}(t) \rangle dt \\ &\quad \leq \int_0^T \langle f(t) - g(t), u'(t) \rangle dt - \tilde{\varphi}_\lambda(u(T)) + \tilde{\varphi}_\lambda(u_0) - \lambda \int_0^T |u'(t)|_H^2 dt \\ &\quad = \int_0^T \langle f(t) - g(t) - \partial_H \tilde{\varphi}_\lambda(u(t)) - \lambda u'(t), u'(t) \rangle dt = \int_0^T \langle \eta(t), u'(t) \rangle dt,\end{aligned}$$

which together with (4.7) and (4.8) implies $\eta \in \partial_{L^p(0,T;V)} \Psi(u')$ (see Proposition 2.1). Consequently, we can deduce from (3.4) that $[u'(t), \eta(t)] \in \partial_V \psi^t$ for a.e. $t \in (0, T)$. This completes our proof for Theorem 4.2. □

4.2. Perturbed problem

This subsection is devoted to proving the existence of local (in time) strong solutions for $(\mathrm{CP})_\lambda$. As was mentioned in the beginning of Sect. 4, we shall obtain a fixed point g_* of the mapping $\mathcal{F}_{T_0} : L^{p'}(0, T_0; V^*) \to 2^{L^{p'}(0,T_0;V^*)}$ for some $T_0 \in (0, T]$ independent of λ by using the following Kakutani-Fan's fixed point theorem for multi-valued mappings (see Corollary 2 to Theorem 6.3 of [22, p. 75] for more detail):

PROPOSITION 4.8. *Let K be a non-empty compact convex subset of a locally convex topological vector space E. Let $\mathcal{T}$ be an upper semicontinuous mapping from K into 2^E such that $\mathcal{T}x$ is a closed convex subset of E and $\mathcal{T}x \cap K \neq \emptyset$ for each $x \in K$. Then, $\mathcal{T}$ has a fixed point $x_* \in K$, that is, $\mathcal{T}x_* \ni x_*$.*

We also emphasize that T_0 is independent of λ, and this fact plays a crucial role in the limiting process, which will be described in Sect. 5.

Now our goal of this subsection is the following:

THEOREM 4.9. *Let $T > 0$ and $p \in (1, \infty)$ be fixed. Suppose that* (A1)–(A3), $(\Phi 1)$ *and* $(\mathrm{B1})_\varepsilon$–(B3) *are all satisfied with a sufficiently small $\varepsilon > 0$ (the smallness of ε is determined only from p, C_1, C_3 and C_H). Then, for any $f \in L^{p'}(0, T; V^*)$ and $u_0 \in D(\varphi)$, there exists $T_0 = T_0(\|f\|_{L^{p'}(0,T;V^*)} + \varphi(u_0) + |u_0|_H) > 0$ such that for each $\lambda \in (0, 1]$, the Cauchy problem* $(\mathrm{CP})_\lambda$ *admits at least one strong solution $u \in W^{1,p}(0, T_0; V) \cap W^{1,2}(0, T_0; H)$ on $[0, T_0]$ satisfying*

$$J_\lambda u(\cdot) \in C([0, T_0]; V) \cap W^{1,p}(0, T_0; H), \tag{4.16}$$

$$\tilde{\varphi}_\lambda(u(\cdot)) \in W^{1,1}(0, T_0), \quad \eta, g \in L^{p'}(0, T_0; V^*), \tag{4.17}$$

where $\eta(t)$ and $g(t)$ stand for the sections of $\partial_V \psi^t(u'(t))$ and $B(t, J_\lambda u(t))$, respectively, such that $\lambda u'(t) + \eta(t) + \partial_H \tilde{\varphi}_\lambda(u(t)) + g(t) = f(t)$ for a.e. $t \in (0, T_0)$.

Proof. Repeating the same argument as in the proof of Lemma 4.4, we can immediately derive the following lemma. □

LEMMA 4.10. *There exists a constant $M \geq 0$ such that for all $S \in (0, T]$ and $g \in L^{p'}(0, S; V^*)$, every strong solution u of* $(\mathrm{CP})_{\lambda,g}$ *on $[0, S]$ satisfies*

$$\lambda \int_0^S |u'(t)|_H^2 dt + \int_0^S \psi^t(u'(t))dt + \sup_{t\in[0,S]} \tilde{\varphi}_\lambda(u(t))$$
$$\leq M \left\{ \varphi(u_0) + C_2 S + |N_{\frac{1}{2}}|_{L^1(0,S)} + \|f - g\|^{p'}_{L^{p'}(0,S;V^*)} \right\} \tag{4.18}$$

with a constant $M = M(p, C_1)$ depending only on p and C_1.

By Theorem 4.2, $(\mathrm{B1})'_\varepsilon$ and (B3), we can assert that $\mathcal{F}_S g$ is non-empty for every $S \in (0, T]$ and $g \in L^{p'}(0, S; V^*)$. We next prove the closedness of the graph of $\mathcal{F}_S$ in $L^{p'}(0, S; V^*)$.

LEMMA 4.11. *Let $S \in (0, T]$ be arbitrarily given. Let $[g_n, h_n] \in \mathcal{F}_S$ be such that $g_n \to g$ and $h_n \to h$ strongly in $L^{p'}(0, S; V^*)$ as $n \to \infty$. Then, it follows that $[g, h] \in \mathcal{F}_S$.*

Proof. Let $u_n := \mathcal{S}_S g_n$ and let $\eta_n(t) := f(t) - g_n(t) - \partial_H \tilde{\varphi}_\lambda(u_n(t)) - \lambda u_n'(t) \in \partial_V \psi^t(u_n'(t))$. Then, by Lemma 4.10, we have

$$\lambda \int_0^S |u_n'(t)|_H^2 dt + \int_0^S \psi^t(u_n'(t)) dt + \sup_{t \in [0,S]} \tilde{\varphi}_\lambda(u_n(t)) \le C, \tag{4.19}$$

which also implies

$$\sup_{t \in [0,S]} |u_n(t)|_V + \int_0^S |u_n'(t)|_V^p dt \le C, \quad \sup_{t \in [0,S]} |\partial_H \tilde{\varphi}_\lambda(u_n(t))|_H \le C_\lambda, \tag{4.20}$$

$$\int_0^S |\eta_n(t)|_{V^*}^{p'} dt \le C, \tag{4.21}$$

$$\sup_{t \in [0,S]} |J_\lambda u_n(t)|_H + \int_0^S \left| \frac{d}{dt} J_\lambda u_n(t) \right|_H^p dt \le C. \tag{4.22}$$

Hence by $(\Phi 1)$, the sequence $\{J_\lambda u_n(\cdot)\}$ is bounded in $L^\infty(0, S; X)$. Thus, just as in the proof of Lemma 4.6, by virtue of Theorem 5 of [50], there exists a subsequence $\{n'\}$ of $\{n\}$ such that $J_\lambda u_{n'}(\cdot) \to J_\lambda u(\cdot)$ strongly in $C([0, S]; V)$ as $n' \to \infty$, and moreover, we can also obtain

$$\begin{aligned}
u_{n'} &\to u && \text{weakly in } W^{1,p}(0, S; V) \cap W^{1,2}(0, S; H),\\
u_{n'}(t) &\to u(t) && \text{weakly in } V \text{ for each } t \in [0, S],\\
\partial_H \tilde{\varphi}_\lambda(u_{n'}(\cdot)) &\to \partial_H \tilde{\varphi}_\lambda(u(\cdot)) && \text{weakly in } L^2(0, S; H),\\
J_\lambda u_{n'}(\cdot) &\to J_\lambda u(\cdot) && \text{weakly in } W^{1,p}(0, S; H),\\
\eta_{n'} &\to \eta && \text{weakly in } L^{p'}(0, S; V^*)
\end{aligned}$$

for some $u \in W^{1,p}(0, S; V) \cap W^{1,2}(0, S; H)$ and $\eta \in L^{p'}(0, S; V^*)$. Furthermore, since $g_n \to g$ strongly in $L^{p'}(0, S; V^*)$, we get $\eta(t) = f(t) - g(t) - \partial_H \tilde{\varphi}_\lambda(u(t)) - \lambda u'(t) \in \partial_V \psi^t(u'(t))$ for a.e. $t \in (0, S)$. Therefore, u becomes a strong solution of $(\mathrm{CP})_{\lambda,g}$ on $[0, S]$. Hence, we have $u = \mathcal{S}_S g$.

Now we recall the assumptions that $h_n \to h$ strongly in $L^{p'}(0, S; V^*)$ and $h_n(t) \in B(t, J_\lambda u_n(t))$ for a.e. $t \in (0, S)$. Therefore, noting that (4.19) gives

$$\sup_{t \in [0,S]} \varphi(J_\lambda u_n(t)) \le \sup_{t \in [0,S]} \tilde{\varphi}_\lambda(u_n(t)) \le C$$

and that $\partial_H \tilde{\varphi}_\lambda(u_n(t)) \in \partial_H \tilde{\varphi}(J_\lambda u_n(t)) \subset \partial_V \varphi(J_\lambda u_n(t))$, we can derive that $h(t) \in B(t, J_\lambda u(t))$ for a.e. $t \in (0, S)$ from (4.20), (4.22) and (B2). Consequently, we can deduce that $[g, h] \in \mathcal{F}_S$. □

Thus, we have:

LEMMA 4.12. *The following* (i)–(iii) *hold true.*

(i) *Let* $R \geq \|f\|^{p'}_{L^{p'}(0,T;V^*)} + \varphi(u_0) + |m_1|_{L^1(0,T)} + |N_{\frac{1}{2}}|_{L^1(0,T)} + T(C_2+1)$ *be fixed. Then, there exists a constant* $T_0 = T_0(\|f\|_{L^{p'}(0,T;V^*)} + R + \varphi(u_0) + |u_0|_H) \in (0, T]$ *independent of* $\lambda \in (0, 1]$ *such that* $\mathcal{F}_{T_0} B^{T_0}_R \subset B^{T_0}_R$, *where*

$$B^{T_0}_R := \left\{ g \in L^{p'}(0, T_0; V^*);\ \int_0^{T_0} |g(t)|^{p'}_{V^*} dt \leq R \right\};$$

(ii) *Let* $Q^{T_0}_R := \overline{\mathrm{conv}(\mathcal{F}_{T_0} B^{T_0}_R)}$ *be the closed convex hull of* $\mathcal{F}_{T_0} B^{T_0}_R$ *in* $L^{p'}(0, T_0; V^*)$. *Then,* $\mathcal{F}_{T_0} Q^{T_0}_R \subset Q^{T_0}_R$, *and* $Q^{T_0}_R$ *is compact in* $L^{p'}(0, T_0; V^*)$;

(iii) *The restriction of* $\mathcal{F}_{T_0}$ *to* $Q^{T_0}_R$ *is an upper semicontinuous mapping from* $Q^{T_0}_R$ *into* $2^{L^{p'}(0,T_0;V^*)}$.

Proof. **Proof of (i).** Let $T_0 \in (0, T]$ be a number which will be determined later. Let $g \in B^{T_0}_R$ and let $u = \mathcal{S}_{T_0} g$, i.e., u is a strong solution of $(\mathrm{CP})_{\lambda,g}$ on $[0, T_0]$. Putting $\sigma := \min\{2, p'\}$, we get, by $(\mathrm{CP})_{\lambda,g}$ and (A2),

$$\int_0^{T_0} |\partial_H \tilde{\varphi}_\lambda(u(t))|^{\sigma}_{V^*} dt \leq c_1 \left\{ \int_0^{T_0} |f(t)|^{p'}_{V^*} dt + \int_0^{T_0} |g(t)|^{p'}_{V^*} dt + \lambda^2 \int_0^{T_0} |u'(t)|^2_H dt \right.$$
$$\left. + \int_0^{T_0} \psi^t(u'(t)) dt + |m_1|_{L^1(0,T_0)} + T_0 \right\},$$

where $c_1 = c_1(p, C_3, C_H)$ is a constant depending only on p, C_3 and C_H.

Let $h \in \mathcal{F}_{T_0} g$ be arbitrarily given, that is, $h \in L^{p'}(0, T_0; V^*)$ and $h(t) \in B(t, J_\lambda u(t))$ for a.e. $t \in (0, T_0)$. Since $\partial_H \tilde{\varphi}_\lambda(u(t)) \in \partial_V \varphi(J_\lambda u(t))$, by $(\mathrm{B1})'_\varepsilon$ and Lemma 4.10, it follows that

$$\int_0^{T_0} |h(t)|^{p'}_{V^*} dt$$
$$\leq \varepsilon c_2 \left\{ \|f\|^{p'}_{L^{p'}(0,T;V^*)} + R + \varphi(u_0) + |m_1|_{L^1(0,T)} + |N_{\frac{1}{2}}|_{L^1(0,T)} + T(C_2 + 1) \right\}$$
$$+ \left(\int_0^{T_0} |m^\varepsilon_2(t)| dt \right) \ell^\varepsilon_3 \left(C \left\{ \|f\|^{p'}_{L^{p'}(0,T;V^*)} + R + \varphi(u_0) + |u_0|_H + 1 \right\} \right),$$

where $c_2 = c_2(p, C_1, C_3, C_H)$ is a constant depending only on p, C_1, C_3 and C_H. Here, we also remark that the constant C above is independent of $\lambda \in (0, 1]$ and T_0. We set $\varepsilon > 0$ such that

$$\varepsilon c_2 \leq \frac{1}{4}, \tag{4.23}$$

then $\varepsilon c_2\{\|f\|^{p'}_{L^{p'}(0,T;V^*)} + R + \varphi(u_0) + |m_1|_{L^1(0,T)} + |N_{\frac{1}{2}}|_{L^1(0,T)} + T(C_2+1)\} \leq R/2$. Since $m^\varepsilon_2 \in L^1(0, T)$, we can take $T_0 \in (0, T]$ independent of λ such that

$$\left(\int_0^{T_0} |m^\varepsilon_2(t)| dt \right) \ell^\varepsilon_3 \left(C\{\|f\|^{p'}_{L^{p'}(0,T;V^*)} + R + \varphi(u_0) + |u_0|_H + 1\} \right) \leq R/2.$$

It then follows that

$$\int_0^{T_0} |h(t)|_{V^*}^{p'} dt \leq R,$$

which proves (i).

Proof of (ii). Since $B_R^{T_0}$ is convex and closed in $L^{p'}(0, T_0; V^*)$, (i) gives

$$Q_R^{T_0} := \overline{\mathrm{conv}\left(\mathcal{F}_{T_0} B_R^{T_0}\right)} \subset \overline{\mathrm{conv}\left(B_R^{T_0}\right)} = B_R^{T_0}.$$

Hence, it follows that $\mathcal{F}_{T_0} Q_R^{T_0} \subset \mathcal{F}_{T_0} B_R^{T_0} \subset Q_R^{T_0}$.

It now remains to prove that $Q_R^{T_0}$ is compact in $L^{p'}(0, T_0; V^*)$. To this end, we claim that $\mathcal{F}_{T_0} B_R^{T_0}$ is precompact in $L^{p'}(0, T_0; V^*)$. Indeed, let $\{h_n\}$ be a sequence in $\mathcal{F}_{T_0} B_R^{T_0}$. Then (i) implies that $\{h_n\}$ is bounded in $L^{p'}(0, T_0; V^*)$. We can take a sequence $\{g_n\}$ in $B_R^{T_0}$ such that $h_n \in \mathcal{F}_{T_0} g_n$, i.e., $h_n(t) \in B(t, J_\lambda u_n(t))$ for a.e. $t \in (0, T_0)$, where $u_n := \mathcal{S}_{T_0} g_n$. Since $\{g_n\}$ is bounded in $L^{p'}(0, T_0; V^*)$, by Lemma 4.10, we can derive that $\{J_\lambda u_n(\cdot)\}$ and $\{\varphi(J_\lambda u_n(\cdot))\}$ are bounded in $W^{1,p}(0, T_0; H)$ and $L^\infty(0, T_0)$, respectively, for all $n \in \mathbb{N}$, and that $\{J_\lambda u_n(\cdot)\}$ is precompact in $C([0, T_0]; V)$. Moreover, by $(\mathrm{CP})_{\lambda, g_n}$, we find that $\{\partial_H \tilde{\varphi}_\lambda(u_n(\cdot))\}$ is bounded in $L^\sigma(0, T_0; V^*)$. Thus, (B2) implies that $\{h_n\}$ is precompact in $L^{p'}(0, T_0; V^*)$, and so is $\mathcal{F}_{T_0} B_R^{T_0}$. Therefore by Mazur's theorem (see, e.g, (C.4) Theorem of [29, p. 603]), $Q_R^{T_0}$ becomes compact in $L^{p'}(0, T_0; V^*)$.

Proof of (iii). Applying Lemma 4.11 with $g_n \equiv g$ and $S = T_0$, we can deduce that the set $\mathcal{F}_{T_0} g$ is closed in $L^{p'}(0, T_0; V^*)$. Hence, by virtue of Lemma 4.11 and the following proposition (see Proposition 6.2 of [22, p. 77] for its proof), it follows from (ii) that $\mathcal{F}_{T_0}$ is upper semicontinuous from $Q_R^{T_0}$ into $2^{Q_R^{T_0}}$.

PROPOSITION 4.13. *Let K and K_1 be two compact topological spaces, and let $\mathcal{T}$ be a mapping from K into 2^{K_1} such that $\mathcal{T}x$ is closed for each $x \in K$. Then, $\mathcal{T}$ is upper semicontinuous from K into 2^{K_1} if and only if the graph of $\mathcal{T}$ is a closed subset in $K \times K_1$.*

Furthermore, since the topology of $Q_R^{T_0}$ is induced by $L^{p'}(0, T_0; V^*)$, it also holds true that $\mathcal{F}_{T_0}$ is upper semicontinuous from $Q_R^{T_0}$ into $2^{L^{p'}(0, T_0; V^*)}$. □

Now, let $g \in Q_R^{T_0}$ be fixed. Then, for arbitrary $h_1, h_2 \in \mathcal{F}_{T_0} g$ and $\theta \in [0, 1]$, we have $(1 - \theta)h_1 + \theta_2 h_2 \in L^{p'}(0, T_0; V^*)$, and moreover, by (B3), we see $(1 - \theta)h_1(t) + \theta h_2(t) \in B(t, J_\lambda(\mathcal{S}_{T_0} g)(t))$ for a.e. $t \in (0, T_0)$. Hence, it follows that $(1 - \theta)h_1 + \theta h_2 \in \mathcal{F}_{T_0} g$, which implies that the set $\mathcal{F}_{T_0} g$ is convex. Therefore by Lemma 4.12, we can apply Proposition 4.8 to the mapping $\mathcal{F}_{T_0}$ restricted to $Q_R^{T_0}$, so that there exists a fixed point $g_* \in Q_R^{T_0}$ of $\mathcal{F}_{T_0}$, i.e., $g_* \in \mathcal{F}_{T_0} g_*$. This completes our proof of Theorem 4.9. □

5. Convergence of approximate solutions

In this section, we derive the convergence of strong solutions $u_\lambda \in W^{1,p}(0, T_0; V) \cap W^{1,2}(0, T_0; H)$ for $(\mathrm{CP})_\lambda$ on $[0, T_0]$ by establishing a priori estimates. Here, we recall the fact that T_0 is independent of λ (see Lemma 4.12). By Theorem 4.9, for each $\lambda \in (0, 1]$, there exist $g_\lambda, \eta_\lambda \in L^{p'}(0, T_0; V^*)$ such that

$$\lambda u_\lambda'(t) + \eta_\lambda(t) + \partial_H \tilde{\varphi}_\lambda(u_\lambda(t)) + g_\lambda(t) = f(t) \text{ in } V^*, \tag{5.1}$$

$$\eta_\lambda(t) \in \partial_V \psi^t(u_\lambda'(t)), \quad g_\lambda(t) \in B(t, J_\lambda u_\lambda(t)) \quad \text{for a.e. } t \in (0, T_0). \tag{5.2}$$

Throughout this section, every constant denoted by C will be independent of λ. Since $\partial_H \tilde{\varphi}_\lambda(u_\lambda(t)) \in \partial_V \varphi(J_\lambda u_\lambda(t))$ for a.e. $t \in (0, T_0)$, by (A2), $(\mathrm{B1})'_\varepsilon$ and (5.1), it follows that

$$|g_\lambda(t)|_{V^*}^{p'} \le \varepsilon c_3 \left\{ |f(t)|_{V^*}^{p'} + |g_\lambda(t)|_{V^*}^{p'} + \lambda^2 |u_\lambda'(t)|_H^2 + \psi^t(u_\lambda'(t)) + |m_1(t)| + 1 \right\} \\ + |m_2^\varepsilon(t)| \ell_3^\varepsilon \left(\tilde{\varphi}_\lambda(u_\lambda(t)) + |J_\lambda u_\lambda(t)|_H \right)$$

with some constant $c_3 = c_3(p, C_3, C_H)$ depending only on p, C_3 and C_H. We can then deduce that

$$(1 - \varepsilon c_3) |g_\lambda(t)|_{V^*}^{p'} \le \varepsilon c_3 \left\{ |f(t)|_{V^*}^{p'} + \lambda^2 |u_\lambda'(t)|_H^2 + \psi^t(u_\lambda'(t)) + |m_1(t)| + 1 \right\} \\ + |m_2^\varepsilon(t)| \ell_3^\varepsilon \left(\tilde{\varphi}_\lambda(u_\lambda(t)) + |J_\lambda u_\lambda(t)|_H \right). \tag{5.3}$$

Multiplying (5.1) by $u_\lambda'(t)$ and using $(\mathrm{A2})'$ with $\zeta = 1/2$ and (A1), we observe

$$\lambda |u_\lambda'(t)|_H^2 + \frac{1}{2} \psi^t(u_\lambda'(t)) + \frac{d}{dt} \tilde{\varphi}_\lambda(u_\lambda(t)) \\ \le N_{\frac{1}{2}}(t) + c_4 \left(|g_\lambda(t)|_{V^*}^{p'} + |f(t)|_{V^*}^{p'} \right) + \frac{1}{4} \psi^t(u_\lambda'(t)) + \frac{C_2}{4}$$

with a constant $c_4 = c_4(p, C_1)$ depending only on p and C_1. Moreover, fixing ε so small that

$$\frac{\varepsilon c_3 c_4}{1 - \varepsilon c_3} \le \frac{1}{8}, \tag{5.4}$$

we have

$$\frac{\lambda}{2} |u_\lambda'(t)|_H^2 + \frac{1}{8} \psi^t(u_\lambda'(t)) + \frac{d}{dt} \tilde{\varphi}_\lambda(u_\lambda(t)) \\ \le C \left(|f(t)|_{V^*}^{p'} + |N_{\frac{1}{2}}(t)| + |m_1(t)| + 1 \right) \\ + C |m_2^\varepsilon(t)| \ell_3^\varepsilon \left(\tilde{\varphi}_\lambda(u_\lambda(t)) + |J_\lambda u_\lambda(t)|_H \right) \tag{5.5}$$

for a.e. $t \in (0, T_0)$. Here by (A1), we note that

$$\frac{d}{dt} |J_\lambda u_\lambda(t)|_H \le \left| \frac{d}{dt} J_\lambda u_\lambda(t) \right|_H \le |u_\lambda'(t)|_H \le C_H |u_\lambda'(t)|_V \le \frac{1}{8} \psi^t(u_\lambda'(t)) + C.$$

Hence, it follows that

$$\frac{d}{dt}\{\tilde{\varphi}_\lambda(u_\lambda(t)) + |J_\lambda u_\lambda(t)|_H\}$$
$$\leq C\left(|f(t)|_{V^*}^{p'} + |N_{\frac{1}{2}}(t)| + |m_1(t)| + 1\right) + C|m_2^\varepsilon(t)|\ell_3^\varepsilon(\tilde{\varphi}_\lambda(u_\lambda(t)) + |J_\lambda u_\lambda(t)|_H)$$

for a.e. $t \in (0, T_0)$. Then, we employ the following standard fact:

PROPOSITION 5.1. *Let $T > 0$, let $\rho, m \in L^1(0, T)$ and let ϕ be an absolutely continuous function from $[0, T]$ into $\mathbb{R}$ such that*

$$\frac{d\phi}{dt}(t) \leq \rho(t) + |m(t)|\ell(\phi(t)) \quad \text{for a.e. } t \in (0, T) \tag{5.6}$$

with some non-decreasing function ℓ on $[0, \infty)$. Then, it follows that

$$\sup_{t\in[0,T_*]} \phi(t) \leq \phi(0) + |\rho|_{L^1(0,T)} + 1 \tag{5.7}$$

with a constant $T_* \in (0, T]$ satisfying

$$\int_0^{T_*} |m(t)|dt \leq \frac{1}{1 + \ell(\phi(0) + |\rho|_{L^1(0,T)} + 1)}. \tag{5.8}$$

Therefore, by Propositions 2.2 and 5.1, we can take $T_* = T_*(\varphi(u_0) + |u_0|_H + \|f\|_{L^{p'}(0,T;V^*)}) \in (0, T_0]$ independent of λ such that

$$\sup_{t\in[0,T_*]} \{\tilde{\varphi}_\lambda(u_\lambda(t)) + |J_\lambda u_\lambda(t)|_H\} \leq C. \tag{5.9}$$

Furthermore, integrating (5.5) over $(0, T_*)$, we can obtain

$$\lambda \int_0^{T_*} |u_\lambda'(t)|_H^2 dt + \int_0^{T_*} \psi^t(u_\lambda'(t))dt \leq C, \tag{5.10}$$

which together with (A1) and (A2) also implies

$$\int_0^{T_*} |u_\lambda'(t)|_V^p dt \leq C, \quad \int_0^{T_*} |\eta_\lambda(t))|_{V^*}^{p'} dt \leq C.$$

Moreover, it follows from (5.3) and (5.1) that

$$\int_0^{T_*} |g_\lambda(t)|_{V^*}^{p'} dt \leq C, \quad \int_0^{T_*} |\partial_H \tilde{\varphi}_\lambda(u_\lambda(t))|_{V^*}^\sigma dt \leq C$$

with $\sigma = \min\{2, p'\}$.

Therefore, we can obtain the following convergences by taking a sequence $\{\lambda_n\}$ in $(0, 1)$ such that $\lambda_n \to +0$. There exist $u \in W^{1,p}(0, T_*; V)$, $\eta, g \in L^{p'}(0, T_*; V^*)$

and $\xi \in L^\sigma(0, T_*; V^*)$ such that

$$\begin{aligned} u_{\lambda_n} &\to u \quad \text{weakly in } W^{1,p}(0, T_*; V), \\ \eta_{\lambda_n} &\to \eta \quad \text{weakly in } L^{p'}(0, T_*; V^*), \\ g_{\lambda_n} &\to g \quad \text{weakly in } L^{p'}(0, T_*; V^*), \\ \partial_H \tilde{\varphi}_{\lambda_n}(u_{\lambda_n}(\cdot)) &\to \xi \quad \text{weakly in } L^\sigma(0, T_*; V^*), \\ \lambda_n u'_{\lambda_n} &\to 0 \quad \text{strongly in } L^2(0, T_*; H). \end{aligned}$$

Here, we also find that $\xi = f - \eta - g \in L^{p'}(0, T_*; V^*)$. Furthermore, since $\{u'_{\lambda_n}\}$ is bounded in $L^p(0, T_*; V)$, it follows that $\sqrt{\lambda_n} u'_{\lambda_n} \to 0$ strongly in $L^p(0, T_*; V)$. Hence by (5.10), we can assert that

$$\sqrt{\lambda_n} u'_{\lambda_n} \to 0 \quad \text{weakly in } L^2(0, T_*; H). \tag{5.11}$$

Moreover, note that

$$\int_0^{T_*} \left| \frac{d}{dt} J_\lambda u_\lambda(t) \right|_H^p dt \le \int_0^{T_*} |u'_\lambda(t)|_H^p dt \le C. \tag{5.12}$$

Therefore, by (5.9) and (Φ1), Theorem 5 of [50] implies

$$J_{\lambda_n} u_{\lambda_n} \to v \quad \text{strongly in } C([0, T_*]; V) \tag{5.13}$$

for some $v \in C([0, T_*]; V)$. Furthermore, we can also prove $v = u$ by using the definition of $\partial_H \tilde{\varphi}_\lambda$ and the fact that $\{\partial_H \tilde{\varphi}_\lambda(u_\lambda(\cdot))\}$ is bounded in $L^\sigma(0, T_*; V^*)$. Since $\partial_H \tilde{\varphi}_{\lambda_n}(u_{\lambda_n}(t)) \in \partial_V \varphi(J_{\lambda_n} u_{\lambda_n}(t))$ for a.e. $t \in (0, T_*)$, by Proposition 1.1 of [32] and Proposition 2.1, we assert that

$$\xi(t) \in \partial_V \varphi(u(t)) \quad \text{for a.e. } t \in (0, T_*),$$

which also yields that $\varphi(u(\cdot)) \in W^{1,1}(0, T_*)$ and $d\varphi(u(t))/dt = \langle \xi(t), u'(t) \rangle$ for a.e. $t \in (0, T_*)$. Moreover, we can also deduce from (B2) that

$$\begin{aligned} g_{\lambda_n} &\to g \quad \text{strongly in } L^{p'}(0, T_*; V^*), \\ g(t) &\in B(t, u(t)) \quad \text{for a.e. } t \in (0, T_*). \end{aligned}$$

Furthermore, we claim that $\eta(t) \in \partial_V \psi^t(u'(t))$ for a.e. $t \in (0, T_*)$. Indeed, we see

$$\begin{aligned} &\int_0^{T_*} \langle \eta_{\lambda_n}(t), u'_{\lambda_n}(t) \rangle dt \\ &\quad \le \int_0^{T_*} \langle f(t), u'_{\lambda_n}(t) \rangle dt - \int_0^{T_*} |\sqrt{\lambda_n} u'_{\lambda_n}(t)|_H^2 dt - \varphi(J_{\lambda_n} u_{\lambda_n}(T_*)) + \varphi(u_0) \\ &\qquad - \int_0^{T_*} \langle g_{\lambda_n}(t), u'_{\lambda_n}(t) \rangle dt. \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&\limsup_{n\to\infty} \int_0^{T_*} \langle \eta_{\lambda_n}(t), u'_{\lambda_n}(t)\rangle dt \\
&\quad \le \lim_{n\to\infty} \int_0^{T_*} \langle f(t), u'_{\lambda_n}(t)\rangle dt - \liminf_{n\to\infty} \int_0^{T_*} |\sqrt{\lambda_n} u'_{\lambda_n}(t)|_H^2 dt \\
&\qquad - \liminf_{n\to\infty} \varphi(J_{\lambda_n} u_{\lambda_n}(T_*)) + \varphi(u_0) - \lim_{n\to\infty} \int_0^{T_*} \langle g_{\lambda_n}(t), u'_{\lambda_n}(t)\rangle dt \\
&\quad \le \int_0^{T_*} \langle f(t) - \xi(t) - g(t), u'(t)\rangle dt,
\end{aligned}
$$

which implies $\eta(t) = f(t) - \xi(t) - g(t) \in \partial_V \psi^t(u'(t))$ for a.e. $t \in (0, T_*)$.

Finally, we check the initial condition, $u(0) = u_0$. We observe that

$$
\begin{aligned}
|u(t) - u_0|_H &\le |u(t) - J_{\lambda_n} u_{\lambda_n}(t)|_H + |J_{\lambda_n} u_{\lambda_n}(t) - J_{\lambda_n} u_0|_H + |J_{\lambda_n} u_0 - u_0|_H \\
&\le C_H \sup_{t\in[0,T_*]} |u(t) - J_{\lambda_n} u_{\lambda_n}(t)|_V \\
&\quad + \left(\int_0^T \left| \frac{d}{d\tau} J_{\lambda_n} u_{\lambda_n}(\tau) \right|_H^p d\tau \right)^{1/p} t^{1/p'} + |J_{\lambda_n} u_0 - u_0|_H .
\end{aligned}
$$

Hence, passing to the limit as $n \to \infty$, we can deduce from (5.12) and (5.13) that

$$
|u(t) - u_0|_H \le C t^{1/p'} \to 0 \quad \text{as } t \to +0.
$$

Thus, from the fact that $u \in C([0, T_*]; V)$, we also conclude that $u(t) \to u_0$ strongly in V as $t \to +0$. Consequently, u becomes a strong solution of (CP) on $[0, T_*]$, and our proof of Theorem 3.6 is complete. □

6. Global existence

In this section, we give proofs of Theorems 3.7 and 3.8.

6.1. Proof of Theorem 3.7

Let $S \in (0, T]$ and let u be a strong solution of (CP) on $[0, S]$. In this proof, every constant denoted by C is independent of S. Multiplying (CP) by $u'(t)$ and using (A1) and (A2)$'$ with $\zeta = 1/2$, we get

$$
\frac{1}{2}\psi^t(u'(t)) - N_{\frac{1}{2}}(t) + \frac{d}{dt}\varphi(u(t)) \le c_4 \left(|f(t)|_{V^*}^{p'} + |g(t)|_{V^*}^{p'} \right) + \frac{1}{4}\psi^t(u'(t)) + \frac{C_2}{4},
$$

where $g(t)$ denotes the section of $B(t, u(t))$ as in (3.2), for a.e. $t \in (0, S)$. Now, by (B4)$_\varepsilon$, we see that

$$
|g(t)|_{V^*}^{p'} \le \varepsilon |\xi(t)|_{V^*}^{\sigma} + |m_3^\varepsilon(t)| \left\{ \varphi(u(t)) + |u(t)|_V^p + 1 \right\},
$$

where $\xi(t)$ denotes the section of $\partial_V \varphi(u(t))$ as in (3.2), and moreover, as in (5.3),

$$(1-\varepsilon c_5)|g(t)|_{V^*}^{p'} \le \varepsilon c_5 \left\{ |f(t)|_{V^*}^{p'} + \psi^t(u'(t)) + |m_1(t)| + 1 \right\} + |m_3^\varepsilon(t)| \left\{ \varphi(u(t)) + |u(t)|_V^p + 1 \right\}$$

with some constant $c_5 = c_5(p, C_3)$ depending only on p and C_3. Hence, choosing $\varepsilon > 0$ so small that

$$\frac{\varepsilon c_5}{1-\varepsilon c_5} \le \frac{1}{8c_4},$$

we can derive

$$|g(t)|_{V^*}^{p'} \le C \left(|f(t)|_{V^*}^{p'} + |m_1(t)| + |m_3^\varepsilon(t)| + 1 \right) + C|m_3^\varepsilon(t)| \left\{ \varphi(u(t)) + |u(t)|_V^p \right\} + \frac{1}{8c_4} \psi^t(u'(t))$$

for a.e. $t \in (0, S)$. Furthermore, by (A1), we observe that

$$\frac{d}{dt}|u(t)|_V^p \le p|u(t)|_V^{p-1}|u'(t)|_V \le C|u(t)|_V^p + \frac{1}{8}\psi^t(u'(t)) + \frac{C_2}{8}.$$

Thus,

$$\frac{d}{dt}\left\{ \varphi(u(t)) + |u(t)|_V^p \right\} \le C \left(|f(t)|_{V^*}^{p'} + |m_1(t)| + |m_3^\varepsilon(t)| + |N_{\frac{1}{2}}(t)| + 1 \right) + C \left(|m_3^\varepsilon(t)| + 1 \right) \left\{ \varphi(u(t)) + |u(t)|_V^p \right\}$$

for a.e. $t \in (0, S)$. Hence, integrating this over $(0, t)$ and applying Gronwall's inequality, we can deduce that

$$\sup_{t \in [0,S]} \left\{ \varphi(u(t)) + |u(t)|_V^p \right\} \le C \tag{6.1}$$

with some constant C independent of S.

We find that $(B4)_\varepsilon$ implies $(B1)'_\varepsilon$ with m_2^ε replaced by m_3^ε. By virtue of Theorem 3.6, there exists a strong solution u of (CP) on $[0, T_0]$ for some $T_0 \in (0, T]$, and moreover, (6.1) holds with $S = T_0$. In case $T_0 = T$, we obtain our desired conclusion. In case $T_0 < T$, recall the proof of Theorem 3.6 (particularly, the choice of T_0 and T_*) and note that the function $I \subset [0, T] \mapsto \int_I |m_3^\varepsilon(t)| dt$ is absolutely continuous by $m_3^\varepsilon \in L^1(0, T)$.

Then, due to (6.1), we can extend u onto $[0, T]$ as a strong solution of (CP), by using Theorem 3.6.

6.2. Proof of Theorem 3.8

We first prepare the following Lemma 6.1 (see Lemma 4.4 of [6] for its proof).

LEMMA 6.1. *Let $T > 0$, let $\rho \in L^1(0, T)$ and let ϕ be a non-negative absolutely continuous function from $[0, T]$ into $\mathbb{R}$ such that*

$$\frac{d\phi}{dt}(t) + \alpha\phi^{q-1}(t) \le K|\rho(t)| \quad \text{for a.e. } t \in (0, T), \tag{6.2}$$

where $\alpha > 0$, $K > 0$ and $q > 1$. Let $r > 0$ and suppose that $\phi(0) \le r$ and $\|\rho\| \le r^{q-1}$, where $\|\rho\|$ is given by

$$\|\rho\| := \begin{cases} \sup_{t\in[1,T]} \int_{t-1}^{t} |\rho(\tau)|d\tau & \text{if } 1 \le T, \\ \int_0^T |\rho(\tau)|d\tau & \text{if } 0 < T < 1. \end{cases}$$

Then, there exists a non-decreasing function $M_{\alpha,K,q}(\cdot)$ on $[0, \infty)$ depending only on α, K, q such that

$$\phi(t) \le M_{\alpha,K,q}(r)r \quad \text{for all } t \in [0, T].$$

Now, we proceed to prove Theorem 3.8. We first fix $\varepsilon > 0$ satisfying (6.10), which will be given later, and assume (A6)$_\varepsilon$. Since $\lim_{s\to+0} \ell_i(s) = 0$ for each $i = 4, 5$ and $\lim_{s\to+0} \ell_6^\varepsilon(s) = 0$, we next choose $\delta_0 > 0$ satisfying (6.5) and (6.13), which will be stated below.

Let $S \in (0, T]$ and let u be a strong solution of (CP) on $[0, S]$ with u_0 and f satisfying

$$\|f\|_\star + \varphi(u_0) < \delta \tag{6.3}$$

for an enough small constant $\delta \in (0, \delta_0)$, which will be determined by (6.14) and will not depend on S and T. Then, we shall prove that

$$\sup_{t\in[0,S]} \varphi(u(t)) \le \delta_0 \tag{6.4}$$

by contradiction. Assume $\sup_{t\in[0,S]} \varphi(u(t)) > \delta_0$, so that there exists $T_1 \in (0, S)$ such that $\varphi(u(T_1)) = \delta_0$ and $\varphi(u(t)) < \delta_0$ for all $t \in [0, T_1)$.

Set $\delta_0 > 0$ such that

$$\ell_4(\delta_0) < \frac{C_4}{2}, \quad \ell_5(\delta_0) < \frac{C_4 p}{8}. \tag{6.5}$$

By (B5), it then follows that

$$\frac{C_4}{2}\varphi(u(t)) \le \langle \xi(t) + g(t), u(t)\rangle, \tag{6.6}$$

$$|u(t)|_V^p \le \frac{C_4 p}{8}\varphi(u(t)), \tag{6.7}$$

where $\xi(t)$ and $g(t)$ stand for the sections of $\partial_V \varphi(u(t))$ and $B(t, u(t))$, respectively, as in (3.2), for a.e. $t \in (0, T_1)$. Hence, we have

$$\begin{aligned}\frac{C_4}{2}\varphi(u(t)) &\le \langle f(t) - \eta(t), u(t)\rangle \\ &\le \frac{1}{p'}\left(|f(t)|_{V^*}^{p'} + |\eta(t)|_{V^*}^{p'}\right) + \frac{C_4}{4}\varphi(u(t)),\end{aligned}$$

where $\eta(t) \in \partial_V \psi^t(u'(t))$ as in (3.2). Thus, (A2) with $m_1 \equiv 0$ gives

$$\frac{C_4}{4}\varphi(u(t)) \le \frac{1}{p'}\left\{|f(t)|_{V^*}^{p'} + C_3\psi^t(u'(t))\right\} \quad \text{for a.e. } t \in (0, T_1). \tag{6.8}$$

On the other hand, since $\psi^t(0) \equiv 0$ and $m_1 \equiv 0$, we can take $N_\zeta \equiv 0$ in (A2)$'$. Hence, multiplying (CP) by $u'(t)$ and using (A2)$'$ with $\zeta = 0$ and (A1) with $C_2 = 0$, we get

$$\psi^t(u'(t)) + \frac{d}{dt}\varphi(u(t)) \le c_6\left(|f(t)|_{V^*}^{p'} + |g(t)|_{V^*}^{p'}\right) + \frac{1}{2}\psi^t(u'(t)), \tag{6.9}$$

where $c_6 = c_6(p, C_1)$ is a constant depending only on p and C_1. Moreover, we get, by (3.8) and (A2) with $m_1 \equiv 0$,

$$(1 - \varepsilon c_7)|g(t)|_{V^*}^{p'} \le \varepsilon c_7\left(|f(t)|_{V^*}^{p'} + C_3\psi^t(u'(t))\right) + \ell_6^\varepsilon(\delta_0)\varphi(u(t)),$$

where $c_7 = 3^{p'-1}$, for a.e. $t \in (0, T_1)$. Now, fixing ε so small that

$$0 < \frac{\varepsilon c_7}{1 - \varepsilon c_7} \le \frac{1}{4c_6C_3} \tag{6.10}$$

(hence, the smallness of ε depends only on p, C_1 and C_3), we have

$$|g(t)|_{V^*}^{p'} \le \frac{1}{4c_6C_3}|f(t)|_{V^*}^{p'} + \frac{1}{4c_6}\psi^t(u'(t)) + \frac{1}{1 - \varepsilon c_7}\ell_6^\varepsilon(\delta_0)\varphi(u(t)) \tag{6.11}$$

for a.e. $t \in (0, T_1)$. Hence, (6.9) yields

$$\begin{aligned}&\frac{1}{2}\psi^t(u'(t)) + \frac{d}{dt}\varphi(u(t)) \\ &\quad \le \left(c_6 + \frac{1}{4C_3}\right)|f(t)|_{V^*}^{p'} + \frac{1}{4}\psi^t(u'(t)) + \frac{c_6}{1 - \varepsilon c_7}\ell_6^\varepsilon(\delta_0)\varphi(u(t))\end{aligned} \tag{6.12}$$

for a.e. $t \in (0, T_1)$.

Therefore, adding (6.8) multiplied by $p'/(4C_3)$ to (6.12), we can obtain

$$\begin{aligned}&\frac{1}{4}\psi^t(u'(t)) + \frac{d}{dt}\varphi(u(t)) + 2\alpha\varphi(u(t)) \\ &\quad \le K|f(t)|_{V^*}^{p'} + \frac{c_6}{1 - \varepsilon c_7}\ell_6^\varepsilon(\delta_0)\varphi(u(t)) + \frac{1}{4}\psi^t(u'(t)),\end{aligned}$$

where

$$\alpha := \frac{p' C_4}{32 C_3}, \quad K := \left(c_6 + \frac{1}{2C_3} \right)$$

for a.e. $t \in (0, T_1)$. Hence, we can deduce that

$$\frac{d}{dt}\varphi(u(t)) + \alpha\varphi(u(t)) \leq K|f(t)|_{V^*}^{p'} \quad \text{for a.e. } t \in (0, T_1),$$

since $\delta_0 > 0$ satisfies

$$\ell_6^\varepsilon(\delta_0) < \alpha \frac{1 - \varepsilon c_7}{c_6}. \tag{6.13}$$

Thus, by Lemma 6.1, since u_0 and f satisfies (6.3) with an enough small constant $\delta \in (0, \delta_0)$ determined by

$$M_{\alpha,K,2}(\delta)\, \delta < \frac{\delta_0}{2}, \tag{6.14}$$

it follows that

$$\varphi(u(t)) < \frac{\delta_0}{2} \quad \text{for all } t \in [0, T_1],$$

which contradicts the fact that $\varphi(u(T_1)) = \delta_0$. Hence (6.4) follows.

Thus, since δ_0 and δ are independent of S, as in the proof of Theorem 3.7, we can prove the existence of strong solutions of (CP) on $[0, T]$. □

7. Applications to nonlinear PDEs

In this section, we apply the preceding abstract theory to doubly nonlinear parabolic equations.

7.1. Doubly nonlinear parabolic equations of degenerate type

In this subsection, we treat doubly nonlinear parabolic equations of degenerate type, for which (1.7) is a typical example, and we finally provide sufficient conditions for the existence of local and global (in time) solutions of the initial-boundary value problems. Let $T > 0$ and let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. We first deal with the following initial-boundary value problem,

$$\alpha(x, t, u_t(x, t)) - \operatorname{div} \mathbf{a}(x, \nabla u(x, t)) + g(x, t, u(x, t)) \ni f(x, t),$$
$$(x, t) \in \Omega \times (0, T), \tag{7.1}$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \tag{7.2}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega \tag{7.3}$$

with functions $\alpha : \Omega \times (0, T) \times \mathbb{R} \to 2^{\mathbb{R}}$, $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $g : \Omega \times (0, T) \times \mathbb{R} \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$ and $f : \Omega \times (0, T) \to \mathbb{R}$. To discuss the existence of solutions for (7.1)–(7.3), we introduce the following assumptions for $p \in [2, \infty)$.

(H1) (i) There exists a function $j : \Omega \times [0, T] \times \mathbb{R} \to [0, \infty)$ such that
- $j(x, t, \cdot) \in \Phi(\mathbb{R})$ for a.e. $x \in \Omega$ and all $t \in [0, T]$,
- $\partial_{\mathbb{R}} j(x, t, \cdot) = \alpha(x, t, \cdot)$ for a.e. $(x, t) \in \Omega \times (0, T)$,
- $j(\cdot, t, r)$ is continuous in Ω for a.e. $t \in (0, T)$ and all $r \in \mathbb{R}$.

(ii) For each $v \in L^p(\Omega)$, the function $j(\cdot, t, v(\cdot))$ is measurable in Ω for all $t \in [0, T]$, and there exists a function $\eta : \Omega \times (0, T) \to \mathbb{R}$ such that $\eta(\cdot, t) \in \alpha(\cdot, t, v(\cdot))$ and $\eta(\cdot, t)$ is measurable in Ω for a.e. $t \in (0, T)$. Furthermore, for all $t \in [0, T]$, there exists $v_0 \in L^p(\Omega)$ such that $j(\cdot, t, v_0(\cdot)) \in L^1(\Omega)$.

(iii) There exist $\rho, \sigma \in W^{1,1}(0, T)$, $b_1 \in L^1(\Omega)$ and a constant $\delta > 0$ with the following property: for all $t_0 \in [0, T]$ and $r_0 \in \mathbb{R}$, there exists a function $\pi : I_\delta(t_0) \times \Omega \times \mathbb{R} \to \mathbb{R}$, where $I_\delta(t_0) := [t_0 - \delta, t_0 + \delta] \cap [0, T]$, such that

$$|\pi(t; x, r_0) - r_0| \leq |\rho(t) - \rho(t_0)| \{j(x, t_0, r_0) + b_1(x)\}^{1/p},$$
$$j(x, t, \pi(t; x, r_0)) \leq j(x, t_0, r_0) + |\sigma(t) - \sigma(t_0)| \{j(x, t_0, r_0) + b_1(x)\}$$

for a.e. $x \in \Omega$ and all $t \in I_\delta(t_0)$, and $\pi(t; \cdot, v(\cdot))$ is measurable in Ω for all $t \in [0, T]$ and $v \in L^p(\Omega)$.

(H2) There exist a constant $C_5 \geq 0$, $a_1 \in L^1(\Omega)$ and $a_2 \in L^1(\Omega \times (0, T))$ such that the following (i), (ii) hold.

(i) $|r|^p \leq C_5 j(x, t, r) + a_1(x)$ for a.e. $x \in \Omega$ and all $(t, r) \in [0, T] \times \mathbb{R}$.

(ii) $|\eta|^{p'} \leq C_5 j(x, t, r) + a_2(x, t)$ for a.e. $(x, t) \in \Omega \times (0, T)$ and all $r \in \mathbb{R}$, $\eta \in \alpha(x, t, r)$.

(H3) (i) There exists a function $\phi = \phi(x, \mathbf{p}) : \Omega \times \mathbb{R}^N \to \mathbb{R}$ such that $\phi(\cdot, \mathbf{p})$ is measurable in Ω for all $\mathbf{p} \in \mathbb{R}^N$, $\phi(x, \cdot)$ is convex and Fréchet differentiable in $\mathbb{R}^N$ and its derivative $\partial_{\mathbb{R}^N} \phi(x, \cdot)$ coincides with $\mathbf{a}(x, \cdot)$ for a.e. $x \in \Omega$.

(ii) For all $\mathbf{v} \in L^p(\Omega; \mathbb{R}^N)$, the function $\mathbf{a}(\cdot, \mathbf{v}(\cdot))$ is measurable in Ω. Moreover, there exists $\mathbf{v}_0 \in L^p(\Omega; \mathbb{R}^N)$ such that $\phi(\cdot, \mathbf{v}_0(\cdot)) \in L^1(\Omega)$.

There exist constants $m > 1$, $C_6 \geq 0$ and $a_3, b_2 \in L^1(\Omega)$ such that

(iii) $|\mathbf{p}|^m \leq C_6 \phi(x, \mathbf{p}) + a_3(x)$ for a.e. $x \in \Omega$ and all $\mathbf{p} \in \mathbb{R}^N$;

(iv) $|\mathbf{a}(x, \mathbf{p})|^{m'} \leq C_6 \phi(x, \mathbf{p}) + b_2(x)$ for a.e. $x \in \Omega$ and all $\mathbf{p} \in \mathbb{R}^N$.

(H4) (i) There exist constants $q > 1 + 1/p'$, $C_7 \geq 0$ and $a_4 \in L^1(\Omega \times (0, T))$ such that
$|g(x, t, r)|^{p'} \leq C_7 |r|^{p'(q-1)} + a_4(x, t)$ for a.e. $(x, t) \in \Omega \times (0, T)$ and all $r \in \mathbb{R}$.

(ii) The function $g = g(x, t, r)$ is a Carathéodory function in $\Omega \times (0, T) \times \mathbb{R}$ (i.e., measurable in (x, t) and continuous in r).

REMARK 7.1. (i) By (i) of (H3), we can deduce that $\phi(x, \cdot)$ is continuous in $\mathbb{R}^N$ for a.e. $x \in \Omega$. Hence, $\phi(\cdot, \mathbf{v}(\cdot))$ becomes measurable in Ω for each measurable function $\mathbf{v} : \Omega \to \mathbb{R}^N$.

(ii) Let us give simple examples of functions α which satisfy (H1) and (H2) with $p \geq 2$. The following is concerned with the case where α is single-valued:

$$\alpha(x, t, r) = k(x, t)|r|^{p-2}r,$$

where k is an absolutely continuous function from $[0, T]$ into $C(\overline{\Omega})$ such that $k(x, t) \geq k_0 > 0$ for all $(x, t) \in \Omega \times [0, T]$ with a positive number k_0.

As for the case where α is multi-valued, we give

$$\alpha(x, t, r) = \begin{cases} \{|r - c(t)|^{p-2}(r - c(t))\} & \text{if } 1 < |r - c(t)|, \\ \left\{\frac{r-c(t)}{|r-c(t)|}\right\} & \text{if } 0 < |r - c(t)| \leq 1, \\ [-1, 1] & \text{if } r = c(t) \end{cases}$$

with $c \in W^{1,1}(0, T)$.

(iii) Typical examples of $\mathbf{a}(x, \mathbf{p})$ and $g(x, t, r)$ satisfying (H3) and (H4) are $\mathbf{a}(x, \mathbf{p}) = |\mathbf{p}|^{m-2}\mathbf{p}$ and $g(x, t, r) = \lambda(x, t)|r|^{q-2}r$ with $\lambda \in L^\infty(\Omega \times (0, T))$, respectively. Then, div $\mathbf{a}(x, \nabla u(x))$ coincides with $\Delta_m u(x) :=$ div $(|\nabla u(x)|^{m-2}\nabla u(x))$, where Δ_m is the so-called m-Laplacian.

We are concerned with solutions of the initial-boundary value problem (7.1)–(7.3) defined as follows:

DEFINITION 7.2. For each $T > 0$, a function $u : \Omega \times (0, T) \to \mathbb{R}$ is said to be a solution of the initial-boundary value problem (7.1)–(7.3) on $[0, T]$ if the following conditions are all satisfied:

- $u \in W^{1,p}(0, T; L^p(\Omega)) \cap C([0, T]; W_0^{1,m}(\Omega))$;
- there exists a function $\eta \in L^{p'}(0, T; L^{p'}(\Omega))$ such that $\eta(x, t) \in \alpha(x, t, u_t(x, t))$ for a.e. $(x, t) \in \Omega \times (0, T)$;
- it holds that div $\mathbf{a}(\cdot, \nabla u(\cdot, t))$, $g(\cdot, t, u(\cdot, t)) \in L^{p'}(\Omega)$ and

$$\eta(x, t) - \text{div}\, \mathbf{a}(x, \nabla u(x, t)) + g(x, t, u(x, t)) = f(x, t)$$

for a.e. $(x, t) \in \Omega \times (0, T)$;
- $u(\cdot, t) \to u_0$ strongly in $L^p(\Omega)$ as $t \to +0$.

To apply the preceding abstract theory to (7.1)–(7.3), we suppose that

$$2 \leq p < m^* := \begin{cases} \frac{Nm}{N-m} & \text{if } m < N, \\ +\infty & \text{if } m \geq N \end{cases} \quad \text{and} \quad q < \frac{m^*}{p'} + 1. \tag{7.4}$$

Moreover, set $X := W_0^{1,m}(\Omega)$ with the norm $|\cdot|_X := |\nabla \cdot|_{L^m(\Omega)}$ and set $V := L^p(\Omega)$ and $H := L^2(\Omega)$. Then, V is continuously and densely embedded in H, and by the Rellich-Kondrachov compact embedding theorem, X is compactly embedded in V.

Define the operator $B : (0, T) \times V \to V^*$ by

$$B(t, u) := g(\cdot, t, u(\cdot)) \quad \text{for all } t \in (0, T) \quad \text{and} \quad u \in D(B(t, \cdot))$$

with the domain $D(B(t,\cdot)) := \{u \in V; g(\cdot,t,u(\cdot)) \in V^*\}$. By (H4) and (7.4), we then infer that $X \subset L^{p'(q-1)}(\Omega) \cap V \subset D(B(t,\cdot))$ for each $t \in (0,T)$. Furthermore, we also define the functions $\psi^t, \varphi : V \to (-\infty, +\infty]$ by

$$\psi^t(u) := \begin{cases} \displaystyle\int_\Omega j(x,t,u(x))dx & \text{if } j(\cdot,t,u(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{7.5}$$

for every $t \in [0,T]$, and

$$\varphi(u) := \begin{cases} \displaystyle\int_\Omega \phi(x,\nabla u(x))dx & \text{if } u \in X \quad \text{and} \quad \phi(\cdot,\nabla u(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{7.6}$$

Then, by (H1), it follows that $\psi^t \in \Phi(V)$ for all $t \in [0,T]$; by (H1) and (H2), $D(\psi^t) = V$ for a.e. $t \in (0,T)$. Since $j(\cdot,t,r)$ is upper semicontinuous in Ω for each $(t,r) \in (0,T) \times \Omega$, we can prove $\partial_V \psi^t(u) = \alpha(\cdot,t,u(\cdot))$ for a.e. $t \in (0,T)$ and all $u \in D(\partial_V \psi^t)$ by modifying the proof of Proposition 1.1 of [32], and moreover, $D(\partial_V \psi^t) = V$ for a.e. $t \in (0,T)$. By (H3), we have $\varphi \in \Phi(V)$ and $D(\varphi) = X$, and moreover, the restriction $\varphi|_X$ of φ to X becomes Gâteaux differentiable in X and its derivative $\partial_X(\varphi|_X)(u)$ at u coincides with $-\mathrm{div}\,\mathbf{a}(\cdot,\nabla u(\cdot))$ in the sense of distributions. Hence, since $\partial_V\varphi(u) = \partial_X(\varphi|_X)(u)$ for each $u \in D(\partial_V\varphi)$, the subdifferential $\partial_V\varphi(u)$ of φ at $u \in D(\partial_V\varphi)$ also coincides with $-\mathrm{div}\,\mathbf{a}(\cdot,\nabla u(\cdot))$. Therefore, (7.1)–(7.3) is transcribed into the Cauchy problem (CP).

Furthermore, we prepare the following lemma.

LEMMA 7.3. *Let $T>0$ and let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary $\partial\Omega$.*

(i) *If* (H1) *and* (H2) *are satisfied for some $p \in [2,\infty)$, then* (A1)–(A3) *hold.*

(ii) *If* (H3), (H4) *and* (7.4) *are satisfied for some $p \in [2,\infty)$, then* (Φ1) *and* $(\mathrm{B1})_\varepsilon$–(B3) *hold for any $\varepsilon > 0$.*

Proof. **Proof of (i).** Both (A1) and (A2) are immediately derived from (H2). Let $t_0 \in [0,T]$ and $v_0 \in D(\psi^{t_0})$ be fixed. Define a function $v : \Omega \times I_\delta(t_0) \to \mathbb{R}$ by $v(x,t) := \pi(t; x, v_0(x))$. Then $v(\cdot,t)$ is measurable in Ω for each $t \in I_\delta(t_0)$, and

$$|v(x,t) - v_0(x)| \le |\rho(t) - \rho(t_0)| \{j(x,t_0,v_0(x)) + b_1(x)\}^{1/p},$$
$$j(x,t,v(x,t)) \le j(x,t_0,v_0(x)) + |\sigma(t) - \sigma(t_0)| \{j(x,t_0,v_0(x)) + b_1(x)\}$$

for all a.e. $x \in \Omega$ and $t \in I_\delta(t_0)$. Hence $v(\cdot,t) \in V$ for all $t \in I_\delta(t_0)$, and $j(\cdot,t,v(\cdot,t))$ is measurable in Ω by (H1), and moreover,

$$|v(\cdot,t) - v_0|_V \le |\rho(t) - \rho(t_0)| \left\{\psi^{t_0}(v_0) + |b_1|_{L^1(\Omega)}\right\}^{1/p},$$
$$\psi^t(v(\cdot,t)) \le \psi^{t_0}(v_0) + |\sigma(t) - \sigma(t_0)| \left\{\psi^{t_0}(v_0) + |b_1|_{L^1(\Omega)}\right\},$$

which implies (A3).

Proof of (ii). By (iii) of (H3), we can derive

$$|\nabla u|^m_{L^m(\Omega)} \leq C_6 \varphi(u) + |a_3|_{L^1(\Omega)} \quad \text{for all } u \in X. \tag{7.7}$$

Hence, (Φ1) follows since X is compactly embedded in V by (7.4). As for $(\text{B}1)_\varepsilon$, we obtain, by (H4)

$$|B(t,u)|^{p'}_{V^*} = |g(\cdot,t,u(\cdot))|^{p'}_{V^*} \leq C_7 |u|^{p'(q-1)}_{L^{p'(q-1)}(\Omega)} + |a_4(\cdot,t)|_{L^1(\Omega)}$$
$$\text{for a.e. } t \in (0,T) \text{ and all } u \in D(B(t,\cdot)). \tag{7.8}$$

By (7.4) and (7.7),

$$|B(t,u)|^{p'}_{V^*} \leq C \left\{ \varphi(u)^{p'(q-1)/m} + |a_3|^{p'(q-1)/m}_{L^1(\Omega)} \right\} + |a_4(\cdot,t)|_{L^1(\Omega)}$$
$$\text{for a.e. } t \in (0,T) \text{ and all } u \in D(\varphi), \tag{7.9}$$

which implies $(\text{B}1)_\varepsilon$ for any $\varepsilon > 0$.

By virtue of Theorem 1.27 of [45], the mapping

$$u \mapsto g(\cdot,t,u(\cdot)); \quad L^{p'(q-1)}(\Omega) \to V^*$$

becomes continuous for a.e. $t \in (0,T)$. Moreover, the mapping

$$t \mapsto g(\cdot,t,u(\cdot)); \quad (0,T) \to V^*$$

is strongly measurable in $(0,T)$ for any fixed $u \in L^{p'(q-1)}(\Omega)$. Indeed, by (ii) of (H4), the function $(x,t) \mapsto g(x,t,u(x))$ is measurable in $\Omega \times (0,T)$, so Fubini's theorem ensures that the mapping $t \mapsto \int_\Omega g(x,t,u(x))v(x)dx$ is also measurable in $(0,T)$ whenever v is measurable in Ω. Hence, the mapping $t \mapsto g(\cdot,t,u(\cdot))$ becomes weakly measurable in $(0,T)$ with values in V^*. Thus, since V^* is separable, by Pettis's theorem, we can deduce that it also becomes strongly measurable in $(0,T)$.

Let $S \in (0,T]$ be fixed and define the operator $\mathcal{B} : L^{p'(q-1)}(0,S; L^{p'(q-1)}(\Omega)) \to L^{p'}(0,S;V^*)$ by

$$(\mathcal{B}u)(t) := g(\cdot,t,u(t)(\cdot))$$
$$\text{for all } u \in L^{p'(q-1)}(0,S; L^{p'(q-1)}(\Omega)) \text{ and a.e. } t \in (0,S).$$

Then recalling (7.8) and employing Theorem 1.43 of [45], we can deduce that

$$\mathcal{B} \text{ is continuous from } L^{p'(q-1)}(0,S; L^{p'(q-1)}(\Omega)) \text{ into } L^{p'}(0,S;V^*). \tag{7.10}$$

Since $X \subset L^{p'(q-1)}(\Omega)$, this particularly yields (B3).

We finally prove (B2). Let $\{u_n\}$ be a sequence in $C([0,S];V)$ such that $u_n \to u$ strongly in $C([0,S];V)$ and

$$\sup_{t\in[0,S]} \varphi(u_n(t)) + \int_0^S |u_n'(t)|^p_H dt \quad \text{is bounded for all } n \in \mathbb{N}. \tag{7.11}$$

Hence, since (7.4) implies that X is compactly embedded in $L^{p'(q-1)}(\Omega)$, by Theorem 5 of [50], we can take a subsequence $\{n'\}$ of $\{n\}$ such that

$$u_{n'} \to u \quad \text{strongly in } C([0, S]; L^{p'(q-1)}(\Omega)).$$

Therefore, we can deduce from (7.10) that

$$\mathcal{B}u_{n'} \to \mathcal{B}u \quad \text{strongly in } L^{p'}(0, S; V^*).$$

Thus, (B2) is proved. □

The existence of local (in time) solutions for the initial-boundary value problem (7.1)–(7.3) follows immediately from Lemma 7.3 and Theorem 3.6.

THEOREM 7.4. *Let $T > 0$ and let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary $\partial\Omega$. Suppose that* (H1)–(H4) *and* (7.4) *are satisfied for some $p \in [2, \infty)$. Then, for all $f \in L^{p'}(0, T; L^{p'}(\Omega))$ and $u_0 \in W_0^{1,m}(\Omega)$, there exists $T_* = T_*\left(\int_\Omega \phi(x, \nabla u_0(x))dx + |u_0|_{L^2(\Omega)} + \|f\|_{L^{p'}(0,T;L^{p'}(\Omega))}\right) \in (0, T]$ such that the initial-boundary value problem* (7.1)–(7.3) *admits at least one solution u on* $[0, T_*]$.

As for the global existence, our result is stated as follows.

THEOREM 7.5. *Let $T > 0$ and let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary $\partial\Omega$. Suppose that* (H1)–(H4) *and* (7.4) *are satisfied for some $p \in [2, \infty)$. In addition, assume that*

$$q \leq \max\left\{p, \frac{m}{p'} + 1\right\}. \tag{7.12}$$

Then, for all $f \in L^{p'}(0, T; L^{p'}(\Omega))$ and $u_0 \in W_0^{1,m}(\Omega)$, the initial-boundary value problem (7.1)–(7.3) *admits at least one solution u on* $[0, T]$.

Proof. In order to prove this theorem, it suffices to check $(\mathrm{B4})_\varepsilon$ (see also Theorem 3.7 and Lemma 7.3). Noting that (7.12) yields

$$|u|^{p'(q-1)}_{L^{p'(q-1)}(\Omega)} \leq C\left(\varphi(u) + |u|_V^p + 1\right) \quad \text{for all } u \in X,$$

we can derive $(\mathrm{B4})_\varepsilon$ for any $\varepsilon > 0$ from (7.8). □

Furthermore, the following theorem is concerned with the existence of global (in time) solutions for small data u_0 and f.

THEOREM 7.6. *Let $T > 0$ and let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary $\partial\Omega$. Suppose that* (H1)–(H4) *and* (7.4) *are satisfied with $p \in [2, \infty)$, $a_1 \equiv 0$, $a_2 \equiv 0$, $a_3 \equiv 0$, $a_4 \equiv 0$, $j(\cdot, \cdot, 0) \equiv 0$ and $\phi(\cdot, \mathbf{0}) \equiv 0$. In addition, assume that*

$$m < p \ \ \text{and} \ \ \frac{m}{p'} + 1 < q. \tag{7.13}$$

Then, there exists $\delta > 0$ *independent of* T *such that for all* $f \in L^{p'}(0, T; L^{p'}(\Omega))$ *and* $u_0 \in W_0^{1,m}(\Omega)$ *satisfying* $\|f\|_\star + \int_\Omega \phi(x, u_0(x))dx < \delta$*, where* $\|f\|_\star$ *is given as in* (3.9)*, the initial-boundary value problem* (7.1)–(7.3) *admits at least one solution* u *on* $[0, T]$.

Proof. It follows that $\psi^t(0) = \int_\Omega j(x, t, 0)dx = 0$ for all $t \in [0, T]$. Since $a_1 \equiv 0$ and $a_2 \equiv 0$, we obtain $C_2 = 0$ and $m_1 \equiv 0$ in (A1) and (A2), respectively. Hence, due to Theorem 3.8 and Lemma 7.3, it suffices to prove (3.6)–(3.8).

By (H3) with $a_3 \equiv 0$, it follows that

$$\varphi(u) = \int_\Omega \phi(x, \nabla u(x))dx \geq C_6^{-1} \int_\Omega |\nabla u(x)|^m dx = C_6^{-1} |u|_X^m$$

for all $u \in D(\varphi)$. Hence, we have

$$|u|_V^p \leq C\varphi(u)^{p/m} \quad \text{for all } u \in D(\varphi). \tag{7.14}$$

Therefore, (3.7) follows with a non-decreasing function $\ell_5(s) = O(s^{\frac{p}{m}-1})$ from the fact that $m < p$. Moreover, combining (7.9) with $a_3 = a_4 \equiv 0$ and noting that $p'(q-1) > m$, we can obtain (3.8) with $\ell_6^\varepsilon(s) = O(s^{\frac{p'(q-1)}{m}-1})$ for any $\varepsilon > 0$.

Finally, we shall derive (3.7). Let $t \in (0, T)$ and let $u \in D(\partial_V \varphi)$ be arbitrary given. We can then derive

$$\langle \partial_V \varphi(u) + B(t, u), u \rangle \geq \varphi(u) - \varphi(0) - |B(t, u)|_{V^*} |u|_V \geq \varphi(u) - C\varphi(u)^{\frac{q}{m}}$$

from the fact that $\varphi(0) = \int_\Omega \phi(x, \mathbf{0})dx = 0$. Hence, since (7.13) implies

$$\sigma := \frac{q}{m} > \frac{1}{m}\left(\frac{m}{p'} + 1\right) > \frac{1}{m}\left(\frac{m}{m'} + 1\right) = 1,$$

we conclude that (3.6) holds with a non-decreasing function $\ell_4(s) = O(s^{\sigma-1})$. This completes our proof. □

7.2. Semilinear parabolic equations with gradient nonlinearities

We next deal with the following inclusion instead of (7.1),

$$\alpha(x, t, u_t(x, t)) - \sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) + h(x, t, u(x, t), \nabla u(x, t)) \ni f(x, t) \tag{7.15}$$

with functions $a_{ij} : \Omega \to \mathbb{R}$ $(i, j = 1, 2, \ldots, N)$ and $h : \Omega \times (0, T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. Then, (1.8) is reduced to (7.15) as a special case. To state our existence result, we introduce the following (H3)′ and (H4)′.

(H3)′(i) $a_{ij} \in W^{1,\infty}(\Omega)$ and $a_{ij} = a_{ji}$ for each $i, j = 1, 2, \ldots, N$.

(ii) There exists a constant $\lambda_0 > 0$ such that

$$\lambda_0|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \quad \text{for a.e. } x \in \Omega \text{ and all } \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N.$$

(H4)′ (i) There exist constants $q_1, q_2 > 1 + 1/p'$, $C_8 \geq 0$ and $a_5 \in L^1(\Omega \times (0, T))$ such that

$$|h(x, t, r, \mathbf{p})|^{p'} \leq C_8 \left(|r|^{p'(q_1-1)} + |\mathbf{p}|^{p'(q_2-1)}\right) + a_5(x, t)$$

for a.e. $(x, t) \in \Omega \times (0, T)$ and all $(r, \mathbf{p}) \in \mathbb{R} \times \mathbb{R}^N$.

(ii) The function $h = h(x, t, r, \mathbf{p})$ is a Carathéodory function in $\Omega \times (0, T) \times \mathbb{R} \times \mathbb{R}^N$ (i.e., measurable in (x, t) and continuous in $(r, \mathbf{p})$).

Then, we have:

THEOREM 7.7. *Let $T > 0$ and let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$. Suppose that* (H1), (H2), (H3)′, (H4)′ *and the following* (7.16) *are satisfied for some $p \in [2, \infty)$.*

$$2 \leq p < 2^* := \begin{cases} \frac{2N}{N-2} & \text{if } 2 < N, \\ +\infty & \text{if } 2 \geq N, \end{cases} \quad q_1 < 2^*, \quad q_2 < 2 + \frac{2}{N}. \tag{7.16}$$

Then, for all $f \in L^{p'}(0, T; L^{p'}(\Omega))$ and $u_0 \in H_0^1(\Omega)$, there exists $T_* = T_*\left(|u_0|_{H_0^1(\Omega)} + \|f\|_{L^{p'}(0,T;L^{p'}(\Omega))}\right) \in (0, T]$ such that the initial-boundary value problem {(7.15), (7.2), (7.3)*} admits at least one solution u on* $[0, T_*]$.

Proof. We set $V = L^p(\Omega)$, $H = L^2(\Omega)$ and set $X = H_0^1(\Omega)$ with the norm $|\cdot|_X := |\nabla \cdot|_{L^2}$. Then X is compactly embedded in V by (7.16). Moreover, we define the functional $\varphi : V \to [0, \infty]$ by

$$\varphi(u) := \begin{cases} \displaystyle\frac{1}{2}\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial u}{\partial x_j}(x)dx & \text{if } u \in X, \\ +\infty & \text{otherwise.} \end{cases} \tag{7.17}$$

Then by (H3)′, it follows that

$$\frac{1}{2}\lambda_0|u|_X^2 \leq \varphi(u) \leq C \max_{i,j} |a_{ij}|_{L^\infty(\Omega)}|u|_X^2 \quad \text{for all } u \in X,$$

which implies $D(\varphi) = X$ and (Φ1). Moreover, $\varphi|_X$ becomes Gâteaux differentiable in X and its derivative $\partial_X(\varphi|_X)(u)$ at $u \in X$ coincides with

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right) \tag{7.18}$$

in the sense of distribution. Hence, $\partial_V\varphi(u)$ also coincides with (7.18) in V^* for each $u \in D(\partial_V\varphi)$. Furthermore, thanks to Theorem 9.15 and Lemma 9.17 of [28], we can derive that $D(\partial_V\varphi) = W^{2,p'}(\Omega) \cap H_0^1(\Omega)$ and

$$|u|_{W^{2,p'}(\Omega)} \leq C|\partial_V\varphi(u)|_{V^*} \quad \text{for all } u \in D(\partial_V\varphi). \tag{7.19}$$

Let us check the assumptions of Theorem 3.6. By (i) of Lemma 7.3, (A1)–(A3) hold with ψ^t given by (7.5). Define the operator $B : (0, T) \times V \to V^*$ by

$$B(t, u) := h(\cdot, t, u(\cdot), \nabla u(\cdot)) \quad \text{for all } t \in (0, T) \text{ and } u \in D(B(t, \cdot))$$

with the domain $D(B(t, \cdot)) := \{u \in V;\ h(\cdot, t, u(\cdot), \nabla u(\cdot)) \in V^*\}$. By (H4)$'$, we then obtain

$$|B(t, u)|_{V^*}^{p'} \leq C_8 \left(|u|_{L^{p'(q_1-1)}(\Omega)}^{p'(q_1-1)} + |\nabla u|_{L^{p'(q_2-1)}(\Omega)}^{p'(q_2-1)}\right) + |a_5(\cdot, t)|_{L^1(\Omega)}$$
$$\text{for a.e. } t \in (0, T) \text{ and all } u \in D(B(t, \cdot)).$$

Here for the case where $p'(q_1 - 1) > 2^*$, Gagliardo–Nirenberg's inequality and (7.16) yield

$$|u|_{L^{p'(q_1-1)}(\Omega)} \leq C|u|_{W^{2,p'}(\Omega)}^{\theta_1} |\nabla u|_H^{1-\theta_1} \quad \text{for all } u \in D(\partial_V\varphi) \tag{7.20}$$

with some $\theta_1 \in (0, 1)$ satisfying $\theta_1(q_1 - 1) < 1$; for the case where $p'(q_1 - 1) \leq 2^*$, it follows that $|u|_{L^{p'(q_1-1)}(\Omega)} \leq C|\nabla u|_H$ for all $u \in X$. Moreover, for the case where $p'(q_2 - 1) > 2$, we also have

$$|\nabla u|_{L^{p'(q_2-1)}(\Omega)} \leq C|u|_{W^{2,p'}(\Omega)}^{\theta_2} |\nabla u|_H^{1-\theta_2} \quad \text{for all } u \in D(\partial_V\varphi) \tag{7.21}$$

with some $\theta_2 \in (0, 1)$ satisfying $\theta_2(q_2 - 1) < 1$; for the case where $p'(q_2 - 1) \leq 2$, we get $|\nabla u|_{L^{p'(q_2-1)}(\Omega)} \leq C|\nabla u|_H$ for all $u \in X$. Therefore by (7.19), for all $\varepsilon > 0$, there exists $C_\varepsilon \geq 0$ such that

$$|B(t, u)|_{V'}^{p'} \leq \varepsilon|\partial_V\varphi(u)|_{V^*}^{p'} + C_\varepsilon \ell_7(\varphi(u)) \quad \text{for all } u \in D(\partial_V\varphi) \text{ and a.e. } t \in (0, T)$$

with some non-decreasing function ℓ_7 on $[0, \infty)$, which implies (B1)$_\varepsilon$ for any $\varepsilon > 0$.

By Theorem 1.27 of [45], the Nemytskii mapping

$$[u, \mathbf{v}] \mapsto h(\cdot, t, u(\cdot), \mathbf{v}(\cdot)); \quad L^{p'(q_1-1)}(\Omega) \times L^{p'(q_2-1)}(\Omega; \mathbb{R}^N) \to V^*$$

is continuous for a.e. $t \in (0, T)$, and moreover, the function $t \mapsto h(\cdot, t, u(\cdot), \mathbf{v}(\cdot))$ becomes strongly measurable in $(0, T)$ with values in V^* for any fixed $u \in L^{p'(q_1-1)}(\Omega)$ and $\mathbf{v} \in L^{p'(q_2-1)}(\Omega; \mathbb{R}^N)$. Let $S \in (0, T]$ be fixed. Then by Theorem 1.43 of [45], the mapping $\mathcal{N} : L^{p'(q_1-1)}(0, S; L^{p'(q_1-1)}(\Omega)) \times L^{p'(q_2-1)}(0, S; L^{p'(q_2-1)}(\Omega; \mathbb{R}^N)) \to L^{p'}(0, S; V^*)$ given by

$$(\mathcal{N}(u, \mathbf{v}))(t) := h(\cdot, t, u(t)(\cdot), \mathbf{v}(t)(\cdot)) \quad \text{for a.e. } t \in (0, S)$$
$$\text{for all } [u, \mathbf{v}] \in L^{p'(q_1-1)}(0, S; L^{p'(q_1-1)}(\Omega)) \times L^{p'(q_2-1)}\left(0, S; L^{p'(q_2-1)}(\Omega; \mathbb{R}^N)\right)$$

also becomes continuous; particularly, (B3) follows from (7.19), (7.20) and (7.21).

We next check (B2). Let $\{u_n\}$ be a sequence such that

$$\sup_{t\in[0,S]} \varphi(u_n(t)) + \int_0^S |u_n'(t)|_H^p dt + \int_0^S |\partial_V \varphi(u_n(t))|_{V^*}^{p'} dt \text{ is bounded} \tag{7.22}$$

for all $n \in \mathbb{N}$. Then $\{u_n\}$ is bounded in $L^{p'}(0, S; W^{2,p'}(\Omega)) \cap L^\infty(0, S; H_0^1(\Omega)) \cap W^{1,p}(0, S; H)$ (see (7.19)). Moreover, it follows from (7.16) that $W^{2,p'}(\Omega)$ is compactly embedded in $L^{p'(q_1-1)}(\Omega)$ and also in $W^{1,p'(q_2-1)}(\Omega)$. Hence, Theorem 5 of [50] implies that $\{u_n\}$ is precompact in $L^{p'}(0, S; L^{p'(q_1-1)}(\Omega))$ and also in $L^{p'}(0, S; W^{1,p'(q_2-1)}(\Omega))$. Therefore, extracting a subsequence $\{n'\}$ of $\{n\}$ if necessary, and recalling (7.20) and (7.21), we can deduce that

$$u_{n'} \to u \quad \text{strongly in } L^{p'(q_1-1)}\left(0, S; L^{p'(q_1-1)}(\Omega)\right),$$
$$\nabla u_{n'} \to \nabla u \quad \text{strongly in } L^{p'(q_2-1)}\left(0, S; L^{p'(q_2-1)}(\Omega; \mathbb{R}^N)\right).$$

Hence, the continuity of $\mathcal{N}$ yields that

$$B(\cdot, u_{n'}(\cdot)) \to B(\cdot, u(\cdot)) \quad \text{strongly in } L^{p'}(0, S; V^*),$$

which implies (B2). Thus, by Theorem 3.6, we obtain our desired conclusion. □

We can also prove the existence of global (in time) solutions of the initial-boundary value problem for (7.15) as in Theorem 7.5.

THEOREM 7.8. *Let* $T > 0$ *and let* Ω *be a bounded domain in* $\mathbb{R}^N$ *with* C^2 *boundary* $\partial\Omega$. *Suppose that* (H1), (H2), (H3)′, (H4)′ *and* (7.16) *are satisfied for some* $p \in [2, \infty)$. *In addition, assume that*

$$q_1 \le p \quad \text{and} \quad q_2 \le \frac{2}{p'} + 1. \tag{7.23}$$

Then, for all $f \in L^{p'}(0, T; L^{p'}(\Omega))$ *and* $u_0 \in H_0^1(\Omega)$, *the initial-boundary value problem* {(7.15), (7.2), (7.3)} *admits at least one solution* u *on* $[0, T]$.

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