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# Approximate controllability of a class of semilinear systems with boundary degeneracy

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*Abstract.* In this paper we consider the approximate controllability of a class of degenerate semilinear systems. The equations may be weakly degenerate and strongly degenerate on a portion of the lateral boundary. We prove that the control systems are approximately controllable and the controls can be taken to be of quasi bang-bang form.

## 1. Introduction

Controllability theory has been widely investigated for nondegenerate linear and semilinear parabolic equations over the past 40 years and there have been a great number of results (see for instance [2,11–13] and the references therein for a detailed account). The null controllability and the approximate controllability have been shown to be consistent and the sufficient conditions and necessary conditions have been obtained. Particularly, it has been shown that the approximate controllability is a consequence of the null controllability for the control systems governed by nondegenerate linear parabolic equations [11,12]. However, the study on the controllability of degenerate parabolic equations just began several years ago and very few results have been known [1,3–8,14,17,20–22,24]. Among these, some authors have investigated the null controllability of one-dimensional linear and semilinear equations with boundary degeneracy. In particular, the null controllability of the following degenerate semilinear equation has been considered:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(x^\alpha \frac{\partial u}{\partial x}\right) + g(x,t,u) = h(x,t)\chi_\omega, \quad (x,t) \in (0,1) \times (0,T), \tag{1.1}$$

where $\alpha > 0$, $h$ is the control function, $\chi_\omega$ is the characteristic function of $\omega$, a nonempty subinterval of (0, 1), while $g$ is locally Lipschitz continuous with respect to $u$ and satisfies some structural conditions, whose linear case is just

$$g(x,t,u) = c(x,t)u, \quad (x,t,u) \in (0,1) \times (0,T) \times \mathbb{R} \quad (c \in L^\infty((0,1) \times (0,T))). \tag{1.2}$$

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Equation (1.1) can be used to describe some physical models. For instance, in [4,5,21] we can find a motivating example of a Crocco-type equation coming from the study on the velocity field of a laminar flow on a flat plate. It is noted that Eq. (1.1) is degenerate at $\{0\} \times (0, T)$, a portion of the lateral boundary. As we know, the well-posed problems for parabolic equations with boundary degeneracy are different from the common ones [23,26]. In [1,3–8,22], the degeneracy of Eq. (1.1) is divided into weak one and strong one according to the value of $\alpha$, and different boundary conditions are proposed for the two cases. More precisely, the boundary value condition is

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T) \tag{1.3}$$

in the weakly degenerate case with $0 < \alpha < 1$, while is

$$\left(x^\alpha \frac{\partial u}{\partial x}\right)(0,t) = u(1,t) = 0, \quad t \in (0,T) \tag{1.4}$$

in the strongly degenerate case with $\alpha \geq 1$. Indeed, the following initial value condition is proposed for both cases

$$u(x,0) = u_0(x), \quad x \in (0,1). \tag{1.5}$$

Then, the null controllability problem of the semilinear system (1.1), (1.3) or (1.4), (1.5) is defined as follows: for any $u_0 \in L^2((0,1))$, is there a control function $h$ such that the solution of the system becomes null at the time $T$? The answer is that the system is null controllable if $0 < \alpha < 2$, while not if $\alpha \geq 2$. Here, the proof of the null controllability is based on Carleman estimates.

Since the semilinear system (1.1), (1.3) or (1.4), (1.5) may be not null controllable, a natural question is whether the system is approximately controllable. That is to say, for any given initial datum $u_0 \in L^2((0,1))$, the desired datum $u_d \in L^2((0,1))$ and the admissible error value $\varepsilon > 0$, whether there exists a control function $h$ such that the solution $u$ of the problem (1.1), (1.3) or (1.4), (1.5) approximately approaches the desired datum $u_d$ at time $T$, i.e.,

$$\|u(\cdot,T) - u_d(\cdot)\|_{L^2((0,1))} \leq \varepsilon. \tag{1.6}$$

For the degenerate linear system (1.1) with (1.2), (1.3) or (1.4), (1.5), (1.6), it has been shown via a variational approach in [24] that the system is approximately controllable and the control can be taken to be of quasi bang-bang form for each $\alpha > 0$. Therefore, different from the control systems governed by nondegenerate parabolic equations, the null controllability and the approximate controllability are inconsistent for the control systems governed by degenerate parabolic equations.

In this paper, we investigate the approximate controllability of the semilinear system (1.1), (1.3) or (1.4), (1.5), (1.6). In general, we consider the multi-dimensional case, i.e., the equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x,t)\nabla u) + g(x,t,u) = h(x,t)\chi_D, \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.7}$$

where $a \in C(\overline{Q}_T) \cap C^1(Q_T)$ and is positive in $\Omega \times [0, T]$, $\frac{1}{a}\frac{\partial a}{\partial t} \in L^\infty(Q_T)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $D$ is an open and nonempty subset which is compactly embedded in $\Omega$, $h$ is the control function, $\chi_D$ is the characteristic function of $D$, and $g$ is a measurable function in $Q_T \times \mathbb{R}$ satisfying

$$|g(x, t, u) - g(x, t, v)| \le C_0|u - v|, \quad (x, t) \in Q_T,\ u, v \in \mathbb{R} \tag{1.8}$$

and

$$g(x, t, \cdot) \text{ is differentiable at } u = 0 \text{ uniformly in } Q_T \text{ and } \left\| \frac{\partial g}{\partial u}(x, t, 0) \right\|_{L^\infty(Q_T)} \le C_0 \tag{1.9}$$

with some $C_0 > 0$. It is noted that $a$ can be allowed to vanish at some points on the lateral boundary $\partial\Omega \times (0, T)$, and thus Eq. (1.7) is degenerate on the set $\{(x, t) \in \partial\Omega \times (0, T) : a(x, t) = 0\}$, a portion of the lateral boundary. However, $D$, the set where controls are supported, is away from the region where Eq. (1.7) is degenerate since it is compactly embedded in $\Omega$. As mentioned in [24], we cannot apply the classical theory by Fichera and Oleǐnik to Eq. (1.7) since there is a restriction $a \in W^{2,\infty}(Q_T)$ in the classical theory [23]. Different from [1,3–8,22], in the present paper we do not prescribe the Neumann boundary condition for Eq. (1.7) on the boundary where the equation is strongly degenerate, but describe this boundary condition via restricting the solution space just as done in [24]. That is to say, we propose the following boundary and initial value conditions and desired terminal control condition

$$u(x, t) = 0, \quad (x, t) \in \Sigma, \tag{1.10}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{1.11}$$

$$\|u(\cdot, T) - u_d(\cdot)\|_{L^2(\Omega)} \le \varepsilon, \tag{1.12}$$

where $\Sigma$ is the nondegenerate and weakly degenerate parts of the lateral boundary, i.e.,

$$\begin{aligned}\Sigma = &\{(x, t) \in \partial\Omega \times (0, T) : a(x, t) > 0\} \\ &\cup \Bigg\{(x, t) \in \partial\Omega \times (0, T) : a(x, t) = 0 \text{ and there exists } 0 < \delta < \min\{t, T - t\} \\ &\quad \text{such that } \int_{t-\delta}^{t+\delta} \int_{\Omega \cap B_\delta(x)} \frac{1}{a(y, s)} \mathrm{d}y\mathrm{d}s < +\infty \Bigg\}\end{aligned} \tag{1.13}$$

with $B_\delta(x)$ being the ball in $\mathbb{R}^n$ centered at $x$ and with radius $\delta$ (see [24] for details).

We will prove that the degenerate semilinear system (1.7), (1.10)–(1.12) is approximately controllable in this paper. Our method is inspired by Fabre et al. [10], where the authors studied the approximate controllability of the following nondegenerate semilinear system

$$\frac{\partial u}{\partial t} - \Delta u + g(x,t,u) = h(x,t)\chi_D, \quad (x,t) \in Q_T, \tag{1.14}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.15}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.16}$$

$$\|u(\cdot,T) - u_d(\cdot)\|_{L^2(\Omega)} \le \varepsilon. \tag{1.17}$$

It was shown that for fixed $u_0 \in L^2(\Omega)$, $u_d \in L^2(\Omega)$ and $\varepsilon > 0$, there exists a control function $h \in L^2(Q_T)$ for the nondegenerate semilinear system (1.14)–(1.17) by using the approximate controllability of the linear systems and the Kakutani fixed point theorem. In the present paper, we establish the approximate controllability of the degenerate semilinear system (1.7), (1.10)–(1.12) in a similar way. However, since Eq. (1.7) can be degenerate on a portion of the lateral boundary, weak solutions with poor regularity should be considered and some compact estimates for solutions of non-degenerate equations are missing. For example, there is a $L^2$ estimate for the gradient of the solution to the problem (1.14)–(1.16), which plays an important role in study controllability, while it fails for the problem (1.7), (1.10), (1.11) due to the boundary degeneracy for Eq. (1.7). Therefore, we have to seek techniques to establish necessary compact estimates. It is noted that (1.8) implies

$$|g(x,t,u)| \le C(|u|+1), \quad (x,t,u) \in Q_T \times \mathbb{R}$$

for some $C > 0$. This growth condition is optimal in the sense that the semilinear system (1.7), (1.10)–(1.12) is not approximately controllable if $g$ is superlinear [10].

The paper is organized as follows. In Sects. 2 and 3, we do some necessary compact estimates of solutions to the linear problem and prove the well-posedness of the semilinear problem, respectively. In Sect. 4, we recall the approximate controllability of the linear system and do some preliminaries to study the semilinear system. The approximate controllability of the semilinear system is proved in Sect. 5 subsequently.

## 2. Well-posedness of the linear problem and some compact estimates

In this section, we first recall the well-posedness of the linear problem and then do some necessary compact estimates of solutions.

Consider the linear problem

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x,t)\nabla u) + c(x,t)u = f(x,t), \quad (x,t) \in Q_T, \tag{2.1}$$

$$u(x,t) = 0, \quad (x,t) \in \Sigma, \tag{2.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2.3}$$

where $c \in L^\infty(Q_T)$, $f \in L^2(Q_T)$, $u_0 \in L^2(\Omega)$, and $\Sigma$ is the nondegenerate and weakly degenerate parts of the lateral boundary given by (1.13).

The following two definitions are given in [24].

DEFINITION 2.1. Define $\mathscr{B}$ to be the closure of the set $C_0^\infty(Q_T)$ with respect to the norm

$$\|u\|_{\mathscr{B}} = \left(\iint_{Q_T} a(x,t)(|u(x,t)|^2 + |\nabla u(x,t)|^2)\mathrm{d}x\mathrm{d}t\right)^{1/2}, \quad u \in \mathscr{B}.$$

As to the set $\mathscr{B}$, we give the following remark whose proof can be found in [26, Corollary 2.1 and Remark 2.1].

REMARK 2.1. If $u \in \mathscr{B}$, then $u|_\Sigma = 0$ in the trace sense, while there is no trace on $(\partial\Omega \times (0,T))\backslash\Sigma$ in general.

DEFINITION 2.2. A function $u$ is called to be a weak solution of the problem (2.1)–(2.3), if $u \in L^\infty((0,T);L^2(\Omega))\cap\mathscr{B}$ and for any function $\varphi \in L^\infty((0,T);L^2(\Omega))\cap\mathscr{B}$ with $\frac{\partial\varphi}{\partial t} \in L^2(Q_T)$ and $\varphi(\cdot,T)|_\Omega = 0$, the following integral equality holds

$$\iint_{Q_T}\left(-u(x,t)\frac{\partial\varphi}{\partial t}(x,t)+a(x,t)\nabla u(x,t)\cdot\nabla\varphi(x,t)+c(x,t)u(x,t)\varphi(x,t)\right)\mathrm{d}x\mathrm{d}t$$
$$= \iint_{Q_T} f(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t + \int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x.$$

REMARK 2.2. Assume that $u \in L^\infty((0,T);L^2(\Omega))\cap\mathscr{B}$ with $\frac{\partial u}{\partial t} \in L^2(Q_T)$. Then $u$ is a weak solution of the problem (2.1)–(2.3), if and only if the integral equality

$$\iint_{Q_T}\left(\frac{\partial u}{\partial t}(x,t)\varphi(x,t)+a(x,t)\nabla u(x,t)\cdot\nabla\varphi(x,t)+c(x,t)u(x,t)\varphi(x,t)\right)\mathrm{d}x\mathrm{d}t$$
$$= \iint_{Q_T} f(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t$$

holds for any function $\varphi \in L^\infty((0,T);L^2(\Omega))\cap\mathscr{B}$, and (2.3) holds in the trace sense.

The problem (2.1)–(2.3) is well-posed.

PROPOSITION 2.1. *For any $c \in L^\infty(Q_T)$, $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$, there exists uniquely a weak solution $u \in L^\infty((0,T);L^2(\Omega))\cap\mathscr{B}$ to the problem* (2.1)–(2.3). *Furthermore, the solution u satisfies*

(i) *It holds that*

$$\|u\|^2_{L^\infty((0,T);L^2(\Omega))} \le \mathrm{e}^{2T\|c\|_{L^\infty(Q_T)}}\left(2\|fu\|_{L^1(Q_T)} + \|u_0\|^2_{L^2(\Omega)}\right)$$

*and*

$$\|u\|^2_{L^\infty((0,T);L^2(\Omega))} + \|a|\nabla u|^2\|_{L^1(Q_T)} \le C_1\mathrm{e}^{2T\|c\|_{L^\infty(Q_T)}}$$
$$\times\left(\|f\|^2_{L^2(Q_T)} + \|u_0\|^2_{L^2(\Omega)}\right),$$

*where $C_1 > 0$ is a constant depending only on $T$;*

(ii) *For any* $0 < \tau < T$, *it holds that* $\frac{\partial u}{\partial t} \in L^2(\Omega \times (\tau, T))$, $a|\nabla u|^2 \in L^\infty((\tau, T); L^1(\Omega))$ *and*

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega\times(\tau,T))} + \|a|\nabla u|^2\|_{L^\infty((\tau,T);L^1(\Omega))} \le C_2 \left(1 + \|c\|^2_{L^\infty(Q_T)}\right) \mathrm{e}^{4T\|c\|_{L^\infty(Q_T)}} \left(\|f\|^2_{L^2(Q_T)} + \|u_0\|^2_{L^2(\Omega)}\right),$$

*where* $C_2 > 0$ *is a constant depending only on* $T$, $\left\| \frac{1}{a}\frac{\partial a}{\partial t} \right\|_{L^\infty(Q_T)}$ *and* $\tau$;

(iii) *If* $a|\nabla u_0|^2 \in L^1(\Omega)$ *additionally, then* $\frac{\partial u}{\partial t} \in L^2(Q_T)$, $a|\nabla u|^2 \in L^\infty((0, T); L^1(\Omega))$ *and*

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(Q_T)} + \|a|\nabla u|^2\|_{L^\infty((0,T);L^1(\Omega))} \le C_3 \left(1+\|c\|^2_{L^\infty(Q_T)}\right) \times \mathrm{e}^{2T\|c\|_{L^\infty(Q_T)}} \left(\|f\|^2_{L^2(Q_T)}+\|u_0\|^2_{L^2(\Omega)}+\|a|\nabla u_0|^2\|_{L^1(\Omega)}\right),$$

*where* $C_3 > 0$ *is a constant depending only on* $T$ *and* $\left\| \frac{1}{a}\frac{\partial a}{\partial t} \right\|_{L^\infty(Q_T)}$;

(iv) *If* $u_0 \in L^\infty(\Omega)$ *and* $f \in L^\infty(Q_T)$ *additionally, then* $u \in L^\infty(Q_T)$ *and*

$$\|u\|_{L^\infty(Q_T)} \le \mathrm{e}^{T\|c\|_{L^\infty(Q_T)}} \left(T\|f\|_{L^\infty(Q_T)} + \|u_0\|_{L^\infty(\Omega)}\right).$$

*Proof.* The proof is similar to [24, Theorem 2.1], where the existence is proved by parabolic regularization method and the uniqueness is proved by the Holmgren method.

For any positive integer number $k$, choose $a_k, c_k, f_k, u_0^{(k)} \in C^\infty(\overline{Q}_T)$ satisfying

$$a(x,t) + \frac{1}{k} \le a_k(x,t) \le a(x,t) + \frac{2}{k}, \quad \left\| \frac{1}{a_k}\frac{\partial a_k}{\partial t} \right\|_{L^\infty(Q_T)} \le \left\| \frac{1}{a}\frac{\partial a}{\partial t} \right\|_{L^\infty(Q_T)}, \quad k = 1, 2, \ldots,$$

$$\|c_k\|_{L^\infty(Q_T)} \le \|c\|_{L^\infty(Q_T)}, \quad \|f_k\|_{L^2(Q_T)} \le \|f\|_{L^2(Q_T)}, \quad \|u_0^{(k)}\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)}, \quad k = 1, 2, \ldots$$

and

$$c_k \to c \text{ and } f_k \to f \text{ in } L^2(Q_T), \quad u_0^{(k)} \to u_0 \text{ in } L^2(\Omega), \qquad \text{as } k \to \infty;$$

further,

$$\left\| a_k|\nabla u_0^{(k)}|^2 \right\|_{L^1(\Omega)} \le \left\| a|\nabla u_0|^2 \right\|_{L^1(\Omega)}, \quad k = 1, 2, \ldots$$

if $a|\nabla u_0|^2 \in L^1(\Omega)$ additionally, and

$$\|u_0^{(k)}\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}, \quad \|f_k\|_{L^\infty(Q_T)} \le \|f\|_{L^\infty(Q_T)}, \quad k = 1, 2, \ldots$$

if $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(Q_T)$ additionally. Consider the problem

$$\frac{\partial u^{(k)}}{\partial t} - \operatorname{div}\left(a_k(x,t)\nabla u^{(k)}\right) + c_k(x,t)u^{(k)} = f_k(x,t), \quad (x,t) \in Q_T, \tag{2.4}$$

$$u^{(k)}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.5}$$

$$u^{(k)}(x,0) = u_0^{(k)}(x), \quad x \in \Omega. \tag{2.6}$$

According to the classical theory on parabolic equations, the problem (2.4)–(2.6) admits a unique classical solution $u^{(k)}$. Multiply Eq. (2.4) by $u^{(k)}$ and then integrate over $Q_s$ $(0 < s < T)$ by parts to get

$$\iint_{Q_s} \left(\frac{1}{2}\frac{\partial}{\partial t}\left(|u^{(k)}|^2\right) + a_k|\nabla u^{(k)}|^2 + c_k|u^{(k)}|^2\right) \mathrm{d}x\mathrm{d}t = \iint_{Q_s} f_k u^{(k)} \mathrm{d}x\mathrm{d}t.$$

Therefore,

$$\iint_{Q_s} \left(\frac{1}{2}\frac{\partial}{\partial t}(|u^{(k)}|^2) + a_k|\nabla u^{(k)}|^2\right) \mathrm{d}x\,\mathrm{d}t \le \|c\|_{L^\infty(Q_T)} \iint_{Q_s} |u^{(k)}|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{Q_s} f_k u^{(k)} \mathrm{d}x\,\mathrm{d}t, \quad 0 < s < T.$$

Using the Hölder inequality and the Gronwall inequality, we can get by a standard process (see for example [25,26]) that

$$\int_\Omega |u^{(k)}(x,t)|^2 \mathrm{d}x \le 2\iint_{Q_t} \left(\mathrm{e}^{2\|c\|_{L^\infty(Q_T)}(t-s)} - 1\right) f_k(x,s)u^{(k)}(x,s)\mathrm{d}x\,\mathrm{d}s + \int_\Omega |u_0^{(k)}|^2 \mathrm{d}x + 2\iint_{Q_t} f_k(x,s)u^{(k)}(x,s)\mathrm{d}x\,\mathrm{d}s, \quad 0 \le t \le T \tag{2.7}$$

and

$$\|u^{(k)}\|^2_{L^\infty((0,T);L^2(\Omega))} + \left\|a_k|\nabla u^{(k)}|^2\right\|_{L^1(Q_T)} \le C\mathrm{e}^{2T\|c\|_{L^\infty(Q_T)}} \left(\|f_k\|^2_{L^2(Q_T)} + \|u_0^{(k)}\|^2_{L^2(\Omega)}\right). \tag{2.8}$$

Since $a \in C(\overline{Q}_T)$ and is positive in $\Omega \times [0,T]$, (2.8) implies that $\{u^{(k)}\}_{k=1}^\infty$ is uniformly bounded in $L^\infty((0,T);L^2(\Omega))$ and $L^2((0,T);H^1_{loc}(\Omega))$. It follows from the diagonal principle that there exist a subsequence of $\{u^{(k)}\}_{k=1}^\infty$, denoted by itself for convenience, and a function $u \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1_{loc}(\Omega))$ such that

$$u^{(k)} \rightharpoonup u \text{ weakly in } L^2(Q_T), \quad \nabla u^{(k)} \rightharpoonup \nabla u \text{ weakly in } L^2((0,T);L^2_{loc}(\Omega)), \quad \text{as} \quad k \to \infty. \tag{2.9}$$

Further, one gets from (2.7)–(2.9) that $u \in L^\infty((0,T); L^2(\Omega)) \cap \mathscr{B}$ satisfies

$$\begin{aligned}\int_\Omega |u(x,t)|^2 \mathrm{d}x \le 2 \iint_{Q_t} \left(\mathrm{e}^{2\|c\|_{L^\infty(Q_T)}(t-s)} - 1\right) f(x,s)u(x,s)\mathrm{d}x\mathrm{d}s \\ + \int_\Omega |u_0|^2 dx + 2\iint_{Q_t} f(x,s)u(x,s)\mathrm{d}x\mathrm{d}s \\ \le \mathrm{e}^{2T\|c\|_{L^\infty(Q_T)}} \left(2\|fu\|_{L^1(Q_T)} + \|u_0\|^2_{L^2(\Omega)}\right), \quad 0 \le t \le T\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\|u\|^2_{L^\infty((0,T);L^2(\Omega))} + \|a|\nabla u|^2\|_{L^1(Q_T)} \le C\mathrm{e}^{2T\|c\|_{L^\infty(Q_T)}} \\ \left(\|f\|^2_{L^2(Q_T)} + \|u_0\|^2_{L^2(\Omega)}\right).\end{aligned} \tag{2.11}$$

Now let us show that $u$ is just a weak solution to the problem (2.1)–(2.3). For any function $\varphi \in C^1(\overline{Q}_T)$ satisfying $\varphi(x,t) = 0$ for $x$ near $\partial\Omega$ or $t = T$, multiply Eq. (2.4) by $\varphi$ and then integrate by parts over $Q_T$ to get

$$\begin{aligned}\iint_{Q_T} \left(-u^{(k)}\frac{\partial\varphi}{\partial t} + a_k \nabla u^{(k)} \cdot \nabla\varphi + c_k u^{(k)}\varphi\right) \mathrm{d}x\, \mathrm{d}t \\ = \iint_{Q_T} f_k\varphi \mathrm{d}x\mathrm{d}t + \int_\Omega u_0^{(k)}(x)\varphi(x,0)\mathrm{d}x.\end{aligned}$$

Letting $k \to \infty$ and using (2.9), one gets

$$\iint_{Q_T} \left(-u\frac{\partial\varphi}{\partial t} + a\nabla u \cdot \nabla\varphi + cu\varphi\right) \mathrm{d}x\, \mathrm{d}t = \iint_{Q_T} f\varphi \mathrm{d}x\, \mathrm{d}t + \int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x.$$

For any function $\varphi \in L^\infty((0,T); L^2(\Omega)) \cap \mathscr{B}$ with $\frac{\partial\varphi}{\partial t} \in L^2(Q_T)$ and $\varphi(\cdot,T)\Big|_\Omega = 0$, the above integral equality still holds after an approximate procedure since $u \in L^\infty((0,T); L^2(\Omega)) \cap \mathscr{B}$. Therefore, $u$ is a weak solution to the problem (2.1)–(2.3).

Let us prove the estimates in (i)–(iv). First, (2.10) and (2.11) are just the estimates in (i). Second, if $a|\nabla u_0|^2 \in L^1(\Omega)$ additionally, multiplying Eq. (2.4) by $\frac{\partial u^{(k)}}{\partial t}$ and then integrating over $Q_s$ $(0 < s < T)$ by parts, we get

$$\iint_{Q_s} \left(\left|\frac{\partial u^{(k)}}{\partial t}\right|^2 + \frac{1}{2}a_k\frac{\partial}{\partial t}(|\nabla u^{(k)}|^2) + c_k u^{(k)}\frac{\partial u^{(k)}}{\partial t}\right) \mathrm{d}x\mathrm{d}t = \iint_{Q_s} f_k\frac{\partial u^{(k)}}{\partial t}\mathrm{d}x\mathrm{d}t,$$

i.e.

$$\begin{aligned}\iint_{Q_s} \left(\left|\frac{\partial u^{(k)}}{\partial t}\right|^2 + \frac{1}{2}\frac{\partial}{\partial t}(a_k|\nabla u^{(k)}|^2) + c_k u^{(k)}\frac{\partial u^{(k)}}{\partial t}\right) \mathrm{d}x\, \mathrm{d}t \\ = \iint_{Q_s} \left(f_k\frac{\partial u^{(k)}}{\partial t} + \frac{1}{2}\frac{\partial a_k}{\partial t}|\nabla u^{(k)}|^2\right)\mathrm{d}x\, \mathrm{d}t.\end{aligned}$$

Using the Hölder inequality and the Gronwall inequality, together with the estimate (2.8), we can get that

$$\begin{aligned}&\left\|\frac{\partial u^{(k)}}{\partial t}\right\|^2_{L^2(Q_T)}+\left\|a_k|\nabla u^{(k)}|^2\right\|_{L^\infty((0,T);L^1(\Omega))}\\&\le C\left(1+\|c_k\|^2_{L^\infty(Q_T)}\right)\mathrm{e}^{2T\|c\|_{L^\infty(Q_T)}}\\&\quad\times\left(\|f_k\|^2_{L^2(Q_T)}+\|u_0^{(k)}\|^2_{L^2(\Omega)}+\|a_k|\nabla u_0^{(k)}|^2\|_{L^1(\Omega)}\right),\end{aligned}\tag{2.12}$$

which leads to the estimate in (iii). Third, if $u_0\in L^\infty(\Omega)$ and $f\in L^\infty(Q_T)$ additionally, then it follows from the maximum principle that

$$\|u^{(k)}\|_{L^\infty(Q_T)}\le \mathrm{e}^{T\|c_k\|_{L^\infty(Q_T)}}\left(T\|f_k\|_{L^\infty(Q_T)}+\|u_0^{(k)}\|_{L^\infty(\Omega)}\right),$$

which yields the estimate in (iv). Now let us show the estimate in (ii). Fix $0<\tau<T$. From the mean value theorem and the estimate (2.8), there exists a $\tau_k\in(0,\tau)$ such that

$$\begin{aligned}\left\|a_k|\nabla u^{(k)}(\cdot,\tau_k)|^2\right\|_{L^1(\Omega)}&=\frac{1}{\tau}\iint_{Q_\tau}a_k|\nabla u^{(k)}|^2\mathrm{d}x\mathrm{d}t\\&\le\frac{C}{\tau}\mathrm{e}^{2T\|c\|_{L^\infty(Q_T)}}\left(\|f_k\|^2_{L^2(Q_T)}+\|u_0^{(k)}\|^2_{L^2(\Omega)}\right).\end{aligned}\tag{2.13}$$

Similar to the proof of (2.12), we can get that

$$\begin{aligned}&\left\|\frac{\partial u^{(k)}}{\partial t}\right\|^2_{L^2(\Omega\times(\tau_k,T))}+\left\|a_k|\nabla u^{(k)}|^2\right\|_{L^\infty((\tau_k,T);L^1(\Omega))}\\&\le C\left(1+\|c_k\|^2_{L^\infty(Q_T)}\right)\mathrm{e}^{2T\|c\|_{L^\infty(\Omega\times(\tau_k,T))}}\\&\quad\times\left(\|f_k\|^2_{L^2(\Omega\times(\tau_k,T))}+\|u^{(k)}(\cdot,\tau_k)\|^2_{L^2(\Omega)}+\|a_k|\nabla u^{(k)}(\cdot,\tau_k)|^2\|_{L^1(\Omega)}\right).\end{aligned}\tag{2.14}$$

It follows from (2.14), (2.13) and (2.8) that

$$\begin{aligned}&\left\|\frac{\partial u^{(k)}}{\partial t}\right\|^2_{L^2(\Omega\times(\tau,T))}+\|a_k|\nabla u^{(k)}|^2\|_{L^\infty((\tau,T);L^1(\Omega))}\\&\le C\left(1+\frac{1}{\tau}\right)\left(1+\|c_k\|^2_{L^\infty(Q_T)}\right)\mathrm{e}^{4T\|c\|_{L^\infty(Q_T)}}\left(\|f_k\|^2_{L^2(Q_T)}+\|u_0^{(k)}\|^2_{L^2(\Omega)}\right),\end{aligned}$$

which leads to the estimate in (ii).

Finally, let us prove the uniqueness by the Holmgren method. Let $\bar{u}$ and $\tilde{u}$ be two weak solutions of the problem (2.1)–(2.3) and denote

$$w(x,t)=\bar{u}(x,t)-\tilde{u}(x,t),\quad (x,t)\in Q_T.$$

Then $w\in L^\infty((0,T);L^2(\Omega))\cap\mathscr{B}$ and for any function $\varphi\in L^\infty((0,T);L^2(\Omega))\cap\mathscr{B}$ with $\frac{\partial\varphi}{\partial t}\in L^2(Q_T)$ and $\varphi(\cdot,T)\Big|_\Omega=0$, the following integral equality holds

$$\iint_{Q_T}\left(-w\frac{\partial\varphi}{\partial t}+a\nabla w\cdot\nabla\varphi+cw\varphi\right)\mathrm{d}x\mathrm{d}t=0. \tag{2.15}$$

For any $\xi\in L^2(Q_T)$, the above existence result shows that the problem

$$\begin{aligned}&-\frac{\partial\psi}{\partial t}-\operatorname{div}(a(x,t)\nabla\psi)+c(x,t)\psi=\xi(x,t),\quad (x,t)\in Q_T,\\&\psi(x,t)=0,\quad (x,t)\in\Sigma,\\&\psi(x,T)=0,\quad x\in\Omega\end{aligned}$$

admits a weak solution $\psi\in L^\infty((0,T);L^2(\Omega))\cap\mathscr{B}$ with $\frac{\partial\psi}{\partial t}\in L^2(Q_T)$, which implies that

$$\iint_{Q_T}\left(-\frac{\partial\psi}{\partial t}\varphi+a\nabla\psi\cdot\nabla\varphi+c\psi\varphi\right)\mathrm{d}x\mathrm{d}t=\iint_{Q_T}\xi\varphi\mathrm{d}x\,\mathrm{d}t \tag{2.16}$$

for any function $\varphi\in L^\infty((0,T);L^2(\Omega))\cap\mathscr{B}$. Taking $\varphi=\psi$ in (2.15) and $\varphi=w$ in (2.16), we get

$$\iint_{Q_T}\xi w\mathrm{d}x\,\mathrm{d}t=0.$$

This leads to

$$w(x,t)=0,\quad \text{a.e. } (x,t)\in Q_T$$

owing to the arbitrariness of $\xi\in L^2(Q_T)$. Therefore,

$$\bar{u}(x,t)=\tilde{u}(x,t),\quad \text{a.e. } (x,t)\in Q_T.$$

The proof is complete. □

COROLLARY 2.1. *Assume that* $\|c_k\|_{L^\infty(Q_T)}$, $\|f_k\|_{L^2(Q_T)}$ *and* $\|u_0^{(k)}\|_{L^2(\Omega)}$ *are uniformly bounded and*

$$\begin{aligned}&c_k\rightharpoonup c \text{ weakly} * \text{ in } L^\infty(Q_T),\quad f_k\rightharpoonup f \text{ weakly in } L^2(Q_T),\\&u_0^{(k)}\rightharpoonup u_0 \text{ weakly in } L^2(\Omega).\end{aligned} \tag{2.17}$$

*Then there exists a subsequence of* $\{u^{(k)}\}_{k=1}^\infty$, *which converges to* $u$ *weakly in* $L^2(Q_T)$ *and strongly in* $L^1(Q_T)$, *where* $u$ *is the solution of the problem* (2.1)–(2.3), *while* $u^{(k)}$ *is the solution of the problem* (2.1)–(2.3) *with* $c=c_k$, $f=f_k$ *and* $u_0=u_0^{(k)}$ *for* $k=1,2,\ldots$.

*Proof.* From Proposition 2.1 (i) and (ii), one gets that $u^{(k)} \in L^\infty((0,T); L^2(\Omega)) \cap \mathscr{B}$ with $\frac{\partial u^{(k)}}{\partial t} \in L^2(\Omega \times (\tau, T))$ for any $0 < \tau < T$ satisfies

$$\|u^{(k)}\|_{L^\infty((0,T);L^2(\Omega))} + \|a|\nabla u^{(k)}|^2\|_{L^1(Q_T)} + \left\|\frac{\partial u^{(k)}}{\partial t}\right\|_{L^2(\Omega\times(\tau,T))} \leq C \quad (2.18)$$

with some $C > 0$ independent of $k$. Therefore, there exist a subsequence of $\{u^{(k)}\}_{k=1}^\infty$, denoted by itself for convenience, and a function $u \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1_{loc}(\Omega))$ and a $n$-dimensional vector function $\vec{\zeta} \in L^2(Q_T)$ such that

$$\begin{aligned} &u^{(k)} \rightharpoonup u \quad \text{and} \quad a^{1/2}\nabla u^{(k)} \rightharpoonup \vec{\zeta} \quad \text{weakly in } L^2(Q_T), \\ &\nabla u^{(k)} \rightharpoonup \nabla u \quad \text{weakly in } L^2((0,T); L^2_{loc}(\Omega)) \end{aligned} \quad (2.19)$$

and

$$u^{(k)} \to u \quad \text{in } L^1(Q_T). \quad (2.20)$$

Here, (2.19) is derived from (2.18) directly, while (2.20) is derived from (2.18) via the following detailed discussion. Fix a positive integer $\bar{m} > 1/T$ satisfying $\{x \in \Omega : \text{dist}(x, \partial\Omega) > 1/\bar{m}\} \neq \emptyset$. For any integer $m \geq \bar{m}$, denote

$$\Omega_m = \{x \in \Omega : \text{dist}(x, \partial\Omega) > 1/m\}, \quad Q_T^{(m)} = \Omega_m \times (1/m, T).$$

On the one hand, it follows from the embedding theorem and (2.18) that there exists a subsequence of $\{u^{(k)}\}_{k=1}^\infty$, denoted by $\{u^{(k_{\bar{m}}(l))}\}_{l=1}^\infty$, such that

$$u^{(k_{\bar{m}}(l))} \to u \text{ in } L^2(Q_T^{(\bar{m})}) \qquad \text{as } l \to \infty.$$

Similarly, for $m \geq \bar{m} + 1$, there exists a subsequence of $\{u^{(k_{m-1}(l))}\}_{l=1}^\infty$, denoted by $\{u^{(k_m(l))}\}_{l=1}^\infty$, such that

$$u^{(k_m(l))} \to u \text{ in } L^2(Q_T^{(m)}) \qquad \text{as } l \to \infty. \quad (2.21)$$

On the other hand, it follows from the Hölder inequality that

$$\begin{aligned} \left(\iint_{Q_T \setminus Q_T^{(m)}} |u^{(k)} - u| \mathrm{d}x\mathrm{d}t\right)^2 &\leq \text{meas}\left(Q_T \setminus Q_T^{(m)}\right) \iint_{Q_T \setminus Q_T^{(m)}} |u^{(k)} - u|^2 \mathrm{d}x\mathrm{d}t \\ &\leq 2\text{meas}\left(Q_T \setminus Q_T^{(m)}\right) \iint_{Q_T} \left(|u^{(k)}|^2 + u^2\right) \mathrm{d}x\mathrm{d}t \\ &\leq 4TC^2\text{meas}\left(Q_T \setminus Q_T^{(m)}\right) \to 0, \quad \text{as } m \to \infty. \end{aligned} \quad (2.22)$$

Give $\varepsilon > 0$. Owing to (2.22), there exists a positive integer $m_0 \geq \bar{m} + 1$ such that

$$\iint_{Q_T \setminus Q_T^{(m_0)}} |u^{(k)} - u| \mathrm{d}x\mathrm{d}t < \frac{\varepsilon}{2}, \quad k = 1, 2, \ldots. \quad (2.23)$$

Due to (2.21), there exists a positive integer $l_0$ such that for the so fixed $m_0$ and any $l \geq l_0$,

$$\iint_{Q_T^{(m_0)}} |u^{(k_{m_0}(l))} - u| \mathrm{d}x\mathrm{d}t < \frac{\varepsilon}{2}. \tag{2.24}$$

Therefore, for any $m \geq m_0 + l_0$, we get from (2.23) and (2.24) that

$$\iint_{Q_T} \left| u^{(k_m(m))} - u \right| \mathrm{d}x\mathrm{d}t = \iint_{Q_T \setminus Q_T^{(m_0)}} \left| u^{(k_m(m))} - u \right| \mathrm{d}x\mathrm{d}t + \iint_{Q_T^{(m_0)}} \left| u^{(k_m(m))} - u \right| \mathrm{d}x\mathrm{d}t < \varepsilon.$$

Hence

$$\lim_{m\to\infty} \iint_{Q_T} \left| u^{(k_m(m))} - u \right| \mathrm{d}x\mathrm{d}t = 0.$$

Finally, let us show that $u$ is just the solution of the problem (2.1)–(2.3). It is not hard to verify from (2.19) that $u \in \mathscr{B}$ and

$$\vec{\zeta}(x,t) = (a(x,t))^{1/2} \nabla u(x,t), \quad \text{a.e. } (x,t) \in Q_T. \tag{2.25}$$

For any function $\varphi \in L^\infty(Q_T) \cap \mathscr{B}$ with $\frac{\partial \varphi}{\partial t} \in L^2(Q_T)$ and $\varphi(\cdot, T)|_\Omega = 0$, it follows from the definition of weak solutions that

$$\iint_{Q_T} \left( -u^{(k)} \frac{\partial \varphi}{\partial t} + a \nabla u^{(k)} \cdot \nabla \varphi + c_k u^{(k)} \varphi \right) \mathrm{d}x\mathrm{d}t = \iint_{Q_T} f_k \varphi \mathrm{d}x\mathrm{d}t + \int_\Omega u_0^{(k)}(x) \varphi(x,0) \mathrm{d}x.$$

Letting $k \to \infty$ with (2.17), (2.19), (2.20) and (2.25), we get that

$$\iint_{Q_T} \left( -u \frac{\partial \varphi}{\partial t} + a \nabla u \cdot \nabla \varphi + c u \varphi \right) \mathrm{d}x\mathrm{d}t = \iint_{Q_T} f \varphi \mathrm{d}x\mathrm{d}t + \int_\Omega u_0(x) \varphi(x,0) \mathrm{d}x.$$

Since $c \in L^\infty(Q_T)$ and $u \in L^\infty((0,T); L^2(\Omega))$, the above integral equality still holds for each $\varphi \in L^\infty((0,T); L^2(\Omega)) \cap \mathscr{B}$ with $\frac{\partial \varphi}{\partial t} \in L^2(Q_T)$ and $\varphi(\cdot, T)\big|_\Omega = 0$. The proof is complete. □

For $0 < t \leq T$, since $a$ is allowed to vanish at some points on the lateral boundary $\partial\Omega \times (0,T)$, it seems impossible to get a subsequence of $\{u^{(k)}(\cdot,t)\}_{k=1}^\infty$, which converges to $u(\cdot,t)$ strongly in $L^2(\Omega)$, in Corollary 2.1. In the following three lemmas, we establish this convergence under some additional conditions.

COROLLARY 2.2. *Assume that* $\|c_k\|_{L^\infty(Q_T)}$, $\|f_k\|_{L^2(Q_T)}$ *and* $\|u_0^{(k)}\|_{L^2(\Omega)}$ *are uniformly bounded and*

$$c_k \rightharpoonup c \text{ weakly} * \text{ in } L^\infty(Q_T), \quad f_k \to f \text{ in } L^2(Q_T), \quad u_0^{(k)} \to u_0 \text{ in } L^2(\Omega). \tag{2.26}$$

*Then there exists a subsequence of* $\{u^{(k)}\}_{k=1}^{\infty}$, *which converges to* $u$ *in* $L^{\infty}((0,T);L^2(\Omega))$, *where* $u$ *is the solution of the problem* (2.1)–(2.3), *while* $u^{(k)}$ *is the solution of the problem* (2.1)–(2.3) *with* $c=c_k$, $f=f_k$ *and* $u_0=u_0^{(k)}$ *for* $k=1,2,\ldots$.

*Proof.* First, it follows from Proposition 2.1 (i) that

$$\left\|u^{(k)}-u\right\|_{L^2(Q_T)}\le C \tag{2.27}$$

with some $C>0$ independent of $k$. Second, due to Corollary 2.1, there exists a subsequence of $\{u^{(k)}\}_{k=1}^{\infty}$, denoted by itself for convenience, such that

$$u^{(k)}\to u \text{ in } L^1(Q_T)\quad \text{as } k\to\infty. \tag{2.28}$$

Now, from the assumption of this corollary, we get that $u^{(k)}-u\in L^{\infty}((0,T);L^2(\Omega))\cap\mathscr{B}$ is just the weak solution of the problem

$$\frac{\partial(u^{(k)}-u)}{\partial t}-\operatorname{div}\left(a(x,t)\nabla(u^{(k)}-u)\right)+c_k(x,t)(u^{(k)}-u)$$
$$=f_k(x,t)-f(x,t)-(c_k(x,t)-c(x,t))u(x,t),\quad (x,t)\in Q_T, \tag{2.29}$$
$$(u^{(k)}-u)(x,t)=0,\quad (x,t)\in\Sigma, \tag{2.30}$$
$$(u^{(k)}-u)(x,0)=u_0^{(k)}(x)-u_0(x),\quad x\in\Omega. \tag{2.31}$$

It follows from Proposition 2.1 (i) that

$$\begin{aligned}
&\|u^{(k)}-u\|^2_{L^{\infty}((0,T);L^2(\Omega))}\\
&\quad\le C\left(\|(f_k-f-(c_k-c)u)(u^{(k)}-u)\|_{L^1(Q_T)}+\|u_0^{(k)}-u_0\|^2_{L^2(\Omega)}\right)\\
&\quad\le C\left(\|(f_k-f)(u^{(k)}-u)\|_{L^1(Q_T)}+\|(c_k-c)u(u^{(k)}-u)\|_{L^1(Q_T)}\right.\\
&\qquad\left.+\|u_0^{(k)}-u_0\|^2_{L^2(\Omega)}\right)
\end{aligned} \tag{2.32}$$

with some $C>0$ independent of $k$. Let us estimate the terms on the right side of (2.32). On the one hand, it follows from (2.26) and (2.27) that

$$\left\|(f_k-f)(u^{(k)}-u)\right\|_{L^1(Q_T)}\le\|f_k-f\|_{L^2(Q_T)}\|u^{(k)}-u\|_{L^2(Q_T)}\to 0,\quad \text{as } k\to\infty \tag{2.33}$$

and

$$\lim_{k\to\infty}\left\|u_0^{(k)}-u_0\right\|_{L^2(\Omega)}=0. \tag{2.34}$$

On the other hand, for any $M > 0$, one gets that

$$\begin{aligned}
&\|(c_k - c)u(u^{(k)} - u)\|_{L^1(Q_T)} \\
&\quad \le \|c_k - c\|_{L^\infty(Q_T)}\|u(u^{(k)} - u)\|_{L^1(Q_T)} \\
&\quad = \|c_k - c\|_{L^\infty(Q_T)} \iint_{\{(x,t)\in Q_T:|u(x,t)|>M\}} \left|u(x,t)(u^{(k)}(x,t) - u(x,t))\right| \mathrm{d}x\mathrm{d}t \\
&\qquad + \|c_k - c\|_{L^\infty(Q_T)} \iint_{\{(x,t)\in Q_T:|u(x,t)|\le M\}} \left|u(x,t)(u^{(k)}(x,t) - u(x,t))\right| \mathrm{d}x\mathrm{d}t \\
&\quad \le \|c_k - c\|_{L^\infty(Q_T)}\|u^{(k)} - u\|_{L^2(Q_T)} \left(\iint_{\{(x,t)\in Q_T:|u(x,t)|>M\}} |u(x,t)|^2 \mathrm{d}x\mathrm{d}t\right)^{1/2} \\
&\qquad + M\|c_k - c\|_{L^\infty(Q_T)}\|u^{(k)} - u\|_{L^1(Q_T)},
\end{aligned}$$

which implies

$$\lim_{k\to\infty} \left\|(c_k - c)u(u^{(k)} - u)\right\|_{L^1(Q_T)} = 0 \tag{2.35}$$

owing to $u \in L^2(Q_T)$ with (2.27) and (2.28). Letting $k \to \infty$ in (2.32), we get from (2.33)–(2.35) that

$$\lim_{k\to\infty} \left\|u^{(k)} - u\right\|_{L^\infty((0,T);L^2(\Omega))} = 0.$$

The proof is complete. □

COROLLARY 2.3. *Assume that* $\|c_k\|_{L^\infty(Q_T)}$, $\|f_k\|_{L^\infty(Q_T)}$ *and* $\|u_0^{(k)}\|_{L^\infty(\Omega)}$ *are uniformly bounded and*

$$\begin{aligned}
&c_k \rightharpoonup c \text{ weakly} * \text{ in } L^\infty(Q_T), \quad f_k \rightharpoonup f \text{ weakly} * \text{ in } L^\infty(Q_T), \\
&u_0^{(k)} \rightharpoonup u_0 \text{ weakly} * \text{ in } L^\infty(\Omega).
\end{aligned}$$

*Then there exists a subsequence of* $\{u^{(k)}\}_{k=1}^\infty$*, which converges to* $u$ *in* $L^2(Q_T)$*, where* $u$ *is the solution of the problem* (2.1)–(2.3)*, while* $u^{(k)}$ *is the solution of the problem* (2.1)–(2.3) *with* $c = c_k$*,* $f = f_k$ *and* $u_0 = u_0^{(k)}$ *for* $k = 1, 2, \ldots$*.*

*Proof.* From Proposition 2.1 (i), (ii) and (iv), one gets that $u^{(k)} \in L^\infty(Q_T) \cap \mathscr{B}$ with $\frac{\partial u^{(k)}}{\partial t} \in L^2(\Omega \times (\tau, T))$ for any $0 < \tau < T$ satisfies

$$\|u^{(k)}\|_{L^\infty(Q_T)} + \left\|a|\nabla u^{(k)}|^2\right\|_{L^1(Q_T)} + \left\|\frac{\partial u^{(k)}}{\partial t}\right\|_{L^2(\Omega\times(\tau,T))} \le C \tag{2.36}$$

with some $C > 0$ independent of $k$. Therefore, there exist a subsequence of $\{u^{(k)}\}_{k=1}^\infty$, denoted by itself for convenience, and a function $u \in L^\infty(Q_T) \cap L^2((0,T); H^1_{loc}(\Omega))$ and a $n$-dimensional vector function $\vec{\zeta} \in L^2(Q_T)$ such that

$$u^{(k)} \rightharpoonup u \text{ weakly} * \text{ in } L^\infty(Q_T), \quad a^{1/2}\nabla u^{(k)} \rightharpoonup \vec{\zeta} \text{ weakly in } L^2(Q_T), \tag{2.37}$$

$$\nabla u^{(k)} \rightharpoonup \nabla u \text{ weakly in } L^2((0,T); L^2_{loc}(\Omega)) \tag{2.38}$$

and

$$u^{(k)} \to u \text{ in } L^2(Q_T). \tag{2.39}$$

Here, (2.37) and (2.38) are derived from (2.36) directly, while (2.39) is derived from (2.36) via a similar process as the proof of (2.20). On the one hand, (2.21) holds. On the other hand, instead of (2.22), one gets from (2.36) that

$$\begin{aligned}\iint_{Q_T\setminus Q_T^{(m)}} |u^{(k)} - u|^2 \mathrm{d}x\mathrm{d}t &\le \text{meas}\left(Q_T\setminus Q_T^{(m)}\right) \|u^{(k)} - u\|^2_{L^\infty(Q_T)} \\ &\le 4C^2 \text{meas}\left(Q_T\setminus Q_T^{(m)}\right) \to 0, \quad \text{as } m \to \infty.\end{aligned} \tag{2.40}$$

Then, (2.39) follows from (2.21) and (2.40). Finally, we can prove that $u$ is just the solution of the problem (2.1)–(2.3) via the same process as the one in the proof of Corollary 2.1. The proof is complete. □

COROLLARY 2.4. *Assume that* $\|c_k\|_{L^\infty(Q_T)}$, $\|f_k\|_{L^2(Q_T)}$ *and* $\|u_0^{(k)}\|_{L^2(\Omega)}$ *are uniformly bounded and*

$$(c_k - c)^2 \rightharpoonup 0 \text{ weakly} * \text{ in } L^\infty(Q_T), \quad f_k \to f \text{ in } L^2(Q_T), \quad u_0^{(k)} \to u_0 \text{ in } L^2(\Omega). \tag{2.41}$$

*Then* $\{u^{(k)}\}_{k=1}^\infty$ *converges to* $u$ *in* $L^\infty((0,T); L^2(\Omega))$, *where* $u$ *is the solution to the problem* (2.1)–(2.3), *while* $u^{(k)}$ *is the solution to the problem* (2.1)–(2.3) *with* $c = c_k$, $f = f_k$ *and* $u_0 = u_0^{(k)}$ *for* $k = 1, 2, \ldots$.

*Proof.* As shown in Corollary 2.2, $u^{(k)} - u \in L^\infty((0,T); L^2(\Omega)) \cap \mathscr{B}$ is just the weak solution to the problem (2.29)–(2.31). It follows from Proposition 2.1 (i) that

$$\begin{aligned}\left\|u^{(k)} - u\right\|^2_{L^\infty((0,T);L^2(\Omega))} &\le C\left(\|f_k - f - (c_k - c)u\|^2_{L^2(Q_T)} + \|u_0^{(k)} - u_0\|^2_{L^2(\Omega)}\right) \\ &\le C\left(\|f_k - f\|^2_{L^2(Q_T)} + \|(c_k - c)u\|^2_{L^2(Q_T)} \right.\\ &\quad \left. + \|u_0^{(k)} - u_0\|^2_{L^2(\Omega)}\right) \\ &= C\left(\|f_k - f\|^2_{L^2(Q_T)} + \|(c_k - c)^2 u^2\|_{L^1(Q_T)} \right.\\ &\quad \left. + \|u_0^{(k)} - u_0\|^2_{L^2(\Omega)}\right)\end{aligned}$$

with some $C > 0$ independent of $k$. Then, one gets from (2.41) that

$$\lim_{k\to\infty} \|u^{(k)} - u\|_{L^\infty((0,T);L^2(\Omega))} = 0.$$

The proof is complete. □

## 3. Well-posedness of the semilinear problem

In this section, we prove the well-posedness of the semilinear problem

$$\frac{\partial u}{\partial t} - \mathrm{div}(a(x,t)\nabla u) + g(x,t,u) = f(x,t), \quad (x,t) \in Q_T, \tag{3.1}$$

$$u(x,t) = 0, \quad (x,t) \in \Sigma, \tag{3.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{3.3}$$

where $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$.

DEFINITION 3.1. A function $u$ is called to be a weak solution to the problem (3.1)–(3.3), if $u \in L^\infty((0,T); L^2(\Omega)) \cap \mathscr{B}$ and for any function $\varphi \in L^\infty((0,T); L^2(\Omega)) \cap \mathscr{B}$ with $\frac{\partial \varphi}{\partial t} \in L^2(Q_T)$ and $\varphi(\cdot, T)\Big|_\Omega = 0$, the following integral equality holds

$$\iint_{Q_T} \left( -u(x,t)\frac{\partial \varphi}{\partial t}(x,t) + a(x,t)\nabla u(x,t)\cdot\nabla\varphi(x,t) + g(x,t,u(x,t))\varphi(x,t)\right) \mathrm{d}x\mathrm{d}t$$
$$= \iint_{Q_T} f(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t + \int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x.$$

As shown in the introduction, $g$ is a measurable function in $Q_T \times \mathbb{R}$ satisfying (1.8) and (1.9). Define the function

$$\sigma(x,t,u) = \begin{cases} \dfrac{g(x,t,u) - g(x,t,0)}{u}, & (x,t) \in Q_T,\ 0 \neq u \in \mathbb{R}, \\ \dfrac{\partial g}{\partial u}(x,t,0), & (x,t) \in Q_T,\ u = 0. \end{cases}$$

Then $\sigma \in L^\infty(Q_T \times \mathbb{R})$. Furthermore, $\sigma$ satisfies the following property.

LEMMA 3.1. *Assume that* $\{w_k\}_{k=1}^\infty$ *converges to* $w$ *in* $L^1(Q_T)$. *Then*

$$\sigma(x,t,w_k(x,t)) \rightharpoonup \sigma(x,t,w(x,t)) \text{ weakly} * \text{ in } L^\infty(Q_T) \quad \text{as } k \to \infty$$

*and*

$$(\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t)))^2 \rightharpoonup 0 \text{ weakly} * \text{ in } L^\infty(Q_T) \quad \text{as } k \to \infty.$$

*Proof.* We first prove

$$\lim_{k\to\infty} \iint_{Q_T} |\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t))|\mathrm{d}x\mathrm{d}t = 0. \tag{3.4}$$

For any $\varepsilon > 0$, it follows from (1.9) that there exists a $\delta > 0$ such that

$$\left|\sigma(x,t,u) - \frac{\partial g}{\partial u}(x,t,0)\right| \le \varepsilon, \quad |u| \le \delta,\ (x,t) \in Q_T. \tag{3.5}$$

For any $u, v \in \mathbb{R}$ with $|u|, |v| \geq \delta$, one gets from (1.8) that

$$
\begin{aligned}
&|\sigma(x,t,u) - \sigma(x,t,v)| \\
&= \left| \frac{v(g(x,t,u) - g(x,t,v)) - (u-v)(g(x,t,v) - g(x,t,0))}{uv} \right| \\
&\leq \left| \frac{g(x,t,u) - g(x,t,v)}{u} \right| + \left| \frac{u-v}{u} \right| \left| \frac{g(x,t,v) - g(x,t,0)}{v} \right| \\
&\leq C_0 \left| \frac{u-v}{u} \right| + C_0 \left| \frac{u-v}{u} \right| \leq 2\frac{C_0}{\delta}|u-v|, \quad (x,t) \in Q_T. \qquad (3.6)
\end{aligned}
$$

Fix $k \geq 1$. Denote

$$
\begin{aligned}
G_1^{(k)} &= \{(x,t) \in Q_T : |w_k(x,t)| < \delta, |w| < \delta\}, \\
G_2^{(k)} &= \{(x,t) \in Q_T : |w_k(x,t)| \geq \delta, |w| \geq \delta\}, \\
G_3^{(k)} &= \{(x,t) \in Q_T : |w_k(x,t)| < \delta, |w| \geq \delta\}, \\
G_4^{(k)} &= \{(x,t) \in Q_T : |w_k(x,t)| \geq \delta, |w| < \delta\}.
\end{aligned}
$$

Let us estimate the integers of $|\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t))|$ over $G_1^{(k)}$, $G_2^{(k)}$, $G_3^{(k)}$, $G_4^{(k)}$, respectively. First, it follows from (3.5) that

$$
\begin{aligned}
&\iint_{G_1^{(k)}} |\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t))| \mathrm{d}x\mathrm{d}t \\
&\leq \iint_{G_1^{(k)}} \left| \sigma(x,t,w_k(x,t)) - \frac{\partial g}{\partial u}(x,t,0) \right| \mathrm{d}x\mathrm{d}t \\
&\quad + \iint_{G_1^{(k)}} \left| \sigma(x,t,w(x,t)) - \frac{\partial g}{\partial u}(x,t,0) \right| \mathrm{d}x\mathrm{d}t \\
&\leq 2\varepsilon \mathrm{meas}(G_1^{(k)}). \qquad (3.7)
\end{aligned}
$$

Second, (3.6) leads to

$$
\begin{aligned}
&\iint_{G_2^{(k)}} |\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t))| \mathrm{d}x\mathrm{d}t \\
&\leq 2\frac{C_0}{\delta} \iint_{G_2^{(k)}} |w_k(x,t) - w(x,t)| \mathrm{d}x\mathrm{d}t. \qquad (3.8)
\end{aligned}
$$

Third, we get from (3.5) and (3.6) that

$$
\begin{aligned}
&\iint_{G_3^{(k)}} |\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t))| \mathrm{d}x\mathrm{d}t \\
&\leq \iint_{G_3^{(k)}} |\sigma(x,t,w_k(x,t)) - \sigma(x,t,\delta \mathrm{sgn}(w(x,t)))| \mathrm{d}x\mathrm{d}t \\
&\quad + \iint_{G_3^{(k)}} |\sigma(x,t,\delta \mathrm{sgn}(w(x,t))) - \sigma(x,t,w(x,t))| \mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\le \iint_{G_3^{(k)}} \left|\sigma(x,t,w_k(x,t)) - \frac{\partial g}{\partial u}(x,t,0)\right| \mathrm{d}x\mathrm{d}t \\
&\quad + \iint_{G_3^{(k)}} \left|\sigma(x,t,\delta \mathrm{sgn}(w(x,t))) - \frac{\partial g}{\partial u}(x,t,0)\right| \mathrm{d}x\mathrm{d}t \\
&\quad + 2\frac{C_0}{\delta} \iint_{G_2^{(k)}} |\delta \mathrm{sgn}(w(x,t)) - w(x,t)| \mathrm{d}x\mathrm{d}t \\
&\le 2\varepsilon \mathrm{meas}\left(G_3^{(k)}\right) + 2\frac{C_0}{\delta} \iint_{G_3^{(k)}} |w_k(x,t) - w(x,t)| \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.9}
$$

Similar to the proof of (3.9), one can prove

$$
\begin{aligned}
&\iint_{G_4^{(k)}} |\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t))| \mathrm{d}x\mathrm{d}t \\
&\le \iint_{G_4^{(k)}} |\sigma(x,t,w_k(x,t)) - \sigma(x,t,\delta \mathrm{sgn}(w_k(x,t)))| \mathrm{d}x\mathrm{d}t \\
&\quad + \iint_{G_4^{(k)}} |\sigma(x,t,\delta \mathrm{sgn}(w_k(x,t))) - \sigma(x,t,w(x,t))| \mathrm{d}x\mathrm{d}t \\
&\le 2\frac{C_0}{\delta} \iint_{G_4^{(k)}} |w_k(x,t) - w(x,t)| \mathrm{d}x\mathrm{d}t + 2\varepsilon \mathrm{meas}\left(G_4^{(k)}\right).
\end{aligned} \tag{3.10}
$$

It follows from (3.7)–(3.10) that

$$
\begin{aligned}
&\iint_{Q_T} |\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t))| \mathrm{d}x\mathrm{d}t \\
&\quad \le 2\varepsilon \mathrm{meas}(Q_T) + 2\frac{C_0}{\delta} \iint_{Q_T} |w_k(x,t) - w(x,t)| \mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

which leads to (3.4).

Give $\varphi \in L^1(Q_T)$. For any $M > 0$, we get that

$$
\begin{aligned}
&\iint_{Q_T} (\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t)))\varphi(x,t) \mathrm{d}x\mathrm{d}t \\
&= \iint_{\{(x,t)\in Q_T : |\varphi(x,t)| > M\}} (\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t)))\varphi(x,t) \mathrm{d}x\mathrm{d}t \\
&\quad + \iint_{\{(x,t)\in Q_T : |\varphi(x,t)| \le M\}} (\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t)))\varphi(x,t) \mathrm{d}x\mathrm{d}t \\
&\le 2\|\sigma\|_{L^\infty(Q_T\times\mathbb{R})} \iint_{\{(x,t)\in Q_T : |\varphi(x,t)| > M\}} |\varphi(x,t)| \mathrm{d}x\mathrm{d}t \\
&\quad + M \iint_{\{(x,t)\in Q_T : |\varphi(x,t)| \le M\}} |\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t))| \mathrm{d}x\mathrm{d}t
\end{aligned}
$$

and

$$\begin{aligned}
&\iint_{Q_T} (\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t)))^2 \varphi(x,t)\mathrm{d}x\mathrm{d}t \\
&\quad = \iint_{\{(x,t)\in Q_T:|\varphi(x,t)|>M\}} (\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t)))^2 \varphi(x,t)\mathrm{d}x\mathrm{d}t \\
&\qquad + \iint_{\{(x,t)\in Q_T:|\varphi(x,t)|\le M\}} (\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t)))^2 \varphi(x,t)\mathrm{d}x\mathrm{d}t \\
&\quad \le 4\|\sigma\|^2_{L^\infty(Q_T\times\mathbb{R})} \iint_{\{(x,t)\in Q_T:|\varphi(x,t)|>M\}} |\varphi(x,t)|\mathrm{d}x\mathrm{d}t \\
&\qquad + 2M\|\sigma\|_{L^\infty(Q_T\times\mathbb{R})} \iint_{\{(x,t)\in Q_T:|\varphi(x,t)|\le M\}} |\sigma(x,t,w_k(x,t)) \\
&\qquad\qquad - \sigma(x,t,w(x,t))|\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

From these estimates, together with $\varphi \in L^1(Q_T)$ and (3.4), we get that

$$\lim_{k\to\infty} \iint_{Q_T} (\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t)))\varphi(x,t)\mathrm{d}x\mathrm{d}t = 0$$

and

$$\lim_{k\to\infty} \iint_{Q_T} (\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t)))^2\varphi(x,t)\mathrm{d}x\mathrm{d}t = 0.$$

The proof is complete. □

Let us establish the existence and uniqueness results for the weak solution to the problem (3.1)–(3.3).

THEOREM 3.1. *For any $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$, there exists uniquely a weak solution to the problem* (3.1)–(3.3).

*Proof.* Let us prove the existence by the Schauder fixed point theorem. For any $w \in L^1(Q_T)$, we get that $\sigma(x,t,w(x,t)) \in L^\infty(Q_T)$ owing to $\sigma \in L^\infty(Q_T \times \mathbb{R})$. It follows from Proposition 2.1 that the problem

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x,t)\nabla u) + \sigma(x,t,w(x,t))u = f(x,t) - g(x,t,0), \quad (x,t) \in Q_T, \tag{3.11}$$

$$u(x,t) = 0, \quad (x,t) \in \Sigma, \tag{3.12}$$

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{3.13}$$

admits a unique weak solution $u \in L^\infty((0,T);L^2(\Omega)) \cap \mathscr{B}$. Define the mapping $\Lambda : L^1(Q_T) \to L^1(Q_T)$ as follows:

$$\Lambda(w) = u, \quad w \in L^1(Q_T),$$

where $u$ is the solution of the problem (3.11)–(3.13).

On the one hand, let us show that the range of $\Lambda$ is precompact. Give $\{w_k\}_{k=1}^{\infty} \subset L^1(Q_T)$. Then, $\{\sigma(x,t,w_k(x,t))\}_{k=1}^{\infty}$ is uniformly bounded in $L^{\infty}(Q_T)$ owing to $\sigma \in L^{\infty}(Q_T \times \mathbb{R})$. Therefore, there exists a subsequence of $\{\sigma(x,t,w_k(x,t))\}_{k=1}^{\infty}$, which converges weakly $*$ in $L^{\infty}(Q_T)$. From this convergence and Corollary 2.1, there exists a subsequence of $\{\Lambda(w_k)\}_{k=1}^{\infty}$, which converges in $L^1(Q_T)$. Hence the range of $\Lambda$ is precompact.

On the other hand, let us show that $\Lambda$ is continuous. Assume that $\{w_k\}_{k=1}^{\infty}$ converges to $w$ in $L^1(Q_T)$. It follows from Lemma 3.1 that

$$(\sigma(x,t,w_k(x,t)) - \sigma(x,t,w(x,t)))^2 \rightharpoonup 0 \text{ weakly } * \text{ in } L^{\infty}(Q_T) \quad \text{as } k \to \infty.$$

Then, $\{\Lambda(w_k)\}_{k=1}^{\infty}$ converges to $\Lambda(w)$ in $L^{\infty}((0,T); L^2(\Omega))$ and thus in $L^1(Q_T)$ due to Corollary 2.4. Therefore, $\Lambda$ is continuous.

From the above discussion, one gets that the restriction of the mapping $\Lambda$ to the close and convex hull of the range of $\Lambda$ satisfies the hypotheses of the Schauder fixed point theorem [15, Theorem 11.1]. Therefore, $\Lambda$ admits a fixed point $u \in L^1(Q_T)$. That is to say, $u = \Lambda(u) \in L^{\infty}((0,T); L^2(\Omega)) \cap \mathscr{B}$ is just a weak solution to the problem (3.1)–(3.3).

Finally, the uniqueness can be proved by the Holmgren method (see for example [25,27]). Let $\bar{u}$ and $\tilde{u}$ be two weak solutions to the problem (3.1)–(3.3) and denote

$$w(x,t) = \bar{u}(x,t) - \tilde{u}(x,t), \quad (x,t) \in Q_T.$$

Then $w \in L^{\infty}((0,T); L^2(\Omega)) \cap \mathscr{B}$ and for any function $\varphi \in L^{\infty}((0,T); L^2(\Omega)) \cap \mathscr{B}$ with $\frac{\partial \varphi}{\partial t} \in L^2(Q_T)$ and $\varphi(\cdot, T)|_{\Omega} = 0$, the following integral equality holds:

$$\iint_{Q_T} \left( -w\frac{\partial \varphi}{\partial t} + a\nabla w \cdot \nabla \varphi + cw\varphi \right) \mathrm{d}x\mathrm{d}t = 0, \tag{3.14}$$

where

$$c(x,t) = \begin{cases} \dfrac{g(x,t,\bar{u}(x,t)) - g(x,t,\tilde{u}(x,t))}{\bar{u}(x,t) - \tilde{u}(x,t)}, & \bar{u}(x,t) \neq \tilde{u}(x,t), \\ C_0, & \bar{u}(x,t) = \tilde{u}(x,t), \end{cases} \qquad (x,t) \in Q_T.$$

It follows from (1.8) that $c \in L^{\infty}(Q_T)$. For any $\xi \in L^2(Q_T)$, the existence result of Proposition 2.1 shows that the problem

$$\begin{aligned} &-\frac{\partial \psi}{\partial t} - \operatorname{div}(a(x,t)\nabla \psi) + c(x,t)\psi = \xi(x,t), \quad (x,t) \in Q_T, \\ &\psi(x,t) = 0, \quad (x,t) \in \Sigma, \\ &\psi(x,T) = 0, \quad x \in \Omega \end{aligned}$$

admits a weak solution $\psi \in L^{\infty}((0,T); L^2(\Omega)) \cap \mathscr{B}$ with $\frac{\partial \psi}{\partial t} \in L^2(Q_T)$, which implies that

$$\iint_{Q_T} \left( -\frac{\partial \psi}{\partial t}\varphi + a\nabla \psi \cdot \nabla \varphi + c\psi\varphi \right) \mathrm{d}x\mathrm{d}t = \iint_{Q_T} \xi\varphi \mathrm{d}x\mathrm{d}t \tag{3.15}$$

for any function $\varphi \in L^\infty((0,T);L^2(\Omega)) \cap \mathscr{B}$. Taking $\varphi = \psi$ in (3.14) and $\varphi = w$ in (3.15), we get

$$\iint_{Q_T} \xi w \mathrm{d}x\mathrm{d}t = 0.$$

This leads to

$$w(x,t) = 0, \quad \text{a.e. } (x,t) \in Q_T$$

owing to the arbitrariness of $\xi \in L^2(Q_T)$. Therefore,

$$\bar{u}(x,t) = \tilde{u}(x,t), \quad \text{a.e. } (x,t) \in Q_T.$$

That is to say, the weak solution of the problem (3.1)–(3.3) is unique. The proof is complete. □

## 4. Approximate controllability of the linear system and some preliminaries

In this section, we recall the approximate controllability of the linear system, which was proved in [24], and do some preliminaries to study the semilinear system.

For convenience, we just consider the following linear control system with null initial data:

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x,t)\nabla u) + c(x,t)u = h(x,t)\chi_D, \quad (x,t) \in Q_T, \tag{4.1}$$

$$u(x,t) = 0, \quad (x,t) \in \Sigma, \tag{4.2}$$

$$u(x,0) = 0, \quad x \in \Omega, \tag{4.3}$$

$$\|u(\cdot,T) - u_d(\cdot)\|_{L^2(\Omega)} \le \varepsilon. \tag{4.4}$$

The study on the approximate controllability of the system (4.1)–(4.4) is related to its conjugate problem

$$-\frac{\partial v}{\partial t} - \operatorname{div}(a(x,t)\nabla v) + c(x,t)v = 0, \quad (x,t) \in Q_T, \tag{4.5}$$

$$v(x,t) = 0, \quad (x,t) \in \Sigma, \tag{4.6}$$

$$v(x,T) = v_0(x), \quad x \in \Omega. \tag{4.7}$$

Define the mapping

$$\mathscr{L} : L^2(\Omega) \times L^\infty(Q_T) \to L^1(Q_T), \quad (v_0, c) \longmapsto v,$$

where $v$ is the weak solution to the conjugate problem (4.5)–(4.7). This mapping is of the following two properties. On the one hand, it is obvious from Corollary 2.4 that $\mathscr{L}$ is a continuous linear operator from $L^2(\Omega) \times L^\infty(Q_T)$ to $L^1(Q_T)$. On the other hand, the weak solution of the conjugate problem (4.5)–(4.7) has the following property of unique continuation:

$$\mathscr{L}(v_0, c) = 0 \text{ a.e. } (x,t) \in D_T \Longrightarrow \mathscr{L}(v_0, c) = 0 \text{ a.e. } (x,t) \in Q_T, \tag{4.8}$$

where $D$ is an open and nonempty subset which is compactly embedded in $\Omega$ as mentioned in introduction, $D_T = D \times (0, T)$. Here (4.8) is deduced from the property of unique continuation for nondegenerate equation. Assume that $\mathscr{L}(v_0, c) = 0$ a.e. in $D_T$. For any domain $G$ satisfying $D \subset G \subset\subset \Omega$, Eq. (4.5) is uniformly parabolic in $G \times (0, T)$ and thus we get from [9, Theorem 1.1] that $\mathscr{L}(v_0, c) = 0$ a.e. in $G \times (0, T)$. Then $\mathscr{L}(v_0, c) = 0$ a.e. in $Q_T$ owing to the choice of $G$.

Fix $\varepsilon > 0$. For any $u_d \in L^2(\Omega)$ and $c \in L^\infty(Q_T)$, define the functional

$$J(v_0; u_d, c) = \frac{1}{2}\left(\iint_{D_T} |\mathscr{L}(v_0, c)(x,t)|\mathrm{d}x\mathrm{d}t\right)^2 + \varepsilon\|v_0\|_{L^2(\Omega)} - \int_\Omega u_d(x)v_0(x)\mathrm{d}x,$$
$$v_0 \in L^2(\Omega).$$

This functional possesses the following property [24, Proposition 3.1]:

LEMMA 4.1. *For any $u_d \in L^2(\Omega)$ and $c \in L^\infty(Q_T)$, the functional $J(\cdot; u_d, c)$ is strictly convex and achieves its minimum at a unique point $\hat{v}_0$ in $L^2(\Omega)$.*

It has been shown in [24, Theorem 3.1 and Remark 3.1] that the linear system (4.1)–(4.4) is approximately controllable and the control can be constructed by the conjugate problem (4.5)–(4.7) with $v_0 = \hat{v}_0$. This construction should be owed to Lions [18,19].

LEMMA 4.2. *For any $u_d \in L^2(\Omega)$ and $c \in L^\infty(Q_T)$, there exists $z \in \operatorname{sgn}(\hat{v})\chi_D$ such that the weak solution $u$ of the problem (4.1)–(4.3) with*

$$h(x,t) = \|\hat{v}\|_{L^1(D_T)} z(x,t), \quad (x,t) \in Q_T$$

*satisfies (4.4), where $\hat{v}$ is the weak solution to the conjugate problem (4.5)–(4.7) with $v_0 = \hat{v}_0$ and $\hat{v}_0$ is the unique minimum of $J(\cdot; u_d, c)$. In the present paper, we say $z \in \operatorname{sgn}(\hat{v})$, if $z(x,t) = \hat{v}(x,t)/|\hat{v}(x,t)|$ when $\hat{v}(x,t) \neq 0$, while $|z(x,t)| \leq 1$ when $\hat{v}(x,t) = 0$.*

In the rest of this section, let us investigate the properties of $\hat{v}_0$ with respect to $u_d$ and $c$, which are preliminaries for the semilinear system. For convenience, we introduce a mapping defined as follows:

$$\mathscr{M} : L^2(\Omega) \times L^\infty(Q_T) \to L^2(\Omega), \quad (u_d, c) \longmapsto \hat{v}_0,$$

where $\hat{v}_0$ is the unique minimum point of the functional $J(\cdot; u_d, c)$.

PROPOSITION 4.1. *Assume that $K$ is a compact subset of $L^2(\Omega)$ and $B$ is a bounded subset of $L^\infty(Q_T)$. Then $\mathscr{M}(K \times B)$ is a bounded subset of $L^2(\Omega)$.*

*Proof.* For any $(u_d, c) \in K \times B$, it holds that

$$J(0; u_d, c) = 0.$$

Therefore, it suffices to prove that

$$\lim_{\|v_0\|_{L^2(\Omega)} \to +\infty} \inf \frac{J(v_0; u_d, c)}{\|v_0\|_{L^2(\Omega)}} \geq \varepsilon \text{ uniformly in } (u_d, c) \in K \times B. \tag{4.9}$$

Let us prove (4.9) by contradiction. Otherwise, there exist two sequences $\{(u_d^{(k)}, c_k)\}_{k=1}^{\infty} \subset K \times B$ and $\{v_0^{(k)}\}_{k=1}^{\infty} \subset L^2(\Omega)$ satisfying

$$\lim_{k \to \infty} \|v_0^{(k)}\|_{L^2(\Omega)} = +\infty, \quad \lim_{k \to \infty} \frac{J\left(v_0^{(k)}; u_d^{(k)}, c_k\right)}{\|v_0^{(k)}\|_{L^2(\Omega)}} < \varepsilon. \tag{4.10}$$

Define

$$\tilde{v}_0^{(k)} = \frac{v_0^{(k)}}{\|v_0^{(k)}\|_{L^2(\Omega)}}, \quad k = 1, 2, \ldots.$$

Since $\{u_d^{(k)}\}_{k=1}^{\infty} \subset K$ is compact in $L^2(\Omega)$, $\{c_k\}_{k=1}^{\infty} \subset B$ is bounded in $L^{\infty}(Q_T)$ and $\{\tilde{v}_0^{(k)}\}_{k=1}^{\infty}$ is bounded in $L^2(\Omega)$, there exists a subsequence of $\{(u_d^{(k)}, c_k, \tilde{v}_0^{(k)})\}_{k=1}^{\infty}$, denoted by itself for convenience, such that

$$u_d^{(k)} \to u_d \text{ in } L^2(\Omega), \quad c_k \rightharpoonup c \text{ weakly } * \text{ in } L^{\infty}(Q_T), \quad \tilde{v}_0^{(k)} \rightharpoonup \tilde{v}_0 \text{ weakly in } L^2(\Omega), \tag{4.11}$$

where $u_d \in L^2(\Omega)$, $c \in L^{\infty}(Q_T)$ and $\tilde{v}_0 \in L^2(\Omega)$ with $\|\tilde{v}_0\|_{L^2(\Omega)} \leq 1$. It follows from Corollary 2.1 with (4.11) that there exists a subsequence of $\{\mathscr{L}(\tilde{v}_0^{(k)}, c_k)\}_{k=1}^{\infty}$, denoted by itself for convenience, which converges to $\mathscr{L}(\tilde{v}_0, c)$ in $L^1(Q_T)$. Additionally, (4.10) yields

$$\lim_{k \to \infty} \iint_{D_T} |\mathscr{L}(\tilde{v}_0^{(k)}, c_k)(x, t)| \mathrm{d}x \mathrm{d}t = 0.$$

Hence

$$\iint_{D_T} |\mathscr{L}(\tilde{v}_0, c)(x, t)| \mathrm{d}x \mathrm{d}t = 0.$$

This and (4.8) lead to $\mathscr{L}(\tilde{v}_0, c) = 0$ a.e. in $Q_T$ and thus $\tilde{v}_0 = 0$ a.e. in $\Omega$ from the uniqueness result in Proposition 2.1, which, together with (4.11), implies

$$\lim_{k \to \infty} \frac{\int_{\Omega} u_d^{(k)}(x) v_0^{(k)}(x) \mathrm{d}x}{\|v_0^{(k)}\|_{L^2(\Omega)}} = \lim_{k \to \infty} \int_{\Omega} u_d^{(k)}(x) \tilde{v}_0^{(k)}(x) \mathrm{d}x = 0.$$

Hence

$$\lim_{k\to\infty} \frac{J\left(v_0^{(k)}; u_d^{(k)}, c_k\right)}{\|v_0^{(k)}\|_{L^2(\Omega)}} \geq \varepsilon,$$

which contradicts (4.10) and shows that (4.9) holds. The proof is complete. □

PROPOSITION 4.2. *Assume that* $u_d^{(k)}$ *converges to* $u_d$ *in* $L^2(\Omega)$, $\|c_k\|_{L^\infty(Q_T)}$ *is uniformly bounded and* $c_k$ *converges to* $c$ *weakly* $*$ *in* $L^\infty(Q_T)$. *Then there exists a subsequence of* $\{\mathscr{M}(u_d^{(k)}, c_k)\}_{k=1}^\infty$, *which converges to* $\mathscr{M}(u_d, c)$ *in* $L^2(\Omega)$.

*Proof.* For convenience, we denote

$$\hat{v}_0 = \mathscr{M}(u_d, c), \quad \hat{v}_0^{(k)} = \mathscr{M}(u_d^{(k)}, c_k), \quad k = 1, 2, \ldots.$$

It follows from Proposition 4.1 that $\{\hat{v}_0^{(k)}\}_{k=1}^\infty$ is bounded in $L^2(\Omega)$. Therefore, there exist a subsequence of $\{\hat{v}_0^{(k)}\}_{k=1}^\infty$, denoted by itself for convenience, and a function $\check{v}_0 \in L^2(\Omega)$ such that $\hat{v}_0^{(k)}$ converges to $\check{v}_0$ weakly in $L^2(\Omega)$. Then, it follows from Corollary 2.1 that there exists a subsequence of $\{\mathscr{L}(\hat{v}_0^{(k)}, c_k)\}_{k=1}^\infty$, denoted by itself for convenience, such that

$$\mathscr{L}(\hat{v}_0^{(k)}, c_k) \to \mathscr{L}(\check{v}_0, c) \text{ in } L^1(Q_T) \quad \text{as } k \to \infty. \tag{4.12}$$

Therefore,

$$\lim_{k\to\infty} J\left(\hat{v}_0^{(k)}; u_d^{(k)}, c_k\right) \geq J(\check{v}_0; u_d, c). \tag{4.13}$$

Via a same argument, there exists a subsequence of $\{\mathscr{L}(\hat{v}_0, c_k)\}_{k=1}^\infty$, denoted by itself for convenience, such that

$$\mathscr{L}(\hat{v}_0, c_k) \to \mathscr{L}(\hat{v}_0, c) \text{ in } L^1(Q_T) \quad \text{as } k \to \infty,$$

which leads to

$$\lim_{k\to\infty} J\left(\hat{v}_0; u_d^{(k)}, c_k\right) = J(\hat{v}_0; u_d, c). \tag{4.14}$$

Letting $k \to \infty$ in

$$J\left(\hat{v}_0^{(k)}; u_d^{(k)}, c_k\right) \leq J\left(\hat{v}_0; u_d^{(k)}, c_k\right), \quad k = 1, 2, \ldots$$

and using (4.13) and (4.14), we get that

$$J(\check{v}_0; u_d, c) \leq J(\hat{v}_0; u_d, c),$$

which implies from the uniqueness result of Lemma 4.1 that

$$\check{v}_0 = \hat{v}_0 \quad \text{and} \quad \lim_{k\to\infty} J\left(\hat{v}_0^{(k)}; u_d^{(k)}, c_k\right) = J(\hat{v}_0; u_d, c). \tag{4.15}$$

From (4.12) and (4.15), we get furthermore that

$$\lim_{k\to\infty}\iint_{D_T}\left|\mathscr{L}(\hat{v}_0^{(k)},c_k)(x,t)\right|\mathrm{d}x\mathrm{d}t=\iint_{D_T}|\mathscr{L}(\check{v}_0,c)(x,t)|\mathrm{d}x\mathrm{d}t$$
$$=\iint_{D_T}|\mathscr{L}(\hat{v}_0,c)(x,t)|\mathrm{d}x\mathrm{d}t. \tag{4.16}$$

Additionally, since $u_d^{(k)}$ converges to $u_d$ in $L^2(\Omega)$ and $\hat{v}_0^{(k)}$ converges to $\check{v}_0=\hat{v}_0$ weakly in $L^2(\Omega)$, one gets that

$$\lim_{k\to\infty}\int_\Omega u_d^{(k)}(x)\hat{v}_0^{(k)}(x)\mathrm{d}x=\int_\Omega u_d(x)\hat{v}_0(x)\mathrm{d}x. \tag{4.17}$$

It follows from (4.15)–(4.17) that

$$\lim_{k\to\infty}\|\hat{v}_0^{(k)}\|_{L^2(\Omega)}=\|\hat{v}_0\|_{L^2(\Omega)},$$

which implies furthermore that $\hat{v}_0^{(k)}$ converges to $\hat{v}_0$ in $L^2(\Omega)$ and completes the proof. □

## 5. Approximate controllability of the semilinear system

In this section, we prove the approximate controllability of the semilinear system (1.7), (1.10)–(1.12) by using the Kakutani fixed point theorem.

For any $w\in L^2(Q_T)$, it holds that $\sigma(x,t,w(x,t))\in L^\infty(Q_T)$ due to $\sigma\in L^\infty(Q_T\times\mathbb{R})$. Let $\tilde{u}$ be the weak solution of the problem

$$\frac{\partial\tilde{u}}{\partial t}-\operatorname{div}(a(x,t)\nabla\tilde{u})+\sigma(x,t,w(x,t))\tilde{u}=-g(x,t,0),\quad (x,t)\in Q_T, \tag{5.1}$$

$$\tilde{u}(x,t)=0,\quad (x,t)\in\Sigma, \tag{5.2}$$

$$\tilde{u}(x,0)=u_0(x),\quad x\in\Omega. \tag{5.3}$$

Consider the control system

$$\frac{\partial\check{u}}{\partial t}-\operatorname{div}(a(x,t)\nabla\check{u})+\sigma(x,t,w(x,t))\check{u}=h(x,t)\chi_D,\quad (x,t)\in Q_T, \tag{5.4}$$

$$\check{u}(x,t)=0,\quad (x,t)\in\Sigma, \tag{5.5}$$

$$\check{u}(x,0)=0,\quad x\in\Omega, \tag{5.6}$$

$$\|\check{u}(\cdot,T)-(u_d(\cdot)-\tilde{u}(\cdot,T))\|_{L^2(\Omega)}\le\varepsilon. \tag{5.7}$$

It follows from Lemma 4.2 that the control system (5.4)–(5.7) is approximately controllable with a control given by

$$h(x,t)=\|\hat{v}\|_{L^1(D_T)}z(x,t),\quad (x,t)\in Q_T, \tag{5.8}$$

where $z \in \mathrm{sgn}(\hat{v})\chi_D$ and $\hat{v}$ is the weak solution of the conjugate problem

$$-\frac{\partial \hat{v}}{\partial t} - \mathrm{div}(a(x,t)\nabla \hat{v}) + \sigma(x,t,w(x,t))\hat{v} = 0, \quad (x,t) \in Q_T, \tag{5.9}$$

$$\hat{v}(x,t) = 0, \quad (x,t) \in \Sigma, \tag{5.10}$$

$$\hat{v}(x,T) = \hat{v}_0(x), \quad x \in \Omega \tag{5.11}$$

with $\hat{v}_0 = \mathscr{M}(u_d(x) - \tilde{u}(x,T), \sigma((x,t,w(x,t)))$ being the unique minimum point of the functional

$$\begin{aligned} &J(v_0(x); u_d(x) - \tilde{u}(x,T), \sigma((x,t,w(x,t))) \\ &= \frac{1}{2}\left(\iint_{D_T} |\mathscr{L}(v_0(x); \sigma((x,t,w(x,t)))(x,t)|\mathrm{d}x\mathrm{d}t\right)^2 + \varepsilon\|v_0\|_{L^2(\Omega)} \\ &\quad - \int_\Omega (u_d(x) - \tilde{u}(x,T))v_0(x)\mathrm{d}x, \quad v_0 \in L^2(\Omega). \end{aligned}$$

Let

$$u(x,t) = \tilde{u}(x,t) + \check{u}(x,t), \quad (x,t) \in Q_T.$$

Then $u$ is just the weak solution of the problem

$$\frac{\partial u}{\partial t} - \mathrm{div}(a(x,t)\nabla u) + \sigma(x,t,w(x,t))u = -g(x,t,0) + \|\hat{v}\|_{L^1(D_T)} z(x,t)\chi_D, \quad (x,t) \in Q_T, \tag{5.12}$$

$$u(x,t) = 0, \quad (x,t) \in \Sigma, \tag{5.13}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{5.14}$$

and satisfies

$$\|u(\cdot,T) - u_d(\cdot)\|_{L^2(\Omega)} = \|\check{u}(\cdot,T) - (u_d(\cdot) - \tilde{u}(\cdot,T))\|_{L^2(\Omega)} \le \varepsilon.$$

For convenience, we denote

$$\tilde{u}(x,t) = \tilde{u}[w](x,t), \quad (x,t) \in Q_T,$$

$$\check{u}(x,t) = \check{u}[w,z](x,t), \quad (x,t) \in Q_T,$$

$$\hat{v}(x,t) = \hat{v}[w](x,t), \quad (x,t) \in Q_T,$$

$$u(x,t) = u[w,z](x,t), \quad (x,t) \in Q_T,$$

where $\tilde{u}$, $\check{u}$, $\hat{v}$ and $u$ are the solutions to the problems (5.1)–(5.3), (5.4)–(5.8), (5.9)–(5.11) and (5.12)–(5.14), respectively.

Now we define the following mapping $\Lambda$ with set values

$$\Lambda(w) = \left\{u = u[w,z] \in L^2(Q_T) : \|u(\cdot,T) - u_d(\cdot)\|_{L^2(\Omega)} \le \varepsilon, z \in \mathrm{sgn}(\hat{v}[w])\chi_D, \right\}, \quad w \in L^2(Q_T).$$

According to the above discussion, $\Lambda(w)$ is a nonempty subset of $L^2(Q_T)$ for any $w \in L^2(Q_T)$. Furthermore, the mapping $\Lambda$ possesses the following properties:

PROPOSITION 5.1. *The mapping* $\Lambda$ *satisfies that*

(i) *There exists a compact subset* $X \subset L^2(Q_T)$ *such that*

$$\Lambda(w) \subset X, \quad w \in L^2(Q_T);$$

(ii) $\Lambda(w)$ *is a nonempty convex compact subset of* $L^2(Q_T)$ *for any* $w \in L^2(Q_T)$.

*Proof.* (i) We first prove that

$$Y = \left\{\tilde{u}[w](\cdot, T) : w \in L^2(Q_T)\right\}$$

is a precompact set in $L^2(\Omega)$. Give $\{w_k\}_{k=1}^{\infty} \subset L^2(Q_T)$. Owing to $\sigma \in L^\infty(Q_T \times \mathbb{R})$, there exists a subsequence of $\{\sigma((x,t,w_k(x,t))\}_{k=1}^{\infty}$, which converges weakly $*$ in $L^\infty(Q_T)$. Then, it follows from Corollary 2.2 that there exists a subsequence of $\{\tilde{u}[w_k]\}_{k=1}^{\infty}$, which converges in $L^\infty((0,T);L^2(\Omega))$. Hence $Y \subset L^2(\Omega)$ is precompact and the closure of $Y$ in $L^2(\Omega)$ is compact. Using Proposition 4.1 and $\sigma \in L^\infty(Q_T \times \mathbb{R})$, one gets that

$$\left\{\mathscr{M}(u_d(x) - \tilde{u}(x,T), \sigma((x,t,w(x,t))) : w \in L^2(Q_T)\right\}$$

is bounded in $L^2(\Omega)$, which implies from Proposition 2.1 (i) that

$$Z = \left\{\|\hat{v}[w]\|_{L^1(D_T)} z\chi_D : z \in \operatorname{sgn}(\hat{v}[w])\chi_D, w \in L^2(Q_T)\right\}$$

is a bounded set in $L^\infty(Q_T)$. Now let us show that the set

$$X_0 = \left\{u[w,z] : z \in \operatorname{sgn}(\hat{v}[w])\chi_D, w \in L^2(Q_T)\right\}$$

is precompact. For any $u = u[w,z] \in X_0$, we can rewrite

$$u[w,z](x,t) = \tilde{u}[w](x,t) + \check{u}[w,z](x,t), \quad (x,t) \in Q_T.$$

Since $\sigma \in L^\infty(Q_T \times \mathbb{R})$ and $Z$ is bounded in $L^\infty(Q_T)$, it follows from Corollary 2.2 and Corollary 2.3 that

$$\tilde{X}_0 = \left\{\tilde{u}[w] : w \in L^2(Q_T)\right\}$$

is precompact in $L^\infty((0,T);L^2(\Omega))$, while

$$\check{X}_0 = \left\{\check{u}[w,z] : z \in \operatorname{sgn}(\hat{v}[w])\chi_D, w \in L^2(Q_T)\right\}$$

is precompact in $L^2(Q_T)$. Thus, $X_0$ is precompact and we can take $X$ as the closure of $X_0$ in $L^2(Q_T)$ to complete the proof of (i).

(ii) Fix $w \in L^2(Q_T)$. As mentioned before this proposition, $\Lambda(w)$ is a nonempty subset of $L^2(Q_T)$. It is easy to verify that $\Lambda(w)$ is convex. Therefore, we just need to show that $\Lambda(w)$ is compact. Furthermore, as $\Lambda(w) \subset X$ with $X \subset L^2(Q_T)$ being a compact set, it suffices to prove that $\Lambda(w)$ is closed. Assume that $\{u[w, z_k]\}_{k=1}^{\infty} \subset \Lambda(w)$ converging to a function $v \in X$ in $L^2(Q_T)$, where $z_k \in \operatorname{sgn}(\hat{v}[w])\chi_D$ for $k = 1, 2, \ldots$. It is not difficult to show that there exists a subsequence of $\{z_k\}_{k=1}^{\infty}$, denoted by itself for convenience, such that

$$z_k \rightharpoonup z \in \operatorname{sgn}(\hat{v}[w])\chi_D \text{ weakly} * \text{ in } L^{\infty}(Q_T) \quad \text{as } k \to \infty.$$

Then, it follows from Corollary 2.1 and Proposition 2.1 (ii) that there exists a subsequence of $\{u[w, z_k]\}_{k=1}^{\infty}$, which converges weakly to $u[w, z] \in \Lambda(w)$ in $L^2(Q_T)$. This yields $v = u[w, z] \in \Lambda(w)$ and completes the proof of (ii). □

LEMMA 5.1. *Assume that $\{w_k\}_{k=1}^{\infty}$ converges to $w$ in $L^2(Q_T)$. Then, there exists a subsequence of $\{\hat{v}[w_k]\}_{k=1}^{\infty}$, which converges to $\hat{v}[w]$ in $L^{\infty}((0, T); L^2(\Omega))$.*

*Proof.* It follows from Lemma 3.1 that

$$\sigma(x, t, w_k(x, t)) \rightharpoonup \sigma(x, t, w(x, t)) \text{ weakly} * \text{ in } L^{\infty}(Q_T) \quad \text{as } k \to \infty \tag{5.15}$$

and

$$(\sigma(x, t, w_k(x, t)) - \sigma(x, t, w(x, t)))^2 \rightharpoonup 0 \text{ weakly} * \text{ in } L^{\infty}(Q_T) \quad \text{as } k \to \infty. \tag{5.16}$$

Using Corollary 2.4, one gets from (5.16) that $\{\tilde{u}[w_k]\}_{k=1}^{\infty}$ converges to $\tilde{u}[w]$ in $L^{\infty}((0, T); L^2(\Omega))$. From this convergence and (5.15), by Proposition 4.2 we can extract a subsequence of $\{\mathscr{M}(u_d(x) - \tilde{u}[w_k](x, T), \sigma((x, t, w_k(x, t)))\}_{k=1}^{\infty}$, denoted by itself for convenience, such that

$$\begin{aligned} &\mathscr{M}(u_d(x) - \tilde{u}[w_k](x, T), \sigma((x, t, w_k(x, t))) \to \mathscr{M}(u_d(x) - \tilde{u}[w](x, T), \\ &\quad \sigma((x, t, w(x, t))) \text{ in } L^2(Q_T). \end{aligned} \tag{5.17}$$

Finally, using Corollary 2.4 again, it follows from (5.16) and (5.17) that $\{\hat{v}[w_k]\}_{k=1}^{\infty}$ converges to $\hat{v}[w]$ in $L^{\infty}((0, T); L^2(\Omega))$. □

PROPOSITION 5.2. *The mapping $\Lambda$ is upper hemicontinuous in $L^2(Q_T)$. That is to say, for any $\xi \in L^2(Q_T)$, the functional*

$$\lambda(w; \xi) = \sup_{u \in \Lambda(w)} \iint_{Q_T} u(x, t)\xi(x, t)\mathrm{d}x\mathrm{d}t, \quad w \in L^2(Q_T)$$

*is upper hemicontinuous in $L^2(Q_T)$.*

*Proof.* For fixed $\xi \in L^2(Q_T)$, let us prove that $\lambda(\cdot; \xi)$ is upper hemicontinuous in $L^2(Q_T)$. Otherwise, there exists a sequence $\{w_k\}_{k=1}^{\infty} \subset L^2(Q_T)$ and a function $w \in L^2(Q_T)$, such that

$$w_k \to w \text{ in } L^2(Q_T) \quad \text{as } k \to \infty \tag{5.18}$$

and

$$\lim_{k\to\infty} \lambda(w_k;\xi) > \lambda(w;\xi). \tag{5.19}$$

Owing to (5.18), one gets by Lemma 3.1 that

$$\sigma(x,t,w_k(x,t)) \rightharpoonup \sigma(x,t,w(x,t)) \text{ weakly } * \text{ in } L^\infty(Q_T) \qquad \text{as } k\to\infty. \tag{5.20}$$

Since $\Lambda(w_k)$ is compact owing to Proposition 5.1 (ii), there exists $u_k = u_k[w_k, z_k] \in \Lambda(w_k)$ such that

$$\lambda(w_k;\xi) = \iint_{Q_T} u_k(x,t)\xi(x,t)\mathrm{d}x\mathrm{d}t \tag{5.21}$$

for each $k = 1, 2, \ldots$, where

$$z_k \in \operatorname{sgn}(\hat{v}[w_k])\chi_D, \quad k = 1, 2, \ldots.$$

From Lemma 5.1 with (5.18), there exists a subsequence of $\{\hat{v}[w_k]\}_{k=1}^\infty$, denoted by itself for convenience, such that

$$\hat{v}[w_k] \to \hat{v}[w] \text{ in } L^\infty((0,T);L^2(\Omega)) \qquad \text{as } k\to\infty. \tag{5.22}$$

From this convergence, we can extract a subsequence of $\{z_k\}_{k=1}^\infty$, denoted by itself for convenience, such that

$$z_k \rightharpoonup z \in \operatorname{sgn}(\hat{v}[w])\chi_D \text{ weakly } * \text{ in } L^\infty(Q_T) \qquad \text{as } k\to\infty. \tag{5.23}$$

Using Corollary 2.1 with (5.20), (5.22) and (5.23) and using Proposition 2.1 (ii), we get that there exists a subsequence of $\{u_k[w_k, z_k]\}_{k=1}^\infty$, denoted by itself for convenience, which converges weakly to $u = u[w, z] \in \Lambda(w)$ in $L^2(Q_T)$. From this convergence and (5.21), we get that

$$\lim_{k\to\infty} \lambda(w_k;\xi) = \iint_{Q_T} u(x,t)\xi(x,t)\mathrm{d}x\mathrm{d}t \le \lambda(w;\xi),$$

which contradicts (5.19). The proof is complete. □

From these two propositions, we can prove the approximate controllability of the semilinear system (1.7), (1.10)–(1.12) by using the Kakutani fixed point theorem.

THEOREM 5.1. *The semilinear system* (1.7)*,* (1.10)–(1.12) *is approximately controllable. More precisely, for any* $u_0 \in L^2(\Omega)$*,* $u_d \in L^2(\Omega)$ *and* $\varepsilon > 0$*, there exist* $v \in L^\infty((0,T);L^2(\Omega)) \cap \mathscr{B}$ *and* $z \in \operatorname{sgn}(v)\chi_D$ *such that the weak solution* $u$ *of the problem* (1.7)*,* (1.10)*,* (1.11) *with*

$$h(x,t) = \|v\|_{L^1(D_T)} z(x,t), \quad (x,t)\in Q_T$$

*satisfies* (1.12)*.*

*Proof.* For fixed $u_0 \in L^2(\Omega)$, $u_d \in L^2(\Omega)$ and $\varepsilon > 0$, it follows from Propositions 5.1 and 5.2 that the restriction of the mapping $\Lambda$ to the convex hull of $X$ satisfies the hypotheses of the Kakutani fixed point theorem [16]. Therefore, $\Lambda$ admits a fixed point $u \in L^2(Q_T)$. That is to say, $u \in L^\infty((0,T); L^2(\Omega)) \cap \mathscr{B}$ is a weak solution to the problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \operatorname{div}(a(x,t)\nabla u) + g(x,t,u) = \|\hat{v}\|_{L^1(D_T)} z(x,t)\chi_D, \quad (x,t) \in Q_T,\\
&u(x,t) = 0, \quad (x,t) \in \Sigma,\\
&u(x,0) = u_0(x), \quad x \in \Omega
\end{aligned}$$

and satisfies

$$\|u(\cdot,T) - u_d(\cdot)\|_{L^2(\Omega)} \leq \varepsilon,$$

where $z \in \operatorname{sgn}(v)\chi_D$, while $v \in L^\infty((0,T); L^2(\Omega)) \cap \mathscr{B}$ is the weak solution of the conjugate problem

$$\begin{aligned}
&-\frac{\partial v}{\partial t} - \operatorname{div}(a(x,t)\nabla v) + \sigma(x,t,u(x,t))v = 0, \quad (x,t) \in Q_T,\\
&v(x,t) = 0, \quad (x,t) \in \Sigma,\\
&v(x,T) = \mathscr{M}(u_d(x) - \tilde{u}(x,T), \sigma((x,t,u(x,t))), \quad x \in \Omega
\end{aligned}$$

with $\tilde{u} \in L^\infty((0,T); L^2(\Omega)) \cap \mathscr{B}$ being the weak solution to the problem

$$\begin{aligned}
&\frac{\partial \tilde{u}}{\partial t} - \operatorname{div}(a(x,t)\nabla \tilde{u}) + \sigma(x,t,u(x,t))\tilde{u} = -g(x,t,0), \quad (x,t) \in Q_T,\\
&\tilde{u}(x,t) = 0, \quad (x,t) \in \Sigma,\\
&\tilde{u}(x,0) = u_0(x), \quad x \in \Omega.
\end{aligned}$$

The proof of the theorem is complete. □

REMARK 5.1. The controls obtained in Theorem 5.1 are quasi bang-bang controls.

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## REFERENCES

[1] ALABAU-BOUSSOUIRA, F., CANNARSA, P. and FRAGNELLI, G., *Carleman estimates for degenerate parabolic operators with applications to null controllability*, J. Evol. Equ., *6*(2)(2006), 161–204.

[2] BARBU, V., *Controllability of parabolic and Navier–Stokes equations*, Sci. Math. Jpn., *56*(1)(2002), 143–211.

[3] CANNARSA, P. and FRAGNELLI, G., *Null controllability of semilinear degenerate parabolic equations in bounded domains*, Electron. J. Differential Equations, *2006*(136)(2006), 1–20.

[4] Cannarsa, P., Martinez, P. and Vancostenoble, J., *Persistent regional null controllability for a class of degenerate parabolic equations*, Commun. Pure Appl. Anal., *3*(4)(2004), 607–635.

[5] Cannarsa, P., Martinez, P. and Vancostenoble J., *Null controllability of degenerate heat equations*, Adv. Differential Equations, *10*(2)(2005), 153–190.

[6] Cannarsa, P., Fragnelli, G. and Vancostenoble, J., *Linear degenerate parabolic equations in bounded domains: controllability and observability, Systems, control, modeling and optimization*, 163–173, IFIP Int. Fed. Inf. Process., 202, Springer, New York, 2006.

[7] Cannarsa, P., Fragnelli, G. and Vancostenoble, J., *Regional controllability of semilinear degenerate parabolic equations in bounded domains*, J. Math. Anal. Appl., *320*(2)(2006), 804–818.

[8] Cannarsa, P., Martinez, P. and Vancostenoble, J., *Carleman estimates for a class of degenerate parabolic operators*, SIAM J. Control Optim., *47*(1)(2008), 1–19.

[9] Escauriaza, L. and Fernández, F. J., *Unique continuation for parabolic operators*, Ark. Mat., *41*(1)(2003), 35–60.

[10] Fabre, C., Puel, J. and Zuazua, E., *Approximate controllability of a semilinear heat equation*, Proc. Roy. Soc. Edinburgh Sect. A, *125*(1)(1995), 31–61.

[11] Fernández- Cara, E. and Zuazua, E., *The cost of approximate controllability for heat equations: The linear case*, Adv. Differential Equations, *5*(2000), 465–514.

[12] Fernández- Cara, E. and Zuazua, E., *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré Anal. Non Lineaire, *17*(5)(2000), 583–616.

[13] Fursikov, A. V. and Imanuvilov, O. Y., *Controllability of evolution equations*, Lecture Notes Series 34, Seoul National University, Seoul, Korea, 1996.

[14] Gao, H., Hou, X. and Pavel, N. H., *Optimal control and controllability problems for a class of nonlinear degenerate diffusion equations*, Panamer. Math. J., *13*(1)(2003), 103–126.

[15] Gilbarg, D. and Trudinger, N. S., *Elliptic partial differential equations of second order*, Second edition, Springer-Verlag, Berlin, 1983.

[16] Kakutani, S., *A generalization of Brouwer's fixed point theorem*, Duke Math J, *8*(1941), 457–459.

[17] Lin, P., Gao, H. and Liu, X., *Some results on controllability of a nonlinear degenerate parabolic system by bilinear control*, J. Math. Anal. Appl., *326*(2)(2007), 1149–1160.

[18] Lions, J. L., *Remarques sur la contrôlabilité approchée*, Proceedings of "Jornadas Hispano-Francesas sobre Control de Sistemas Distribuidos", University of Málaga, Spain, October, 1990.

[19] Lions, J. L., *Remarks on approximate controllability*, J. Anal. Math., *59*(1992), 103–116.

[20] Liu, X. and Gao, H., *Controllability of a class of Newtonian filtration equations with control and state constraints*, SIAM J. Control Optim., *46*(6)(2007), 2256–2279.

[21] Martinez, P., Raymond, J. P. and Vancostenoble, J., *Regional null controllability of a linearized Crocco-type equation*, SIAM J. Control Optim., *42*(2)(2003), 709–728.

[22] Martinez, P. and Vancostenoble, J., *Carleman estimates for one-dimensional degenerate heat equations*, J. Evol. Equ., *6*(2)(2006), 325–362.

[23] Oleinik, O. A. and Radkevic, E. V., *Second order differential equations with nonnegative characteristic form*, American Mathematical Society, Rhode Island and Plenum Press, New York, 1973.

[24] Wang, C., *Approximate controllability of a class of degenerate systems*, Appl. Math. Comput., *203*(1)(2008), 447–456.

[25] Wang, C., Wang, L., Yin, J. and Zhou, S., *Hölder continuity of weak solutions of a class of linear equations with boundary degeneracy*, J. Differential Equations, *239*(1)(2007), 99–131.

[26] Yin, J. and Wang, C., *Evolutionary weighted p-Laplacian with boundary degeneracy*, J. Differential Equations, *237*(2)(2007), 421–445.

[27] Wu, Z., Zhao, J., Yin, J. and Li, H., *Nonlinear diffusion equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

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