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Dirichlet forms for singular one-dimensional operators and on graphs

ULRIKE KANT, TOBIAS KLAUSS, JÜRGEN VOIGT AND MATTHIAS WEBER

Abstract. We treat the time evolution of states on a finite directed graph, with singular diffusion on the edges of the graph and glueing conditions at the vertices. The operator driving the evolution is obtained by the method of quadratic forms on a suitable Hilbert space. Using the Beurling–Deny criteria we describe glueing conditions leading to positive and to submarkovian semigroups, respectively.

0. Introduction

The intentions of this paper are twofold. The first aim is to present a treatment of one-dimensional "singular" diffusion in the framework of Dirichlet forms. The second is to present suitable boundary or glueing conditions on graphs (quantum graphs) leading to positive or submarkovian C_0-semigroups.

Concerning the first topic we assume that μ is a finite Borel measure on a bounded interval $[a, b]$. We assume that particles move in $[a, b]$ according to "Brownian motion" but are only allowed to be located in the support of μ, and in fact are slowed down or accelerated by the "speed measure" μ. (Incidentally, the support of μ is allowed to have gaps, what sometimes is referred to by "gap diffusion".) More generally, instead of starting with Brownian motion, one also can include a drift in the diffusion. This leads to including a scale function. The treatment of the corresponding process has a longer history (cf. [1,3–5,12–14,19,20,22]), but there appears to be no treatment of the arising evolution in the context of Dirichlet forms.

Concerning the second topic, we assume that finitely many intervals, with diffusion as described above, are arranged in a graph, and we treat the question how boundary conditions (glueing conditions) at the vertices can be posed in a way to describe diffusing particles. These topics have also been treated in a recent paper by Kostrykin et al. [16]. Since we pose the glueing conditions in a different form (following [17]) it does not seem evident to us to establish the connection between the conditions given in [16] and our conditions.

There are many motivations from applications for this work. Since we were not primarily motivated by specific applications we refer to [17,18,23] for some of these

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motivations. An evident motivation is the description of (electric) currents in networks. In [10] it is explained how diffusion processes on graphs can arise from various limiting or averaging procedures of diffusion processes on higher dimensional sets. This theory has been worked out in more specific situations in [6–9,11]. To close the introduction we describe the contents of the paper in more detail.

In Sect. 1 we present the treatment of one-dimensional singular diffusion mentioned initially on an interval $[a, b]$. We assume that μ is a finite Borel measure on $[a, b]$, with $a, b \in \operatorname{spt} \mu$ but $\mu(\{a, b\}) = 0$. We define the classical Dirichlet form τ,

$$\tau(f, g) := \int f(x)\overline{g(x)}\,\mathrm{d}x,$$

in the Hilbert space $\mathcal{H} = L_2((a, b), \mu)$, on a suitable domain $D(\tau)$. The interesting (and touchy) point in this combination of data is the circumstance that in the form the values of f, g may occur on intervals where μ does not have mass. We show that the C_0-semigroup associated with the form is submarkovian (Theorem 1.7), and we describe the self-adjoint operator associated with the form (Theorem 1.9).

In Sect. 2 we treat a slightly more complicated operator on an interval. We assume that, besides the data prescribed in Sect. 1, there also is a scale $s\colon [a, b] \to \mathbb{R}$, continuous and strictly monotonically increasing. We treat the second order differential expression $\partial_\mu \partial_s$, interpreted in the sense of distributions. We show that—by using $s\colon [a, b] \to [s(a), s(b)]$ as a homeomorphism—the problem can be transformed to the situation treated in Sect. 1. In view of this reduction we did not include a scale in our further investigations in this paper.

In Sect. 3 we start the treatment of diffusion processes on graphs. We consider a finite directed metric graph. The graph Γ is given by finite sets of vertices (nodes) and edges. Each edge has a starting vertex and an end vertex and otherwise is considered as a bounded interval in $\mathbb{R}$. The vertices carry no mass, and on each of the edges a formal differential expression like in Sect. 1 is given. From [17] we recall a method how to prescribe glueing conditions at the vertices in terms of forms, and we give the description of the operators associated with the forms (Theorem 3.3). We then single out those glueing conditions giving rise to positive and to submarkovian C_0-semigroups (Theorem 3.5).

In Sect. 4 we treat the case of a graph as in Sect. 3 but where additionally the vertices may carry a mass. In this case we only treat the case where the functions in the domain of the form are continuous on the whole graph, which in particular means that the traces of functions at the end points of the edges coincide with the values of the functions at the corresponding vertices. In this case the corresponding C_0-semigroup is always submarkovian (Theorem 4.2). In the description of the corresponding operator (Theorem 4.3) the boundary condition also occurs in the value of the operator applied to functions at the corresponding vertices.

In an Appendix we indicate the structure of (Stonean) sublattices of $\mathbb{K}^n$ and of positive or submarkovian operators on these sublattices. These results are needed in Sect. 3.

1. The singular Dirichlet form on an interval

Let $a, b \in \mathbb{R}, a < b$. Let $\mu \neq 0$ be a finite Borel measure on $[a, b]$, $a, b \in \operatorname{spt} \mu$, but $\mu(\{a, b\}) = 0$. In order to define the classical Dirichlet form τ in $\mathcal{H} := L_2((a, b), \mu)$ we need some notation. Our function spaces will consist of $\mathbb{K}$-valued functions, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We note that $[a, b] \setminus \operatorname{spt} \mu$ is an open subset of $\mathbb{R}$, and we define

$$C_\mu[a, b] := \big\{f \in C[a, b];\ f \text{ affine linear on the components of } [a, b] \setminus \operatorname{spt} \mu\big\}.$$

In order to pose boundary conditions we choose $\alpha, \beta \in [0, \frac{\pi}{2}]$. We define

$$C_{\mu,\alpha,\beta}[a, b] := \begin{cases} \big\{f \in C_\mu[a, b];\ f(a) = f(b) = 0\big\} & \text{if } \alpha = 0,\ \beta = 0, \\ \big\{f \in C_\mu[a, b];\ f(a) = 0\big\} & \text{if } \alpha = 0,\ 0 < \beta \le \frac{\pi}{2}, \\ \big\{f \in C_\mu[a, b];\ f(b) = 0\big\} & \text{if } 0 < \alpha \le \frac{\pi}{2},\ \beta = 0, \\ C_\mu[a, b] & \text{otherwise,} \end{cases}$$

$$W^1_{2,\mu}(a, b) := W^1_2(a, b) \cap C_\mu[a, b].$$

The spaces $C_\mu[a, b]$ and $C_{\mu,\alpha,\beta}[a, b]$ are closed subspaces of $(C[a, b], \|\cdot\|_\infty)$.

We define the form τ by

$$D(\tau) := W^1_{2,\mu,\alpha,\beta}(a, b) := C_{\mu,\alpha,\beta}[a, b] \cap W^1_2(a, b),$$

$$\tau(f, g) := \int_a^b f'(x)\overline{g'(x)}\,\mathrm{d}x + Q_0(f, g) + Q_1(f, g),$$

where $Q_0, Q_1 \colon D(\tau) \times D(\tau) \to \mathbb{K}$ are defined by

$$Q_0(f, g) := \begin{cases} 0 & \text{if } \alpha = 0, \\ \frac{\cos\alpha}{\sin\alpha} f(a)\overline{g(a)} & \text{if } 0 < \alpha \le \frac{\pi}{2}, \end{cases}$$

$$Q_1(f, g) := \begin{cases} 0 & \text{if } \beta = 0, \\ \frac{\cos\beta}{\sin\beta} f(b)\overline{g(b)} & \text{if } 0 < \beta \le \frac{\pi}{2}. \end{cases}$$

REMARK 1.1. The value $\alpha = 0$ corresponds to the zero Dirichlet boundary condition at a, $\alpha = \frac{\pi}{2}$ to the Neumann boundary condition, and $\alpha \in (0, \frac{\pi}{2})$ to Robin boundary conditions. Similarly for β and b.

We note that the form τ is well-defined, i.e., $f \in D(\tau)$, $f = 0$ as an element of $\mathcal{H}$ (or equivalently, $f(x) = 0$ μ-a.e.) implies $\tau(f, g) = 0$ for all $g \in D(\tau)$. Indeed, if $f = 0$ μ-a.e. and f is continuous, then $f(x) = 0$ for all $x \in \operatorname{spt} \mu$. Since f is affine linear on the components of $[a, b] \setminus \operatorname{spt} \mu$ we conclude that $f(x) = 0$ for all $x \in [a, b]$, and thus $\tau(f, g) = 0$ for all $g \in D(\tau)$.

The definition of τ immediately shows that τ is positive and symmetric.

The following lemma is needed in order to obtain the denseness of $D(\tau)$ in $\mathcal{H}$, and later for the Beurling–Deny criteria. In its proof we will need the fact that the open set $[a, b] \setminus \operatorname{spt} \mu$ can be decomposed as

$$[a, b] \setminus \operatorname{spt} \mu = \bigcup_{j \in N} (a_j, b_j),$$

where $N \subseteq \mathbb{N}$ and $\big((a_j, b_j)\big)_{j \in N}$ is a countable family of mutually disjoint open intervals. This decomposition will be used repeatedly.

By the Sobolev embedding theorem the space $W_2^1(a, b)$ can be considered as a subspace of $C[a, b]$; for elements $f \in W_2^1(a, b)$ we will always choose the continuous representative.

LEMMA 1.2. *Let $f \in W_2^1(a, b)$. Define*

$$\tilde{f}(x) := \begin{cases} f(a_j) + \frac{f(b_j)-f(a_j)}{b_j-a_j}(x - a_j) & \text{if } x \in (a_j, b_j),\ j \in N, \\ f(x) & \text{if } x \in \operatorname{spt} \mu. \end{cases}$$

Then $\tilde{f} \in W_{2,\mu}^1(a, b)$, and $\tilde{f} = f$ as elements of $\mathcal{H}$. Moreover,

$$\tilde{f}'(x) = \begin{cases} \frac{f(b_j)-f(a_j)}{b_j-a_j} & \text{if } x \in (a_j, b_j),\ j \in N, \\ f'(x) & \text{if } x \in \operatorname{spt} \mu. \end{cases}$$

Proof. For $n \in \mathbb{N}$ we define

$$f_n(x) := \begin{cases} f(a_j) + \frac{f(b_j)-f(a_j)}{b_j-a_j}(x - a_j) & \text{if } x \in (a_j, b_j),\ j \in N,\ j \le n, \\ f(x) & \text{otherwise}, \end{cases}$$

$$g_n(x) := \begin{cases} \frac{f(b_j)-f(a_j)}{b_j-a_j} & \text{if } x \in (a_j, b_j),\ j \in N,\ j \le n, \\ f'(x) & \text{otherwise}. \end{cases}$$

It is easy to see that then $f_n \in W_2^1(a, b)$, and $f_n' = g_n$, for all $n \in \mathbb{N}$. Obviously $\tilde{f} = f_n = f$ as elements of $\mathcal{H}$, for all $n \in \mathbb{N}$.

Moreover $\tilde{f} \in C_\mu[a, b]$, and $f_n \to \tilde{f}$ $(n \to \infty)$ uniformly on $[a, b]$. We define

$$g(x) := \lim_{n\to\infty} g_n(x) \quad (x \in (a, b)).$$

From

$$\int_{a_j}^{b_j} |g(x)|^2 \,\mathrm{d}x = (b_j - a_j) \left| \frac{f(b_j) - f(a_j)}{b_j - a_j} \right|^2$$
$$= \frac{1}{b_j - a_j} \left| \int_{a_j}^{b_j} f'(y) \,\mathrm{d}y \right|^2 \le \int_{a_j}^{b_j} |f'(y)|^2 \,\mathrm{d}y,$$

for all $j \in N$, we obtain $g \in L_2(a, b)$, and therefore the dominated convergence theorem implies $g_n \to g$ in $L_2(a, b)$. This shows $\tilde{f} \in W_2^1(a, b)$, $\tilde{f}' = g$. □

THEOREM 1.3. *The set $C_c^1(a, b)$, as a subset of $\mathcal{H}$, is contained in $D(\tau)$. As a consequence, $D(\tau)$ is dense in $\mathcal{H}$.*

Proof. According to Lemma 1.2, each element $f \in C_c^1(a, b)$ possesses a representative $\tilde{f} \in D(\tau)$. (Note that $\tilde{f}(a) = \tilde{f}(b) = 0$, and therefore $\tilde{f} \in C_{\mu,\alpha,\beta}[a, b]$.)

The set $C_c(a, b)$ is dense in $\mathcal{H}$. Using mollifiers one obtains that $C_c^1(a, b)$ is dense in $\mathcal{H}$. □

In order to show the closedness of τ we need the continuity of the embedding $(D(\tau), \|\cdot\|_\tau) \hookrightarrow C[a,b]$, where the form norm $\|\cdot\|_\tau$ is defined by

$$\|f\|_\tau := \Big(\tau(f) + \|f\|_{\mathcal{H}}^2\Big)^{\frac{1}{2}} \quad (f \in D(\tau)).$$

LEMMA 1.4. *There exists a constant* $C > 0$ *such that*

$$\|f\|_\infty \le C\left(\|f'\|_{L_2(a,b)}^2 + \|f\|_{\mathcal{H}}^2\right)^{\frac{1}{2}}$$

for all $f \in W_2^1(a,b)$.

Proof. Let $x, y \in (a,b)$. The Cauchy–Schwarz inequality yields

$$|f(x) - f(y)| = \left|\int_y^x f'(z)\,\mathrm{d}z\right| \le \|f'\|_{L_2(a,b)}(b-a)^{\frac{1}{2}},$$
$$|f(x)| \le |f(y)| + \|f'\|_{L_2(a,b)}(b-a)^{\frac{1}{2}}.$$

Integrating this inequality with respect to μ one obtains

$$\begin{aligned}|f(x)|\mu((a,b)) &\le \int_{(a,b)} |f(y)|\,\mathrm{d}\mu(y) + \|f'\|_{L_2(a,b)}(b-a)^{\frac{1}{2}}\mu((a,b))\\ &\le \|f\|_{\mathcal{H}}\,\mu((a,b))^{\frac{1}{2}} + \|f'\|_{L_2(a,b)}(b-a)^{\frac{1}{2}}\mu((a,b)).\end{aligned}$$

This inequality shows the assertion. □

THEOREM 1.5. *The form* τ *is closed.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subseteq D(\tau)$, $f_n \to f$ $(n\to\infty)$ in $\mathcal{H}$, $\tau(f_n - f_m) \to 0$ $(m,n\to\infty)$. We have to show that $f \in D(\tau)$, $\tau(f_n - f) \to 0$ $(n\to\infty)$.

The sequence (f_n) is a $\|\cdot\|_\tau$-Cauchy sequence, and therefore Lemma 1.4 implies that (f_n) is a $\|\cdot\|_\infty$-Cauchy sequence. This implies that there exists $\tilde{f} \in C_{\mu,\alpha,\beta}[a,b]$ such that $\|f_n - \tilde{f}\|_\infty \to 0$ $(n\to\infty)$. We further conclude that $f_n \to \tilde{f}$ in $\mathcal{H}$, and therefore we may assume that $f = \tilde{f}$ belongs to $C_{\mu,\alpha,\beta}[a,b]$.

Since (f_n) is uniformly convergent to f and (f_n') is a Cauchy sequence in $L_2(a,b)$, thus convergent, we obtain that $f \in W_2^1(a,b)$, $f_n' \to f'$ in $L_2(a,b)$ $(n\to\infty)$. We conclude that $f \in C_{\mu,\alpha,\beta}[a,b] \cap W_2^1(a,b) = D(\tau)$,

$$\begin{aligned}\tau(f_n - f) &= \int_a^b |f_n'(x) - f'(x)|^2\,\mathrm{d}x + Q_0(f_n - f) + Q_1(f_n - f)\\ &\to 0 \quad (n\to\infty).\end{aligned}$$

□

REMARKS 1.6. (a) Summarising, we have shown that τ is a densely defined closed positive form in $\mathcal{H}$. The "first representation theorem" implies that there is a unique positive definite self-adjoint operator H in $\mathcal{H}$ associated with τ; cf. [15, VI, Theorem 2.1]. Accordingly, the operator $-H$ is the generator of a C_0-semigroup $(e^{-tH})_{t\ge 0}$ on $\mathcal{H}$.

In order to show that the operators e^{-tH} are positive (in the sense of the order on $\mathcal{H} = L_2((a,b),\mu)$) and moreover submarkovian we will show that the conditions of the Beurling–Deny criteria are satisfied.

(b) We recall that the first Beurling–Deny criterion states (amongst others) that positivity of the C_0-semigroup $(e^{-tH})_{t\geq 0}$ is equivalent to:

For all $f \in D(\tau)$ one has $|f| \in D(\tau)$, $\tau(|f|) \leq \tau(f)$ (cf. [25, Corollary 2.18]).

(c) The second Beurling–Deny criterion states (amongst others) that, assuming positivity of $(e^{-tH})_{t\geq 0}$, the property that $(e^{-tH})_{t\geq 0}$ is submarkovian (i.e., positive and L_∞-contractive) is equivalent to:

For all $0 \leq f \in D(\tau)$ one has $f \wedge 1 \in D(\tau)$, $\tau(f \wedge 1) \leq \tau(f)$ (cf. [25, Corollary 2.18]).

(d) A mapping $F : \mathbb{K} \to \mathbb{K}$ is called a *normal contraction* if $F(0) = 0$ and $|F(x) - F(y)| \leq |x-y|$ $(x, y \in \mathbb{K})$. The C_0-semigroup $(e^{-tH})_{t\geq 0}$ is submarkovian if and only if the following condition is satisfied:

For all normal contractions $F : \mathbb{K} \to \mathbb{K}$ and all $f \in D(\tau)$ one has $F \circ f \in D(\tau)$, $\tau(F \circ f) \leq \tau(f)$ (cf. [25, Theorem 2.25]).

THEOREM 1.7. *Let $F : \mathbb{K} \to \mathbb{K}$ be a normal contraction. Let $f \in D(\tau)$ $(= C_{\mu,\alpha,\beta}[a,b] \cap W_2^1(a,b))$. Then $F \circ f \in D(\tau)$, $\tau(F \circ f) \leq \tau(f)$. As a consequence, the C_0-semigroup $(e^{-tH})_{t\geq 0}$ is (positive and) submarkovian.*

Proof. It is well-known that $F \circ f \in W_2^1(a,b)$ and $|(F \circ f)'| \leq |f'|$. (See Remark 1.8 below.) Using Lemma 1.2 we obtain that the function $\tilde{f}$, defined by

$$\tilde{f}(x) := \begin{cases} F(f(a_j)) + \frac{F(f(b_j)) - F(f(a_j))}{b_j - a_j}(x - a_j) & \text{if } x \in (a_j, b_j),\ j \in N, \\ F(f(x)) & \text{if } x \in \operatorname{spt} \mu, \end{cases}$$

belongs to $C_{\mu,\alpha,\beta}[a,b]$. Evidently $\tilde{f} = F \circ f$ as elements of $\mathcal{H}$. For $j \in N$, $x \in (a_j, b_j)$ one obtains the estimate

$$|\tilde{f}'(x)| = \left| \frac{F(f(b_j)) - F(f(a_j))}{b_j - a_j} \right| \leq \left| \frac{f(b_j) - f(a_j)}{b_j - a_j} \right| = |f'(x)|,$$

and therefore $F \circ f \in D(\tau)$,

$$\begin{aligned} \tau(F \circ f) &= \tau(\tilde{f}) \\ &= \int_{\operatorname{spt} \mu} |(F \circ f)'(x)|^2 \, \mathrm{d}x + \sum_{j \in N} \int_{a_j}^{b_j} |\tilde{f}'(x)|^2 \, \mathrm{d}x \\ &\quad + Q_0(F \circ f) + Q_1(F \circ f) \\ &\leq \tau(f). \end{aligned}$$

Now the second last assertion of the theorem follows from Remark 1.6(d). □

REMARK 1.8. A function $f \in C[a,b]$ belongs to $W_1^2(a,b)$ if and only if f is absolutely continuous, and then f is differentiable almost everywhere, and f' is the distributional derivative. Then, if f is absolutely continuous and $F\colon \mathbb{K} \to \mathbb{K}$ is a normal contraction, it is immediate that $F \circ f$ is absolutely continuous, and therefore $F \circ f \in W_1^1(a,b)$. For those points x where f and $F \circ f$ are differentiable (a set of full measure) it is immediate that $|(F \circ f)'(x)| \le |f'(x)|$, and therefore $|(F \circ f)'| \le |f'|$. (For similar properties in more dimensions we refer to [21, Appendix, Corollary 1].)

To conclude this section we describe the operator H associated with the form τ. In order to do so we introduce some additional notation.

If $f \in L_{1,\mathrm{loc}}(a,b)$, $g \in L_1((a,b),\mu)$ are such that $f' = g\mu$ (where $f' = \partial f$ denotes the distributional derivative of f), then we call g *distributional derivative of f with respect to μ*, and we write

$$\partial_\mu f := g.$$

Note that then necessarily $f' = 0$ on $[a,b] \setminus \operatorname{spt}\mu$, i.e., f is constant on each of the components of $[a,b] \setminus \operatorname{spt}\mu$. It is easy to see that this is equivalent to

$$f(x) = c + \int_{(a,x)} g(y)\,\mathrm{d}\mu(y) \quad \text{a.e.,} \tag{1.1}$$

with some $c \in \mathbb{K}$. Thus, the function f has representatives of bounded variation and these have one-sided limits (not depending on the representative) at all points of $[a,b]$.

We define the "maximal operator" $\hat{H}$ in $\mathcal{H}$, associated with the differential expression $-\partial_\mu \partial$, by

$$\begin{aligned} D(\hat{H}) &:= \big\{f \in C_\mu[a,b];\ f' \in L_1(a,b),\ \partial_\mu f' \text{ exists, } \partial_\mu f' \in \mathcal{H}\big\}, \\ \hat{H} f &:= -\partial_\mu f' \quad (f \in D(\hat{H})). \end{aligned}$$

(Note that $\hat{H}$ is well-defined as an operator in $\mathcal{H}$: If $f \in C_\mu[a,b]$ vanishes on $\operatorname{spt}\mu$, then $f = 0$, and therefore $\partial_\mu f' = 0$.)

Using (1.1) one easily obtains that $D(\hat{H}) \subseteq W_2^1(a,b)$.

THEOREM 1.9. *The operator H is given by*

$$\begin{aligned} D(H) &= \big\{f \in D(\hat{H});\ \cos\alpha f(a) - \sin\alpha f'(a+) = \cos\beta f(b) + \sin\beta f'(b-) = 0\big\}, \\ Hf &= -\partial_\mu f' \quad (f \in D(H)). \end{aligned}$$

Proof. As a preliminary step we supply an identity. Let $f \in D(\hat{H})$, $g \in D(\tau)$. From (1.1) we obtain

$$f'(x) = f'(a+) + \int_{(a,x)} \partial_\mu f'(y)\,\mathrm{d}\mu(y) \quad \text{a.e.}$$

Using Fubini's theorem, we now obtain

$$\begin{aligned}
&\int_a^b f'(x)\overline{g'(x)}\,\mathrm{d}x \\
&= \int_a^b \left(f'(a+) + \int_{(a,x)} \partial_\mu f'(y)\,\mathrm{d}\mu(y) \right) \overline{g'(x)}\,\mathrm{d}x \\
&= f'(a+)\overline{(g(b)-g(a))} + \int_{(a,b)} \int_{(a,x)} \partial_\mu f'(y)\,\mathrm{d}\mu(y)\overline{g'(x)}\,\mathrm{d}x \\
&= f'(a+)\overline{(g(b)-g(a))} + \int_{(a,b)} \partial_\mu f'(y) \int_{(y,b)} \overline{g'(x)}\,\mathrm{d}x\,\mathrm{d}\mu(y) \\
&= f'(a+)\overline{(g(b)-g(a))} + \int_{(a,b)} \partial_\mu f'(y)\overline{(g(b)-g(y))}\,\mathrm{d}\mu(y) \\
&= f'(b-)\overline{g(b)} - f'(a+)\overline{g(a)} + (\hat{H} f \,|\, g)_{\mathcal{H}}.
\end{aligned} \tag{1.2}$$

Let $f \in D(H)$ ($\subseteq D(\tau)$), $g \in D(\tau)$. Then

$$\begin{aligned}
\int_{(a,b)} Hf(x)\overline{g(x)}\,\mathrm{d}\mu(x) &= \tau(f,g) \\
&= \int_a^b f'(x)\overline{g'(x)}\,\mathrm{d}x + Q_0(f,g) + Q_1(f,g).
\end{aligned}$$

Choosing $g \in C_c^1(a,b)$, $\tilde{g} \in C_\mu[a,b]$ such that $\tilde{g}|_{\operatorname{spt}\mu} = g|_{\operatorname{spt}\mu}$ (recall Lemma 1.2), we obtain

$$\begin{aligned}
\int_{(a,b)} Hf(x)\overline{g(x)}\,\mathrm{d}\mu(x) &= \int_a^b f'(x)\overline{\tilde{g}'(x)}\,\mathrm{d}x \\
&= \int_{\operatorname{spt}\mu} f'(x)\overline{\tilde{g}'(x)}\,\mathrm{d}x + \sum_{j\in N} f'(a_j+) \int_{a_j}^{b_j} \overline{\frac{g(b_j)-g(a_j)}{b_j-a_j}}\,\mathrm{d}x \\
&= \int_{\operatorname{spt}\mu} f'(x)\overline{g'(x)}\,\mathrm{d}x + \sum_{j\in N} f'(a_j+) \int_{a_j}^{b_j} \overline{g'(x)}\,\mathrm{d}x \\
&= \int_a^b f'(x)\overline{g'(x)}\,\mathrm{d}x.
\end{aligned}$$

This implies $f \in D(\hat{H})$, $Hf = \hat{H}f = -\partial_\mu f'$.

For general $g \in D(\tau)$ we conclude, using (1.2),

$$\begin{aligned}
\tau(f,g) - (Q_0(f,g) + Q_1(f,g)) &= \int_a^b f'(x)\overline{g'(x)}\,\mathrm{d}x \\
&= f'(b-)\overline{g(b)} - f'(a+)\overline{g(a)} + (Hf \,|\, g)_{\mathcal{H}}.
\end{aligned}$$

This means that the equation

$$Q_0(f,g) - f'(a+)\overline{g(a)} + Q_1(f,g) + f'(b-)\overline{g(b)} = 0$$

must be satisfied. If $\alpha = 0$, then $\cos\alpha f(a) - \sin\alpha f'(a+) = f(a) = 0$ because of $f \in D(H) \subseteq D(\tau) \subseteq C_{\mu,\alpha,\beta}[a,b]$. If $0 < \alpha \le \frac{\pi}{2}$, then we define $g \in D(\tau)$ by $g(x) := x - b$ $(x \in [a,b])$ and conclude that $\frac{\cos\alpha}{\sin\alpha} f(a) - f'(a+) = 0$. This shows that f satisfies the boundary condition at a. The argument for the boundary condition at b is analogous.

Conversely, let $f \in D(\hat{H})$, and let the boundary conditions

$$\begin{aligned} \cos\alpha f(a) - \sin\alpha f'(a+) &= 0, \\ \cos\beta f(b) + \sin\beta f'(b-) &= 0, \end{aligned} \tag{1.3}$$

be satisfied. We note that this implies $f \in C_{\mu,\alpha,\beta}[a,b]$, and therefore $f \in D(\tau)$. Let $g \in D(\tau)$. Inserting the boundary conditions (1.3) into (1.2) (or using $g(a) = 0$ if $\alpha = 0$, $g(b) = 0$ if $\beta = 0$, respectively) we obtain

$$(\hat{H} f \,|\, g)_{\mathcal{H}} = \int_a^b f'(x)\overline{g'(x)}\,\mathrm{d}x + Q_0(f,g) + Q_1(f,g) = \tau(f,g).$$

Now the definition of H implies $f \in D(H)$, $Hf = \hat{H} f$. □

The following observation is a preparation for the subsequent examples. Let $c \in (a,b)$, $\mu_c := \mu(\{c\}) > 0$, and let $f \in L_{1,\mathrm{loc}}(a,b)$ be such that $\partial_\mu f \in L_1((a,b),\mu)$ exists. Then the equation $\partial_\mu f \mu = f'$ implies $\partial_\mu f(c) = \frac{f(c+) - f(c-)}{\mu_c}$.

EXAMPLES 1.10. (a) Let $a < a' < b' < b$. Assume that (a',b') is a component of $[a,b] \setminus \operatorname{spt}\mu$, and that $\mu(\{a',b'\}) = 0$

Let $f \in D(H)$. Then f' is continuous at a' and b', and

$$f'(a') = f'(b') = \frac{f(b') - f(a')}{b' - a'}.$$

In Example 3.7 we will transform this example into an example on a graph.

(b) Let $a < c < b$, $\mu_c := \mu(\{c\}) > 0$, and $\mu((c-\varepsilon, c)) > 0$, $\mu((c, c+\varepsilon)) > 0$ for all $\varepsilon > 0$.

Let $f \in D(H)$. Then

$$Hf(c) = -\partial_\mu f'(c) = -\frac{f'(c+) - f'(c-)}{\mu_c}, \tag{1.4}$$

according to Theorem 1.9 and the preceding remark. In Example 4.6 we will give an isomorphic description in the context of graphs.

(c) Let $a < a' < c < b' < b$, and assume that (a',c), (c,b') are components of $[a,b] \setminus \operatorname{spt}\mu$.

Let $f \in D(H)$. Then

$$Hf(c) = -\partial_\mu f'(c) = -\frac{f'(c+) - f'(c-)}{\mu_c} = -\frac{1}{\mu_c}\left(\frac{f(b') - f(c)}{b' - c} - \frac{f(c) - f(a')}{c - a'}\right).$$

2. The Dirichlet form including a scaling function

In this section we transfer the results of Sect. 1 to more general one-dimensional diffusion operators. We suppose that $a', b' \in \mathbb{R}$, $a' < b'$, and that $s\colon [a', b'] \to \mathbb{R}$ is a *scale* (or *scaling function*), i.e., s is a continuous strictly monotonically increasing function. As in Sect. 1 we assume that μ' is a finite Borel measure on $[a', b']$, $a', b' \in \operatorname{spt} \mu'$, but $\mu'(\{a', b'\}) = 0$. The aim is to associate an operator in $\mathcal{H}' := L_2([a', b'], \mu')$ with the differential expression $-\partial_{\mu'}\partial_s$, and to present the corresponding Dirichlet form. This aim will be achieved by transforming the problem to the case treated in Sect. 1.

We define $a := s(a')$, $b := s(b')$. Then $s\colon [a', b'] \to [a, b]$ is a homeomorphism. Let $\mu := \mu'_s$ denote the image measure of μ under s (i.e., $\mu(A) := \mu'(s^{-1}(A))$, for all Borel sets $A \subseteq [a, b]$). Then μ has the properties required in Sect. 1; we further note that $\operatorname{spt} \mu = s(\operatorname{spt} \mu')$.

We recall the following fact concerning image measures. A function $f\colon [a, b] \to \mathbb{K}$ is μ-integrable if and only if $f \circ s$ is μ'-integrable, and then the *substitution rule*

$$\int_{[a,b]} f(x)\, \mathrm{d}\mu(x) = \int_{[a',b']} f \circ s(x')\, \mathrm{d}\mu'(x') \tag{2.1}$$

holds.

For each $p \in [1, \infty]$ the mapping Ψ,

$$\Psi f := f \circ s$$

(mapping functions on $[a, b]$ to functions on $[a', b']$) is an isometric Banach lattice isomorphism $\Psi\colon L_p([a, b], \mu) \to L_p([a', b'], \mu')$. In particular, $\Psi\colon \mathcal{H} \to \mathcal{H}'$ is a unitary operator and a Hilbert lattice isomorphism. The inverse mapping is given by $\Psi^{-1} g = g \circ s^{-1}$.

Let τ be the form treated in Sect. 1. We define the form τ' in $\mathcal{H}'$ by

$$D(\tau') := \Psi(D(\tau)), \quad \tau'(f, g) := \tau(\Psi^{-1} f, \Psi^{-1} g). \tag{2.2}$$

Then it is standard to show that τ' is a densely defined, closed positive form, and that the self-adjoint operator H' associated with τ' is given by

$$D(H') = \Psi(D(H)), \quad H' = \Psi H \Psi^{-1}. \tag{2.3}$$

The generated C_0-semigroups are related by

$$e^{-tH'} = \Psi e^{-tH} \Psi^{-1} \quad (t \geq 0).$$

The form τ' satisfies the conditions of the Beurling–Deny criteria, and the C_0-semigroup $(e^{-tH'})_{t \geq 0}$ is (positive and) submarkovian.

It remains to determine the form τ' and the operator H' more explicitly. In order to describe $D(\tau') = \Psi(D(\tau))$ we first describe the spaces $\Psi(C_\mu[a, b])$, $\Psi(C_{\mu,\alpha,\beta}[a, b])$,

$\Psi(W_2^1(a,b))$. A function $f\colon [a',b'] \to \mathbb{K}$ will be called *affine linear in s on* (a_0', b_0'), an open subinterval of $[a',b']$, if there exist constants $c, d \in \mathbb{K}$ such that

$$f(x') = c + ds(x') \quad (x' \in (a_0', b_0')).$$

We define

$$C_{\mu',s}[a',b'] := \big\{ f \in C[a',b'];\ f \text{ affine linear in } s \text{ on the components of } [a',b'] \setminus \operatorname{spt} \mu' \big\}.$$

It is not difficult to see that then $C_{\mu',s}[a',b'] = \Psi(C_\mu[a,b])$). Also, for $\alpha, \beta \in [0, \frac{\pi}{2}]$, the space

$$C_{\mu',s,\alpha,\beta}[a',b'] := \Psi(C_{\mu,\alpha,\beta}[a,b])$$

is the space analogous to $C_{\mu,\alpha,\beta}[a,b]$.

We denote by ds the Borel-Stieltjes measure on $[a',b']$ generated by the function s. Then the image measure of ds under s is the Borel-Lebesgue measure on $[a,b]$. Let $f\colon (a',b') \to \mathbb{K}$ be a bounded Borel measurable function, $g \in L_1((a',b'), ds)$. If $f' = g\,ds$ in the sense of distributions, then we write $g = \partial_s f$. We note that, equivalently,

$$f(x') = c + \int_{(a,x']} g(y')\,\mathrm{d}s(y') \quad \text{a.e.,}$$

for a suitable constant $c \in \mathbb{K}$.

Let $f \in W_2^1(a,b)$. Then

$$f(x) = c + \int_a^x f'(y)\,\mathrm{d}y \quad \text{a.e.}$$

Applying the substitution rule (2.1) to ds and the Borel-Lebesgue measure one obtains, for $x' \in (a',b')$,

$$f \circ s(x') = c + \int_{a'}^{x'} f' \circ s(y')\,\mathrm{d}s(y'),$$

i.e., $\partial_s(f \circ s) = f' \circ s \in L_2(a',b')$; and the argument can also be reversed.

These remarks explain already the first part of the following result.

THEOREM 2.1. *The form τ' is given by*

$$D(\tau') = \big\{ f \in C_{\mu',s,\alpha,\beta}[a',b'];\ \partial_s f \text{ exists on } (a',b'), \text{ and } \partial_s f \in L_2((a',b'), \mathrm{d}s) \big\},$$

$$\tau'(f,g) = \int_{a'}^{b'} \partial_s f(x') \overline{\partial_s g(x')}\,\mathrm{d}s(x') + Q_0'(f,g) + Q_1'(f,g) \quad (f, g \in D(\tau'),$$

where Q_0', Q_1' are defined analogously to Q_0, Q_1 in Sect. 1.

Proof. In view of (2.2) and the previous considerations we only have to explain the integral part in the formula for τ'.

Let $f, g \in D(\tau)$. As explained before, we then have $f' \circ s = \partial_s(f \circ s)$, $g' \circ s = \partial_s(g \circ s)$. Therefore the substitution rule yields

$$\begin{aligned}\int_a^b f'(x)\overline{g'(x)}\,\mathrm{d}x &= \int_{a'}^{b'} f' \circ s(x')\overline{g' \circ s(x')}\,\mathrm{d}s(x') \\ &= \int_{a'}^{b'} \partial_s(f \circ s)(x')\overline{\partial_s(g \circ s)(x')}\,\mathrm{d}s(x'). \end{aligned}$$ □

In order to describe $H' = \Psi H \Psi^{-1}$ we first determine the "maximal operator" $\hat{H}' = \Psi \hat{H} \Psi^{-1}$. Let $f \in D(\hat{H})$. As above, the condition $f' \in L_1(a, b)$ translates to $\partial_s(f \circ s) = f' \circ s \in L_1((a', b'), \mathrm{d}s)$. From $\partial_\mu f' \in L_2((a, b), \mu)$ we conclude that f' is bounded, and therefore $\partial_s(f \circ s) = f' \circ s$ is bounded. Further, the condition that $\partial_\mu f' \in L_2((a, b), \mu)$ translates to

$$\begin{aligned} f'(x) &= f'(a+) + \int_{(a,x)} \partial_\mu f'(y)\,\mathrm{d}\mu(y) \quad \text{a.e.}, \\ f' \circ s(x') &= f' \circ s(a'+) + \int_{(a',x']} \partial_\mu f' \circ s(y')\,\mathrm{d}\mu'(y') \quad \text{a.e.}, \end{aligned}$$

i.e., $\partial_{\mu'}\partial_s(f \circ s) = \partial_{\mu'}(f' \circ s)$ exists and $\partial_{\mu'}\partial_s(f \circ s) = \partial_\mu f' \circ s \in L_2((a', b'), \mu')$. As the computations are also valid in the reverse direction we have shown that

$$\begin{aligned} D(\hat{H}') = \big\{ f \in C_{\mu',s}[a', b'];\ &\partial_s f \text{ exists and is bounded}, \\ &\partial_{\mu'}\partial_s f \text{ exists and } \partial_{\mu'}\partial_s f \in L_2((a', b'), \mu')\big\}, \\ \hat{H}' f = -\partial_{\mu'}\partial_s f \quad &(f \in D(\hat{H}')). \end{aligned}$$

Now the following description of H' is an immediate consequence of Theorem 1.9.

THEOREM 2.2. *The operator H' is given by*

$$\begin{aligned} D(H') = \big\{ f \in D(\hat{H}');\ &\cos\alpha f(a') - \sin\alpha\,\partial_s f(a'+) \\ &= \cos\beta f(b') + \sin\beta\,\partial_s f(b'-) = 0\big\}, \\ H' f = -\partial_{\mu'}\partial_s f \quad &(f \in D(H')). \end{aligned}$$

3. Dirichlet forms for singular operators on graphs

Let $\Gamma = (V, E, \eta)$ be a finite directed graph. Here, V and E are finite sets (and $V \cap E = \emptyset$). The elements of V are the *vertices* of Γ, those of E the *edges*, and they are related by the mapping $\eta = (\eta_0, \eta_1) \colon E \to V \times V$, where $\eta_0(e)$ should denote the "starting vertex" of e, and $\eta_1(e)$ the "end vertex". (Loops, i.e., $\eta_1(e) = \eta_0(e)$, are allowed.)

For $v \in V$, the sets

$$E_{v,j} := \{e \in E;\ \eta_j(e) = v\}, \quad \text{for } j = 0, 1,$$

describe the sets of all edges starting or ending at v, respectively. We also will need the set

$$E_v := (E_{v,0} \times \{0\}) \cup (E_{v,1} \times \{1\})$$

of all edges connected with v (but where loops starting and ending at v yield two contributions).

Each edge $e \in E$ corresponds to an interval $[a_e, b_e] \subseteq \mathbb{R}$ (where $a_e, b_e \in \mathbb{R}$, $a_e < b_e$), and we assume that μ_e is a Borel measure on $[a_e, b_e]$ satisfying $a_e, b_e \in \operatorname{spt} \mu_e$, $\mu_e(\{a_e, b_e\}) = 0$. The form and operator will be defined in the Hilbert space

$$\mathcal{H}_\Gamma := \bigoplus_{e \in E} L_2((a_e, b_e), \mu_e),$$

with scalar product

$$(f \,|\, g)_{\mathcal{H}_\Gamma} := \sum_{e \in E} (f_e \,|\, g_e)_{L_2((a_e,b_e),\mu_e)}.$$

In the present section the vertices will not have mass; a special case where masses are attributed to the vertices will be treated in the following section.

In order to define the classical Dirichlet form in $\mathcal{H}_\Gamma$ we introduce the following notation. We define

$$\begin{aligned}
C(E) &:= \{(f_e)_{e \in E};\ f_e \in C[a_e, b_e]\ (e \in E)\},\\
C_\mu(E) &:= \{f \in C(E);\ f_e \in C_{\mu_e}[a_e, b_e]\ (e \in E)\},\\
W_2^1(E) &:= \{(f_e)_{e \in E};\ f_e \in W_2^1(a_e, b_e)\ (e \in E)\},\\
W_{2,\mu}^1(E) &:= W_2^1(E) \cap C_\mu(E).
\end{aligned}$$

For $v \in V$ we define a *trace operator* $\operatorname{tr}_v \colon C(E) \to \mathbb{K}^{E_v}$, by

$$\operatorname{tr}_v f(e, j) := \begin{cases} f_e(a_e) & \text{if } j = 0,\ e \in E_{v,0}, \\ f_e(b_e) & \text{if } j = 1,\ e \in E_{v,1} \end{cases} \qquad (f \in C(E)).$$

Also, it will be convenient to use the notation

$$\tau_E(f, g) := \sum_{e \in E} \int_{a_e}^{b_e} f_e'(x) \overline{g_e'(x)}\, \mathrm{d}x,$$

for $f, g \in W_{2,\mu}^1(E)$.

REMARK 3.1. In this remark we recall from [17] how forms are described, giving rise to self-adjoint operators subject to boundary conditions at the vertices. For each $v \in V$ we prescribe a subspace X_v of $\mathbb{K}^{E_v}$ and a self-adjoint operator L_v on X_v.

We define τ by

$$\begin{aligned} D(\tau) &:= \big\{ f \in W^1_{2,\mu}(E);\ \operatorname{tr}_v f \in X_v\ (v \in V) \big\}, \\ \tau(f,g) &:= \sum_{e\in E} \int_{a_e}^{b_e} f_e'(x)\overline{g_e'(x)}\,\mathrm{d}x + \sum_{v\in V} (L_v(\operatorname{tr}_v f) \mid \operatorname{tr}_v g)_{\mathbb{K}^{E_v}} \\ &= \tau_E(f,g) + \sum_{v\in V} (L_v(\operatorname{tr}_v f) \mid \operatorname{tr}_v g)_{X_v}. \end{aligned}$$

Then τ is a densely defined closed semi-bounded (below) symmetric form. (See Remarks 3.2.) The conditions $\operatorname{tr}_v f \in X_v$ and the second part of the form are responsible for the glueing conditions at the vertices.

Denoting by Q_v the orthogonal projection onto X_v and by P_v the complementary orthogonal projection, the condition $\operatorname{tr}_v f \in X_v$ can also be expressed as the equation $P_v(\operatorname{tr}_v f) = 0$.

We refer to [17, Theorem 9] for this description, in the case that all the measures μ_e are the Lebesgue-measure. (Note that we changed the sign of the matrices L_v with respect to [17, Theorem 9]; this is more convenient in our later development.)

REMARKS 3.2. (a) We consider the special case where $P_v = 0$, $L_v = 0$ for all $v \in V$. We denote the corresponding form by τ_N (the index N indicating Neumann boundary conditions). The form τ_N decomposes as the sum of the Neumann forms on each of the edges, and therefore the closedness of τ_N follows from Sect. 1. Obviously, τ_N is positive.

(b) The domain $D(\tau)$ contains the dense set $\big\{ f \in \mathcal{H}_\Gamma;\ f_e \in C_c^1(a_e,b_e)\ (e \in E) \big\}$ (compare Theorem 1.3), and therefore is dense. In order to obtain the closedness and semi-boundedness of τ in Remark 3.1 it is now sufficient to show that the trace mappings tr_v are infinitesimally form small with respect to τ_N. This, however, follows from the last estimate in the proof of Lemma 1.4, which can be rewritten as

$$|f(a)| \le r^{\frac12} \|f'\|_{L_2(a,a+r)} + \|f\|_{L_2([a,a+r],\mu)}\, \mu((a,a+r))^{-\frac12},$$

for arbitrary $r \in (0, b-a)$, and correspondingly for b.

In order to describe the self-adjoint operator H associated with the form τ of Remark 3.1 we define the "maximal operator" $\hat{H}$ in $\mathcal{H}_\Gamma$, by

$$\begin{aligned} D(\hat{H}) &:= \big\{ f \in C_\mu(E);\ f_e' \in L_1(a,b),\ \partial_{\mu_e} f_e' \text{ exists}, \\ &\qquad \partial_{\mu_e} f_e' \in L_2((a_e,b_e),\mu_e)\ (e \in E) \big\}, \\ \hat{H} f &:= (-\partial_{\mu_e} f_e')_{e\in E} \quad (f \in D(\hat{H}), \end{aligned}$$

with the notation introduced in Sect. 1. From Sect. 1 we recall that $D(\hat{H}) \subseteq D(\tau_N)$. For $f \in D(\hat{H})$ we know that the limits $f_e'(a_e+)$, $f_e'(b_e-)$ exist. For the formulation of the boundary conditions it is convenient to define the notion of *signed traces*, as follows. For $v \in V$ we define $\operatorname{str}_v g \in \mathbb{K}^{E_v}$ by

$$\operatorname{str}_v g(e, j) := \begin{cases} g_e(a_e+) & \text{if } j = 0,\ e \in E_{v,0}, \\ -g_e(b_e-) & \text{if } j = 1,\ e \in E_{v,1}, \end{cases}$$

if g is a function for which the one-sided limits exist. Then, for $v \in V$, $f \in D(\hat{H})$, the vector of the *outgoing derivatives from* v is obtained by $\operatorname{str}_v f' \in \mathbb{K}^{E_v}$.

THEOREM 3.3. *Let X_v, L_v $(v \in V)$ and τ be as in Remark 3.1, and let H be the self-adjoint operator associated with τ. Then*

$$D(H) = \big\{f \in D(\hat{H});\ \operatorname{tr}_v f \in X_v,\ Q_v(\operatorname{str}_v f') = L_v(\operatorname{tr}_v f)\ (v \in V)\big\},$$
$$Hf = \hat{H} f \quad (f \in D(H)).$$

Proof. (analogous to the proof of Theorem 1.9) Let $f \in D(\hat{H})$, $g \in D(\tau)$. Summing Eq. (1.2) (from the proof of Theorem 1.9) over the edges we obtain

$$\sum_{e\in E} \int_{a_e}^{b_e} f_e'(x)\overline{g_e'(x)}\,\mathrm{d}x = -\sum_{v\in V} (\operatorname{str}_v f' \mid \operatorname{tr}_v g)_{\mathbb{K}^{E_v}} + (\hat{H} f \mid g)_{\mathcal{H}_\Gamma}. \tag{3.1}$$

Let $f \in D(H)$. From $D(H) \subseteq D(\tau)$ we conclude that $\operatorname{tr}_v f \in X_v$ $(v \in V)$. As in the proof of Theorem 1.9 one obtains $f \in D(\hat{H})$, $Hf = \hat{H} f$. Let $g \in D(\tau)$. Using (3.1) we obtain

$$\begin{aligned}\tau_E(f, g) + \sum_{v\in V} (\operatorname{str}_v f' \mid \operatorname{tr}_v g)_{\mathbb{K}^{E_v}} &= (Hf \mid g)_{\mathcal{H}_\Gamma} \\ &= \tau(f, g) = \tau_E(f, g) + \sum_{v\in V} (L_v(\operatorname{tr}_v f) \mid \operatorname{tr}_v g)_{\mathbb{K}^{E_v}},\end{aligned}$$

i.e.,

$$\sum_{v\in V} (\operatorname{str}_v f' - L_v(\operatorname{tr}_v f) \mid \operatorname{tr}_v g)_{\mathbb{K}^{E_v}} = 0.$$

Let $v \in V$, $\xi \in X_v$. There exists $g \in D(\tau)$ such that $\operatorname{tr}_v g = \xi$, $\operatorname{tr}_w g = 0$ for all $w \in V\backslash\{v\}$; indeed, one only has to connect the prescribed traces affine linearly on all edges. This shows that $\operatorname{str}_v f' - L_v(\operatorname{tr}_v f)$ is orthogonal to $X_v = R(Q_v)$, i.e.,

$$0 = Q_v(\operatorname{str}_v f' - L_v(\operatorname{tr}_v f)) = Q_v(\operatorname{str}_v f') - L_v(\operatorname{tr}_v f).$$

Conversely, let $f \in D(\hat{H})$, and let the boundary conditions $\operatorname{tr}_v f \in X_v$, $Q_v(\operatorname{str}_v f') = L_v(\operatorname{tr}_v f)$ $(v \in V)$ be satisfied. Then $f \in D(\tau)$, by the first part of the boundary conditions. Let $g \in D(\tau)$. We note that then

$$\begin{aligned}(\operatorname{str}_v f' \mid \operatorname{tr}_v g)_{\mathbb{K}^{E_v}} &= (\operatorname{str}_v f' \mid Q_v(\operatorname{tr}_v g))_{\mathbb{K}^{E_v}} \\ &= (Q_v(\operatorname{str}_v f') \mid \operatorname{tr}_v g)_{\mathbb{K}^{E_v}} = (L_v(\operatorname{tr}_v f) \mid \operatorname{tr}_v g)_{\mathbb{K}^{E_v}}\end{aligned}$$

for all $v \in V$. Using (3.1) we obtain

$$(\hat{H} f \mid g)_{\mathcal{H}_\Gamma} = \tau_E(f, g) + \sum_{v \in V} (\mathrm{str}_v\, f' \mid \mathrm{tr}_v\, g)_{\mathbb{K}^{E_v}}$$
$$= \tau_E(f, g) + \sum_{v \in V} (L_v(\mathrm{tr}_v\, f) \mid \mathrm{tr}_v\, g)_{\mathbb{K}^{E_v}} = \tau(f, g).$$

Now the definition of H implies that $f \in D(H)$, $Hf = \hat{H} f$. □

EXAMPLE 3.4. We refer to [17] for a variety of examples. We only want to mention a very simple example, where the graph consists of one vertex 0 and one loop [0, 1]. We let $X_0 := \mathrm{lin}\{(1, 1)\}$, $L_0 := 0$. Then the boundary conditions contained in the description of the domain of H are those of periodicity, $f(0) = f(1)$, $f'(0) - f'(1) = 0$.

We now come to the main object of this section, i.e., the investigation under what conditions the generated C_0-semigroup is positive or submarkovian. In this analysis we make use of the description of (Stonean) sublattices of $\mathbb{K}^n$ and operators on such sublattices. These topics are treated in the Appendix.

THEOREM 3.5. *Let X_v, L_v $(v \in V)$ and τ be as in Remark 3.1, and let H be the self-adjoint operator associated with τ.*

(a) *Assume additionally that X_v is a sublattice of $\mathbb{K}^{E_v}$ and that $(e^{-tL_v})_{t \geq 0}$ is a positive C_0-semigroup on X_v, for all $v \in V$. Then $(e^{-tH})_{t \geq 0}$ is a positive C_0-semigroup on $\mathcal{H}_\Gamma$.*

(b) *Assume additionally that X_v is a Stonean sublattice of $\mathbb{K}^{E_v}$ and that $(e^{-tL_v})_{t \geq 0}$ is a submarkovian C_0-semigroup on X_v, for all $v \in V$. Then $(e^{-tH})_{t \geq 0}$ is a submarkovian C_0-semigroup on $\mathcal{H}_\Gamma$.*

Proof. (a) The proof is given by verifying the condition of the first Beurling–Deny criterion stated in Remark 1.6(b). Thus, let $f \in D(\tau)$. Then $|f| \in D(\tau_N)$ (as an element of $\mathcal{H}_\Gamma$) and $\tau_E(|f|) \leq \tau_E(f)$, by Theorem 1.7. Moreover $\mathrm{tr}_v\, |f| = |\mathrm{tr}_v\, f| \in X_v$, by the hypothesis that X_v is a sublattice of $\mathbb{K}^{E_v}$, for all $v \in V$, and this shows that $|f| \in D(\tau)$. Finally, the hypothesis on L_v implies that $\big(L_v(\mathrm{tr}_v\, |f|) \big| \mathrm{tr}_v\, |f|\big) \leq (L_v(\mathrm{tr}_v\, f) \mid \mathrm{tr}_v\, f)$ for all $v \in V$, by Lemma A.3(a), and this shows $\tau(|f|) \leq \tau(f)$.

(b) From part (a) we already know that the C_0-semigroup $(e^{-tH})_{t \geq 0}$ is positive. In order to show that it is submarkovian we check the condition of the second Beurling–Deny criterion mentioned in Remark 1.6(c). Thus, let $0 \leq f \in D(\tau)$. Then $f \wedge 1 \in D(\tau_N)$ (as an element of $\mathcal{H}_\Gamma$) and $\tau_E(f \wedge 1) \leq \tau_E(f)$, by Theorem 1.7. Moreover $\mathrm{tr}_v(f \wedge 1) = (\mathrm{tr}_v\, f) \wedge 1 \in X_v$, by the hypothesis that X_v is a Stonean sublattice of $\mathbb{K}^{E_v}$, for all $v \in V$, and this shows that $f \wedge 1 \in D(\tau)$. Finally, the hypothesis on L_v implies that $(L_v(\mathrm{tr}_v(f \wedge 1)) \mid \mathrm{tr}_v(f \wedge 1)) \leq (L_v(\mathrm{tr}_v\, f) \mid \mathrm{tr}_v\, f)$ for all $v \in V$, by Lemma A.3(b), and this shows $\tau(f \wedge 1) \leq \tau(f)$. □

REMARK 3.6. The conditions at the vertices stated in Theorem 3.3 are also necessary for the positivity or submarkovian property, respectively. In order to see this it is sufficient to observe that, for any vertex $v_0 \in V$ and any $x \in \mathbb{K}^{E_{v_0}}$, there exists a function $f \in D(\tau)$ such that $\operatorname{tr}_{v_0} f = x$ and $\operatorname{tr}_v f = 0$ for all $v \in V \setminus \{v_0\}$. Such an element is easily constructed by taking f affine linear on all edges. (Observe that for this element one obtaines $\tau_E(|f|) = \tau_E(f)$.)

EXAMPLE 3.7. We turn Example 1.10(a) into a graph with edges $e_1 := [a, a']$, $e_2 := [b', b]$ and vertices a, b and v, where $\eta_1(e_1) = \eta_0(e_2) = v$. In the vertex v we define $X_v := \mathbb{K}^2$ and $L_v := \frac{1}{b'-a'} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. These definitions give rise to the glueing condition

$$\begin{pmatrix} -f'(a') \\ f'(b') \end{pmatrix} = L_v \begin{pmatrix} f(a') \\ f(b') \end{pmatrix} = \frac{1}{b'-a'} \begin{pmatrix} f(a') - f(b') \\ -f(a') + f(b') \end{pmatrix}.$$

Using the restriction of μ to e_1 and e_2 as μ_1 and μ_2, respectively (and formulating the boundary conditions at a and b in the graph version) we see that the two descriptions give rise to the same evolution.

4. Graphs with masses on the vertices

In this section we treat the case of graphs as in the preceding section, but additionally we assume that (some of) the vertices carry a mass, i.e., in the language of stochastic processes, particles visiting these vertices have the tendency to stick to these vertices. Otherwise, the assumption concerning the graph $\Gamma = (V, E, \eta)$ and the measures μ_e $(e \in E)$ are as in Sect. 3.

So, additionally to the assumption of the previous section, we assume that, for $v \in V$, we are given $\mu_v \geq 0$, and we define

$$V_0 := \{v \in V;\ \mu_v = 0\}.$$

Then the form τ will be given the Hilbert space

$$\mathcal{H}_\Gamma := \bigoplus_{e \in E} L_2([a_e, b_e], \mu_e) \oplus \ell_2(V, (\mu_v)_{v \in V}).$$

(By $\ell_2(V, (\mu_v)_{v\in V})$ we understand $\mathbb{K}^{V \setminus V_0}$, equipped with the scalar product $(x \mid y) := \sum_{V \setminus V_0} x_v \overline{y_v} \mu_v$.) In the present paper, we will only treat the case where the functions in the domain are continuous on the graph, which, in view of the treatment presented in Sect. 3, is a special case. Accordingly, we define

$$\begin{aligned} C(\Gamma) &:= \big\{ f \in \mathbb{K}^{V \cup E} \cap C(E);\ f(v) = (\operatorname{tr}_v f)_{(e,j)}\ ((e, j) \in E_v,\ v \in V) \big\}, \\ C_\mu(\Gamma) &:= C(\Gamma) \cap C_\mu(E). \end{aligned}$$

For the definition of τ we further assume that, for all $v \in V$, we are given a coefficient $l_v \geq 0$. We define

$$D(\tau) := C_\mu(\Gamma) \cap W_2^1(E),$$
$$\tau(f, g) := \sum_{e \in E} \int_{a_e}^{b_e} f_e'(x)\overline{g_e'(x)}\,\mathrm{d}x + \sum_{v \in V} l_v f(v)\overline{g(v)}$$
$$\left(= \tau_E(f, g) + \sum_{v \in V} l_v f(v)\overline{g(v)} \right).$$

LEMMA 4.1. *The form τ is a closed positive symmetric form and is densely defined.*

Proof. Positivity and symmetry are obvious. The closedness is obtained as in Remark 3.2(b).

For $v \in V$ we define the element $g^v \in D(\tau)$ by $g^v(v) := 1$, $g^v(w) := 0$ for all $w \in V \setminus \{v\}$, and by affine linear connection of the corresponding traces. Now, let $f \in \mathcal{H}_\Gamma$. Then the element $f - \sum_{\{v \in V;\ \mu_v \neq 0\}} f(v) g^v$ can be approximated in $\mathcal{H}_\Gamma$ by functions in $\left\{ f \in D(\tau);\ f_e \in C_c^1(a_e, b_e)\ (e \in E),\ f(v) = 0\ (v \in V) \right\} \subseteq D(\tau)$, and therefore $D(\tau)$ is dense in $\mathcal{H}_\Gamma$. □

Let H be the self-adjoint operator associated with τ.

THEOREM 4.2. *The C_0-semigroup $(e^{-tH})_{t \geq 0}$ is submarkovian.*

Proof. The proof that τ satisfies the Beurling–Deny criteria mentioned in Remarks 1.6(b), (c) is analogous to (but easier than) the proof of Theorem 3.5, and is therefore omitted. □

It remains to describe the operator H associated with τ.

THEOREM 4.3. *The operator H is given by*

$$D(H) = \Bigg\{ f \in C_\mu(\Gamma) f_e' \in L_1(a, b),\ \partial_{\mu_e} f_e' \text{ exists,}$$
$$\partial_{\mu_e} f_e' \in L_2((a_e, b_e), \mu_e)\ (e \in E),$$
$$\sum_{e \in E_{v,0}} f_e'(a_e+) - \sum_{e \in E_{v,1}} f_e'(b_e-) = l_v f(v)\ (v \in V_0) \Bigg\},$$
$$(Hf)_e = -\partial_{\mu_e} f_e' \quad (e \in E),$$
$$Hf(v) = \frac{1}{\mu_v} \left(-\sum_{e \in E_{v,0}} f_e'(a_e+) + \sum_{e \in E_{v,1}} f_e'(b_e-) + l_v f(v) \right) \quad (v \in V \setminus V_0),$$

for $f \in D(H)$.

Proof. (analogous to the proofs of Theorem 1.9 and Theorem 3.3) Let $f \in C_\mu(\Gamma)$ be such that $f_e' \in L_1(a,b)$, $\partial_{\mu_e} f_e'$ exists and belongs to $L_2((a_e,b_e),\mu_e)$, for all $e \in E$, and let $g \in D(\tau)$. Then, summing Eq. (1.2) (from the proof of Theorem 1.9) over the edges we obtain

$$\tau_E(f,g) = -\sum_{v\in V} (\mathrm{str}_v\, f' \,|\, \mathrm{tr}_v\, g)_{\mathbb{K}^{E_v}} - \sum_{e\in E} \int_{a_e}^{b_e} \partial_{\mu_e} f_e'(x)\overline{g(x)}\,\mathrm{d}\mu_e(x). \quad (4.1)$$

Let $f \in D(H)$ $(\subseteq D(\tau))$. As in the proof of Theorem 1.9 one concludes that $f_e' \in L_1(a,b)$, $\partial_{\mu_e} f_e'$ exists, $\partial_{\mu_e} f_e' \in L_2((a_e,b_e),\mu_e)$, and that $(Hf)_e = -\partial_{\mu_e} f_e'$, for all $e \in E$. Let $g \in D(\tau)$. Using (4.1) we obtain

$$\begin{aligned}\tau_E(f,g) + \sum_{v\in V} (\mathrm{str}_v\, f' \,|\, \mathrm{tr}_v\, g)_{\mathbb{K}^{E_v}} + \sum_{v\in V} Hf(v)\overline{g(v)}\mu_v = (Hf\,|\,g)_{\mathcal{H}_\Gamma}\\ = \tau(f,g) = \tau_E(f,g) + \sum_{v\in V} l_v f(v)\overline{g(v)},\end{aligned}$$

i.e.,

$$\sum_{v\in V}\left(\sum_{e\in E_{v,0}} f_e'(a_e+) - \sum_{e\in E_{v,1}} f_e'(b_e-) + Hf(v)\mu_v - l_v f(v)\right)\overline{g(v)} = 0.$$

Let $v \in V$, and let $g^v \in D(\tau)$ be as in the proof of Lemma 4.1. Then we obtain that

$$-\sum_{e\in E_{v,0}} f_e'(a_e+) + \sum_{e\in E_{v,1}} f_e'(b_e-) + l_v f(v) = Hf(v)\mu_v.$$

If $\mu_v = 0$, then this equality yields the boundary condition included in the domain of H. If $\mu_v > 0$, then we obtain $Hf(v)$ as stated in the assertion.

Conversely, let $\tilde{H}$ be the operator defined by the right hand sides of the assertion, and let $f \in D(\tilde{H})$. This implies $f \in D(\tau)$. Let $g \in D(\tau)$. Then

$$(\mathrm{str}_v\, f' \,|\, \mathrm{tr}_v\, g)_{\mathbb{K}^{E_v}} = (l_v f(v) - \tilde{H}f(v)\mu_v)\overline{g(v)}$$

for all $v \in V$. Using (4.1) we obtain

$$\begin{aligned}(\tilde{H}f\,|\,g)_{\mathcal{H}_\Gamma} = \tau_E(f,g) + \sum_{v\in V} (\mathrm{str}_v\, f' \,|\, \mathrm{tr}_v\, g)_{\mathbb{K}^{E_v}} + \sum_{v\in V} \tilde{H}f(v)\overline{g(v)}\mu_v\\ = \tau_E(f,g) + \sum_{v\in V} l_v f(v)\overline{g(v)} = \tau(f,g)\end{aligned}$$

Now the definition of H implies that $f \in D(H)$, $Hf = \tilde{H}f$. □

REMARK 4.4. Under different hypotheses, and in a different formulation, Theorem 4.2 and Theorem 4.3 have already been obtained in [23, Lemma 3.3, Lemma 4.1, Proposition 5.3].

REMARKS 4.5. (a) For $v \in V_0$, the boundary condition in the description of $D(H)$ in Theorem 4.3 is of the kind treated in Sect. 3. Indeed, defining $X_v := \mathrm{lin}\{\mathbf{1}_{E_v}\}$, we obtain $Q_v = \frac{1}{n_v}\begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$, where $n_v := \#E_v$ is the number of elements of E_v. Further we obtain $l_v f(v)\overline{g(v)} = l_v \frac{1}{n_v}(\mathrm{tr}_v f \mid \mathrm{tr}_v g)_{X_v}$, i.e., L_v is multiplication by $\frac{l_v}{n_v}$ on X_v.

(b) For $v \in V \setminus V_0$, the value of the application of H to f is part of the boundary condition. This is typical for Wentzell boundary conditions; cf. [2,26].

EXAMPLE 4.6. We turn Example 1.10(b) into a graph with edges $e_1 := [a, c]$, $e_2 := [c, b]$ and vertices a, b and v, where $\eta_1(e_1) = \eta_0(e_2) = v$. For the vertex v we define $\mu_v := \mu(\{c\})$, $l_v := 0$. Then Eq. (1.4) of Example 1.10(b) corresponds to the value of $Hf(v)$ given in Theorem 4.3. Example 1.10(c) does not enter the context of graphs within the framework treated so far.

Appendix: Sublattices of $\mathbb{K}^n$

The following observations are preparations for the analysis of the glueing conditions in Sect. 3.

We will use the lattice structure of $\mathbb{K}^n$, i.e., $\mathbb{K}^n$ should be considered as the function space $C(\{1, \dots, n\}; \mathbb{K})$. Accordingly $|x| = (|x_1|, \dots, |x_n|)$, for $x \in \mathbb{K}^n$, and $x \wedge y = (x_1 \wedge y_1, \dots, x_n \wedge y_n)$, for $x, y \in \mathbb{R}^n$. The p-norm on $\mathbb{K}^n$ will be denoted by $|\cdot|_p$, for $1 \le p \le \infty$.

LEMMA A.1. *Let $X \subseteq \mathbb{K}^n$ be a subspace, $m := \dim X$.*

(a) *The following properties are equivalent.*

(i) *X is a sublattice (i.e., $x \in X$ implies $|x| \in X$);*

(ii) *there exist $x^1, \dots, x^m \in X_+$, $x^j \wedge x^k = 0$ $(j \neq k)$, such that $X = \mathrm{lin}\{x^j;\ j = 1, \dots, m\}$.*

(b) *The following properties are equivalent.*

(iii) *X is a Stonean sublattice of $\mathbb{K}^n$, i.e., a sublattice satisfying $x \wedge 1 \in X$ for all real $x \in X$;*

(iv) *X is invariant under all normal contractions $F\colon \mathbb{K} \to \mathbb{K}$ (i.e., $F \circ x \in X$ for all $x \in X$);*

(v) *there exists a partition $C_1, \dots, C_m$ of a subset of $\{1, \dots, n\}$ such that $X = \mathrm{lin}\{\mathbf{1}_{C_j};\ j = 1, \dots, m\}$.*

Proof. (a) It was shown by Yudin (cf. [27, Theorem III.14.1]) that an m-dimensional Archimedean vector lattice is linearly and lattice isomorphic to $\mathbb{K}^m$. This shows that (i) implies (ii). It is obvious that (ii) implies (i).

(b) It is obvious that (v) implies (iv) and that (iv) implies (iii). In order to show that (iii) implies (v) we note that (iii) implies (ii), by part (a) above. For $j = 1, \dots, m$ we define $C_j := \{k \in \{1, \dots, n\};\ x^j_k \neq 0\}$. Assume that for some $j \in \{1, \dots, m\}$ the element x^j is not a multiple of $\mathbf{1}_{C_j}$. Then there exists $\gamma > 0$ such that $(\gamma x^j) \wedge 1$ is not a multiple of x^j, and consequently $(\gamma x^j) \wedge 1 \notin X$, a contradiction. □

REMARKS A.2. (a) In the following we will assume that X is a sublattice of $\mathbb{K}^n$, $m = \dim X$, and that $x^1, \dots, x^m \in X_+$ are as in property (ii) of Lemma A.1 and an orthonormal basis of X. Then the mapping

$$J\colon \mathbb{K}^m \to X, \quad \alpha = (\alpha_j)_{j=1,\dots,m} \mapsto \sum_{j=1}^{m} \alpha_j x^j,$$

is a Hilbert lattice isomorphism. If L is an operator in X, then it possesses a matrix representation $(l_{jk})_{j,k=1,\dots,m}$ with respect to the orthonormal basis $x^1, \dots, x^m$. The operator L is self-adjoint if and only if the matrix $(l_{jk})_{j,k=1,\dots,m}$ is self-adjoint.

(b) If X is a Stonean sublattice of $\mathbb{K}^n$, and $C_1, \dots, C_m$ are as in property (v) of Lemma A.1, then we define $n_j := \# C_j$ as the number of elements of C_j, for $j = 1, \dots, m$. In this case we use the mapping

$$J_S\colon \mathbb{K}^n \to X, \quad \alpha \mapsto \sum_{j=1}^{m} \alpha_j \mathbf{1}_{C_j},$$

which again is a lattice isomorphism, and also an isometric isomorphism of $\ell_p(\{1, \dots, m\}, (n_j)_{j=1,\dots,m})$ (i.e., $\mathbb{K}^m$ provided with the weighted norm $\|x\|_p := \left(\sum_{j=1}^m |x_j|^p n_j\right)^{1/p}$, for $1 \le p < \infty$) and $(X, |\cdot|_p)$, for $1 \le p \le \infty$. The mapping J_S also has the property that it commutes with the composition of vectors with functions $F\colon \mathbb{K} \to \mathbb{K}$.

LEMMA A.3. *Let X be a sublattice of $\mathbb{K}^n$, L as above, with associated matrix $(l_{jk})_{j,k=1,\dots,m}$, and L self-adjoint.*

(a) *The following properties are equivalent.*

(i) *For all $x \in X$ one has $\big(L|x| \,\big|\, |x|\big) \le (Lx \,|\, x)$;*

(ii) *the C_0-semigroup $(e^{-tL})_{t\ge 0}$ on X is positive;*

(iii) *$l_{jk} \le 0$ for all $j, k \in \{1, \dots, m\}$ with $j \ne k$.*

(b) *Assume additionally that X is Stonean. Then the following properties are equivalent.*

(iv) *Property* (i) *holds, and for all $x \in X_+$ one has $(L(x \wedge 1) \,|\, x \wedge 1) \le (Lx \,|\, x)$;*

(v) *for all normal contractions $F\colon \mathbb{K} \to \mathbb{K}$ one has $(L(F \circ x) \,|\, F \circ x) \le (Lx \,|\, x)$ $(x \in X)$;*

(vi) *the C_0-semigroup $(e^{-tL})_{t\ge 0}$ on X is submarkovian;*

(vii) *property* (iii) *holds, and $\sum_{j=1}^m \sqrt{n_j}\, l_{jk} \ge 0$ for all $k = 1, \dots, n$, where n_j and l_{jk} are defined in Remarks A.2.*

Proof. (a) In view of the mapping J of Remark A.2(a), the equivalence of (i) and (ii) is part of the first Beurling–Deny criterion. In order to see that (ii) implies (iii) let $j \ne k$. Then the function $t \mapsto (e^{-tL} x^k \,|\, x^j)$ is non-negative, and zero for $t = 0$. Therefore

$0 \le \frac{\mathrm{d}}{\mathrm{d}t}(e^{-tL}x^k \mid x^j)|_{t=0} = (-Lx^k \mid x^j) = -l_{jk}$. In order so show that (iii) implies (ii) we note that there exists $\gamma \in \mathbb{R}$ such that all entries of the matrix $L - \gamma E_m$ are ≤ 0 (where E_m is the m-dimensional unit matrix). Therefore $e^{-tL} = e^{-\gamma t} e^{-t(L-\gamma E_m)}$ is positive for all $t \ge 0$.

(b) In view of the mapping J_S of Remark A.2(b), the equivalence of (iv), (v) and (vi) is part of the second Beurling–Deny criterion. We refer to [24, C-II, Theorem 1.11] for the equivalence of (ii) and (iii) in a more general context.

In order to show the equivalence of (vi) and (vii) we assume that the C_0-semigroup $(e^{-tL})_{t\ge 0}$ is positive. Let $0 \le x \in X$, i.e., $x = \sum_{j=1}^m \alpha_j x^j$, with $\alpha_1, \ldots, \alpha_m \ge 0$. Then

$$\left|e^{-tL}x\right|_1 = \Big|\sum_{j,k=1}^m (e^{-tL})_{jk}\alpha_k x^j\Big|_1 = \sum_{k=1}^m \left(\sum_{j=1}^m (e^{-tL})_{jk}\sqrt{n_j}\right)\alpha_k,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|e^{-tL}x\right|_1\Big|_{t=0} = -\sum_{k=1}^m \left(\sum_{j=1}^m (l_{jk}\sqrt{n_j})\right)\alpha_k.$$

This shows that the C_0-semigroup $(e^{-tL})_{t\ge 0}$ is substochastic in $(X, |\cdot|_1)$ (or equivalently, $(e^{-tL})_{t\ge 0}$ is submarkovian) if and only if $\sum_{j=1}^m \sqrt{n_j}\, l_{jk} \ge 0$ for all $k = 1, \ldots, m$. □

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U. Kant
Fachbereich Mathematik,
Universität Kassel,
Heinrich-Plett-Str. 40,
34132 Kassel, Germany
E-mail: ukant@mathematik.uni-kassel.de

T. Klauss
Neureuther Straße,
80799 Munich, Germany
E-mail: tobias.klausz@gmx.de

J. Voigt
Fachrichtung Mathematik,
Technische Universität Dresden,
01062 Dresden, Germany
E-mail: juergen.voigt@tu-dresden.de

M. Weber
Hochschule für Technik und
Wirtschaft Dresden,
Fachbereich Informatik/Mathematik,
PF 120701,
01008 Dresden, Germany
E-mail: weber@informatik.htw-dresden.de