The Neumann problem for nonlocal nonlinear diffusion equations

Fuensanta Andreu, José M. Mazón, Julio D. Rossi and Julián Toledo

Dedicated to I. Peral on the Occasion of His 60th Birthday

Abstract. We study nonlocal diffusion models of the form

$$(\gamma(u))_t(t, x) = \int_\Omega J(x - y)(u(t, y) - u(t, x)) \, dy.$$

Here Ω is a bounded smooth domain and γ is a maximal monotone graph in $\mathbb{R}^2$. This is a nonlocal diffusion problem analogous with the usual Laplacian with Neumann boundary conditions. We prove existence and uniqueness of solutions with initial conditions in $L^1(\Omega)$. Moreover, when γ is a continuous function we find the asymptotic behaviour of the solutions, they converge as $t \to \infty$ to the mean value of the initial condition.

1. Introduction

First, let us introduce what kind of nonlocal diffusion problems we consider. Nonlocal evolution equations of the form

$$u_t(t, x) = J * u - u(t, x) = \int_{\mathbb{R}^N} J(x - y)u(t, y) \, dy - u(t, x), \qquad (1.1)$$

and variations of it, have been recently widely used to model diffusion processes, see [2], [4], [9], [11], [12], [16], [17], [18], [19] and [20]. As stated in [16], if $u(t, x)$ is thought of as the density of a single population at the point x at time t, and $J(x - y)$ is thought of as the probability distribution of jumping from location y to location x, then the convolution $(J * u)(t, x) = \int_{\mathbb{R}^N} J(y - x)u(t, y) \, dy$ is the rate at which individuals are arriving to position x from all other places and $-u(t, x) = -\int_{\mathbb{R}^N} J(y - x)u(t, x) \, dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (1.1). Moreover, a nonlinearity of the form $\int J(x - y)(F(u(y)) - F(u(x))) \, dy$ may also appear in population models, see [9] and references therein.

Equation (1.1) is called nonlocal diffusion equation since the diffusion of the density u at a point x and time t does not only depend on $u(t, x)$, but on all the values of u in a neighborhood of x through the convolution term $J * u$. This equation shares many properties with the classical heat equation, $u_t = cu_{xx}$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and, even if J is compactly supported, perturbations propagate with infinite speed, [16]. However, there is no regularizing effect in general.

In this paper we turn our attention to nonlinear evolutions with Neumann boundary conditions. We study

$$
P_\gamma^J(z_0) \quad
\begin{cases}
z_t(t, x) = \displaystyle\int_\Omega J(x - y)(u(t, y) - u(t, x))\, dy, & x \in \Omega,\ t > 0, \\[2mm]
z(t, x) \in \gamma(u(t, x)), & x \in \Omega,\ t > 0, \\[2mm]
z(0, x) = z_0(x), & x \in \Omega.
\end{cases}
$$

Here Ω is a bounded domain, $z_0 \in L^1(\Omega)$, γ is a maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \gamma(0)$, and $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous radial function with $\int_{\mathbb{R}^N} J(r)dr = 1$ and $0 \in \text{int}[\text{supp}(J)]$.

In this model the right hand side takes into account the diffusion inside Ω. In fact, as we have explained, the integral $\int J(x - y)(u(t, y) - u(t, x))\, dy$ takes into account the individuals arriving or leaving position x from or to other places. Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω. There is no flux of individuals across the boundary. This is the analogous of what is called homogeneous Neumann boundary conditions in the literature.

Solutions to $P_\gamma^J(z_0)$ will be understood in the following sense.

DEFINITION 1.1. A *solution* of $P_\gamma^J(z_0)$ in $[0, T]$ is a function $z \in W^{1,1}(]0, T[; L^1(\Omega))$ which satisfies $z(0, x) = z_0(x)$, $a.e.\ x \in \Omega$, and for which there exists $u \in L^2(0, T; L^2(\Omega))$, $z \in \gamma(u)$ $a.e.$ in $Q_T = \Omega \times]0, T[$, such that

$$
z_t(t, x) = \int_\Omega J(x - y)(u(t, y) - u(t, x))\, dy \quad a.e \text{ in }]0, T[\times\Omega.
$$

Our main results can be summarized as follows.

"Under some natural assumptions about the initial condition z_0, there exists a unique global solution to $P_\gamma^J(z_0)$. Moreover, a contraction principle holds, given two solutions z_i of $P_\gamma^J(z_{i0})$, $i = 1, 2$, then

$$
\int_\Omega (z_1(t) - z_2(t))^+ \le \int_\Omega (z_{10} - z_{20})^+.
$$

Respect to the asymptotic behaviour of the solution we prove that if γ is a continuous function, then

$$\lim_{t \to \infty} z(t) = \frac{1}{|\Omega|} \int_\Omega z_0,$$

strongly in $L^1(\Omega)$".

Concerning previous references for this model, we cite [10], [12] and [13] where only the linear case, $\gamma(s) = s$, was considered. In [10] and [12] the asymptotic behaviour for the linear model is studied and in [13] it is proved that solutions to the linear problem converge to the solution of the classical heat equation with Neumann boundary conditions when the convolution kernel J is rescaled in a suitable way.

We can consider different maximal monotone graphs γ. For example, if $\gamma(r) = r^m$, problem $P_\gamma^J(z_0)$ corresponds to the nonlocal version of the porous medium equation if $0 < m < 1$, or to the fast diffusion equation if $m > 1$. Note also that γ may be multivalued, so we are considering the nonlocal version of various phenomena with phase changes like the multiphase Stefan problem, for which

$$\gamma(r) = \begin{cases} r - 1 & \text{if } r < 0, \\ [-1, 0] & \text{if } r = 0, \\ r & \text{if } r > 0. \end{cases}$$

Even γ can have a domain different from $\mathbb{R}$, which corresponds to obstacle problems.

We have to point out that the right hand side of the equation in $P_\gamma^J(z_0)$ has no regularizing effect. Hence the analysis becomes more involved (due to the lack of compactness) that the one that can be done for the usual Laplacian with Neumann boundary conditions.

Organization of the paper. The rest of the paper is organized as follows: in Section 2 we collect some preliminaries and fix the notation. In Section 3, by means of Nonlinear Semigroup Theory, we construct a mild solution and we prove uniqueness of solutions, in fact we prove a contraction principle. In Section 4 we prove that the mild solution is indeed a solution in the sense of Definition 1.1. Finally, Section 5 deal with the asymptotic behaviour of the solutions.

2. Notations and preliminaries

In this section we collect some preliminaries and notations that will be used in the sequel. For a maximal monotone graph η in $\mathbb{R} \times \mathbb{R}$ and $r \in \mathbb{N}$ we denote by η_r the *Yosida approximation* of η, given by $\eta_r = r(I - (I + \frac{1}{r}\eta)^{-1})$. The function η_r is maximal monotone and Lipschitz. We recall the definition of the *main section* η^0 of η

$$\eta^0(s) := \begin{cases} \text{the element of minimal absolute value of } \eta(s) \text{ if } \eta(s) \neq \emptyset, \\ +\infty \quad \text{if } [s, +\infty) \cap D(\eta) = \emptyset, \\ -\infty \quad \text{if } (-\infty, s] \cap D(\eta) = \emptyset, \end{cases}$$

where $D(\eta)$ denotes the domain of η. The following properties hold: if $s \in D(\eta)$, $|\eta_r(s)| \leq |\eta^0(s)|$ and $\eta_r(s) \to \eta^0(s)$ as $r \to +\infty$, and if $s \notin D(\eta)$, $|\eta_r(s)| \to +\infty$ as $r \to +\infty$.

We will use the following notations, $\eta_- := \inf \operatorname{Ran}(\eta)$ and $\eta_+ := \sup \operatorname{Ran}(\eta)$, where $\operatorname{Ran}(\eta)$ denotes the range of η. If $0 \in D(\eta)$, $j_\eta(r) = \int_0^r \eta^0(s)ds$ defines a convex l.s.c. function such that $\eta = \partial j_\eta$. If j_η^* is the Legendre transform of j_η then $\eta^{-1} = \partial j_\eta^*$.

Also we will denote by J_0 and P_0 the following sets of functions,

$$J_0 = \{j : \mathbb{R} \to [0, +\infty], \text{ convex and lower semi-continuos with } j(0) = 0\},$$

$$P_0 = \left\{p \in C^\infty(\mathbb{R}) : 0 \leq p' \leq 1, \operatorname{supp}(p') \text{ is compact, and } 0 \notin \operatorname{supp}(p)\right\}.$$

In [6] the following relation for $u, v \in L^1(\Omega)$ is defined,

$$u \ll v \text{ if and only if } \int_\Omega j(u)\,dx \leq \int_\Omega j(v)\,dx, \quad \forall j \in J_0,$$

and the following facts are proved.

PROPOSITION 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$.*
(i) For any $u, v \in L^1(\Omega)$, if $\int_\Omega u p(u) \leq \int_\Omega v p(u)$ for all $p \in P_0$, then $u \ll v$.
(ii) If $u, v \in L^1(\Omega)$ and $u \ll v$, then $\|u\|_q \leq \|v\|_q$ for any $q \in [1, +\infty]$.
(iii) If $v \in L^1(\Omega)$, then $\{u \in L^1(\Omega) : u \ll v\}$ is a weakly compact subset of $L^1(\Omega)$.

The following Poincaré's type inequality is given in [10].

PROPOSITION 2.2. *Given J and Ω the quantity*

$$\beta_1 := \beta_1(J, \Omega) = \inf_{u \in L^2(\Omega), \int_\Omega u = 0} \frac{\frac{1}{2} \int_\Omega \int_\Omega J(x - y)(u(y) - u(x))^2 \, dy \, dx}{\int_\Omega (u(x))^2 \, dx} \tag{2.1}$$

is strictly positive. Consequently

$$\beta_1 \int_\Omega \left| u - \frac{1}{|\Omega|} \int_\Omega u \right|^2 \leq \frac{1}{2} \int_\Omega \int_\Omega J(x - y)(u(y) - u(x))^2 \, dy \, dx, \quad \forall u \in L^2(\Omega). \tag{2.2}$$

In order to obtain a generalized Poincaré's type inequality we need the following result.

PROPOSITION 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $k > 0$. There exists a constant $C > 0$ such that for any $K \subset \Omega$ with $|K| > k$, it holds*

$$\|u\|_{L^2(\Omega)} \leq C \left(\left\| u - \frac{1}{|\Omega|} \int_\Omega u \right\|_{L^2(\Omega)} + \left| \int_K u \right| \right), \quad \forall\, u \in L^2(\Omega). \tag{2.3}$$

Proof. Assume the conclusion does not hold. Then, for every $n \in \mathbb{N}$ there exists $K_n \subset \Omega$ with $|K_n| > k$, and $u_n \in L^2(\Omega)$ satisfying

$$\|u_n\|_{L^2(\Omega)} \geq n \left(\left\| u_n - \frac{1}{|\Omega|} \int_\Omega u_n \right\|_{L^2(\Omega)} + \left| \int_{K_n} u_n \right| \right), \quad \forall n \in \mathbb{N}. \tag{2.4}$$

We normalize u_n by $\|u_n\|_{L^2(\Omega)} = 1$ for all $n \in \mathbb{N}$, and consequently we can assume that

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(\Omega). \tag{2.5}$$

Moreover, by (2.4), we have

$$\left\| u_n - \frac{1}{|\Omega|} \int_\Omega u_n \right\|_{L^2(\Omega)} \leq \frac{1}{n}, \quad \text{and} \quad \left| \int_{K_n} u_n \right| \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Hence

$$u_n - \frac{1}{|\Omega|} \int_\Omega u_n \to 0 \quad \text{in } L^2(\Omega),$$

and by (2.5) we get $u(x) = \frac{1}{|\Omega|} \int_\Omega u = \alpha$ for almost all $x \in \Omega$, and $u_n \to \alpha$ strongly in $L^2(\Omega)$. Since $\|u_n\|_{L^2(\Omega)} = 1$ for each $n \in \mathbb{N}$, $\alpha \neq 0$. On the other hand, (2.6) implies

$$\lim_{n \to \infty} \int_{K_n} u_n = 0.$$

Since χ_{K_n} is bounded in $L^2(\Omega)$, we can extract a subsequence (still denoted by χ_{K_n}) such that

$$\chi_{K_n} \rightharpoonup \phi \quad \text{weakly in } L^2(\Omega).$$

Moreover, ϕ is nonnegative and verifies

$$k \leq \lim_{n \to \infty} |K_n| = \lim_{n \to \infty} \int_\Omega \chi_{K_n} = \int_\Omega \phi.$$

Now, since $u_n \to \alpha$ strongly in $L^2(\Omega)$ and $\chi_{K_n} \rightharpoonup \phi$ weakly in $L^2(\Omega)$ we have

$$0 = \lim_{n \to \infty} \int_{K_n} u_n = \lim_{n \to \infty} \int_\Omega \chi_{K_n} u_n = \alpha \int_\Omega \phi \neq 0,$$

a contradiction. $\qquad \square$

To simplify the notation we define the linear self-adjoint operator $A : L^2(\Omega) \to L^2(\Omega)$ by

$$Au(x) = \int_\Omega J(x - y)(u(y) - u(x)) \, dy, \quad x \in \Omega.$$

As a consequence of the above results we have the next proposition, which plays the role of the classical generalized Poincaré's inequality for Sobolev spaces.

PROPOSITION 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $k > 0$. There exists a constant $C = C(J, \Omega, k)$ such that, for any $K \subset \Omega$ with $|K| > k$,*

$$\|u\|_{L^2(\Omega)} \leq C \left(\left(-\int_\Omega Au \, u \right)^{1/2} + \|u\|_{L^2(K)} \right) \quad \forall u \in L^2(\Omega). \tag{2.7}$$

Using the above result and working as in the proof of Lemma 4.2 in [1], we obtain the following lemma, of which we give the proof for the sake of completeness.

LEMMA 2.5. *Let γ be a maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \gamma(0)$. Let $\{u_n\}_{n \in \mathbb{N}} \subset L^2(\Omega)$ and $\{z_n\}_{n \in \mathbb{N}} \subset L^1(\Omega)$ such that, for every $n \in \mathbb{N}$, $z_n \in \gamma(u_n)$ a.e. in Ω.*
Let us suppose that
(i) if $\gamma_+ = +\infty$, there exists $M > 0$ such that

$$\int_\Omega z_n^+ < M, \quad \forall n \in \mathbb{N},$$

(ii) if $\gamma_+ < +\infty$, there exists $M \in \mathbb{R}$ and $h > 0$ such that

$$\int_\Omega z_n < M < \gamma_+ |\Omega|, \quad \forall n \in \mathbb{N}$$

and

$$\int_{\{x \in \Omega : z_n(x) < -h\}} |z_n| < \frac{\gamma_+ |\Omega| - M}{4}, \quad \forall n \in \mathbb{N}.$$

Then, there exists a constant $C = C(M, \Omega)$ in case (i), $C = C(M, \Omega, \gamma, h)$ in case (ii), such that

$$\|u_n^+\|_{L^2(\Omega)} \leq C \left(\left(-\int_\Omega Au_n^+ \, u_n^+ \right)^{1/2} + 1 \right), \quad \forall n \in \mathbb{N}. \tag{2.8}$$

Let us suppose that
(iii) if $\gamma_- = -\infty$, there exists $M > 0$ such that

$$\int_\Omega z_n^- < M, \quad \forall n \in \mathbb{N},$$

(iv) if $\gamma_- > -\infty$, there exists $M \in \mathbb{R}$ and $h > 0$ such that

$$\int_\Omega z_n > M > \gamma_- |\Omega|, \quad \forall n \in \mathbb{N}$$

and

$$\int_{\{x \in \Omega : z_n(x) > h\}} z_n < \frac{M - \gamma_- |\Omega|}{4}, \quad \forall n \in \mathbb{N}.$$

Then, there exists a constant $\tilde{C} = \tilde{C}(M, \Omega)$ in case (iii), $\tilde{C} = \tilde{C}(M, \Omega, \gamma, h)$ in case (iv), such that

$$\|u_n^-\|_{L^2(\Omega)} \le \tilde{C}\left(\left(-\int_\Omega A u_n^- \, u_n^-\right)^{1/2} + 1\right), \quad \forall n \in \mathbb{N}. \qquad (2.9)$$

Proof. Let us only prove (2.8), since the proof of (2.9) is similar. First, consider the case $\gamma_+ = +\infty$. Then, by assumption, there exists $M > 0$ such that

$$\int_\Omega z_n^+ < M, \quad \forall n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ let $K_n = \left\{x \in \Omega : z_n^+(x) < \frac{2M}{|\Omega|}\right\}$. Then

$$0 \le \int_{K_n} z_n^+ = \int_\Omega z_n^+ - \int_{\Omega \setminus K_n} z_n^+ \le M - (|\Omega| - |K_n|)\frac{2M}{|\Omega|} = |K_n|\frac{2M}{|\Omega|} - M.$$

Therefore,

$$|K_n| \ge \frac{|\Omega|}{2},$$

and

$$\|u_n^+\|_{L^2(K_n)} \le |K_n|^{1/2} \sup \gamma^{-1}\left(\frac{2M}{|\Omega|}\right).$$

Then, by Proposition 2.4, for all $n \in \mathbb{N}$,

$$\|u_n^+\|_{L^2(\Omega)} \le \tilde{C}(J, \Omega)\left(\left(-\int_\Omega A u_n^+ \, u_n^+\right)^{1/2} + |\Omega|^{1/2} \sup \gamma^{-1}\left(\frac{2M}{|\Omega|}\right)\right).$$

Now, let us consider the case $\gamma_+ < +\infty$. Let

$$\delta = \gamma_+ |\Omega| - M.$$

By assumption, for every $n \in \mathbb{N}$, we have,

$$\int_\Omega z_n < \gamma_+ |\Omega| - \delta. \qquad (2.10)$$

For $n \in \mathbb{N}$, let $K_n = \left\{x \in \Omega : z_n(x) < \gamma_+ - \frac{\delta}{2|\Omega|}\right\}$. Then, by (2.10),

$$\int_{K_n} z_n = \int_\Omega z_n - \int_{\Omega \setminus K_n} z_n < -\frac{\delta}{2} + |K_n|\left(\gamma_+ - \frac{\delta}{2|\Omega|}\right).$$

Moreover,

$$\int_{K_n} z_n = -\int_{K_n \cap \{x \in \Omega : z_n < -h\}} |z_n| + \int_{K_n \cap \{x \in \Omega : z_n \ge -h\}} z_n \ge -\frac{\delta}{4} - h|K_n|.$$

Therefore,

$$|K_n| \left(h - \frac{\delta}{2|\Omega|} + \gamma_+ \right) \geq \frac{\delta}{4}.$$

Hence $|K_n| > 0$, $h - \frac{\delta}{2|\Omega|} + \gamma_+ > 0$ and

$$|K_n| \geq \frac{\delta}{4 \left(h - \frac{\delta}{2|\Omega|} + \gamma_+ \right)}.$$

Consequently,

$$\|u_n^+\|_{L^2(K_n)} \leq |K_n|^{1/2} \sup \gamma^{-1} \left(\gamma_+ - \frac{\delta}{2|\Omega|} \right).$$

Then, by Proposition 2.4,

$$\|u_n^+\|_{L^2(\Omega)} \leq \tilde{C}(J, \Omega, \gamma, h) \left(\left(- \int_\Omega A u_n^+ \, u_n^+ \right)^{1/2} + |\Omega|^{1/2} \sup \gamma^{-1} \left(\gamma_+ - \frac{\delta}{2|\Omega|} \right) \right).$$

This ends the proof of (2.8). $\square$

Finally, we have the following monotonicity result. Its proof is straightforward.

LEMMA 2.6. *Let* $T : \mathbb{R} \to \mathbb{R}$ *a nondecreasing function. For every* $u \in L^2(\Omega)$ *such that* $T(u) \in L^2(\Omega)$, *it holds*

$$- \int_\Omega A u(x) T(u(x)) \, dx = - \int_\Omega \int_\Omega J(x - y)(u(y) - u(x)) \, dy \, T(u(x)) \, dx$$

$$= \frac{1}{2} \int_\Omega \int_\Omega B(x, y)(u(y) - u(x))^2 \, dy \, dx,$$

where $B(x, y)$ *is the non negative symmetric function given by*

$$B(x, y) = \begin{cases} J(x - y) \dfrac{T(u(y)) - T(u(x))}{u(y) - u(x)} & \text{if } u(y) \neq u(x), \\ 0 & \text{if } u(y) = u(x). \end{cases}$$

In particular we have

$$- \int_\Omega A u(x) \, u(x) \, dx = \frac{1}{2} \int_\Omega \int_\Omega J(x - y)(u(y) - u(x))^2 \, dy \, dx.$$

3. Mild solutions and contraction principle

In this section we obtain a mild solution to our problem studying the associated integral operator.

Given a maximal monotone graph γ in $\mathbb{R}^2$ such that $0 \in \gamma(0)$, $\gamma_- < \gamma_+$, we consider the problem,

$$(S_\phi^\gamma) \quad \gamma(u) - Au \ni \phi \quad \text{in } \Omega.$$

DEFINITION 3.1. Let $\phi \in L^1(\Omega)$. A pair of functions $(u, z) \in L^2(\Omega) \times L^1(\Omega)$ is a *solution* of problem (S_ϕ^γ) if $z(x) \in \gamma(u(x))$ *a.e.* $x \in \Omega$ and $z(x) - Au(x) = \phi(x)$ *a.e.* $x \in \Omega$, that is,

$$z(x) - \int_\Omega J(x - y)(u(y) - u(x))\, dy = \phi(x) \quad a.e.\ x \in \Omega.$$

With respect to uniqueness of problem (S_ϕ^γ), we have the following maximum principle.

THEOREM 3.2. *(i) Let $\phi_1 \in L^1(\Omega)$ and (u_1, z_1) a subsolution of $(S_{\phi_1}^\gamma)$, that is, $z_1(x) \in \gamma(u_1(x))$ a.e. $x \in \Omega$ and $z_1(x) - Au_1(x) \le \phi_1(x)$ a.e. $x \in \Omega$, and let $\phi_2 \in L^1(\Omega)$ and (u_2, z_2) a supersolution of $(S_{\phi_2}^\gamma)$, that is, $z_2(x) \in \gamma(u_2(x))$ a.e. $x \in \Omega$ and $z_2(x) - Au_2(x) \ge \phi_2(x)$ a.e. $x \in \Omega$. Then*

$$\int_\Omega (z_1 - z_2)^+ \le \int_\Omega (\phi_1 - \phi_2)^+.$$

Moreover, if $\phi_1 \le \phi_2$, $\phi_1 \ne \phi_2$, then $u_1(x) \le u_2(x)$ a.e. $x \in \Omega$.

(ii) Let $\phi \in L^1(\Omega)$, and (u_1, z_1), (u_2, z_2) two solutions of (S_ϕ^γ). Then, $z_1 = z_2$ a.e. and there exists a constant c such that $u_1 = u_2 + c$, a.e.

Proof. To prove *(i)*, let (u_1, z_1) a subsolution of $(S_{\phi_1}^\gamma)$ and (u_2, z_2) a supersolution of $(S_{\phi_2}^\gamma)$. Then

$$-(Au_1(x) - Au_2(x)) + z_1(x) - z_2(x) \le \phi_1(x) - \phi_2(x).$$

Multiplying the above inequality by $\frac{1}{k} T_k^+ (u_1 - u_2 + k\, \text{sign}_0^+ (z_1 - z_2))$ and integrating we get,

$$\int_\Omega (z_1 - z_2) \frac{1}{k} T_k^+ (u_1 - u_2 + k\, \text{sign}_0^+ (z_1 - z_2))$$

$$- \int_\Omega (Au_1(x) - Au_2(x)) \frac{1}{k} T_k^+ (u_1(x) - u_2(x) + k\, \text{sign}_0^+ (z_1(x) - z_2(x)))\, dx \quad (3.1)$$

$$\le \int_\Omega (\phi_1 - \phi_2) \frac{1}{k} T_k^+ (u_1 - u_2 + k\, \text{sign}_0^+ (z_1 - z_2)) \le \int_\Omega (\phi_1(x) - \phi_2(x))^+\, dx.$$

Let us write $u = u_1 - u_2$ and $z = \text{sign}_0^+(z_1 - z_2)$, then, by the monotonicity proved in Lemma 2.6,

$$\lim_{k \to 0} \int_\Omega (Au_1(x) - Au_2(x)) \frac{1}{k} T_k^+ (u_1(x) - u_2(x) + k \, \text{sign}_0^+(z_1(x) - z_2(x))) \, dx$$

$$= \lim_{k \to 0} \int_\Omega Au(x) \frac{1}{k} T_k^+ (u(x) + kz(x)) \, dx$$

$$= \lim_{k \to 0} \int_\Omega A(u + kz)(x) \frac{1}{k} T_k^+ (u(x) + kz(x)) \, dx \le 0.$$

Therefore, taking limit as k goes to 0 in (3.1), we obtain

$$\int_\Omega (z_1 - z_2)^+ \le \int_\Omega (\phi_1 - \phi_2)^+.$$

Let us now suppose that $\phi_1 \le \phi_2$, $\phi_1 \ne \phi_2$. By the previous calculations we know that $z_1 \le z_2$. Since

$$\int_\Omega z_1 \le \int_\Omega \phi_1 < \int_\Omega \phi_2 \le \int_\Omega z_2,$$

$z_1 \ne z_2$. Going back to (3.1), if $u = u_1 - u_2$, we get

$$-\int_\Omega Au(x) \, T_k^+(u(x)) \, dx = 0,$$

and therefore,

$$-\int_\Omega Au(x) \, u^+(x) \, dx = 0.$$

Consequently, by Lemma 2.6, there exists a null set $C \subset \Omega \times \Omega$ such that

$$J(x - y)(u^+(y) - u^+(x))(u(y) - u(x)) = 0 \quad \text{for all } (x, y) \in \Omega \times \Omega \setminus C. \quad (3.2)$$

Let B a null subset of Ω such that if $x \notin B$, the section $C_x = \{y \in \Omega : (x, y) \in C\}$ is null. Let $x \notin B$, if $u(x) > 0$ then, since there exists $r_0 > 0$ such that $J(z) > 0$ for every z such that $|z| \le r_0$, by a compactness argument and having in mind (3.2), it is easy to see that $u(y) = u(x) > 0$ for all $y \notin C_x$. Therefore $u_1(y) > u_2(y)$ for all $y \notin C_x$ in Ω and consequently $z_1(y) \ge z_2(y)$ a.e. in Ω which contradicts that $z_1 \le z_2$, $z_1 \ne z_2$.

Let us now prove (ii). As (u_i, z_i) are solutions of (S_ϕ^γ) we have that

$$-(Au_1(x) - Au_2(x)) + z_1(x) - z_2(x) = 0.$$

Now, by (i), $z_1 = z_2$, a.e. Consequently,

$$0 = -(Au_1(x) - Au_2(x)) = -A(u_1 - u_2)(x).$$

Therefore, multiplying the above equation by $u = u_1 - u_2$ and integrating we obtain

$$\frac{1}{2} \int_\Omega \int_\Omega J(x - y)(u(y) - u(x))^2 \, dy \, dx = 0.$$

From here, by (2.2), u is constant $a.e.$ in Ω.   □

In particular we have the following result.

COROLLARY 3.3. *Let $k > 0$ and $u \in L^2(\Omega)$ such that*

$$ku - Au \geq 0 \quad a.e. \text{ in } \Omega,$$

then $u \geq 0$ a.e. in Ω.

Proof. Since (u, ku) is a supersolution of (S_0^γ), where $\gamma(r) = kr$, and $(0, 0)$ is a subsolution of (S_0^γ), by Theorem 3.2, the result follows.   □

To study the existence of solutions of problem (S_ϕ^γ) we start with the following two lemmas.

LEMMA 3.4. *Assume $\gamma : \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz continuous function with $\gamma(0) = 0$ and $\gamma_- < \gamma_+$. Let $\phi \in C(\overline{\Omega})$ such that $\gamma_- < \phi < \gamma_+$. Then, there exists a solution $(u, \gamma(u))$ of problem (S_ϕ^γ). Moreover, $\gamma(u) \ll \phi$.*

Proof. Since $\gamma_- < \phi < \gamma_+$ and $\phi \in C(\overline{\Omega})$, we can find $c_1 \leq c_2$ such that

$$\gamma_- < \gamma(c_1) \leq \phi(x) \leq \gamma(c_2) < \gamma_+ \quad \forall x \in \Omega. \tag{3.3}$$

Since γ is a nondecreasing Lipschitz continuous function there exists $k > 0$ for which the function $s \mapsto ks - \gamma(s)$ is nondecreasing. Let us see by induction that we can find a sequence $\{u_i\} \subset L^2(\Omega)$ such that

$$u_0 = c_1, \qquad u_i \leq u_{i+1} \leq c_2,$$

$$ku_{i+1} - Au_{i+1} = \phi - \gamma(u_i) + ku_i, \quad \forall i \in \mathbb{N}. \tag{3.4}$$

Since $k > 0$, as a consequence of being A self-adjoint, it is easy to see that k does not belong to the spectrum of A, then there exists $u_1 \in L^2(\Omega)$ such that

$$ku_1 - Au_1 = \phi - \gamma(c_1) + kc_1.$$

Then, by (3.3), we have

$$ku_1 - Au_1 = \phi - \gamma(c_1) + kc_1 \geq kc_1 = kc_1 - Ac_1.$$

Hence, from Corollary 3.3 we get that $u_0 = c_1 \leq u_1$. Analogously, there exists u_2 such that

$$ku_2 - Au_2 = \phi - \gamma(u_1) + ku_1.$$

Now, since $c_1 \leq u_1$, we get

$$ku_2 - Au_2 \geq \phi - \gamma(c_1) + kc_1 = ku_1 - Au_1.$$

Again by Corollary 3.3, we get $u_1 \leq u_2$, and by induction we obtain that $u_i \leq u_{i+1}$. On the other hand, since the function $s \mapsto ks - \gamma(s)$ is nondecreasing, $c_1 \leq c_2$ and (3.3), we have

$$kc_2 - Ac_2 \geq \phi - \gamma(c_2) + kc_2 \geq \phi - \gamma(c_1) + kc_1 = ku_1 - Au_1.$$

Applying again Corollary 3.3, we get $c_2 \geq u_1$, and by an inductive argument we deduce that $u_i \leq c_2$ for all $i \in \mathbb{N}$. Hence (3.4) holds. Consequently, there exists $u \in L^\infty(\Omega)$, such that $u(x) = \lim_{i \to +\infty} u_i(x)$ *a.e.* in Ω. Taking limits in (3.4), we obtain that

$$ku - Au = \phi - \gamma(u) + ku,$$

and $(u, \gamma(u))$ is a solution of problem (S_ϕ^γ), that is,

$$\gamma(u) - Au = \phi. \tag{3.5}$$

Finally, given $p \in P_0$, multiplying (3.5) by $p(\gamma(u))$, and integrating in Ω, we get

$$\int_\Omega \gamma(u(x)) p(\gamma(u(x))) \, dx - \int_\Omega Au(x) p(\gamma(u(x))) \, dx = \int_\Omega \phi(x) p(\gamma(u(x))) \, dx.$$

Now, by Lemma 2.6, the second term in the above equality is nonnegative, therefore

$$\int_\Omega \gamma(u(x)) p(\gamma(u(x))) \, dx \leq \int_\Omega \phi(x) p(\gamma(u(x))) \, dx.$$

By Proposition 2.1, we conclude that $\gamma(u) \ll \phi$. $\qquad\square$

LEMMA 3.5. *Assume γ is a maximal monotone graph in $\mathbb{R}^2$, $]-\infty, 0] \subset D(\gamma)$, $0 \in \gamma(0)$, $\gamma_- < \gamma_+$. Let $\tilde{\gamma}(s) = \gamma(s)$ if $s < 0$, $\tilde{\gamma}(s) = 0$ if $s \geq 0$. Assume $\tilde{\gamma}$ is Lipschitz continuous in $]-\infty, 0]$. Let $\phi \in C(\overline{\Omega})$ such that $\gamma_- < \phi < \gamma_+$. Then, there exists a solution (u, z) of (S_ϕ^γ). Moreover, $z \ll \phi$.*

Proof. If $\gamma_- < 0$, let c_1 such that $\gamma(c_1) = \{m_1\}$, $\gamma_- < m_1 < 0$ and $m_1 \leq \phi$. And if $\gamma_- = 0$ let $c_1 = m_1 = 0$. Let γ_r, $r \in \mathbb{N}$, be the Yosida approximation of γ and let the maximal monotone graph

$$\gamma^r(s) = \begin{cases} \gamma(s) & \text{if } s < 0, \\ \gamma_r(s) & \text{if } s \geq 0. \end{cases}$$

Observe that γ^r is a nondecreasing Lipschitz continuous function with $\gamma^r(0) = 0$ and, for r large enough, $\gamma_- = \gamma_-^r < \phi < \gamma_+^r$, $\gamma^r \leq \gamma^{r+1}$, and converges in the sense of maximal monotone graphs to γ. From the previous lemma, for each γ^r we obtain a solution (u_r, z_r) of $(S_\phi^{\gamma^r})$, that is, $z_r = \gamma^r(u_r)$ $a.e.$ and

$$z_r - Au_r = \phi. \tag{3.6}$$

Moreover, $z_r \ll \phi$, and consequently, $z_r \geq m_1$. Moreover, $u_r \geq c_1$. Let

$$\hat{z}_r(x) = \begin{cases} z_r(x) & \text{if } u_r(x) \leq 0, \\ \gamma_{r+1}(u_r(x)) & \text{if } u_r(x) > 0. \end{cases}$$

Then, since γ_r is nondecreasing,

$$\hat{z}_r \geq z_r,$$

and also,

$$\hat{z}_r \in \gamma^{r+1}(u_r).$$

Therefore, $(u_r, \hat{z}_r)$ is a supersolution to $(S_\phi^{\gamma^{r+1}})$. Using Theorem 3.2, we obtain that

$$\hat{z}_r \geq z_{r+1}.$$

Now, if $\hat{z}_r = z_r$ then

$$z_r \geq z_{r+1},$$

and if $\hat{z}_r \neq z_r$, by Theorem 3.2,

$$u_r \geq u_{r+1}.$$

So, there exists a monotone non increasing subsequence of $\{u_r\}$, denoted equal, with $u_r \geq \hat{c}_1$, or there exists a monotone non increasing subsequence of $\{z_r\}$, denoted equal, with $z_r \geq m_1$. In the first case, we have that

$$u_r \to u \quad \text{in } L^2(\Omega),$$

and also, since $z_r \ll \phi$,

$$z_r \to z \quad \text{weakly in } L^1(\Omega).$$

And in the second case, we obtain

$$z_r \to z \quad \text{in } L^1(\Omega). \tag{3.7}$$

In fact, since $z_r \ll \phi$, we get that

$$z_r \to z \quad \text{in } L^2(\Omega). \tag{3.8}$$

Now, in this second case, multiplying (3.6) by $u_r - u_s$ and integrating we get

$$- \int_\Omega A u_r (u_r - u_s) = \int_\Omega \phi (u_r - u_s) - \int_\Omega z_r (u_r - u_s).$$

Moreover,

$$- \int_\Omega A u_s (u_r - u_s) = \int_\Omega \phi (u_r - u_s) - \int_\Omega z_s (u_r - u_s).$$

Hence, since $\int_\Omega z_r = \int_\Omega z_s$,

$$- \int_\Omega A(u_r - u_s)(u_r - u_s) = - \int_\Omega (z_r - z_s)(u_r - u_s)$$

$$= - \int_\Omega (z_r - z_s) \left(u_r - \frac{1}{|\Omega|} \int_\Omega u_r - \left(u_s - \frac{1}{|\Omega|} \int_\Omega u_s \right) \right)$$

and, by Proposition 2.2,

$$\beta_1 \left\| \left(u_r - \frac{1}{|\Omega|} \int_\Omega u_r \right) - \left(u_s - \frac{1}{|\Omega|} \int_\Omega u_s \right) \right\|_{L^2(\Omega)} \le \| z_r - z_s \|_{L^2(\Omega)}.$$

From (3.8) we get,

$$u_r - \frac{1}{|\Omega|} \int_\Omega u_r \to w \quad \text{in } L^2(\Omega).$$

Let us see that $\left\{ \frac{1}{|\Omega|} \int_\Omega u_r \right\}$ is bounded. If not, we can assume, passing to a subsequence if necessary, that it converges to $-\infty$. Then, $u_r \to -\infty$ *a.e.* in Ω. Since $z_r \in \gamma^r(u_r)$, $\gamma^r \to \gamma$ and (3.7), $z = \gamma_-$ *a.e.* in Ω. Consequently, $\int_\Omega \phi = \int_\Omega z = |\Omega| \gamma_-$ which contradicts that $\phi > \gamma_-$. Thus, $\left\{ \frac{1}{|\Omega|} \int_\Omega u_r \right\}$ is bounded and we have that there exists a subsequence of $\{ u_r \}$, denoted equal, such that

$$u_r \to u \quad \text{in } L^2(\Omega).$$

Therefore, in both cases, $z \in \gamma(u)$ *a.e.* in Ω, $z \ll \phi$, and, taking limit in

$$z_r - A u_r \ni \phi,$$

we obtain

$$z - A u \ni \phi,$$

which concludes the proof. $\square$

With this lemma in mind we proceed to extend the result for general monotone graphs.

THEOREM 3.6. *Assume γ is a maximal monotone graph in $\mathbb{R}^2$, $0 \in \gamma(0)$ and $\gamma_- < \gamma_+$. Let $\phi \in C(\overline{\Omega})$ such that $\gamma_- < \phi < \gamma_+$. Then, there exists a solution (u, z) of (S_ϕ^γ). Moreover, $z \ll \phi$.*

Proof. Let γ_r, $r \in \mathbb{N}$, be the Yosida approximation of γ and let the maximal monotone graph

$$\gamma^r(s) = \begin{cases} \gamma(s) & \text{if } s > 0, \\ \gamma_r(s) & \text{if } s \leq 0. \end{cases}$$

Observe that γ^r satisfies the hypothesis of Lemma 3.5, $\gamma_-^r < \phi < \gamma_+^r$ for r large enough, $\gamma^r \geq \gamma^{r+1}$ and converges in the sense of maximal monotone graphs to γ. From the previous lemma, for each γ^r we obtain a solution (u_r, z_r) of $(S_\phi^{\gamma^r})$, $z_r \ll \phi$. Now, we can proceed similarly to the previous lemma passing to the limit to conclude. $\qquad \square$

The natural space to study the problem $P_\gamma^J(z_0)$ from the point of view of Nonlinear Semigroup Theory is $L^1(\Omega)$. In this space we define the following operator,

$$B^\gamma := \left\{ (z, \hat{z}) \in L^1(\Omega) \times L^1(\Omega) \ : \ \exists u \in L^2(\Omega) \text{ such that } (u, z) \text{ is a solution of } (S_{z+\hat{z}}^\gamma) \right\},$$

in other words, $\hat{z} \in B^\gamma(z)$ if and only if there exists $u \in L^2(\Omega)$ such that $z(x) \in \gamma(u(x)) \, a.e.$ in Ω, and

$$- \int_\Omega J(x - y)(u(y) - u(x)) \, dy = \hat{z}(x), \quad a.e. \ x \in \Omega. \tag{3.9}$$

The operator B^γ allows us to rewrite $P_\gamma^J(z_0)$ as the following abstract Cauchy problem in $L^1(\Omega)$,

$$\begin{cases} z'(t) + B^\gamma(z(t)) \ni 0 & t \in (0, T) \\ z(0) = z_0. \end{cases} \tag{3.10}$$

A direct consequence of Theorems 3.2 and 3.6 is the following result.

COROLLARY 3.7. *Assume γ is a maximal monotone graph in $\mathbb{R}^2$, $0 \in \gamma(0)$. Then, the operator B^γ is T-accretive in $L^1(\Omega)$ and satisfies*

$$\left\{ \phi \in C(\overline{\Omega}) : \gamma_- < \phi < \gamma_+ \right\} \subset Ran(I + B^\gamma).$$

The following theorem is a consequence of the above result.

THEOREM 3.8. *Let $T > 0$ and $z_{i\,0} \in L^1(\Omega)$, $i = 1, 2$. Let z_i be a solution in $[0, T]$ of $P_\gamma^J(z_{i\,0})$, $i = 1, 2$. Then*

$$\int_\Omega (z_1(t) - z_2(t))^+ \leq \int_\Omega (z_{10} - z_{20})^+ \tag{3.11}$$

for almost every $t \in]0, T[$.

Proof. Let $(u_i(t), z_i(t))$ be solutions of $P_\gamma^J(z_{0i})$, $i = 1, 2$. Then, since they are strong solutions of (3.10) and B^γ is T-accretive, (3.11) follows from the Nonlinear Semigroup Theory ([7]). $\qquad\square$

In the next result we characterize $\overline{D(B^\gamma)}^{L^1(\Omega)}$.

THEOREM 3.9. *Assume γ is a maximal monotone graph in $\mathbb{R}^2$. Then, we have*

$$\overline{D(B^\gamma)}^{L^1(\Omega)} = \left\{ z \in L^1(\Omega) : \ \gamma_- \le z \le \gamma_+ \right\}.$$

Proof. It is obvious that

$$\overline{D(B^\gamma)}^{L^1(\Omega)} \subset \left\{ z \in L^1(\Omega) : \ \gamma_- \le z \le \gamma_+ \right\}.$$

To obtain the another inclusion, it is enough to take $\phi \in C(\overline{\Omega})$, satisfying $\gamma_- < \phi < \gamma_+$, and to prove that $\phi \in \overline{D(B^\gamma)}^{L^1(\Omega)}$. Let $a, b \in \mathbb{R}$ such that $\gamma_- < a < \phi < b < \gamma_+$. Now, by Theorem 3.6, for any $n \in \mathbb{N}$, there exists $v_n := \left(I + \frac{1}{n} B^\gamma \right)^{-1} \phi \in D(B^\gamma)$. Then, $(v_n, n(\phi - v_n)) \in B^\gamma$, thus there exists $u_n \in L^2(\Omega)$ such that $v_n \in \gamma(u_n)$ *a.e.* in Ω and

$$v_n(x) - \frac{1}{n} \int_\Omega J(x - y)(u_n(y) - u_n(x))\,dy = \phi(x) \qquad \forall x \in \Omega. \qquad (3.12)$$

Moreover, $v_n \ll \phi$. Then,

$$-\infty < \inf \gamma^{-1}(a) \le u_n \le \sup \gamma^{-1}(b) < +\infty. \qquad (3.13)$$

Hence, from (3.12) and (3.13) it follows that $v_n \to \phi$ in $L^1(\Omega)$. $\qquad\square$

As a consequence of the above results we have the following theorem concerning mild solutions (see [7]).

THEOREM 3.10. *Assume γ is a maximal monotone graph in $\mathbb{R}^2$. Let $T > 0$ and let $z_0 \in L^1(\Omega)$ satisfying $\gamma_- \le z_0 \le \gamma_+$. Then, there exists a unique mild solution z of (3.10). Moreover $z(t) \ll z_0$ for all $t \ge 0$.*

Proof. For $n \in \mathbb{N}$, let $\varepsilon = T/n$, and consider a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \varepsilon$. Let $z_0^\varepsilon \in C(\overline{\Omega})$ with

$$\gamma_- < z_0^\varepsilon < \gamma_+$$

and

$$\|z_0^\varepsilon - z_0\|_{L^1(\Omega)} \le \varepsilon.$$

By Theorem 3.6, for n large enough, there exists a solution $(u_i^\varepsilon, z_i^\varepsilon)$ of

$$\gamma(u_i^\varepsilon) - \varepsilon A u_i^\varepsilon \ni z_{i-1}^\varepsilon \tag{3.14}$$

for $i = 1, \ldots, n$, with

$$z_i^\varepsilon \ll z_{i-1}^\varepsilon. \tag{3.15}$$

That is, there exists a unique solution $z_i^\varepsilon \in L^1(\Omega)$ of the time discretized scheme associated with (3.10),

$$z_i^\varepsilon + \varepsilon B^\gamma z_i^\varepsilon \ni \varepsilon z_{i-1}^\varepsilon, \quad \text{for } i = 1, \ldots, n.$$

Therefore, if we define $z_\varepsilon(t)$ by

$$\begin{cases} z_\varepsilon(0) = z_0^\varepsilon, \\ z_\varepsilon(t) = z_i^\varepsilon, \quad \text{for } t \in]t_{i-1}, t_i], \ i = 1, \ldots, n, \end{cases}$$

it is an ε-approximate solution of problem (3.10).

By using now the Nonlinear Semigroup Theory (see [5], [7], [14]), on account of Corollary 3.7 and Theorem 3.9, problem (3.10) has a unique mild-solution $z(t) \in C([0, T] : L^1(\Omega))$, obtained as $z(t) = L^1(\Omega)\text{-}\lim_{\varepsilon \to 0} z_\varepsilon(t)$ uniformly for $t \in [0, T]$. Finally, from (3.15) we get $z \ll z_0$. □

By Crandall-Liggett's Theorem, [14], the mild solution obtained above is given by the well-known exponential formula,

$$e^{-t B^\gamma} z_0 = \lim_{n \to \infty} \left(I + \frac{t}{n} B^\gamma \right)^{-n} z_0. \tag{3.16}$$

The nonlinear contraction semigroup $e^{-t B^\gamma}$ generated by the operator $-B^\gamma$ will be denoted in the sequel by $(S(t))_{t \geq 0}$.

In principle, it is not clear how these mild solutions have to be interpreted respect to $P_\gamma^J(z_0)$. In the next section we will see that they coincide with the solutions defined in the Introduction.

4. Existence of solutions

In this section we prove that the mild solution of (3.10) is in fact a solution in the sense of Definition 1.1 of problem $P_\gamma^J(z_0)$.

THEOREM 4.1. *Let $z_0 \in L^1(\Omega)$ such that $\gamma_- \leq z_0 \leq \gamma_+$, $\gamma_- < \frac{1}{|\Omega|} \int_\Omega z_0 < \gamma_+$ and $\int_\Omega j_\gamma^*(z_0) < +\infty$. Then, there exists a unique solution z to $P_\gamma^J(z_0)$ in $[0, T]$ for every $T > 0$. Moreover, $z(t) \ll z_0$ for all $t \geq 0$.*

Proof. We divide the proof in three steps.

Step 1. First, let us suppose that

$$\text{there exist } c_1, c_2 \text{ such that } c_1 \leq c_2, m_1 \in \gamma(c_1), m_2 \in \gamma(c_2) \tag{4.1}$$
$$\text{and } \gamma_- < m_1 \leq z_0 \leq m_2 < \gamma_+.$$

Let $z(t)$ be the mild solution of (3.10) given by Theorem 3.10. We shall show that z is a solution of problem $P_\gamma^J(z_0)$.

For $n \in \mathbb{N}$, let $\varepsilon = T/n$, and consider a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \varepsilon$. Then, it follows that

$$z(t) = L^1(\Omega)\text{-}\lim_\varepsilon z_\varepsilon(t) \quad \text{uniformly for } t \in [0, T], \tag{4.2}$$

where $z_\varepsilon(t)$ is given, for ε small enough, by

$$\begin{cases} z_\varepsilon(t) = z_0 & \text{for } t \in]-\infty, 0], \\ z_\varepsilon(t) = z_i^n, & \text{for } t \in]t_{i-1}, t_i], \ i = 1, \ldots, n, \end{cases} \tag{4.3}$$

where $(u_i^n, z_i^n) \in L^2(\Omega) \times L^1(\Omega)$ is the solution of

$$-Au_i^n + \frac{z_i^n - z_{i-1}^n}{\varepsilon} = 0, \quad i = 1, 2, \ldots, n. \tag{4.4}$$

Moreover, $z_i^n \ll z_0$. Hence $\gamma_- < m_1 \leq z_i^n \leq m_2 < \gamma_+$ and consequently,

$$\inf \gamma^{-1}(m_1) \leq u_i^n \leq \sup \gamma^{-1}(m_2).$$

Therefore, if we write $u_\varepsilon(t) = u_i^n, t \in]t_{i-1}, t_i], \ i = 1, \ldots, n$, we can suppose that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(0, T; L^2(\Omega)) \text{ as } \varepsilon \to 0^+. \tag{4.5}$$

Since $z_\varepsilon \in \gamma(u_\varepsilon)$ *a.e.* in Q_T, $z_\varepsilon \to z$ in $L^1(Q_T)$, having in mind (4.5), we obtain that $z \in \gamma(u)$ *a.e.* in Q_T. On the other hand, from (4.4),

$$\frac{z_\varepsilon(t) - z_\varepsilon(t - \varepsilon)}{\varepsilon} \rightharpoonup z_t \quad \text{weakly in } L^2(0, T; L^2(\Omega)) \text{ as } \varepsilon \to 0^+.$$

Step 2. Let now $z_0 \in L^1(\Omega)$ such that $\gamma_- \leq z_0 \leq \gamma_+$, $\gamma_-|\Omega| < \int_\Omega z_0 < \gamma_-|\Omega|$, $\int_\Omega j_\gamma^*(z_0) < +\infty$, and

$$\text{there exists } c_1 \text{ and } m_1 \in \gamma(c_1) \text{ with } \gamma_- < m_1 \leq z_0 \tag{4.6}$$
$$\text{and (4.1) is not satisfied.}$$

Let $z_{0n} \in L^\infty(\Omega)$,

$$z_{0n} \nearrow z_0 \quad \text{as } n \text{ goes to } +\infty,$$

such that $\int_\Omega z_{0n} < \int_\Omega z_{0n+1}$ and $z_{0n} \leq m_2(n) < \gamma_+$, $m_2(n) \in \gamma(c_2(n))$ for some $c_2(n)$. By *Step 1*, there exists a solution z_n of problem $P_\gamma^J(z_{0n})$, which is the mild solution of (3.10) with initial datum z_{0n} and satisfies $z_n(t) \ll z_{0n}$ for all $t \geq 0$. It is obvious that

$$\lim_{n \to \infty} z_n = z \quad \text{in } C([0, T] : L^1(\Omega)),$$

being z the mild solution of (3.10) with initial datum z_0, moreover $z(t) \ll z_0$ for all $t \geq 0$. Next we prove that z is the solution of $P_\gamma^J(z_0)$.

Since z_n is a solution of problem $P_\gamma^J(z_{0n})$, there exists $u_n \in L^2(0, T, L^2(\Omega))$, $z_n \in \gamma(u_n)$ *a.e.* in $\Omega \times]0, T[$, such that

$$(z_n)_t - Au_n = 0. \tag{4.7}$$

Moreover, we can suppose that (see Theorem 3.2)

$$u_n \text{ is non decreasing in } n. \tag{4.8}$$

Multiplying (4.7) by u_n, we obtain

$$\frac{d}{dt} \int_\Omega \left(\int_0^{z_n(t)} (\gamma^{-1})^0(s)ds \right) = \int_\Omega Au_n(t)u_n(t)dt \tag{4.9}$$

in $\mathcal{D}'(]0, T[)$. Indeed, since $u_n(t) \in \gamma^{-1}(z_n(t)) = \partial j_\gamma^*(z_n(t))$,

$$(z_n(t + \tau) - z_n(t))u_n(t) \leq \int_{z_n(t)}^{z_n(t+\tau)} (\gamma^{-1})^0(s)ds \quad \text{for all } \tau.$$

Consequently,

$$\int_\Omega (z_n)_t(t)u_n(t) = \frac{d}{dt} \int_\Omega \left(\int_0^{z_n(t)} (\gamma^{-1})^0(s)ds \right)$$

and (4.9) holds.

Integrating now (4.9) between 0 and T we get

$$-\int_0^T \int_\Omega Au_n(t)\, u_n(t)\, dt \leq \int_\Omega j_\gamma^*(z_0). \tag{4.10}$$

Let us see that $\{u_n\}$ is bounded in $L^2(Q_T)$. In the case $\gamma_+ = +\infty$, let

$$M = \sup_{t \in [0,T]} \int_\Omega z^+(t) + 1.$$

Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{t \in [0,T]} \int_\Omega (z_n)^+(t) < M, \qquad \forall n \geq n_0.$$

In the case $\gamma_+ < +\infty$, since we have conservation of mass, there exists $M \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\sup_{t \in [0,T]} \int_\Omega z_n(t) < M < \gamma_+ |\Omega|,$$

moreover, since $z_n \ll z_0$, we have that $z_n \geq m_1$ and, since it is not difficult to see that $|m_1| < \frac{\gamma_+ |\Omega| - M}{4|\Omega|}$, we have

$$\sup_{t \in [0,T]} \int_{\{x \in \Omega : z_n(t)(x) < -4(m_1^2+1)|\Omega|/(\gamma_+|\Omega|-M)\}} |z_n(t)| < \frac{\gamma_+ |\Omega| - M}{4}, \qquad \forall n \in \mathbb{N}.$$

Therefore, in both cases, by Lemma 2.5, there exists $C > 0$ such that

$$\|(u_n(t))^+\|_{L^2(\Omega)} \leq C \left(\left(- \int_\Omega A(u_n(t))^+ (u_n(t))^+ \right)^{1/2} + 1 \right), \qquad \forall t \in [0, T]. \quad (4.11)$$

Hence, by (4.10), since u_n is non decreasing in n, $\{u_n\}$ is bounded in $L^2(Q_T)$.

Passing to a subsequence if necessary, we can assume

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(0, T; L^2(\Omega)) \text{ as } n \to +\infty,$$

and, by (4.8),

$$u_n \to u \quad \text{in } L^2(0, T; L^2(\Omega)) \text{ as } n \to +\infty.$$

Consequently,

$$z \in \gamma(u) \quad a.e. \text{ in } Q_T.$$

Since also $\{Au_n\}$ is bounded in $L^2(Q_T)$, passing to the limit in (4.7) we get

$$z_t - Au = 0.$$

Step 3. Let now $z_0 \in L^1(\Omega)$, $\gamma_- \leq z_0 \leq \gamma_+$ and $\gamma_- |\Omega| < \int_\Omega z_0 < \gamma_- |\Omega|$, $\int_\Omega j_\gamma^*(z_0) < +\infty$ such that (4.6) is not satisfied. Let $z_{0n} \in L^\infty(\Omega)$,

$$z_{0n} \searrow z_0 \quad \text{as } n \text{ goes to } +\infty,$$

such that $\int_\Omega z_{0n} > \int_\Omega z_{0n+1}$ and $z_{0n} \geq m_1(n) > \gamma_-$, $m_1(n) \in \gamma(c_1(n))$ for some $c_1(n)$. By *Step 2*, there exist a solution z_n of problem $P_\gamma^J(z_{0n})$, which is the mild solution of (3.10) with initial datum z_{0n} and satisfies $z_n(t) \ll z_0$ for all $t \geq 0$. It is obvious that

$$\lim_{n \to \infty} z_n = z \quad \text{in } C([0, T] : L^1(\Omega)), \qquad (4.12)$$

being z the mild solution of (3.10) with initial datum z_0. Moreover $z(t) \ll z_0$ for all $t \geq 0$. We shall see that z is the solution of $P_\gamma^J(z_0)$. The proof is similar to the above step and we

only need to take care in the proof of the boundedness of $\{u_n\}$ in $L^2(Q_T)$. To this end we need a formula like (4.11) for u_n^-, that is, we need to prove that there exists $C > 0$ such that

$$\|(u_n(t))^-\|_{L^2(\Omega)} \leq C \left(\left(-\int_\Omega A(u_n(t))^- \, (u_n(t))^- \right)^{1/2} + 1 \right), \qquad \forall\, t \in [0, T]. \quad (4.13)$$

Let us consider first that $\gamma_- = -\infty$, and let

$$M = \sup_{t \in [0,T]} \int_\Omega z^-(t) + 1.$$

Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{t \in [0,T]} \int_\Omega (z_n)^-(t) < M, \qquad \forall n \geq n_0.$$

In the case $\gamma_- > -\infty$, there exists $M \in \mathbb{R}$, $h > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\inf_{t \in [0,T]} \int_\Omega z_n(t) > M > \gamma_-|\Omega| \qquad\qquad (4.14)$$

and

$$\sup_{t \in [0,T]} \int_{\{x \in \Omega: z_n(t)(x) > h\}} z_n(t) < \frac{M - \gamma_-|\Omega|}{4}. \qquad\qquad (4.15)$$

Formula (4.14) is straightforward and (4.15) follows from (4.12). Indeed, by (4.12), there exists $n_0 \in \mathbb{N}$, $\delta > 0$ and $h > 0$ such that, for all $n \geq n_0$ and for all $t \in [0, T]$,

$$\int_E |z_n(t)| < \frac{M - \gamma_-|\Omega|}{4}, \qquad \forall E \subset \Omega, \ |E| < \delta,$$

and we can take h satisfying

$$|\{x \in \Omega : z_n(t)(x) > h\}| < \delta.$$

Therefore, in both cases, by Lemma 2.5, (4.13) is proved.

Uniqueness of solutions follows from Theorem 3.8. $\square$

5. Asymptotic behaviour

In this section we study the asymptotic behaviour of the solutions to $P_\gamma^J(z_0)$. Note that since the solution preserves the total mass it is natural to expect that solutions to our diffusion problem converge to the mean value of the initial condition as $t \to \infty$. We shall see that this is the case, for instance, when γ is a continuous function, nevertheless this fails when γ has jumps.

Let us introduce the $\omega-$limit set for a given initial condition z_0,

$$\omega(z_0) = \left\{ w \in L^1(\Omega) \ : \ \exists t_n \to \infty \text{ with } S(t_n)z_0 \to w, \text{ strongly in } L^1(\Omega) \right\}$$

and the weak $\omega-$limit set

$$\omega_\sigma(z_0) = \left\{ w \in L^1(\Omega) \ : \ \exists t_n \to \infty \text{ with } S(t_n)z_0 \rightharpoonup w, \text{ weakly in } L^1(\Omega) \right\}.$$

Since $S(t)z_0 \ll z_0$, $\omega_\sigma(z_0) \neq \emptyset$ always. Moreover since $S(t)$ preserves the total mass, for all $w \in \omega_\sigma(z_0)$,

$$\int_\Omega w = \int_\Omega z_0.$$

We denote by F the set of fixed points of the semigroup $(S(t))$, that is,

$$F = \left\{ w \in \overline{D(B^\gamma)}^{L^1(\Omega)} \ : \ S(t)w = w \ \ \forall t \geq 0 \right\}.$$

It is easy to see that

$$F = \left\{ w \in L^1(\Omega) \ : \ \exists k \in D(\gamma) \text{ such that } w \in \gamma(k) \right\}. \tag{5.1}$$

THEOREM 5.1. *Let $z_0 \in L^1(\Omega)$ such that $\gamma_- \leq z_0 \leq \gamma_+$, $\gamma_- < \frac{1}{|\Omega|} \int_\Omega z_0 < \gamma_+$ and $\int_\Omega j_\gamma^*(z_0) < +\infty$. Then, $\omega_\sigma(z_0) \subset F$. Moreover, if $\omega(z_0) \neq \emptyset$, then $\omega(z_0)$ consists of a unique $w \in F$, and consequently,*

$$\lim_{t\to\infty} S(t)z_0 = w \quad \text{strongly in } L^1(\Omega).$$

Proof. Along this proof we denote by $z(t) = S(t)z_0$ the solution to problem $P_\gamma^J(z_0)$ and $u(t)$ the corresponding function that appears in Definition 1.1.

Multiplying the equation in $P_\gamma^J(z_0)$ by $u(t)$ and integrating, we deduce

$$-\int_0^{+\infty} \int_\Omega Au(t)\,u(t)\,dt \leq \int_\Omega j_\gamma^*(z_0). \tag{5.2}$$

Therefore, thanks to (2.2), we obtain that there exists a constant C such that

$$\int_0^{+\infty} \int_\Omega \left| u(t) - \frac{1}{|\Omega|} \int_\Omega u(t) \right|^2 dt \leq C. \tag{5.3}$$

Let $w \in \omega_\sigma(z_0)$, then there exists a sequence $t_n \to +\infty$ such that $S(t_n)z_0 \rightharpoonup w$. By (5.3), we have

$$\alpha_n := \int_{t_n}^{+\infty} \int_\Omega \left| u(t) - \frac{1}{|\Omega|} \int_\Omega u(t) \right|^2 dt \to 0.$$

Take $s_n \to 0$ such that

$$\lim_{n\to\infty} \frac{\alpha_n}{s_n} = 0. \tag{5.4}$$

By contradiction it is easy to see that there exists $\bar{t}_n \in [t_n, t_n + \frac{C}{s_n}]$ such that

$$\int_\Omega \left| u(\bar{t}_n) - \frac{1}{|\Omega|} \int_\Omega u(\bar{t}_n) \right|^2 \le s_n,$$

and consequently,

$$\int_\Omega \left| u(\bar{t}_n) - \frac{1}{|\Omega|} \int_\Omega u(\bar{t}_n) \right|^2 \to 0. \tag{5.5}$$

Let us prove that

$$\frac{1}{|\Omega|} \int_\Omega u(\bar{t}_n)$$

is bounded. In fact, assume there exists a subsequence (still denoted by $\bar{t}_n$) such that

$$\frac{1}{|\Omega|} \int_\Omega u(\bar{t}_n) \to +\infty.$$

By (5.5) we get that $u(\bar{t}_n) \to +\infty$ $a.e.$ Since $z(\bar{t}_n) \in \gamma(u(\bar{t}_n))$, then $z(\bar{t}_n) \to \gamma_+$ $a.e.$ Moreover, as $z(\bar{t}_n) \ll z_0$ and $\gamma^+ \ge 0$, we can deduce that $\lim_{n\to\infty} z(\bar{t}_n) = \lim_{n\to\infty} z(\bar{t}_n)^+$ weakly in $L^1(\Omega)$. Hence, applying Fatou's Lemma, we get

$$\int_\Omega z_0 = \lim_{n\to\infty} \int_\Omega z(\bar{t}_n)^+ \ge \gamma_+ |\Omega|,$$

a contradiction. A similar argument shows that $\frac{1}{|\Omega|} \int_\Omega u(\bar{t}_n)$ is bounded from below. Therefore, passing to a subsequence if necessary, we may assume that

$$\frac{1}{|\Omega|} \int_\Omega u(\bar{t}_n) \to k$$

for some constant k. Using again (5.5),

$$u(\bar{t}_n) \to k, \qquad \text{strongly in } L^2(\Omega) \text{ and } a.e. \tag{5.6}$$

Since $z(\bar{t}_n) \ll z_0$, we can assume, taking a subsequence if necessary, that $z(\bar{t}_n) \rightharpoonup \hat{w}$ weakly in $L^1(\Omega)$. Then, from (5.6) it follows that $\hat{w} \in \gamma(k)$, and consequently $\hat{w} \in F$. Let us show now that $w = \hat{w}$. By (5.4), we have

$$\left\| z(\bar{t}_n) - z(t_n) \right\| = \left\| \int_{t_n}^{\bar{t}_n} z_t(s)\, ds \right\|_{L^1(\Omega)} = \left\| \int_{t_n}^{\bar{t}_n} Au(s)\, ds \right\|_{L^1(\Omega)}$$

$$\le M(\bar{t}_n - t_n)^{1/2} \left(\int_{t_n}^{+\infty} \int_\Omega \left| u(s) - \frac{1}{|\Omega|} \int_\Omega u(s) \right|^2 ds \right)^{1/2} \le M \left(C \frac{\alpha_n}{s_n} \right)^{1/2} \to 0,$$

where M is a constant depending of $|\Omega|$. Therefore, taking limit, we get $z(\bar{t}_n) - z(t_n) \to 0$, strongly in $L^1(\Omega)$, since it converges weakly to $\overline{w} - w$, it follows that $w = \overline{w}$, which is a fixed point. Finally, if $\omega(z_0) \neq \emptyset$, since $\omega(z_0) \subset \omega_\sigma(z_0) \subset F$ and $(S(t))$ is a contraction semigroup, we have that $\omega(z_0) = \{w\} \subset F$ and

$$\lim_{t \to \infty} S(t)z_0 = w \quad \text{strongly in } L^1(\Omega).$$

$\square$

REMARK 5.2. Note that in order to proof that $\omega(z_0) \neq \emptyset$, a usual tool is to show that the resolvent of B^γ is compact. In our case this fails in general as the following example shows. Let γ any maximal monotone graph with $\gamma(0) = [0, 1]$, $z_n \in L^\infty(\Omega)$, $0 \leq z_n \leq 1$ such that $\{z_n\}$ is not relatively compact in $L^1(\Omega)$. It is easy to check that $z_n = (I + B^\gamma)^{-1}(z_n)$. Hence $(I + B^\gamma)^{-1}$ is not a compact operator in $L^1(\Omega)$. On the other hand, since the nonlocal operator does not have regularizing effects, here we cannot prove regularity properties of the solutions that would help to find compactness of the orbits. Nevertheless, we shall see in the next result that when γ is a continuous function we are able to prove that $\omega(z_0) \neq \emptyset$.

Let us see now some cases in which $\omega(z_0) \neq \emptyset$ and

$$\lim_{t \to \infty} S(t)z_0 = \frac{1}{|\Omega|} \int_\Omega z_0 \quad \text{strongly in } L^1(\Omega).$$

Given a maximal monotone graph γ in $\mathbb{R} \times \mathbb{R}$, we set

$$\gamma(r+) := \inf \gamma(]r, +\infty[), \quad \gamma(r-) := \sup \gamma(] - \infty, r[)$$

for $r \in \mathbb{R}$, where we use the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. It is easy to see that

$$\gamma(r) = [\gamma(r-), \gamma(r+)] \cap \mathbb{R} \quad \text{for} \quad r \in \mathbb{R}.$$

Moreover, $\gamma(r-) = \gamma(r+)$ except at a countable set of points, which we denote by $J(\gamma)$.

COROLLARY 5.3. Let $z_0 \in L^1(\Omega)$ such that $\gamma_- \leq z_0 \leq \gamma_+$, $\gamma_- < \frac{1}{|\Omega|} \int_\Omega z_0 < \gamma_+$ and $\int_\Omega j_\gamma^*(z_0) < +\infty$. The following statements hold.

1. If $\frac{1}{|\Omega|} \int_\Omega z_0 \notin \gamma(J(\gamma))$ or $\frac{1}{|\Omega|} \int_\Omega z_0 \in \{\gamma(k+), \gamma(k-)\}$ for some $k \in J(\gamma)$, then

$$\lim_{t \to \infty} S(t)z_0 = \frac{1}{|\Omega|} \int_\Omega z_0 \quad \text{strongly in } L^1(\Omega).$$

2. If γ is a continuous function then

$$\lim_{t \to \infty} S(t)z_0 = \frac{1}{|\Omega|} \int_\Omega z_0 \quad \text{strongly in } L^1(\Omega).$$

3. If $\frac{1}{|\Omega|} \int_\Omega z_0 \in]\gamma(k-), \gamma(k+)[$ for some $k \in J(\gamma)$, then

$$\omega_\sigma(z_0) \subset \left\{ w \in L^1(\Omega) \ : w \in [\gamma(k-), \gamma(k+)] \ a.e., \int_\Omega w = \int_\Omega z_0 \right\}.$$

and consequently, for any $w \in \omega_\sigma(z_0)$, there exists a non null set in which $w \in]\gamma(k-), \gamma(k+)[$.

Proof. (1). Along this proof we denote by $z(t) = S(t)z_0$ the solution to problem $P_\gamma^J(z_0)$ and $u(t)$ the corresponding function that appears in Definition 1.1. First, let us assume that $\frac{1}{|\Omega|} \int_\Omega z_0 \notin \gamma(J(\gamma))$ and $z_0 \in L^\infty(\Omega)$. Working as in the above theorem, we have that there exists a constat k such that

$$u(t_n) \to k, \quad \text{strongly in } L^2(\Omega) \text{ and } a.e. \tag{5.7}$$

Since $z(t_n) \ll z_0$, there exists a subsequence such that $z(t_n) \rightharpoonup w$ weakly in $L^1(\Omega)$. Now, from $z(t_n) \in \gamma(u(t_n))$ we deduce that $w \in \gamma(k)$ and consequently, since $\frac{1}{|\Omega|} \int_\Omega z_0 = \frac{1}{|\Omega|} \int_\Omega w, k \notin J(\gamma)$. Then, there exists $\delta > 0$ such that γ is univalued and continuous on $]k - \delta, k + \delta[$. Hence, $w = \gamma(k)$ and $z(t_n) \to \gamma(k)$ $a.e.$ Therefore, Since $z(t_n)$ is bounded in $L^\infty(\Omega)$, $z(t_n) \to \gamma(k) = \frac{1}{|\Omega|} \int_\Omega z_0$ strongly in $L^1(\Omega)$. Then, by the above theorem we get that

$$z(t) \to \frac{1}{|\Omega|} \int_\Omega z_0, \quad \text{as } t \to \infty.$$

The general case $z_0 \in L^1(\Omega)$ follows easily from the previous arguments using again that we deal with a contraction semigroup.

Assume now that $\frac{1}{|\Omega|} \int_\Omega z_0 \in \{\gamma(k+), \gamma(k-)\}$ for some $k \in J(\gamma)$. It is easy to see that we can find $z_{0,n} \in L^1(\Omega)$, with $\gamma_- \leq z_{0,n} \leq \gamma_+, \gamma_- < \frac{1}{|\Omega|} \int_\Omega z_{0,n} < \gamma_+$ and $\int_\Omega j_\gamma^*(z_{0,n}) < +\infty$, such that $z_{0,n} \to z_0$ strongly in $L^1(\Omega)$ and verifying $\frac{1}{|\Omega|} \int_\Omega z_{0,n} \notin \gamma(J(\gamma))$ for all n. Then, by the above step, we have

$$S(t)z_{0,n} \to \frac{1}{|\Omega|} \int_\Omega z_{0,n}, \quad \text{strongly in } L^1(\Omega),$$

from where it follows, using again that $(S(t))$ is a contraction semigroup, that

$$S(t)z_0 \to \frac{1}{|\Omega|} \int_\Omega z_0, \quad \text{strongly in } L^1(\Omega).$$

Statement (2) is an obvious consequence of (1) since in this case $J(\gamma) = \emptyset$.

Finally, we prove (3). Given $w \in \omega_\sigma(z_0)$, by Theorem 5.1, there exists $k_0 \in D(\gamma)$, such that $w \in \gamma(k_0)$. Then, $k_0 = k$. In fact, if we assume, for instance, that $k_0 < k$, then

$$\gamma(k_0+) \geq \frac{1}{|\Omega|} \int_\Omega w = \frac{1}{|\Omega|} \int_\Omega z_0 > \gamma(k-) > \gamma(k_0+),$$

a contradiction. Hence, we have $w \in \gamma(k)$, and

$$\frac{1}{|\Omega|} \int_\Omega w = \frac{1}{|\Omega|} \int_\Omega z_0 \in]\gamma(k-), \gamma(k+)[.$$

Thus, $w \in [\gamma(k-), \gamma(k+)]$ $a.e.$ and, moreover, there exists a non null set in which $w \in]\gamma(k-), \gamma(k+)[.$ $\square$

REMARK 5.4. An alternative proof of the fact that $\omega(z_0) \subset F$ is the following. Let

$$\Psi : L^1(\Omega) \to] - \infty, +\infty]$$

the functional defined by

$$\Psi(z) := \begin{cases} \int_\Omega j_\gamma^*(z) & \text{if } j_\gamma^*(z) \in L^1(\Omega), \\ +\infty & \text{if } j_\gamma^*(z) \notin L^1(\Omega). \end{cases}$$

Since j_γ^* is continuous and convex, Ψ is lower semi-continuous ([8], pag.160). Moreover, since $S(t)z_0 \ll z_0$ for all $t \geq 0$, we have $\Psi(S(t)z_0) \leq \Psi(z_0)$ for all $t \geq 0$. Therefore, Ψ is a lower semi-continuous Liapunov functional for $(S(t))$. Then, by the Invariance Principle of Dafermos ([15]), Ψ is constant on $\omega(z_0)$. Consequently, given $w_0 \in \omega(z_0)$, if $w(t) = S(t)w_0$, we have $\Psi(w(t))$ is constant for all $t \geq 0$. Let $u(t)$ such that $w(t) \in \gamma(u(t))$ and $w_t = A(u(t))$. Working as in the proof of (4.9), we get

$$0 = \frac{d}{dt}\Psi(w(t)) = \frac{d}{dt}\int_\Omega j_\gamma^*(w(t)) = \frac{d}{dt}\int_\Omega j_{\gamma^{-1}}(w(t)) = \int_\Omega Au(t)\,u(t).$$

Then, by Proposition 2.2, we obtain that

$$u(t) = \frac{1}{|\Omega|}\int_\Omega u(t).$$

Hence, $w(t) \in F$ for all $t > 0$, and consequently, $w_0 \in F$.

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Fuensanta Andreu
Departamento de Matemática Aplicada, Universitat de València
Valencia, Spain
e-mail: fuensanta.andreu@uv.es

Jose M. Mazón and J. Julián Toledo-Melero
Departamento de Análisis Matemático, Universitat de València
Valencia, Spain
e-mail: mazon@uv.es, toledojj@uv.es

Julio D. Rossi
Departamento de Matemática, FCEyN UBA (1428)
Buenos Aires, Argentina
e-mail: jrossi@dm.uba.ar