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On the principal eigenvalue of linear cooperating elliptic systems with small diffusion

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Abstract. We use cone methods combined with distribution theory and blow ups to find the asymptotic limit of the principal eigenvalue of a cooperative elliptic linear system when the diffusion is small.

We are interested in the principal eigenvalue of the linear eigenvalue problem

$$-\varepsilon^2\Big(a^i_{mj}\frac{\partial^2 u_i}{\partial x_m \partial x_j}+b^i_j\frac{\partial u_i}{\partial x_j}+c_i u_i\Big)=\sum_{s=1}^{k}F_{is}u_s+\lambda u_i \tag{1}$$

on Ω, $u_i = 0$ on $\partial\Omega$, $i = 1, \dots, k$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, a^i_{mj} are C^1 on $\overline{\Omega}$, b^i_j, c_i and F_{is} are continuous functions on $\overline{\Omega}$, $F_{is}(x) \geq 0$ on Ω if $i \neq s$, $c_i(x) \geq 0$ on Ω and there exists $\mu > 0$ such that $a^i_{mj}\eta_m\eta_j \geq \mu|\eta|^2$ for $\eta \in \mathbb{R}^N$, $x \in \overline{\Omega}$. Here we are using the summation convention on j, m.

It is well known from positive operator theory (that is, cone theory) that for $\varepsilon > 0$, the problem has a least real eigenvalue $\lambda_1(\varepsilon)$ (this is discussed later). More precisely, any other real eigenvalue λ of the system (1) satisfies $\lambda \geq \lambda_1(\varepsilon)$. Our main result is that $\lambda_1(\varepsilon) \to -\sup_{x\in\overline{\Omega}}(\bar{\lambda}(F(x)))$ as $\varepsilon \to 0$, where $F(x)$ is the $k \times k$ matrix $(F_{ij}(x))$ and $\bar{\lambda}(F(x))$ is the principal eigenvalue of the cooperative matrix $F(x)$ (note that the principal eigenvalue depends continuously on x and hence the supremum is achieved). In fact, we can allow other boundary conditions (including Neumann boundary conditions). We discuss this later.

The interest in the problem arises because such equations arise when we study non-linear cooperative population models in mathematical biology when the diffusion is small. In particular, if we are interested in the stability of a stationary solution for small ε, the above formula is very useful. (The principal eigenvalue largely determines the stability.) Indeed, the equation was first posed to me by Professor Yuan Lou of Ohio State University for this reason. I thank him for useful discussions.

We could also apply our methods to the case where a cone $\widetilde{C}$ is preserved (where $\widetilde{C}$ is a cone where some coordinates are non-negative and some are non-positive), with appropriate conditions on F_{ij} for $i \neq j$. This can be reduced to the above case by a simple change of variables. Note that this covers population models where $k - 1$

species cooperate but all compete with the kth species. It seems likely that the formula will have other uses.

Our proof proceeds by sub and supersolutions and a blowing up argument to reduce the problem to a problem for constant coefficients operators on $\mathbb{R}^N$ or half spaces and then using Fourier transforms of distributions and convolutions to complete the proof. We believe our techniques would be useful for other problems. We previously (with Hess) used blow up methods for scalar periodic parabolic eigenvalue problems [5].

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1. Statement of the theorem and reduction to a constant coefficient problem

We write our problem in the abbreviated form

$$\begin{aligned} -\varepsilon^2 A_i u_i &= (Fu)_i + \lambda u_i \qquad &&\text{in } \Omega \\ u_i &= 0 &&\text{on } \partial\Omega \end{aligned}$$

for $1 \leq i \leq k$, or

$$-\varepsilon^2 \bar{A}\bar{u} = F\bar{u} + \lambda\bar{u}.$$

(We stress that the leading order term of $A_i v$ is $a^i_{mj} \frac{\partial^2 v}{\partial x_m \partial x_j}$). First we need to explain why a principal eigenvalue exists. We only sketch this because it is standard (see [15]). Our method, in particular Eq. (2), is useful for us later. Purely for simplicity, assume that $\varepsilon = 1$. By adding a constant to λ, we can assume that $c_i(x) \geq 0$ for each i and $F_{ii} \geq 0$ on Ω for each i, and thus $-\varepsilon^2 A_i$ (with the boundary condition) is invertible, and its inverse V_i preserves positivity in $C(\Omega)$. Hence we have an equivalent system

$$u_i = V_i((Fu)_i + \lambda u_i)$$

for $1 \leq i \leq k$ on $C(\Omega)^k$. This system preserves the cone C of non-negative functions in $C(\Omega)^k$ and is increasing in λ on this cone. Moreover, as we explain below, we arrange our problem (by adding a constant to λ and the c_i's) so that the map $\bar{V}\bar{u} = \{V_i(F\bar{u})_i\}$ has norm less that 1 on $C(\Omega)^k$. Then the principal eigenvalue is

$$\inf_{\lambda > 0} r(\bar{W}(\lambda)) = 1, \tag{2}$$

where r is the spectral radius and $\bar{W}(\lambda) = \bar{V} + \lambda Z$. Here $Z\bar{u} = \{V_i u_i\}$. Note that here we are using that $r(\bar{V}) < 1$. Equivalently, we could take the infimum of $\lambda > 0$ such that $r(\bar{W}(\lambda)) \geq 1$. Similar arguments appear in Hess [9] or [10] and are well known. Note that $r(\bar{W}(\lambda)) > 1$ for λ large positive since $\bar{W}(\lambda) \geq \lambda Z$ and hence $r(\bar{W}(\lambda)) \geq \lambda r(Z)$ if $\lambda > 0$. Note also by the Krein-Rutman theorem (cp. [12]) that if $r(\bar{W}(\lambda)) = 1$, λ is an eigenvalue of (1) to which there corresponds a non-negative eigenfunction.

It remains to prove that $r(\bar{W}(0)) < 1$ if we arrange things carefully. It suffices to prove that we can ensure that each V_i has small norm on $C(\overline{\Omega})$. Equivalently, we need to prove that if $c_i(x) \geq \alpha$ on Ω where α is large positive then $A_i{}^{-1}$ (with the boundary condition) has small norm (in fact norm at most α^{-1}). By positivity, it suffices to estimate the sup norm of the solution $\hat{v}_i$ of $-A_i v_i = 1$ in Ω, $v = 0$ on $\partial\Omega$. Since v is non-negative, it suffices to find an upper bound for v. Since the constant function α^{-1} is easily seen to be a supersolution of this problem, $\|\hat{v}_i\|_\infty \leq \alpha^{-1}$ and hence $\|V_i\| \leq \alpha^{-1}$, as required.

In fact, it is not difficult to use Kato's inequality [12] to prove that the principal eigenvalue is the eigenvalue of smallest real part.

We can prove the existence of a (real) principal eigenvalue for the matrix $F(x)$ for a fixed x by similar arguments. This time it is the eigenvalue of largest real part. This is the well-known Perron-Frobenius Theorem (see [14]). It is denoted by $\bar{\lambda}(F(x))$.

If the infimum in (2) occurs for $\lambda = \tau$, then $r(\bar{W}(\lambda)) > 1$ for $\lambda > \tau$. To see this, we note that since $r(\bar{W}(\lambda))$ increases in λ (by Theorem 2.5 in [13]), either $r(\bar{W}(\lambda)) > 1$ for $\lambda > \tau$ or there is a $\delta > 0$ such that $r(\bar{W}(\lambda)) = 1$ on $(\tau, \tau + \delta)$. In the latter case, every $\lambda \in (\tau, \tau + \delta)$ is an eigenvalue of (1). This is impossible since the spectrum of (1) is discrete by standard theory. The above result implies that τ changes continuously if the terms of our equation are perturbed slightly.

We need one more property of the principal eigenvalue. If $\overline{\Omega}_1 \subset \Omega_2$, then $\lambda_1(\varepsilon, \Omega_1) \geq \lambda_1(\varepsilon, \Omega_2)$, with the obvious notation. Note that $\lambda_1(\varepsilon)$ was defined in the introduction. Once again it suffices to assume that $\varepsilon = 1$. If $\bar{u} = \{u_i\}_{i=1}^k$ is a non-negative eigenfunction corresponding to the eigenvalue $\lambda_1(1, \Omega_1)$ on Ω_1, we extend each u_i to $\overline{\Omega}_2$ by defining

$$\tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if } x \in \Omega_1 \\ 0 & \text{if } x \in \Omega_2 \setminus \Omega_1. \end{cases}$$

By the proof of Lemma I.1 in Berestycki and Lions [2]

$$A_i \tilde{u}_i \leq (F\tilde{u})_i + \lambda_1(1, \Omega_1)\tilde{u}_i$$

on Ω_2 in the sense of distributions for $1 \leq i \leq k$ and hence

$$\bar{A}\tilde{u} \leq F\tilde{u} + \lambda_1(1, \Omega_1)\tilde{u}$$

on Ω_2 in the sense of distributions. Thus by the positivity of inverses,

$$\tilde{u} \leq \bar{V}(F\tilde{u} + \lambda_1(1, \Omega_1)\tilde{u})$$

where the inverses are now on Ω_2. Hence by Theorem 2.5 in [13], $r(\bar{V}(\bar{F} + \lambda_1(1, \Omega_1)I)) \geq 1$ and thus by (2), $\lambda_1(1, \Omega_2) \leq \lambda_1(1, \Omega_1)$, as required. With care, one can prove strict inequality holds.

Our main theorem is then the following.

THEOREM 1. *Under the assumptions of the introduction,*

$$\lambda_1(\varepsilon) \to -\sup_{x \in \overline{\Omega}} \bar{\lambda}(F(x)) \quad \text{as } \varepsilon \to 0.$$

Note that $\lambda_1(\varepsilon)$ and $\bar{\lambda}(F(x))$ were defined in the introduction. In the rest of this section, we reduce the proof of Theorem 1 to an eigenvalue problem for constant coefficient equations on $\mathbb{R}^N$ or a half space. We will then resolve this constant coefficient problem in Sect. 2.

If $x_0 \in \Omega$, there is a $\delta > 0$ such that $B_\delta(x_0) \subset \Omega$. We consider $R > 0$ such that $\varepsilon R \le \delta$ (thus if ε is small, R can be large). By our earlier comparison result, $\lambda_1(\varepsilon, B_{\varepsilon R}(x_0)) \ge \lambda_1(\varepsilon, \Omega)$. Now by rescaling, $\lambda_1(\varepsilon, B_{\varepsilon R}(x_0))$ is the principal eigenvalue of the problem

$$\begin{aligned} -a^i_{mj}(x_0+\varepsilon y)\frac{\partial^2 u_i}{\partial y_m \partial y_j} &- \varepsilon b^i_j(x_0+\varepsilon y)\frac{\partial u_i}{\partial y_j} - \varepsilon^2 c_i(x_0+\varepsilon y)u_i \\ &= F_{is}(x_0+\varepsilon y)u_s + \lambda u_i \quad \text{on } B_R \\ u_i = 0 \quad &\text{on } \partial B_R \end{aligned}$$

Hence, by the continuity of the principal eigenvalue under perturbations, $\lambda_1(\varepsilon, B_{\varepsilon R}(x_0)) \to \tilde{\lambda}(x_0, R)$ as $\varepsilon \to 0$ where $\tilde{\lambda}(x_0, R)$ is the principal eigenvalue of the problem

$$\begin{aligned} -a^i_{mj}(x_0)\frac{\partial^2 u_i}{\partial x_m \partial x_j} &= F_{is}(x_0)u_s + \lambda u_i, \quad 1 \le i \le k, \\ u &= 0 \quad \text{on } \partial B_R. \end{aligned} \tag{3}$$

Note that this new problem has constant coefficients. By our earlier comments, $\tilde{\lambda}(x_0, R)$ decreases in R and by our earlier results, they are bounded below and so $\tilde{\lambda}(x_0) := \lim_{R\to\infty} \tilde{\lambda}(x_0, R)$ exists. Hence we see that $\limsup_{\varepsilon\to 0} \lambda_1(\varepsilon) \le \tilde{\lambda}(x_0)$ for $x_0 \in \Omega$ and hence for $x_0 \in \overline{\Omega}$. Moreover, by rather standard limiting arguments as on p. 435 of [3], either system (3) on $\mathbb{R}^N$ has a non-trivial bounded non-negative solution for $\lambda = \tilde{\lambda}(x_0)$ or system (3) on a half space $\mathbb{R}^{N+}$ with zero boundary condition on the boundary has a non-trivial bounded non-negative solution on $\mathbb{R}^{N+}$ for $\lambda = \tilde{\lambda}(x_0)$ such that $\max\{u_i(x) : 1 \le i \le k,\ x \in \mathbb{R}^{N+}\}$ is achieved (we normalize the non-negative eigenfunction on B_R so that $\max_i\{\|u_i\|_\infty\} = 1$). Which of the two cases occur depends on whether the component u_i with $\|u_i\|_\infty = 1$ has its maximum at a bounded distance from ∂B_R for large R (at least for a subsequence) or the distance tends to infinity with R.

We explain the limit argument a little more. If $R_n \to \infty$ as $n \to \infty$, let $\{u_i^n\}_{i=1}^k$ be a normalized positive eigenfunction for $R = R_n$ and choose $x_n \in B_{R_n}$ such that $u_i^n(x_n) \ge \frac{1}{2}$ for some i (where i depends on n). By standard elliptic estimates as in [8], $\{u_i^n\}$ are bounded in C^2 and hence by choosing a subsequence, we can find a subsequence of n's so that $u_i^n(x - x_n) \to w_i$ in C^1 as $n \to \infty$ on compact sets. Since for large n, the boundary is becoming flatter, the limit function will be defined on a half space or the full space $\mathbb{R}^N$ and will satisfy the natural limit equation. Note that since $u_i^n(x_n) \geq \frac{1}{2}$ for some i for each n, $w_i(0)$ will be positive for some i.

We will prove in Proposition 3 that this implies that $-\tilde{\lambda}(x_0)$ is an eigenvalue of the matrix $F(x_0)$ corresponding to a non-negative eigenfunction. In particular the required upper bound for $\limsup_{\varepsilon\to 0} \lambda_1(\varepsilon)$ follows if $F_{ij}(x_0) > 0$ if $i \neq j$ and $x_0 \in \overline{\Omega}$ since in this case $\tilde{\lambda}(x_0)$ is the only eigenvalue of the matrix $F(x_0)$ to which there corresponds a non-negative eigenfunction (cp. [14], Proposition 1.6.3).

We now obtain the lower bound, once again by a blow up argument. Suppose by way of contradiction there exist $\varepsilon_i \to 0$ and $\delta > 0$ such that $\lambda_1(\varepsilon_i) + \delta \leq -\sup_{x_0\in\overline{\Omega}} \tilde{\lambda}(x_0)$. We first prove $\lambda_1(\varepsilon)$ are uniformly bounded below (in ε). If not, there exist $\varepsilon_n \to 0$ such that $\lambda_1(\varepsilon_n) \to -\infty$ as $n \to \infty$. Now our eigenvalue problem can be written as the system

$$u_i = (-\varepsilon_n A_i - \lambda_n I)^{-1}(F(\bar{u}))_i$$

for $1 \leq i \leq k$, where $\lambda_n = \lambda_1(\varepsilon_n)$. Since $\inf_{x\in\Omega}(c_i(x) - \lambda_n) \to \infty$ as $n \to \infty$, we see as before that $\|(-\varepsilon_n A_i - \lambda_n I)^{-1}\| \to 0$ as $n \to \infty$ and hence $\bar{u}$ satisfies an equation $\bar{u} = S\bar{u}$ where $\|S\| < 1$. This is impossible and so our claim follows.

Suppose now that $\{u_i^n\}_{i=1}^k$ are the non-negative eigenfunctions corresponding to $\lambda_1(\varepsilon_n)$, normalized so that $\sup_i \|u_i^n\|_\infty = 1$ for each i, where the maximum occurs at $x_n \in \Omega$. We choose a subsequence so that $x_n \to \tilde{x} \in \overline{\Omega}$. If we rescale and blow up again much as before we find that there exists $\hat{\lambda} \leq -\delta - \tilde{\lambda}(\tilde{x})$ such that either the system

$$-a_{mj}^i(\tilde{x})\frac{\partial^2 u_i}{\partial x_m \partial x_j} = F_{is}(\tilde{x})u_s + \hat{\lambda}u_i \qquad \text{on } \mathbb{R}^N$$

for $1 \leq i \leq k$ has a non-trivial non-negative bounded solution, or the same equation on a half space T has a non-trivial non-negative bounded solution vanishing on ∂T. In either case, this contradicts Proposition 3 since that result implies $-\hat{\lambda}$ is an eigenvalue of the matrix $F(\tilde{x})$ corresponding to a non-negative eigenfunction. Thus we have proved the lower bound in all cases (assuming the result of Sect. 2). Moreover, we have proved Theorem 1 in the cases of strict cooperativity.

We now remove the strict cooperativity assumption. If $\delta > 0$ is small, denote by $F^\delta(x_0)$ the matrix where we add δ to each of the off diagonal entries of $F(x_0)$. Define $\tilde{\lambda}^\delta(x_0, R)$ to be the corresponding principal eigenvalue on B_R when we replace $F(x_0)$ by $F^\delta(x_0)$. Note that $\tilde{\lambda}(x_0, R)$ is defined just before Eq. (3). By what we have already proved, $\tilde{\lambda}^\delta(x_0, R) \to -\bar{\lambda}(F^\delta(x_0))$ as $R \to \infty$. (Note that by [14], $\bar{\lambda}(F^\delta(x_0))$ is the only eigenvalue of $F^\delta(x_0)$ to which there corresponds a non-negative eigenfunction.) Suppose by way of contradiction $\tilde{\lambda}(x_0, R) \to -\mu > -\bar{\lambda}(F(x_0))$ as $R \to \infty$. Now if δ is small the spectrum of $F^\delta(x_0)$ is close to that of $F(x_0)$ and hence we can find $s \in (-\bar{\lambda}(F(x_0)), -\mu)$ such that $-s$ is not an eigenvalue of $F^\delta(x_0)$ for all $0 \leq \delta \leq \delta_0$ and $s > -\bar{\lambda}(F^\delta(x_0))$ for all $0 \leq \delta \leq \delta_0$. Here δ_0 is small (remember that $\bar{\lambda}(F^\delta(x_0))$

depends continuously on δ). Since $\tilde{\lambda}(x_0, R) \to -\mu$ we find that for all R large, $\tilde{\lambda}(x_0, R) > s$. Fix R large with $\tilde{\lambda}(x_0, R) > s$. Now $\tilde{\lambda}^{\delta}(x_0, R)$ depends continuously on δ for $0 < \delta \leq \delta_0$ and $\tilde{\lambda}^{\delta_0}(x_0, R) \to -\bar{\lambda}(F^{\delta_0}(x_0))$ as $R \to \infty$, by what we have already proved. But $-\bar{\lambda}(F^{\delta_0}(x_0)) < s$. Hence there exists $\delta \in (0, \delta_0)$ with $\tilde{\lambda}^{\delta}(x_0, R) = s$. Repeating, we can find $R_n \to \infty$ and $\delta_n \in (0, \delta_0)$ such that $\tilde{\lambda}^{\delta_n}(x_0, R_n) = s$ for all n. By choosing a subsequence we can assume $\delta_n \to \gamma \in [0, \delta_0]$ as $n \to \infty$. By repeating our blow up argument, we find $\tilde{\lambda}^{\delta_n}(x_0, R_n)$ (or at least a subsequence) converges to s as $n \to \infty$, where (3) (with F replaced by F^{γ} and $\lambda = s$) has a non-trivial non-negative bounded solution on $\mathbb{R}^N$ or a half space T with Dirichlet boundary conditions. Hence by Proposition 3, s is an eigenvalue of $-F^{\gamma}(x_0)$ to which there corresponds a non-negative eigenfunction. This contradicts our choice of δ_0 and hence we have a contradiction. Hence $\tilde{\lambda}(x_0, R) \to -\bar{\lambda}(F(x_0))$ as claimed. This completes the proof of Theorem 1.

REMARK 2. We can easily modify our proof to cover Neumann boundary conditions or Robin boundary conditions $\frac{\partial u_i}{\partial n} + f_i(x)u_i = 0$ provided $f_i(x) \leq 0$ on $\partial\Omega$, where the normal is the outer normal. (Note that after the blow up, Robin boundary conditions become Neumann boundary conditions.) We could allow some components to have Dirichlet boundary conditions and some to have Robin boundary conditions of the above type. This follows because it is easy to use comparison results for linear operators to show that the principal eigenvalue in this case lies between the principal eigenvalue in the case of Robin boundary conditions and the principal eigenvalue with Dirichlet boundary conditions.

2. The technical result for constant coefficient problems

We consider the problem

$$-a^i_{mj}\frac{\partial^2 u}{\partial x_j \partial x_m} = F_{is}u_s + \lambda u_i \tag{4}$$

for $i = 1, \ldots, k$, where a^i_{mj}, F_{is} are *constants*, $F_{is} \geq 0$ if $i \neq s$ and, for each i, a^i_{mj} is elliptic.

PROPOSITION 3.

(i) If there is a constant bounded non-trivial non-negative solution of (4)*, then $-\lambda$ is an eigenvalue of F corresponding to a non-negative eigenfunction.*

(ii) If there is a non-constant bounded non-trivial solution of (4) *on a half space T such that $\bar{u} = 0$ on ∂T (or $\frac{\partial \bar{u}}{\partial n} = 0$ on ∂T), then $-\lambda$ is an eigenvalue of F corresponding to a non-negative eigenfunction.*

REMARK 4. The proof of (i) does not use the cooperativity.

The problem with the proof of (i) is that in many cases we can separate variables (in x) and get many sign changing bounded solutions for an interval of λ's. Because of this and because we do not know solutions decay, we use distributions. We first remind the reader of some notation. The basic theory can be found in Chapter 0 of [6] or Chapters 2 and 7.1 in [11] or in [7].

We let $\mathcal{L}$ denote the set of C^∞ functions ϕ on $\mathbb{R}^N$ for which ϕ and all its partial derivatives of all orders decay faster than any negative power of $\|x\|$. Then $\mathcal{L}'$, the set of tempered distributions is the dual of $\mathcal{L}$ (which we do not write down but can be found in [11]). As an example of a tempered distribution, we can take any C^∞ function f such that f and all its derivatives have at most polynomial growth. This generates a tempered distribution f by

$$f(\phi) = \int f\phi \, \mathrm{d}x \qquad \text{for } \phi \in \mathcal{L}.$$

The integral is easily seen to make sense and is often written $\langle f, \phi\rangle$. More generally, $\langle F, \phi\rangle$ also denotes F evaluated at ϕ if $F \in \mathcal{L}'$, $\phi \in \mathcal{L}$.

Note that the classical Fourier transform is a bijection on $\mathcal{L}$ (if we allow complex-valued functions). We can define the Fourier transform of a tempered distribution F by $\langle \widehat{F}, \phi\rangle = \langle F, \widehat{\phi}\rangle$ for $\phi \in \mathcal{L}$. Then $\widehat{F}$ is a tempered distribution. The map $F \to \widehat{F}$ is one-to-one on the space $\mathcal{L}'$.

Crucial to our proof of Proposition 3 is the following lemma.

LEMMA 5. *If $n > 0$, there exists a smooth non-trivial non-negative $\psi \in \mathcal{L}$ such that $\widehat{\psi}$ has support in $B_{1/n}$.*

Note that we cannot choose ψ to have compact support.

Proof. By scalings, we see that it suffices to find ψ smooth non-negative in $\mathcal{L}$ so $\widehat{\psi}$ has compact support. If $\phi \in \mathcal{L}$ is even and of compact support the convolution $\phi * \phi \in \mathcal{L}$ has compact support, and $\widehat{(\phi * \phi)}(\lambda) = |\phi(\lambda)|^2 \geq 0$ which is non-negative. (Note that if ϕ is even then $\widehat{\phi}(\lambda)$ is real.)

We then simply take $\psi = \widehat{(\phi * \phi)}$. Then except for a complex conjugation and multiplication by a positive number $\widehat{\psi}$ is $\phi * \phi$ and our claim follows (more precisely, if $g = \widehat{\bar{f}}$ where $f \in \mathcal{L}$, then $f = c\widehat{\bar{g}}$ where $c > 0$, cp. [1]. Here at this point and only at this point $-$ denotes complex conjugation). $\square$

We now prove Proposition 3(i). Suppose that $\bar{u} = (u_1, \ldots, u_k)$ is a non-trivial bounded non-negative solution of (4) on $\mathbb{R}^N$. First note that each component is a tempered distribution. By our earlier comments, it suffices to prove that $D^\alpha u_i$ has at most polynomial growth on $\mathbb{R}^N$ for each multi index α. Now by standard elliptic estimates on balls B_i (cp. Gilbarg and Trudinger [8], Section 3.4) $|\nabla u_i|$ is uniformly bounded on $\mathbb{R}^N$ (since u is bounded). Now, by differentiating our system, $\{\frac{\partial u_i}{\partial x_j}\}_{i=1}^k$ is also a solution. Thus we can use an inductive argument on $\mathbb{R}^N$, as required.

Suppose now that ψ_n is as in Lemma 5 with support $\widehat{\psi}_n$ in $B_{1/n}$. Note that ψ_n is non-negative and non-trivial. Define $u_i{}^n := u_i * \psi_n$. It is easy to see that $u_i{}^n$ is bounded and non-negative on $\mathbb{R}^N$ and $u_i{}^n > 0$ on $\mathbb{R}^N$ unless $u_i{}^n \equiv 0$ (by the maximum principle applied to the ith equation of (4), $u_i \equiv 0$ or $u_i(x) > 0$ on $\mathbb{R}^N$). Let $\bar{u}^n = \{u_i{}^n\}_{i=1}^k$. Then $\bar{u}^n$ is a non-trivial non-negative bounded solution of (4). This follows by a simple calculation if we prove $\frac{\partial}{\partial x_j} u_i{}^n = \frac{\partial u_i}{\partial x_j} * \psi_n$. But this follows easily since it is easy to use the dominated convergence theorem to justify differentiating under the integral sign in the definition of the convolution. Now if $E_1 \in \mathcal{L}$ and g is a tempered distribution $(\widehat{E_1 * g}) = \widehat{E}_1 \widehat{g}$ where the right hand side means multiplication of a tempered distribution by a function in $\mathcal{L}$ (cp. [11]). Hence $\widehat{u_i}^n$ have support in $B_{1/n}$. (Note that translating in x does not affect the property that the support of $\widehat{u_i}^n$ lies in $B_{1/n}$.)

Hence the Fourier transform of each component of $\bar{u}^n$ has support in $B_{1/n}$. Hence by translating and multiplying our solution by a constant, we can obtain a non-negative bounded solution $v_i{}^n$ of (4) such that $\sup_i\{\|v_i{}^n\|_\infty\} \le 1$, $\sup_i\{v_i{}^n(0)\} \ge \frac{1}{2}$ and $\widehat{v_i}^n$ has support in $B_{1/n}$ for each i. By the Gilbarg-Trudinger estimate, $v_i{}^n$ are all bounded in C^1 uniformly in n on $\mathbb{R}^N$ and hence we can choose a subsequence converging uniformly on compact sets to $\widetilde{v}_i$ for each i as $n \to \infty$, where $\{\widetilde{v}_i\}$ is a weak (and hence strong) bounded non-negative solution of (4). It is non-trivial since $v_i{}^n(0) \to \tilde{v}_i(0)$ for each i and $\sup_i\{v_i{}^n(0)\} \ge \frac{1}{2}$. Moreover the support of the Fourier transform of $\widetilde{v}_i$ is $\{0\}$ for each i. To see this, note that if $E \in \mathcal{L}$ with support not including zero,

$$\langle \widehat{\widetilde{v}}_i, E \rangle = \langle \widetilde{v}_i, \widehat{E} \rangle = \lim_{n\to\infty} \langle v_i{}^n, \widehat{E} \rangle$$

since the $v_i{}^n$ are uniformly bounded and converge uniformly to $\widetilde{v}_i$ on compact sets (note that $\widehat{E} \in \mathcal{L}$ and thus decays rapidly). Thus $\langle \widehat{\widetilde{v}}_i, E \rangle = \lim_{n\to\infty} \langle v_i{}^n, \widehat{E} \rangle$. But $\widehat{v_i}^n$ has support in $B_{1/n}$ and hence for large n $\langle \widehat{v_i}^n, E \rangle = 0$. Hence $\langle \widehat{\widetilde{v}}_i, E \rangle = 0$ if $E \in \mathcal{L}$ and 0 is not in the support of E. Hence $\widetilde{v}_i$ is a non-negative tempered distribution such that the support of $\widehat{\widetilde{v}}_i$ consists only of zero. Hence $\widehat{\widetilde{v}}_i$ is $C\delta_0$ and thus $\widetilde{v}_i$ is constant. (One proves that $\widehat{\widetilde{v}}_i$ is a finite linear combinations of derivatives of the delta function at 0 by [11, Theorem 2.3.4] and hence $\widetilde{v}_i$ is a polynomial. Since $\widetilde{v}_i$ is bounded, $\widetilde{v}_i$ is constant.)

Hence we have the required non-negative constant solution. This proves (i).

To prove (ii), we may assume that the half space T is $\{x \in \mathbb{R}^N : x_1 > 0\}$. In this case, we reduce to the one dimensional case by proving that we can find a non-negative bounded solution, where the components depend only on x_1.

This step is very similar to the previous case except we do our convolutions etc only in $(x_2, \ldots, x_N)$ and take Fourier transforms purely in $(x_2, \ldots, x_N)$. We can then repeat the earlier arguments to obtain a solution which for each x_1 has Fourier transform (in $x_2, \ldots, x_N$) with support only at $x_2 = \cdots = x_N = 0$ and hence is constant for each x_1 and thus we have the solution we have claimed. There are only two points to note. It is easy to see that the convolution preserves the boundary condition. Second we need to be rather more careful to check that we have a non-trivial limit. As before, we have solutions $\bar{u}^n$ such that $\sup_i \|u_i{}^n\|_\infty = 1$. Let S be the class of non-trivial non-negative solutions with this property. We then have that given $\varepsilon > 0$, there is an

$a > 0$ such that

$$\sup_i \sup_{x_1 \geq a} v_i(x_1, x_2, \ldots, x_N) \leq \varepsilon$$

if $v = (v_1, \ldots, v_k) \in S$ (otherwise we could translate and pass to the limit much as before and obtain a non-trivial non-negative bounded solution on all of $\mathbb{R}^N$ and apply part (i)). Moreover, by elliptic estimates much as before (though this time we also have to use estimates up to the boundary), $\sup\{\nabla v(x) : x \in T,\ v \in S\}$ is finite. This implies that $u_i{}^n(x)$ is small if x is close to ∂T uniformly in n. Hence we see translating in $(x_2, \ldots, x_N)$, we can assume $u_n{}^i(x_1{}^n, 0, 0, \ldots, 0) \geq \frac{1}{2}$ for some i such that $\delta \leq x_1{}^n \leq \delta^{-1}$ where δ is independent of n. If we choose a subsequence so that $\{x_1{}^n\}$ converges, it is easy to see that we have a non-trivial limit as required. Thus we have proved our claim that we can choose our solution to be a function of x_1 only.

There are several ways to complete the proof of (ii). One way is by explicitly solving the ordinary differential equations. Alternatively, we can use moving planes. First, much as earlier each component u_i satisfies $u_i \equiv 0$ on T or $u_i(x) > 0$ on T. By shrinking our system, we can assume $u_i(x) > 0$ for $1 \leq i \leq k$ (this does not affect the cooperativity property). We can then apply the proof of Theorem 2 in [4] to prove $\frac{\partial u_i(x)}{\partial x_1} > 0$ if $x_1 > 0$ and $1 \leq i \leq k$. There are a couple of comments to be made here. First, the derivative term is possibly different positive constants times $-\frac{\partial^2 u}{\partial x_1^2}$ in each equation. However, we can easily reduce to the usual case. Second in [4], the difficulties occur because lateral translates (that is, translates in $x_2, \ldots, x_N$) of our solution may approach zero. This does not occur in our case because solutions are functions of x_1 only. Hence we see that $u_i(x_1)$ has a non-zero limit as $x_1 \to \infty$ for each i. This contradicts what we have already proved. This completes the proof of the proposition.

REMARK 6.

(i) The analogue of Proposition 3(ii) is easier in the case of Neumann boundary conditions. We use our convolution trick again to reduce to the case where our solution is a function of x_1 only and then reduce to the full space case by reflecting evenly across $x_1 = 0$.

(ii) Many of our arguments in this section also seem to apply to higher order elliptic operators.

(iii) There is one case where our proofs can be simplified. If the second order terms in (4) are all multiples of the same operator, we can rotate and stretch axes so that all the second order derivatives are multiples of the Laplacian. We can then use integration over spheres to reduce the proof of part (i) to an ordinary differential equation problem, which can be solved by ordinary differential equation methods.

(iv) It seems that the main use of the positivity is to ensure that the convolution with ψ_n does not vanish identically.

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