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Blowup of solutions to generalized Keller–Segel model

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Abstract. The existence and nonexistence of global in time solutions is studied for a class of equations generalizing the chemotaxis model of Keller and Segel. These equations involve Lévy diffusion operators and general potential type nonlinear terms.

1. Introduction

We consider in this paper the following nonlinear nonlocal evolution equation generalizing the well known Keller–Segel model of chemotaxis

$$\partial_t u + (-\Delta)^{\alpha/2} u + \nabla \cdot (uB(u)) = 0, \tag{1.1}$$

for $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$, where the anomalous diffusion is modeled by a fractional power of the Laplacian, $\alpha \in (1, 2)$, and the linear (vector) operator B is defined (formally) as

$$B(u) = \nabla((-\Delta)^{-\beta/2} u). \tag{1.2}$$

For $\beta \in (1, d]$, and $d \geq 2$, one may express the nonlocal nonlinearity in (1.1) using convolution operators since

$$B(u)(x) = s_{d,\beta} \int_{\mathbb{R}^d} \frac{x - y}{|x - y|^{d-\beta+2}} u(y)\,\mathrm{d}y, \tag{1.3}$$

with some $s_{d,\beta} > 0$, and the assumption $\beta > 1$ is needed for the convergence of this integral.

Of course, the choice $\alpha = 2$, $\beta = 2$ in (1.1)–(1.3) corresponds to the usual Keller–Segel system studied mainly in space dimensions $d = 1, 2, 3$. It is well known that if $\beta = 2$, the one-dimensional system (1.1)–(1.3) possesses global in time solutions not only in the case of classical Brownian diffusion $\alpha = 2$ but also in the fractional diffusion case $1 < \alpha < 2$ as was shown in Escudero [17]. On the other hand, there are many results on the nonexistence of global in time solutions with "large" initial data if $d \geq 2$

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and $\alpha = 2$, $\beta = 2$, see, e.g., [1,2,12,19,22]. Even if $d \geq 2$, $\alpha = 2$ and $1 < \beta \leq d$, there are results on the blowup of solutions with suitably chosen initial data, see [10] (caution: the notation in [10, Prop. 4.2] differs from that in the present paper). Let us finally recall that, in the limit case $\alpha = 2$, $\beta = d$, mass $M = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x$ is the critical parameter for the blowup, see [10, Prop. 4.1] and [14].

The usual method of proving the nonexistence of global in time nonnegative and nontrivial solutions, used in the abovementioned papers, consists in the study the evolution of the second moment of a solution $w_2(t) = \int_{\mathbb{R}^d} |x|^2 u(x,t)\,\mathrm{d}x$ and to show (via suitable differential inequalities) that $w_2(t)$ vanishes for some $t > 0$. The second moment of a typical solution to an evolution equation with fractional Laplacian cannot be finite, see, e.g., [13]. Hence, our goal in this paper is to generalize the classical virial method and to show the blowup of solution system (1.1)–(1.3) by studying moments of lower order $\gamma \in (1,2)$

$$w_\gamma = \int_{\mathbb{R}^d} |x|^\gamma u(x)\,\mathrm{d}x. \tag{1.4}$$

We show the extinction of a moment w_γ for some $T > 0$, which implies $\lim_{t \nearrow T} \|u(t)\|_p = \infty$ for every $p \in (1, \infty]$, see Remark 2.8. Thus, we mean this phenomenon as *blowup* for (1.1).

After this paper was completed, we discovered a recent preprint [24] where the authors show the blowup of solution to system (1.1)–(1.3) with fractional diffusion in the particular case $d = 2$ and $\beta = 2$. Our argument is different than that in [24], shorter, seems to be more direct, and applies in more general situations. Moreover, we are able to formulate a simple condition on the initial data which leads to the blowup in a finite time of the corresponding solution.

Notation. The L^p-norm of a Lebesgue measurable, real-valued function v defined on $\mathbb{R}^d$ is denoted by $\|v\|_p$. The constants (always independent of x, t) will be denoted by the same letter C, even if they may vary from line to line. Sometimes, we write, e.g., $C = C(*)$ when we want to emphasize the dependence of C on a parameter "$*$".

2. Main results

The crucial role in the approach in this paper is played by the following scaling property of system (1.1)–(1.3)

$$u^\lambda(x,t) = \lambda^{\alpha+\beta-2} u(\lambda x, \lambda^\alpha t) \quad \text{for all } \lambda > 0, \tag{2.1}$$

in the sense that if u is a solution to (1.1)–(1.3), then u^λ is so. In particular, in our construction of solutions to (1.1)–(1.3) we use the fact that the usual norm of the Lebesgue space $L^{d/(\alpha+\beta-2)}(\mathbb{R}^d)$ is invariant under the transformation $u_0(x) \mapsto \lambda^{\alpha+\beta-2} u_0(\lambda x)$ for every $\lambda > 0$.

THEOREM 2.1. *Assume that* $d \geq 2$, $\alpha \in (1, 2]$, *and* $\beta \in (1, d]$. *Let*

$$\max\left\{\frac{d}{\alpha+\beta-2}, \frac{2d}{d+\beta-1}\right\} < p \leq d.$$

i) *For every* $u_0 \in L^p(\mathbb{R}^d)$ *there exists* $T = T(\|u_0\|_p)$ *and the unique local in time mild solution* $u \in C([0, T], L^p(\mathbb{R}^d))$ *of system* (1.1)–(1.3) *with* u_0 *as the initial condition.*

ii) *There is* $\varepsilon > 0$ *such that for every* $u_0 \in L^{d/(\alpha+\beta-2)}(\mathbb{R}^d)$ *satisfying*

$$\|u_0\|_{d/(\alpha+\beta-2)} \leq \varepsilon \tag{2.2}$$

there exists a global in time mild solution $u \in C([0, \infty), L^p(\mathbb{R}^d))$ *of system* (1.1)–(1.3) *with* u_0 *as the initial condition.*

Moreover, if $u_0(x) \geq 0$, *then the solution* u *in either* i) *or* ii) *above is nonnegative. Finally, if* $u_0 \in L^1(\mathbb{R}^d)$, *then the corresponding solution conserves mass*

$$\int_{\mathbb{R}^d} u(x, t)\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x \equiv M. \tag{2.3}$$

Recall that $\alpha > 1$ is a usual assumption ([10, Th. 2.2], [8,9]) which permits us to control locally the nonlinearity in (1.1)–(1.3) by the linear term.

The results stated above can be easily generalized for equations with general Lévy diffusion operators considered in [8,9] but we do not pursue this question here. We refer the reader to the abovementioned papers, as well as [10], for physical motivations to study such equations. On the other hand, motivations stemming from probability theory (*propagation of chaos* property for interacting particle systems) can be found in, e.g., [7].

REMARK 2.2. For $\alpha \in (1, 2)$ and $\beta > 1$ satisfying $\alpha + \beta > d + 2$, it can be shown that local in time solutions can be continued to the global in time ones, see [10, Th. 3.2]. However, in [10], another approach (via weak solutions) is used to construct solutions of system (1.1)–(1.3).

The proof of Theorem 2.1 on local and global solutions to (1.1)–(1.3) follows a more or less standard reasoning which we sketch in Sect. 3. Our main goal, however, is to prove the finite time blowup of *nonnegative* solutions to the nonlocal system (1.1)–(1.3) but *a priori* less regular than those in Theorem 2.1.

THEOREM 2.3. *Assume that* $d \geq 2$. *The nonnegative solution of* (1.1)–(1.3) *with a nonnegative and nonzero initial condition* $u(x, 0) = u_0(x)$ *cannot exist globally in time in each of the following cases:*

i) (large mass) *for* $\alpha = 2$, $\beta = d$, $u_0 \in L^1(\mathbb{R}^d, (1 + |x|^2)\,\mathrm{d}x)$, *and if*

$$M = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x > \frac{2d}{s_{d,\beta}},$$

with the constant $s_{d,\beta}$ *defined in* (1.3); *in particular:* $s_{2,2} = \frac{1}{2\pi}$ *so that the threshold value of* M *is* 8π *if* $d = 2$;

ii) (high concentration) *for* $\alpha \in (1, 2]$ *and* $\beta \in (1, d]$ *satisfying* $\alpha + \beta < d + 2$, $u_0 \in L^1(\mathbb{R}^d, (1 + |x|^\gamma)\,\mathrm{d}x)$ *for some* $\gamma \in (1, \alpha)$, *and if*

$$\frac{\int_{\mathbb{R}^d} |x|^\gamma u_0(x)\,\mathrm{d}x}{\int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x} \le c \left(\int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x \right)^{\frac{\gamma}{d+2-\alpha-\beta}} \tag{2.4}$$

for certain (sufficiently small) constant $c > 0$ *independent of* u_0.

The result stated in i) is essentially contained in [14]. The condition for blowup in the form (2.4) appeared already in [2] and [26], of course, for $\alpha = 2$ only. Note that i) is a limit case of ii). Indeed, (2.4) written as

$$\left(\frac{\int_{\mathbb{R}^d} |x|^\gamma u_0(x)\,\mathrm{d}x}{\int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x} \right)^{\frac{d+2-\alpha-\beta}{\gamma}} \le cM,$$

becomes a condition on (sufficiently large) mass: $1 \le cM$, when $(\alpha + \beta) \nearrow (d + 2)$.

We have to emphasize that Theorem 2.3(ii) contains a result which is new even for the classical parabolic-elliptic Keller–Segel model (i.e., equations (1.1)–(1.3) with $\alpha = \beta = 2$). Indeed, for $d \ge 3$, the conditions from part ii) of Theorem 2.3 guarantee the blow up in a finite time if the moment of order γ of the initial condition is finite for some $\gamma \in (1, 2)$. All other known proofs of the blowup required just $\gamma = 2$. The following result for $d = 2$ is also an immediate consequence of Theorem 2.3(ii).

COROLLARY 2.4. *Assume that* $\alpha = \beta = d = 2$ *in equations* (1.1)–(1.3). *Suppose that there exists* $\gamma \in (1, 2)$ *such that* $0 \le u_0 \in L^1(\mathbb{R}^2, (1 + |x|^\gamma)\,\mathrm{d}x)$. *There exists* $M_\gamma > 0$ *such that if*

$$\int_{\mathbb{R}^2} u_0(x)\,\mathrm{d}x > M_\gamma,$$

then the nonnegative solution of (1.1)–(1.3) *with the initial condition* $u(x, 0) = u_0(x)$ *ceases to exist in a finite time.*

It is conjectured that $M_2 = M_\gamma = 8\pi$ for every $\gamma \in (1, 2)$ and we expect that it can be shown by a careful analysis of constants appearing in inequalities (4.6)–(4.10) and in (4.18)–(4.20) in the proof of Theorem 2.3. However, the assumptions in Corollary 2.4 are not optimal: nonnegative solutions to equations (1.1)–(1.3) with $\alpha = \beta = d = 2$ blow up in finite time under the assumption that their mass is larger than 8π and no moment condition imposed on the initial data is necessary, see [22, Th. 1.5] and [4, Prop. 2.2, Th. 3.1] in the radially symmetric case for a detailed presentation.

REMARK 2.5. Due to the translation invariance of problem (1.1)–(1.3), the conditions on moments can be imposed on the quantity

$$\inf_{x_0 \in \mathbb{R}^d} \int_{\mathbb{R}^d} |x - x_0|^\gamma u_0(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} |x - \bar{x}|^\gamma u_0(x)\,\mathrm{d}x,$$

where $\bar{x} = (\int_{\mathbb{R}^d} x u_0(x)\,\mathrm{d}x)/(\int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x)$ is the center of mass of u_0.

REMARK 2.6. Let us observe that the assumptions (2.2) and (2.4) are in a sense complementary due to the following elementary inequality involving the L^p-norms, mass $M = \int_{\mathbb{R}^d} u(x)\,\mathrm{d}x$, and the moment $w_\gamma = \int_{\mathbb{R}^d} |x|^\gamma u(x)\,\mathrm{d}x$ of a nonnegative function u

$$\|u\|_p \ge CM\left(\frac{M}{w_\gamma}\right)^{\frac{d}{\gamma}\left(1-\frac{1}{p}\right)}. \tag{2.5}$$

To prove (2.5) observe that $w_\gamma \ge R^\gamma \int_{\mathbb{R}^d\setminus B_R(0)} u(x)\,\mathrm{d}x$, so that

$$\int_{B_R(0)} u(x)\,\mathrm{d}x = M - \int_{\mathbb{R}^d\setminus B_R(0)} u(x)\,\mathrm{d}x \ge M - R^{-\gamma} w_\gamma.$$

Multiplying both sides of this inequality by $R^{d\left(1-\frac{1}{p}\right)}$ we get with $C = \omega_d^{1-\frac{1}{p}}$

$$\begin{aligned} C\|u\|_p &= \left(\int_{\mathbb{R}^d} u^p(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{B_R(0)} \mathrm{d}x\right)^{1-\frac{1}{p}} R^{d\left(\frac{1}{p}-1\right)} \\ &\ge R^{d\left(\frac{1}{p}-1\right)} \int_{B_R(0)} u(x)\,\mathrm{d}x \ge R^{d\left(\frac{1}{p}-1\right)} M - R^{-\gamma} w_\gamma. \end{aligned}$$

Taking the optimal R, i.e., $R^\gamma = Cw_\gamma/M$, we get (2.5).

Now, it is clear that if condition (2.4) for blowup is satisfied for some $\gamma \in (1,2]$ and a suitable constant c, then for $p = d/(\alpha+\beta-2)$ appearing in Theorem 2.1.ii, we have $\|u\|_p \ge CMM^{-\frac{\gamma}{d+2-\alpha-\beta}\frac{d}{\gamma}\left(1-\frac{1}{p}\right)} = C$, so condition (2.2) for global existence is violated for sufficiently small $\varepsilon > 0$. *Vice versa,* if (2.2) is satisfied, (2.4) cannot be true with small constants c.

REMARK 2.7. Note that one can prove in a similar way the inequality

$$\|u\|_{M^p} \ge CM\left(\frac{M}{w_\gamma}\right)^{\frac{d}{\gamma}\left(1-\frac{1}{p}\right)}, \tag{2.6}$$

where the Morrey space norm $\|u\|_{M^p}$ $(\le C\|u\|_p)$ is defined as

$$\sup_{R>0,\, x_0\in\mathbb{R}^d} R^{d\left(\frac{1}{p}-1\right)} \int_{B_R(x_0)} |u(x)|\,\mathrm{d}x,$$

see [2, (15)] in the particular case $\gamma = 2$. The scale of Morrey spaces is relevant to study mean field type problems related to the Keller–Segel model, since the Morrey norms are particularly well suited to measure the (local) concentration of densities, see, e.g., [3].

REMARK 2.8. We prove Theorem 2.3 and Corollary 2.4 by showing the extinction in a finite time of the function $w(t) = \int_{\mathbb{R}^d} \varphi_\gamma(x)u(x,t)\,\mathrm{d}x$ where $\varphi_\gamma(x)$ is smooth and behaves like $|x|^\gamma$ with some $\gamma \in (1,\alpha)$ for large $|x|$, see (4.1) below. By (2.5) and (2.3) this might be interpreted as the blowup of certain norms (in particular, L^p-norms for each $p \in (1, \infty]$) of nonnegative solutions but we stress on the fact that a nonnegative solution could cease to exist even *before* the critical time $T < \infty$ suggesting by (2.5) that $\lim_{t \nearrow T} \|u(t)\|_p = \infty$.

Note that if $\alpha < 2$, we cannot expect the existence of higher order moments w_γ defined in (1.4) with $\gamma \geq \alpha$. Indeed, even for the linear equation $\partial_t v + (-\Delta)^{\alpha/2} v = 0$, the fundamental solution $p_\alpha(x, t)$ behaves like $p_\alpha(x, t) \sim \left(t^{d/\alpha} + |x|^{d+\alpha}/t\right)^{-1}$, and therefore the moment (1.4) with $\gamma \geq \alpha$ cannot be finite, see [13] and references given there. Thus, we cannot apply the usual reasoning which involves an analysis of the evolution of the second moment w_2 of the solution because the integral defining w_2 may diverge.

We recall in Proposition 3.4 a result showing that the moment (1.4) is finite for a large class of initial conditions in the case of $\gamma < \alpha$.

3. Existence of solutions

We are going to construct solutions to system (1.1)–(1.3) via the following integral formulation

$$u(t) = S_\alpha(t)u_0 - \int_0^t \nabla \cdot S_\alpha(t-\tau)\,(u(\tau)Bu(\tau))\,\mathrm{d}\tau, \tag{3.1}$$

i.e., we consider *mild* solutions. Here $S_\alpha(t)u_0 = p_\alpha(t) * u_0$ is the solution to the linear Cauchy problem

$$\partial_t v + (-\Delta)^{\alpha/2} v = 0, \quad v(x, 0) = u_0, \tag{3.2}$$

and $p_\alpha(x, t)$ is the fundamental solution of (3.2) which can be represented via the Fourier transform $\widehat{p_\alpha}(\xi, t) = e^{-t|\xi|^\alpha}$. In particular,

$$p_\alpha(x, t) = t^{-d/\alpha} P_\alpha(xt^{-1/\alpha}),$$

where P_α is the inverse Fourier transform of $e^{-|\xi|^\alpha}$, see [18, Ch. 3] and [13] for more detail. It is well known that for every $\alpha \in (0, 2)$ the function P_α is smooth, nonnegative, and satisfies the (optimal) estimates

$$0 < P_\alpha(x) \leq C(1+|x|)^{-(\alpha+d)} \quad \text{and} \quad |\nabla P_\alpha(x)| \leq C(1+|x|)^{-(\alpha+d+1)} \tag{3.3}$$

for a constant C and all $x \in \mathbb{R}^d$. Hence, it follows immediately from the Young inequality for the convolution and from the self-similar form of the kernel $p_\alpha(x, t)$ that for every $1 \leq q \leq p \leq \infty$ there exists $C = C(p, q, \alpha) > 0$ such that

$$\|S_\alpha(t)u_0\|_p \leq C t^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_0\|_q \tag{3.4}$$

and

$$\|\nabla S_\alpha(t)u_0\|_p \leq C t^{-\frac{d}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{\alpha}} \|u_0\|_q \tag{3.5}$$

for every $u_0 \in L^q(\mathbb{R}^d)$ and all $t > 0$.

The construction of solution to (3.1) in L^p spaces is based on the following abstract result, see, e.g., [23], [25].

LEMMA 3.1. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space and $H : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ a bounded bilinear form satisfying $\|H(x_1, x_2)\|_{\mathcal{X}} \le \eta \|x_1\|_{\mathcal{X}} \|x_2\|_{\mathcal{X}}$ for all $x_1, x_2 \in \mathcal{X}$ and a constant $\eta > 0$. Then, if $0 < \varepsilon < 1/(4\eta)$ and if $y \in \mathcal{X}$ is such that $\|y\| < \varepsilon$, the equation $u = y + H(u, u)$ has a solution in $\mathcal{X}$ such that $\|u\|_{\mathcal{X}} \le 2\varepsilon$. This solution is the only one in the ball $\bar{B}(0, 2\varepsilon)$.*

We skip an easy proof of this lemma which is a direct consequence of the Banach fixed point theorem.

Sketch of the proof of Theorem 2.1. The proof consists in constructing solutions to the "quadratic" equation (3.1) using Lemma 3.1 with $y = S_\alpha(t)u_0$ and with the bilinear form

$$H(u, v) = -\int_0^t \nabla \cdot S_\alpha(t-\tau)\,(u(\tau)Bv(\tau))\,\mathrm{d}\tau. \tag{3.6}$$

Local existence of solutions It suffices to obtain estimates required by Lemma 3.1 in the Banach space $\mathcal{X}_T^p = C([0, T], L^p(\mathbb{R}^d))$ supplemented with the usual norm $\|u\|_{\mathcal{X}_T^p} = \sup_{t\in[0,T]} \|u(t)\|_p$.

By inequality (3.4), we immediately obtain $\|S_\alpha(\cdot)u_0\|_{\mathcal{X}_T^p} \le \|u_0\|_p$.

Combining the definition (1.3) of the operator B with the Hardy–Littlewood–Sobolev inequality we obtain

$$\|B(u)\|_q \le s_{d,\beta} \big\| |\cdot|^{-d+\beta-1} * u \big\|_q \le C\|u\|_p \tag{3.7}$$

for every $1 < p < q < \infty$ satisfying $\frac{1}{p} - \frac{\beta-1}{d} = \frac{1}{q}$. Moreover, inequality (3.5) and the Hölder inequality lead to

$$\begin{aligned} \|\nabla S_\alpha(t-\tau)(uB(v))\|_p &\le C(t-\tau)^{-\frac{1}{\alpha}-\frac{d}{\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)} \|uB(v)\|_r \\ &\le C(t-\tau)^{-\frac{1}{\alpha}-\frac{d}{\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)} \|u\|_p \|B(v)\|_q. \end{aligned} \tag{3.8}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Hence, by inequalities (3.7) and (3.8), there exists a constant C such that for all $u, v \in \mathcal{X}_T^p$, the bilinear form (3.6) satisfies

$$\begin{aligned} \|H(u, v)\|_{\mathcal{X}_T^p} &\le C \sup_{t\in[0,T]} \int_0^t (t-\tau)^{-\frac{1}{\alpha}-\frac{d}{\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)} \|u(\tau)\|_p \|v(\tau)\|_p \,\mathrm{d}\tau \\ &\le CT^{1-\frac{1}{\alpha}-\frac{d}{\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)} \|u\|_{\mathcal{X}_T^p} \|v\|_{\mathcal{X}_T^p}. \end{aligned} \tag{3.9}$$

In estimates (3.9), we have used the relations $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} = \frac{2}{p} - \frac{\beta-1}{d}$, and we assume that

- $p > d/(\alpha+\beta-2)$ in order to have $1/\alpha + (d/\alpha)(1/r - 1/p) < 1$;
- $p > 2d/(d+\beta-1)$ to guarantee that $r > 1$;
- $p \le d/(\beta-1)$ to be sure that $r \le p$.

Choosing a sufficiently small $T > 0$ in (3.9) we complete the proof of Theorem 2.1.i by an application of Lemma 3.1.

Global in time solutions Here, the reasoning is completely analogous: using inequalities (3.4), (3.5), (3.7)–(3.9) we estimate the bilinear form (3.6) in the Banach space

$$\begin{aligned}\mathcal{X}^p = \; & C([0,\infty), L^{d/(\alpha+\beta-2)}(\mathbb{R}^d)) \\ & \cap \left\{ u \in C((0,\infty), L^p(\mathbb{R}^d)) \; : \; \sup_{t>0} t^{\frac{d}{\alpha}\left(\frac{1}{p}-\frac{\alpha+\beta-2}{d}\right)} \|u(t)\|_p < \infty \right\}\end{aligned}$$

supplemented with the norm

$$\|u\|_{\mathcal{X}^p} = \sup_{t>0} \|u(t)\|_{d/(\alpha+\beta-2)} + \sup_{t>0} t^{\frac{d}{\alpha}\left(\frac{1}{p}-\frac{\alpha+\beta-2}{d}\right)} \|u(t)\|_p.$$

We skip further details of this standard reasoning.

Nonnegativity property To prove that $u_0 \ge 0$ implies $u(t) \ge 0$ for sufficiently regular solutions, it suffices to study the function $u^-(x,t) = \max\{-u(x,t), 0\}$ and to follow the arguments either from [9, Prop. 3.1] or from [16, Prop. 2] in order to show that $u^-(x,t) \equiv 0$. Here, we do not give a detailed presentation because the proof from [24, Lemma 2.7] can be rewritten in this more general case. Thus, the nonnegativity property is natural for solutions of (1.1), and our results show that loss of regularity (*blowup*) and loss of nonnegativity are intimately connected.

Conservation of the integral If $u_0 \in L^1(\mathbb{R}^d)$, one should repeat the fixed point argument from i) (or from ii)) in the space $\mathcal{X}_T^p \cap C([0,T], L^1(\mathbb{R}^d))$ (in $\mathcal{X}^p \cap C([0,\infty), L^1(\mathbb{R}^d))$, resp.) to have $u(t) \in L^1(\mathbb{R}^d)$ for all $t \in [0,T]$ ($t \in (0,\infty)$, resp.). Next, it suffices to integrate over $\mathbb{R}^d$ the both sides of equation (3.1). Using the following consequences of the Fubini theorem

$$\int_{\mathbb{R}^d} p_\alpha(t) * u_0(x)\, \mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\, \mathrm{d}x$$

and

$$\int_{\mathbb{R}^d} \nabla p_\alpha(t) * v(x)\, \mathrm{d}x = 0 \quad \text{for every } v \in L^1(\mathbb{R}^d),$$

we conclude that $\int_{\mathbb{R}^d} u(x,t)\, \mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\, \mathrm{d}x$ for every $t \in [0,T]$. □

REMARK 3.2. The reasoning from the proof of Theorem 2.1 follows the lines of the usual proof of the local in time existence (as well as the global in time existence for small initial conditions) of solutions to system (1.1)–(1.3) with $\alpha = 2$. Since that argument is based on an integral representation analogous to that in (3.1) and on counterparts of decay estimates from (3.4) to (3.5), it can be easily adopted to the more general case of $\alpha \in (1, 2]$. Examples of such a reasoning applied to various semilinear models and realized in miscellaneous Banach spaces can be found in [3,5,6,11,20,21,23,25].

Next, we recall weighted estimates of solutions to the linear Cauchy problem (3.2) which have been proved in, e.g., [5,13]. In the following, we use the weighted L^∞ spaces

$$L^\infty_\vartheta(\mathbb{R}^d) = \left\{ v \in L^\infty(\mathbb{R}^d) \ : \ \|v\|_{L^\infty_\vartheta} \equiv \operatorname{ess\,sup}_{x\in\mathbb{R}^d} (1+|x|)^\vartheta |v(x)| < \infty \right\}$$

for a fixed $\vartheta \geq 0$.

LEMMA 3.3. ([13, Lemma 3.1]) *Assume that* $v_0 \in L^\infty_{\alpha+d}(\mathbb{R}^d)$. *There exists* $C > 0$ *independent of* v_0 *and* t *such that*

$$\|S_\alpha(t)v_0\|_{L^\infty_{\alpha+d}} \leq C(1+t)\|v_0\|_{L^\infty_{\alpha+d}} ,$$
$$\|\nabla S_\alpha(t)v_0\|_{L^\infty_{\alpha+d}} \leq Ct^{-1/\alpha}\|v_0\|_{L^\infty_{\alpha+d}} + Ct^{1-1/\alpha}\|v_0\|_1 ,$$

With these estimates, we are in a position to construct solutions to system (1.1)–(1.3) also in the weighted space $L^\infty_{\alpha+d}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

PROPOSITION 3.4. *Let* $\alpha \in (1, 2]$ *and* $b \in (1, d]$. *Assume that* u *is the solution of system* (1.1)–(1.3)*, constructed in Theorem* 2.1*, supplemented with the initial condition* $u_0 \in L^\infty_{\alpha+d}(\mathbb{R}^d)$. *Then* $u \in C([0, T], L^\infty_{\alpha+d}(\mathbb{R}^d))$. *In particular, for each* $\gamma < \alpha$ *we have* $\int_{\mathbb{R}^d} |x|^\gamma u(x, t)\,\mathrm{d}x < \infty$.

Sketch of proof. Here, it suffices to modify slightly the argument from [13, Proof of Prop. 3.3.i] written the case of a convection equation with the fractional Laplacian. In that reasoning (as well as in the above proof of Theorem 2.1), we construct solutions to equation (3.1) by applying Lemma 3.1 with the Banach space $\mathcal{X}_T = C([0, T], L^\infty_{\alpha+d}(\mathbb{R}^d))$. Obviously, $y = S_\alpha(\cdot)u_0 \in \mathcal{X}_T$ by Lemma 3.3. Next, we show the following estimate of the bilinear form from (3.6)

$$\|H(u, v)\|_{\mathcal{X}_T} \leq CT^{1-1/\alpha}\|u|_{\mathcal{X}_T}\|v\|_{\mathcal{X}_T}$$

for all $u, v \in \mathcal{X}_T$ and a constant C independent of u, v. Here, in order to estimate the singular integral operator (1.3) in the spaces $L^\infty_\vartheta(\mathbb{R}^d)$), one should follow the reasoning from [5, Sec. 2]. Let us omit other details of this classical argument. □

4. Blowup of solutions

The main role in our proof of the blowup of solutions to (1.1)–(1.3) is played by the following smooth nonnegative weight function on $\mathbb{R}^d$

$$\varphi(x) = \varphi_\gamma(x) \equiv (1+|x|^2)^{\gamma/2} - 1 \tag{4.1}$$

with $\gamma \in (1, 2]$. Since $(1 + |x|^2)^\gamma \le (1 + |x|^\gamma)^2$, we have for each $\varepsilon > 0$, suitably chosen $C(\varepsilon) > 0$, and for every $x \in \mathbb{R}^d$

$$\varphi(x) \le |x|^\gamma \le \varepsilon + C(\varepsilon)\varphi(x). \tag{4.2}$$

Next, let us state two auxiliary results concerning the weight function φ which will be used in the proof of Theorem 2.3. Here, for a given $\varphi \in C^2(\mathbb{R}^d)$, we denote by $D^2\varphi$ its Hessian matrix. Moreover, the scalar product of vectors $x, y \in \mathbb{R}^d$ is denoted by $x \cdot y$. If A is either a vector or a matrix, the expression $|A|$ means its Euclidean norm.

LEMMA 4.1. *Let $\alpha \in (1, 2)$, $\gamma \in (1, \alpha)$, and φ be defined by (4.1). Then*

$$(-\Delta)^{\alpha/2}\varphi \in L^\infty(\mathbb{R}^d). \tag{4.3}$$

Proof. First note that by a direct computation we have

$$\nabla\varphi(x) = \gamma(1 + |x|^2)^{\frac{\gamma}{2}-1}x \tag{4.4}$$

and

$$\partial_{x_j}\partial_{x_i}\varphi(x) = \Big(\gamma(1 + |x|^2)\delta_{ij} - \gamma(2-\gamma)x_i x_j\Big)(1 + |x|^2)^{\frac{\gamma}{2}-2}. \tag{4.5}$$

In particular, for every $R > 0$ there exists $C(R, \gamma) > 0$ such that for all $|x| \ge R$ we have

$$|\nabla\varphi(x)| \le C(R, \gamma)|x|^{\gamma-1} \quad \text{and} \quad |D^2\varphi(x)| \le C(R, \gamma)|x|^{\gamma-2}. \tag{4.6}$$

Now, we apply the following Lévy–Khintchine integral representation of the fractional Laplacian

$$(-\Delta)^{\alpha/2}\varphi(x) = C(d, \alpha)\int_{\mathbb{R}^d} \frac{\varphi(x+y) - \varphi(x) - \nabla\varphi(x)\cdot y}{|y|^{d+\alpha}}\,\mathrm{d}y \tag{4.7}$$

(with a suitable constant $C(d, \alpha)$) which is valid for every $\alpha \in (1, 2)$, see, e.g., [16, Th. 1] for a detailed proof of that version of Lévy–Khintchine formula. Using the Taylor expansion and estimates (4.6) one can immediately show that $(-\Delta)^{\alpha/2}\varphi(x)$ is well defined for every $x \in \mathbb{R}^d$ and, moreover, $\sup_{|x|\le R} |(-\Delta)^{\alpha/2}\varphi(x)| < \infty$ for each $R > 0$.

In order to obtain an estimate uniform in $x \in \mathbb{R}^d$, we assume that $|x| \ge 1$ and we shall estimate the integral on the right-hand side of (4.7) for $|y| \le |x|/2$ and $|y| > |x|/2$, separately.

If $|y| \le |x|/2$, by the Taylor formula and the second inequality in (4.6), we obtain

$$\begin{aligned} |\varphi(x+y) - \varphi(x) - \nabla\varphi(x)\cdot y| &\le \frac{1}{2}|y|^2\int_0^1 |D^2\varphi(x+sy)|\,\mathrm{d}s \\ &\le C|y|^2\int_0^1 |x+sy|^{\gamma-2}\,\mathrm{d}s. \end{aligned}$$

Since $|y| \leq |x|/2$ and $s \in [0, 1]$ we can estimate

$$|x + sy| \geq ||x| - s|y|| \geq \big||x| - |y|\big| \geq \frac{1}{2}|x|.$$

Consequently, for $\gamma - 2 < 0$, we obtain

$$\begin{aligned}&\left|\int_{|y|\leq|x|/2} \frac{\varphi(x+y) - \varphi(x) - \nabla\varphi(x)\cdot y}{|y|^{d+\alpha}}\,\mathrm{d}y\right|\\ &\quad\leq C|x|^{\gamma-2}\int_{|y|\leq|x|/2}\frac{\mathrm{d}y}{|y|^{d+\alpha-2}} = C|x|^{\gamma-\alpha}\end{aligned} \tag{4.8}$$

for all $|x| \geq 1$ and a constant $C > 0$ independent of x.

If $|y| \geq |x|/2$ and $|x| \geq 1$, we combine the first inequality from (4.6) (remember that $\gamma - 1 > 0$) with the Taylor expansion to show

$$|\varphi(x+y) - \varphi(x)| \leq |y| \int_0^1 |\nabla\varphi(x+sy)|\,\mathrm{d}s \leq C|y|\left(|x|^{\gamma-1} + |y|^{\gamma-1}\right).$$

Hence,

$$\begin{aligned}&\left|\int_{|y|>|x|/2} \frac{\varphi(x+y) - \varphi(x) - \nabla\varphi(x)\cdot y}{|y|^{d+\alpha}}\,\mathrm{d}y\right|\\ &\quad\leq C\left(|x|^{\gamma-1}\int_{|y|>|x|/2}\frac{\mathrm{d}y}{|y|^{d+\alpha-1}} + \int_{|y|>|x|/2}\frac{\mathrm{d}y}{|y|^{d+\alpha-\gamma}}\right) = C|x|^{\gamma-\alpha}\end{aligned} \tag{4.9}$$

for all $|x| \geq 1$ and a constant $C > 0$ independent of x.

Finally, inequalities (4.8) and (4.9) complete the proof because $\gamma < \alpha$. □

REMARK 4.2. Note that above we have, in fact, proved that

$$\sup_{x\in\mathbb{R}^d}\left(1 + |x|^{\alpha-\gamma}\right)\left|(-\Delta)^{\alpha/2}\varphi(x)\right| < \infty \quad \text{for every } \gamma \in (1, \alpha).$$

LEMMA 4.3. *For every* $\gamma \in (1, 2]$, *the function* φ *defined in* (4.1) *is locally uniformly convex on* $\mathbb{R}^d$. *Moreover, there exists* $K = K(\gamma)$ *such that the following inequality*

$$(\nabla\varphi(x) - \nabla\varphi(y))\cdot(x - y) \geq \frac{K|x-y|^2}{1 + |x|^{2-\gamma} + |y|^{2-\gamma}} \tag{4.10}$$

holds true for all $x,\ y \in \mathbb{R}^d$.

Proof. Using the explicit expression for the Hessian matrix of φ in (4.5) we obtain

$$D^2\varphi(x)y\cdot y = \frac{\gamma(1+|x|^2)|y|^2 - \gamma(2-\gamma)\left(\sum_i x_i^2y_i^2 + \sum_{i\neq j} x_ix_jy_iy_j\right)}{(1+|x|^2)^{2-\gamma/2}} \tag{4.11}$$

for every $x, y \in \mathbb{R}^d$. Now, by the elementary inequality $x_i x_j y_i y_j \leq \frac{1}{2}(x_i^2 y_j^2 + x_i^2 y_j^2)$ we immediately obtain

$$\sum_{i \neq j} x_i x_j y_i y_j \leq \sum_{i \neq j} x_i^2 y_j^2.$$

Consequently,

$$\sum_i x_i^2 y_i^2 + \sum_{i \neq j} x_i x_j y_i y_j \leq \sum_{i,j} x_i^2 y_j^2 = |x|^2 |y|^2. \tag{4.12}$$

Since $\gamma \in (1, 2]$, applying estimate (4.12) to (4.11) we get the inequality

$$D^2 \varphi(x) y \cdot y \geq \frac{\gamma(1 + (\gamma - 1)|x|^2)|y|^2}{(1 + |x|^2)^{2-\gamma/2}}$$

which leads directly to the estimate from below

$$D^2 \varphi(x) y \cdot y \geq \frac{(\gamma - 1)|y|^2}{(1 + |x|^2)^{1-\gamma/2}}. \tag{4.13}$$

Finally, it follows from the integration of the second derivative of φ that

$$(\nabla\varphi(x) - \nabla\varphi(y)) \cdot (x - y) = \int_0^1 D^2 \varphi\,(x + s(y - x))\,(x - y) \cdot (x - y)\,\mathrm{d}s.$$

Hence, using inequality (4.13) and the estimate

$$(1 + |x + s(y - x)|^2)^{1-\gamma/2} \leq C(1 + |x|^{2-\gamma} + |y|^{2-\gamma}),$$

valid for all $x, y \in \mathbb{R}^d$, $s \in [0, 1]$ and a constant $C > 0$ independent of x, y, s, one can easily complete the proof of Lemma 4.3. □

Proof of Theorem 2.3. We consider the function

$$w = w(t) \equiv \int_{\mathbb{R}^d} \varphi(x)\, u(x, t)\, \mathrm{d}x,$$

where φ is defined in (4.1) and $1 < \gamma < \alpha$. Note that, in view of inequalities (4.2), the quantity w is essentially equivalent to the moment w_γ of order γ of the solution u. Moreover, it satisfies the relation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} w &= -\int_{\mathbb{R}^d} (-\Delta)^{\alpha/2} u(x, t)\, \varphi(x)\, \mathrm{d}x - \int_{\mathbb{R}^d} u(x, t) Bu(x, t) \cdot \nabla\varphi(x)\, \mathrm{d}x \\
&= -\int_{\mathbb{R}^d} (-\Delta)^{\alpha/2} \varphi(x)\, u(x, t)\, \mathrm{d}x \\
&\quad - \frac{s_{d,\beta}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\nabla\varphi(x) - \nabla\varphi(y)) \cdot (x - y) \frac{u(x, t) u(y, t)}{|x - y|^{d-\beta+2}}\, \mathrm{d}x\, \mathrm{d}y
\end{aligned} \tag{4.14}$$

after using the definition of the form B in (1.3) and the symmetrization of the double integral. This computation resembles the usual proof of blowup involving the second moments, cf. [2,10,14,15].

i) For $\alpha = 2 = \gamma$ (hence for $\varphi(x) = |x|^2$) and for $\beta = d$, the equality (4.14) can be rewritten as follows

$$\frac{\mathrm{d}}{\mathrm{d}t} w(t) = 2dM - s_{d,\beta} M^2.$$

Evidently, for $M > 2d/s_{d,\beta}$, this implies the inequality $w(T) < 0$ for some $0 < T < \infty$, a contradiction with the global existence of nonnegative solutions. Thus, we recover the result in [10, Prop. 4.1] refined in [14,15].

ii) For $1 < \beta \le d$ and fixed $M > 0$, we are going to use the following simple identity

$$\begin{aligned} M^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x,t)u(y,t)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x,t)u(y,t) \frac{|x-y|^\nu}{\left(1+|x|^{2-\gamma}+|y|^{2-\gamma}\right)^\delta} \\ &\quad \times \frac{\left(1+|x|^{2-\gamma}+|y|^{2-\gamma}\right)^\delta}{|x-y|^\nu}\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

with some $\nu > 0$ and $\delta > 0$. We apply now the Hölder inequality with the exponents $p > 1$ and $p' = \frac{p}{p-1}$ chosen so that

$$\nu p = d - \beta, \quad \delta p = 1, \quad \nu p' + (2-\gamma)\delta p' = \gamma. \tag{4.15}$$

Of course, such a choice of ν, δ, p is possible whenever $\beta < d$ and $\gamma < 2$ because we only need $d - \beta + 2 - \gamma = \gamma(p-1)$. If $\beta = d$, it suffices to take $\nu = 0$ and $p = 2/\gamma > 1$. As a consequence, we get

$$\begin{aligned} M^2 &\le J(t)^{1/p} \\ &\quad \times \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(x,t)u(y,t)|x-y|^{\nu p'} \left(1+|x|^{2-\gamma}+|y|^{2-\gamma}\right)^{\delta p'} \mathrm{d}x\,\mathrm{d}y \right)^{1/p'}, \end{aligned} \tag{4.16}$$

where the integral $J(t)$ satisfies

$$\begin{aligned} J(t) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u(x,t)u(y,t)}{|x-y|^{d-\beta}} \frac{\mathrm{d}x\,\mathrm{d}y}{1+|x|^{2-\gamma}+|y|^{2-\gamma}} \\ &\le \frac{1}{K} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\nabla\varphi(x) - \nabla\varphi(y)) \cdot (x-y) \frac{u(x,t)u(y,t)}{|x-y|^{d-\beta+2}}\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{4.17}$$

by Lemma 4.3.

It follows from relations (4.15) and inequalities (4.2) that there exists a constant $C_1 > 0$ such that

$$|x-y|^{\nu p'} \left(1+|x|^{2-\gamma}+|y|^{2-\gamma}\right)^{\delta p'} \le C_1 \left(1+\varphi(x)+\varphi(y)\right). \tag{4.18}$$

Hence, (4.16) implies

$$M^2 \le C_1^{1/p'} J(t)^{1/p} \left(M^2 + 2Mw(t)\right)^{1/p'}. \tag{4.19}$$

Going back to identity (4.14) we obtain from Lemma 4.1 and from inequalities (4.17)–(4.19) that

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) \le C_2 M - C_3 \frac{M^{2p}}{(M^2 + 2Mw(t))^{p/p'}} \tag{4.20}$$

with $C_2 = \|(-\Delta)^{\alpha/2}\varphi\|_\infty$ and a suitable constant $C_3 > 0$.

Now, we fix for a while $M = M_0$ in (4.20) so large in order to have

$$C_2 M_0 - C_3 \frac{M_0^{2p}}{(M_0^2)^{p/p'}} < 0. \tag{4.21}$$

Hence, there exists $C_4 = C_4(M_0) > 0$ such that for $0 < w(0) \le C_4$ we still have

$$C_2 M_0 - C_3 \frac{M_0^{2p}}{(M_0^2 + 2M_0 w(0))^{p/p'}} < 0.$$

It is clear that if initially $0 < w(0) \le C_4$ then, by inequality (4.20) with $M = M_0$, the function $w(t)$ is decreasing in time. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t) \le C_2 M_0 - C_3 \frac{M_0^{2p}}{(M_0^2 + 2M_0 w(0))^{p/p'}} < 0$$

and, consequently, $w(T) < 0$ for some $0 < T < \infty$. This contradicts the global in time existence of regular nonnegative solutions of (1.1)–(1.3). Finally, note that due to the first inequality in (4.2), it suffices to assume

$$w(0) \le \int_{\mathbb{R}^d} |x|^\gamma u_0(x)\,\mathrm{d}x \le C_4 \quad \text{and} \quad \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x = M_0, \tag{4.22}$$

in order to obtain the blowup in a finite time of the corresponding solution.

Now, assume that $\int_{\mathbb{R}^d} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x = M \ne M_0$. Recall that system (1.1)–(1.3) is invariant under the rescaling (2.1). Choosing $\lambda^{\alpha+\beta-2-d} = M_0/M$ we obtain $\int_{\mathbb{R}^d} u^\lambda(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} u_0^\lambda(x)\,\mathrm{d}x = M_0$ and, by (4.22), the blowup of the solution takes place under the assumption

$$\int_{\mathbb{R}^d} |x|^\gamma u_0^\lambda(x)\,\mathrm{d}x \le C_4.$$

Changing the variables and using the explicit form of λ we obtain the blowup of solutions to (1.1)–(1.3) under the following assumption on the initial condition

$$\int_{\mathbb{R}^d} |x|^\gamma u_0(x)\,\mathrm{d}x \le C_4 M_0^{-1+\frac{\gamma}{\alpha+\beta-2-d}} \left(\int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x\right)^{1+\frac{\gamma}{d+2-\alpha-\beta}}.$$

□

Proof of Corollary 2.4. We follow the proof of Theorem 2.3. In particular, we choose M_γ so large that the inequality (4.21) holds true for all $M_0 > M_\gamma$. This leads to the blowup of the corresponding solution under the assumption (4.22) imposed on the initial data.

To complete the proof, we use the scaling argument again. By (2.1), $u^\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t)$ is a solution for every $\lambda > 0$. Note now that

$$\int_{\mathbb{R}^2} u_0^\lambda(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} u_0(x)\,\mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^2} |x|^\gamma u_0^\lambda(x)\,\mathrm{d}x = \lambda^{-\gamma} \int_{\mathbb{R}^2} |x|^\gamma u_0(x)\,\mathrm{d}x.$$

Hence, each initial data $u_0 \in L^1(\mathbb{R}^2, (1+|x|^\gamma)\,\mathrm{d}x)$ satisfying $\int_{\mathbb{R}^2} u_0(x)\,\mathrm{d}x = M_0 > M^\gamma$ leads to the blowup in a finite time of the corresponding solution because the moment condition in (4.22) can be satisfied replacing u by u^λ and choosing λ large enough. $\square$

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