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Second order evolution equations with time-dependent subdifferentials

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Abstract. We study an abstract second order nonlinear evolution equation in a real Hilbert space. We consider time-dependent convex functions and their subdifferentials operating on the first derivative of the unknown function. Introducing appropriate assumptions on the convex functions and other data, we prove the existence and uniqueness of a strong solution, and give some applications of the abstract theorem to hyperbolic variational inequalities with time-dependent constraints.

1. Introduction

In this paper, we prove the existence and uniqueness of a strong solution of the following evolution equation and initial condition in a real Hilbert space H:

$$u''(t) + Au(t) + \nu Au'(t) + \partial\varphi^t(u'(t)) \ni f(t), \qquad 0 < t < T, \tag{1.1}$$

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{1.2}$$

where A is a non-negative self-adjoint operator in H; $\nu \geq 0$ is a constant; $\partial\varphi^t$ is the subdifferential of a proper, l.s.c. (lower semi-continuous), and convex function φ^t depending on $t \in [0, T]$; f is a given H-valued function; u_0 and u_1 are given initial values. By a strong solution, we mean that all the terms u'', Au and Au' exist, belong to the Hilbert space H and satisfy the equation (1.1) and the initial condition (1.2). A precise definition will be given in Section 2.

J.L. Lions [11] initiated the study of this type of problem which refers to unilateral problems or variational inequalities for hyperbolic equations. Then, Brézis [4], Barbu [2], Sasaki [13] and Bernardi-Luterotti [3] studied various aspects of the problem.

In these studies, except for that of Sasaki [13], the case of a time-independent convex function (or a convex constraint), i.e., $\varphi^t \equiv \varphi$ in (1.1), was treated.

However, as was proposed by Lions [12, Open Problem 9.7] and Duvaut-Lions [8, Introduction], problems with time-dependent convex functions (or constraints) are important and interesting from both a theoretical view point and in respect to application.

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In this respect, Sasaki [13] proved the existence and uniqueness of a strong solution in the case where the convex functions are the indicator functions of time-dependent convex sets $K(t)$, that is, $\varphi^t = I_{K(t)}$, by employing the time-dependence condition on $K(t)$ introduced by Brézis [5] for parabolic variational inequalities (cf. (1.3) below).

The present paper aims to establish the existence of a unique strong solution of the problem {(1.1),(1.2)} with time-dependent subdifferentials $\partial\varphi^t$ in general and thus to unify the relevant precedent results.

For the time-dependence condition on φ^t, we employ the standard in the theory of the first order time-dependent subdifferential evolution equation:

$$u'(t) + \partial\varphi^t(u(t)) \ni f(t), \qquad 0 < t < T, \tag{1.3}$$

as developed by Kenmochi [9], Yamada [14] and Yotsutani [16], in a form as given in Kubo [10, Section 5, (E)]. See the condition (B) given below. This condition, in a slightly weaker form [10, Section 4,(D)], is necessary and sufficient for the equation (1.3) (with an initial condition $u(0) = u_0$) to admit a unique solution obeying a class of energy inequality. See [10, Section 4] for details. In this paper, we employ the framework and the results of [10] to control the time-dependence of φ^t.

We need another condition on φ^t for relating it to the operator A. For this purpose, we generalize the conditions used by Brézis [4, Théorème III.2], [5, (1)] and Sasaki [13, (A.1) and (A.2)]. See the condition (C) below and Lemma 5.1 in Section 5.

The main theorem is given in Section 2, and is proved in Sections 3 and 4 except for the proofs of some technical lemmas. In Section 5, we give some applications of our abstract theorem to variational inequalities for hyperbolic equations. The technical lemmas are proved in Section 6. In Appendix, for the convenience of readers, we give standard notions and notations related to convex functions and their subdifferentials. The basic notations and assumptions in this paper are given below.

Notation and assumptions

Throughout of this paper, H denotes a real Hilbert space with its norm and inner product denoted by $|\cdot|_H$ and $(\cdot,\cdot)$, respectively. For a proper, l.s.c., and convex function $\varphi : H \to \mathbf{R}\cup\{\infty\}$, its subdifferential and their Yosida-approximations are denoted by $\partial\varphi$, φ_λ and $\partial\varphi_\lambda$ $(\lambda > 0)$, respectively. For their definitions and fundamental properties, refer to the Appendix.

The following conditions (A)–(D) for the data A, φ^t, f, u_0 and u_1 are always assumed.

(A) A is a non-negative self-adjoint operator in H. The domains $D(A)$ and $D(A^{1/2})$ of A and of its fractional power $A^{1/2}$ are supposed to be Hilbert spaces with the graph norms. The Yosida-approximation of A is denoted by A_λ $(\lambda > 0)$. We write $A_\lambda^{1/2}$ for the fractional power $(A_\lambda)^{1/2}$ of A_λ.

(B) (cf. [10, Section 5, (E)]) $\{\varphi^t; 0 \le t \le T\}$ is a family of proper, l.s.c., and convex functions on H. There exists a constant $\alpha \in [0, 1]$, and for each $r > 0$ there are a constant $K_r \geq 0$ and two functions $p_r \in W^{1,\beta}(0,T)$ ($\beta = 2$ if $\alpha \in [0, 1/2]$ and $= 1/(1-\alpha)$ if $\alpha \in [1/2, 1]$) and $q_r \in W^{1,1}(0,T)$ such that for each $0 \leq s \leq t \leq T$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$ there exists an element $\tilde{z} \in D(\varphi^t)$ satisfying

$$|\tilde{z} - z|_H \leq |p_r(t) - p_r(s)|(\varphi^s(z) + K_r)^\alpha$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |q_r(t) - q_r(s)|(\varphi^s(z) + K_r).$$

(C) **(a)** There exists a non-negative function $g \in L^2(0,T)$ such that for all $t \in [0,T]$, $z \in H$ and $\lambda > 0$ and there holds

$$(\partial\varphi^t_\lambda(z), A_\lambda z) \geq -g(t)|\partial\varphi^t_\lambda(z)|_H.$$

(b) There exists $h \in W^{1,1}(0,T; D(A^{1/2}))$ such that the function $t \mapsto \varphi^t(h(t))$ belongs to $L^1(0,T)$.

(D) $f \in L^2(0,T; D(A^{1/2}))$, $u_0 \in D(A)$, $u_1 \in D(A^{1/2}) \cap D(\varphi^0)$.

2. Main Theorem

We denote by (E) the problem $\{(1.1),(1.2)\}$. The notion of a strong solution of (E) is defined below.

DEFINITION. A function $u : [0,T] \to H$ is called *a strong solution of* (E) if the following items are satisfied.

(a) $u \in W^{2,2}(0,T;H) \cap W^{1,\infty}(0,T;D(A^{1/2})) \cap L^\infty(0,T;D(A))$. In particular, when $\nu > 0$, $u \in W^{1,2}(0,T;D(A))$.

(b) There holds for a.e. $t \in (0,T)$

$$u''(t) + Au(t) + \nu Au'(t) + \partial\varphi^t(u'(t)) \ni f(t).$$

(c) $u(0) = u_0$ and $u'(0) = u_1$.

The uniqueness of a strong solution can be proved easily by a standard argument using the positivity of A and the monotonicity of $\partial\varphi^t$. Therefore, we omit the proof of uniqueness.

The main result of this paper is stated below.

THEOREM. *Under the conditions* (A)–(D)*, there exists a strong solution of* (E) *satisfying* $\sup_{0\leq t\leq T} |\varphi^t(u'(t))| < \infty$.

We will prove this theorem in the next two sections.

3. Proof of Theorem

Note that our condition (B) is the same as [10, Section 5, (E)]. Therefore, we have the following lemma (cf. [10, Lemma 2.1, Lemmas 3.1–3.3]).

LEMMA 3.1. *(a) There exists a constant $\gamma > 0$ such that for all $t \in [0, T]$, $z \in H$ and $\lambda \in (0, 1]$ there holds*

$$\varphi_\lambda^t(z) + \gamma |z|_H + \gamma \geq 0.$$

(b) For each $z \in H$ and $\lambda \in (0, 1]$, $t \mapsto \varphi_\lambda^t(z)$ and $t \mapsto \partial\varphi_\lambda^t(z)$ are measurable real-valued and H-valued functions, respectively.

(c) There exists a constant $\delta > 0$ such that for all $t \in [0, T]$, $z \in H$ and $\lambda \in (0, 1]$ and there holds

$$|J_\lambda^t z|_H \leq |z|_H + \delta \quad \text{and} \quad |\partial\varphi_\lambda^t(z)|_H \leq \frac{2|z|_H + \delta}{\lambda},$$

where $J_\lambda^t = (I + \lambda\partial\varphi^t)^{-1}$.

By Lemma 3.1 and a standard argument, we can prove the following.

PROPOSITION 3.2. *For each $\lambda \in (0, 1]$, there exists a unique solution $u \in W^{2,2}(0, T; H)$ of the following problem:*

$$u_\lambda''(t) + A_\lambda u_\lambda(t) + \nu A_\lambda u_\lambda'(t) + \partial\varphi_\lambda^t(u_\lambda'(t)) = f(t) \qquad a.e.\ t \in (0, T),$$

$$u_\lambda(0) = u_0, \qquad u_\lambda'(0) = u_1.$$

The crucial step in the proof of the Theorem is to derive the following uniform estimate.

PROPOSITION 3.3. *There exists a constant $M_0 > 0$ independent of $\lambda \in (0, 1]$ and $\nu \in [0, \nu_0]$ with a fixed $\nu_0 > 0$ such that the solution u_λ in Proposition 3.2 has the bound:*

$$|u_\lambda|_{W^{2,2}(0,T;H)} + |A_\lambda^{1/2} u_\lambda|_{W^{1,\infty}(0,T;H)} + |A_\lambda u_\lambda|_{L^\infty(0,T;H)} + \sqrt{\nu}|A_\lambda u_\lambda'|_{L^2(0,T;H)} + \sup_{0\leq t\leq T} |\varphi_\lambda^t(u_\lambda'(t))| \leq M_0.$$

The proof of this proposition will be given in Section 4.

We now prove the main Theorem. First, we have by Proposition 3.3, taking a subsequence of $\lambda \downarrow 0$ if necessary, that

$$u_\lambda \to u \text{weakly in } W^{2,2}(0, T; H),$$

$$A_\lambda^{1/2} u_\lambda \to v_{1/2} \text{weakly-}* \text{ in } W^{1,\infty}(0, T; H),$$

$$A_\lambda u_\lambda \to v_1 \text{weakly-}* \text{ in } L^\infty(0, T; H),$$

$$\partial\varphi_\lambda^{(\cdot)}(u_\lambda') \to u^* \text{weakly in } L^2(0, T; H),$$

and, if $\nu > 0$,

$$A_\lambda u'_\lambda \to w \quad \text{weakly in } L^2(0, T; H)$$

for some u, $v_{1/2}$, v_1, u^* and w.

It is straightforward that

$$u''(t) + v_1(t) + \nu w(t) + u^*(t) = f(t) \qquad \text{a.e. } t \in (0, T)$$

and

$$u_\lambda(0) = u_0, \qquad u'_\lambda(0) = u_1.$$

All we have to show is that $v_{1/2} = A^{1/2}u$, $v_1 = Au$, $w = Au'$, $u^*(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in (0, T)$ and $\sup_{0 \le t \le T} |\varphi^t(u'(t))| < \infty$.

Note first that

$$(I + \lambda A)^{-1}u_\lambda - u = (I + \lambda A)^{-1}u_\lambda - u_\lambda + u_\lambda - u = -\lambda A_\lambda u_\lambda + u_\lambda - u,$$

hence we have

$$(I + \lambda A)^{-1}u_\lambda \to u \quad \text{weakly in } L^2(0, T; H).$$

On the other hand,

$$A(I + \lambda A)^{-1}u_\lambda = A_\lambda u_\lambda \to v_1 \quad \text{weakly in } L^2(0, T; H).$$

Therefore, by the closedness (in the weak topology) of A extended as an operator in $L^2(0, T; H)$, we have $Au \in L^2(0, T; H)$ and $v_1 = Au$. From this and noting $D(A) \subset D(A^{1/2})$, we can also derive $A^{1/2}u \in L^2(0, T; H)$ and $v_{1/2} = A^{1/2}u$. When $\nu > 0$, we can show in the same way

$$w = Au'.$$

In order to prove $u^*(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in (0, T)$, take an arbitrary $v \in L^2(0, T; H)$ with

$$\int_0^T \varphi^t(v(t))dt < \infty.$$

Then, we have by the approximate equation

$$\begin{aligned}
&\int_0^T \varphi^t(v(t))dt - \int_0^T \varphi^t_\lambda(u'_\lambda(t))dt \int_0^T \varphi^t_\lambda(v(t))dt - \int_0^T \varphi^t_\lambda(u'_\lambda(t))dt \\
&\quad \ge \int_0^T (f - u''_\lambda - A_\lambda u_\lambda - \nu A_\lambda u'_\lambda, v - u'_\lambda)dt \\
&\quad = \int_0^T \{(f - u''_\lambda - A_\lambda u_\lambda - \nu A_\lambda u'_\lambda, v) - (f, u'_\lambda)\}dt \\
&\qquad + \int_0^T (u''_\lambda + A_\lambda u_\lambda + \nu A_\lambda u'_\lambda, u'_\lambda)dt \\
&\quad = \int_0^T \{(f - u''_\lambda - A_\lambda u_\lambda - \nu A u'_\lambda, v) - (f, u'_\lambda)\}dt + \frac{1}{2}|u'_\lambda(T)|^2_H - \frac{1}{2}|u_1|^2_H \\
&\qquad + \frac{1}{2}|A_\lambda^{1/2}u_\lambda(T)|^2_H - \frac{1}{2}|A_\lambda^{1/2}u_0|^2_H + \nu \int_0^T |A_\lambda^{1/2}u'_\lambda|^2 dt. \qquad (3.1)
\end{aligned}$$

Since for all $t \in [0, T]$

$$u'_\lambda(t) \to u'(t) \quad \text{and} \quad A_\lambda^{1/2} u_\lambda(t) \to A^{1/2}u(t) \qquad \text{weakly in } H, \tag{3.2}$$

we have

$$|u'(T)|_H^2 \le \liminf_{\lambda\to 0} |u'_\lambda(T)|_H^2 \quad \text{and} \quad |A^{1/2}u_\lambda(T)|_H^2 \le \liminf_{\lambda\to 0} |A_\lambda^{1/2} u_\lambda(T)|_H^2.$$

Similarly, since

$$A_\lambda^{1/2} u'_\lambda \to A^{1/2}u' \quad \text{weakly in } L^2(0, T; H),$$

we have

$$\int_0^T |A^{1/2}u'|^2 dt \le \liminf_{\lambda\to 0} \int_0^T |A_\lambda^{1/2} u'_\lambda|^2 dt.$$

Furthermore by (3.2), (A.1) in Appendix and Fatou's lemma we have

$$\int_0^T \varphi^t(u'(t))dt \le \int_0^T \liminf_{\lambda\to 0} \varphi_\lambda^t(u'_\lambda(t))dt \le \liminf_{\lambda\to 0} \int_0^T \varphi_\lambda^t(u'_\lambda(t))dt.$$

Now, taking these into account, we let $\lambda \to 0$ in (3.1) and we obtain

$$\begin{aligned}\int_0^T \varphi^t(v(t))dt - \int_0^T \varphi^t(u'(t))dt \;\ge\; & \int_0^T \{(f - u'' - Au - \nu Au', v) - (f, u')\}dt \\ & + \frac{1}{2}|u'(T)|_H^2 - \frac{1}{2}|u_1|_H^2 + \frac{1}{2}|A^{1/2}u_\lambda(T)|_H^2 \\ & - \frac{1}{2}|A^{1/2}u_0|_H^2 + \nu \int_0^T |A^{1/2}u'|^2 dt \\ = \; & \int_0^T (f - u'' - Au - \nu Au', v - u')dt \\ = \; & \int_0^T (u^*, v - u')dt.\end{aligned}$$

Therefore, we obtain $u^*(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in (0, T)$ with the aid of the following lemma.

LEMMA 3.4. *Let* $\Phi : L^2(0, T; H) \to \mathbf{R} \cup \{\infty\}$ *be defined by*

$$\Phi(v) := \int_0^T \varphi^t(v(t))dt.$$

Then, Φ *is proper, l.s.c., and convex. Moreover, for* $v, v^* \in L^2(0, T; H)$*,* $v^* \in \partial\Phi(v)$ *if and only if*

$$v^*(t) \in \partial\varphi^t(v(t)) \quad \textit{for a.e. } t \in (0, T).$$

The proof of this lemma is given in Section 6.1.

Finally, by (3.2) and (A.1) again, we have

$$\sup_{0\le t\le T} |\varphi^t(u'(t))| < M_0',$$

where $M_0' > 0$ depends only on M_0 and γ in Proposition 3.3 and Lemma 3.1-(a), respectively.

Thus the proof of Theorem is completed except for the proofs of Proposition 3.3 and Lemma 3.4, which are given in Sections 4 and 6, respectively.

4. Proof of Proposition 3.3

4.1. *Estimate I*

The following lemma will be used in deriving the first estimate.

LEMMA 4.1. *There exists a function* $k \in W^{2,2}(0, T; H) \cap W^{1,\infty}(0, T; D(A^{1/2})) \cap W^{1,2}(0, T; D(A))$ *such that* $\sup_{0\le t\le T} |\varphi^t(k'(t))| < \infty$.

The proof of this lemma is given in Section 6.2.

In what follows, we often write simply u for the approximate solution u_λ. Note first by the approximate equation

$$u'' - k'' + A_\lambda(u-k) + \nu A_\lambda(u'-k') + \partial\varphi^t_\lambda(u') = f_\lambda \qquad (:= f - k'' - A_\lambda k - \nu A_\lambda k').$$

Multiplying this by $u' - k'$ and adding $\gamma|u'|_H + \gamma$ to both sides, we obtain

$$\frac{1}{2}\frac{d}{dt}|u'-k'|_H^2 + \frac{1}{2}\frac{d}{dt}|A_\lambda^{1/2}(u-k)|_H^2 + \nu|A_\lambda^{1/2}(u'-k')|_H^2 + \varphi^t_\lambda(u') + \gamma|u'|_H + \gamma$$
$$\le (f_\lambda, u'-k') + \varphi^t_\lambda(k') + \gamma|u'|_H + \gamma.$$

Hence, by Lemma 4.1, Lemma 3.1-(a) and Gronwall's inequality, we obtain the following estimate.

PROPOSITION 4.2. *There exists a constant* $M_1 > 0$ *such that for all* $\lambda \in (0, 1]$ *there holds*

$$|u_\lambda|_{W^{1,\infty}(0,T;H)} + |A_\lambda^{1/2}u_\lambda|_{L^\infty(0,T;H)} + \sqrt{\nu}|A_\lambda^{1/2}u_\lambda'|_{L^2(0,T;H)} + |\varphi_\lambda^{(\cdot)}(u_\lambda')|_{L^1(0,T)} \le M_1.$$

4.2. *Estimate II*

The second and main estimate is based on the following lemma. For the proof we refer to [10, Lemma 3.4].

LEMMA 4.3. **(a)** *For any* $v \in W^{1,1}(0,T;H)$ *and* $\lambda \in (0,1]$, *the function* $t \mapsto \varphi_\lambda^t(v(t))$ *is of bounded variation and its positive variation is absolutely continuous on* $[0,T]$.

(b) *There exists a family* $\{(a_r, b_r, c_r, d_r); r > 0\} \subset [0,1) \times L^1(0,T) \times L^1(0,T) \times [0,\infty)$ *such that for any* $v \in W^{1,1}(0,T;H)$, $\lambda \in (0,1]$ *and* $r \geq |v|_{L^\infty(0,T;H)}$ *there holds for a.e.* $t \in (0,T)$

$$\frac{d}{dt}\varphi_\lambda^t(v(t)) - (\partial\varphi_\lambda^t(v(t)), v'(t)) \leq a_r|\partial\varphi_\lambda^t(v(t))|_H^2 + b_r(t) + c_r(t)\varphi_\lambda^t(v(t)) + d_r\left(\varphi_\lambda^t(v(t))\right)^2.$$

First, multiply the approximate equation by $A_\lambda u'$. Then, using the condition (C)–(a) and the Schwarz inequality, we have

$$\frac{d}{dt}\left\{\frac{1}{2}|A_\lambda^{1/2}u'|_H^2 + \frac{1}{2}|A_\lambda u|_H^2\right\} + \nu|A_\lambda u'|_H^2 \leq \frac{1}{2}|A_\lambda^{1/2}f|_H^2 + \frac{1}{2}|A_\lambda^{1/2}u'|_H^2 + g(t)|\partial\varphi_\lambda^t(u')|_H. \tag{4.1}$$

Second, by Lemma 4.3 and Proposition 4.2 we have for $r \geq M_1$

$$\frac{d}{dt}\varphi_\lambda^t(u') - (\partial\varphi_\lambda^t(u'), u'') \leq a_r|\partial\varphi_\lambda^t(u')|_H^2 + b_r(t) + \left(c_r(t) + d_r\varphi_\lambda^t(u')\right)\varphi_\lambda^t(u').$$

Hence, by the approximate equation, we have

$$\frac{d}{dt}\varphi_\lambda^t(u') + |u''|_H^2 + (A_\lambda u, u'') + \nu(A_\lambda u', u'') \leq (f, u'') + a_r|u'' + A_\lambda u + \nu A_\lambda u' - f|_H^2 + b_r(t) + \left(c_r(t) + d_r\varphi_\lambda^t(u')\right)\varphi_\lambda^t(u').$$

Notice here that

$$(A_\lambda u, u'') = \frac{d}{dt}(A_\lambda^{1/2}u, A_\lambda^{1/2}u') - |A_\lambda^{1/2}u'|_H^2.$$

Then, we obtain

$$\begin{aligned}&\frac{d}{dt}\left\{\varphi_\lambda^t(u') + (A_\lambda^{1/2}u, A_\lambda^{1/2}u') + \frac{\nu}{2}|A_\lambda^{1/2}u'|_H^2\right\} + |u''|_H^2 \\ &\quad\leq |A_\lambda^{1/2}u'|_H^2 + (f, u'') + a_r|u'' + A_\lambda u + \nu A_\lambda u' - f|_H^2 \\ &\qquad + b_r(t) + \left(c_r(t) + d_r\varphi_\lambda^t(u')\right)\varphi_\lambda^t(u').\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\frac{d}{dt}\left\{\varphi_\lambda^t(u') + (A_\lambda^{1/2}u, A_\lambda^{1/2}u') + \frac{\nu}{2}|A_\lambda^{1/2}u'|_H^2\right\} + (1 - a_r - \varepsilon)|u''|_H^2 \\ &\quad\leq A_\lambda^{1/2}u'|_H^2 + C_\varepsilon(|f|_H^2 + |A_\lambda u|_H^2 + \nu^2|A_\lambda u'|_H^2) \\ &\qquad + b_r(t) + \left(c_r(t) + d_r\varphi_\lambda^t(u')\right)\varphi_\lambda^t(u'),\end{aligned} \tag{4.2}$$

where $\varepsilon > 0$ is chosen so that $1 - a_r - \varepsilon > 0$ and $C_\varepsilon > 0$ depends on ε.

Now, let $C_1 := \max\{1, C_\varepsilon \nu\} + 1$ and calculate $C_1 \times (4.1) + (4.2)$. Then, we have

$$\frac{d}{dt}U + (C_1 - C_\varepsilon \nu)\nu |A_\lambda u'|_H^2 + (1 - a_r - \varepsilon)|u''|_H^2$$
$$\leq C_1 \left(\frac{1}{2}|A_\lambda^{1/2} f|_H^2 + \frac{1}{2}|A_\lambda^{1/2} u'|_H^2 + g(t)|\partial\varphi_\lambda^t(u')|_H \right)$$
$$+|A_\lambda^{1/2} u'|_H^2 + C_\varepsilon(|f|_H^2 + |A_\lambda u|_H^2) + b_r(t) + V(t)|\varphi_\lambda^t(u')|,$$

where

$$U(t) := C_1 \left(\frac{1}{2}|A_\lambda^{1/2} u'|_H^2 + \frac{1}{2}|A_\lambda u|_H^2 \right) + \varphi_\lambda^t(u') + (A_\lambda^{1/2} u, A_\lambda^{1/2} u') + \frac{\nu}{2}|A_\lambda^{1/2} u'|_H^2,$$

$$V(t) := \left| c_r(t) + d_r \varphi_\lambda^t(u'(t)) \right|.$$

Here we again use the Schwarz inequality to the term

$$C_1 g(t)|\partial\varphi_\lambda^t(u')|_H = C_1 g(t)|u'' + A_\lambda u + \nu A_\lambda u' - f|_H.$$

Then, we obtain

$$\frac{d}{dt}U + (C_1 - C_\varepsilon \nu - \varepsilon'\nu)\nu|A_\lambda u'|_H^2 + (1 - a_r - \varepsilon - \varepsilon')|u''|_H^2$$
$$\leq C_{\varepsilon,\varepsilon'} \left(|f|_H^2 + |A_\lambda^{1/2} f|_H^2 + g(t)^2 + W(t) \right) + b_r(t) + V(t)|\varphi_\lambda^t(u')|, \tag{4.3}$$

where $\varepsilon' > 0$ is chosen so that $C_1 - C_\varepsilon \nu - \varepsilon' \nu > 0$ and $1 - a_r - \varepsilon - \varepsilon' > 0$; $C_{\varepsilon,\varepsilon'} > 0$ depends on ε and ε' and is independent of other data, especially of $\lambda \in (0, 1]$ and of $\nu \in [0, \nu_0$ with a fixed $\nu_0 > 0$; and we have put

$$W := |A_\lambda^{1/2} u'|_H^2 + |A_\lambda u|_H^2.$$

Here note by Proposition 4.2 and Lemma 3.1-(a)

$$|A_\lambda^{1/2} u|_H^2 \leq |u|_H^2 + |A_\lambda u|_H^2 \leq M_1^2 + |A_\lambda u|_H^2,$$

$$|\varphi_\lambda^t(u')| \leq \varphi_\lambda^t(u') + 2\gamma|u'|_H + 2\gamma \leq \varphi_\lambda^t(u') + 2\gamma(M_1 + 1).$$

Therefore we have

$$U \geq \varphi_\lambda^t(u') + \frac{C_1 - 1}{2}|A_\lambda^{1/2} u'|_H^2 + + \frac{C_1 - 1}{2}|A_\lambda u|_H^2 - \frac{1}{2}M_1^2.$$

$$\geq |\varphi_\lambda^t(u')| + C_1' \left(|A_\lambda^{1/2} u'|_H^2 + |A_\lambda u|_H^2 \right) - M_1',$$

$$C_1' := \frac{C_1 - 1}{2}, \quad M_1' := \frac{1}{2}M_1^2 + 2\gamma(M_1 + 1).$$

Hence we have

$$W \leq (C_1')^{-1}(U + M_1'), \quad |\varphi_\lambda^t(u')| \leq U + M_1'.$$

Using these inequalities and noting by Proposition 4.2

$$|V|_{L^1(0,T)} \leq |c_r|_{L^1(0,T)} + d_r M_1,$$

we obtain the estimates desired in Proposition 3.3, by applying Gronwall's inequality in (4.3).

5. Application

In this section, we apply the abstract Theorem to some variational inequalities for a wave equation with ($\nu > 0$) or without ($\nu = 0$) the dissipation term:

$$u_{tt} - \Delta u - \nu \Delta u_t = f.$$

The following lemma is useful in verifying the condition (C)–(a).

LEMMA 5.1. *Let φ be a proper, l.s.c., and convex function on H, and A be a non-negative self-adjoint operator. Assume that there exists $g \in H$ such that for all $z \in H$ and $\lambda > 0$*

$$\varphi((I + \lambda A)^{-1}(z + \lambda g)) \leq \varphi(z). \tag{5.1}$$

Then, we have

$$(\partial\varphi_\lambda(z), A_\lambda z) \geq (\partial\varphi_\lambda(z), (I + \lambda A)^{-1}g).$$

Note that $|(I+\lambda A)^{-1}g|_H \leq |g|_H$. Therefore, by this lemma, the condition (5.1) is sufficient for the condition (C)–(a). This lemma will be proved in Section 6.3.

In what follows, Ω denotes a bounded domain in $\mathbf{R}^N$ ($N \geq 1$) with a smooth boundary $\partial\Omega$, and we put $H := L^2(\Omega)$ and $Az := -\Delta z$ for $z \in D(A) := H^2(\Omega) \cap H_0^1(\Omega)$. We give some example of convex functions satisfying the conditions (B) and (C).

5.1. *Unilateral constraints*

Let $j : \mathbf{R} \to \mathbf{R} \cup \{\infty\}$ be proper, l.s.c., and convex and put $\beta := \partial j$. Let $\psi \in W^{1,2}(0, T; H) \cap L^2(0, T; H^1(\Omega))$ satisfy $\Delta\psi \in L^2(0, T; H)$ and $\beta(-\psi(t, x)) \ni 0$ for all $(t, x) \in [0, T] \times \partial\Omega$, and $q \in W^{1,1}(0, T)$ be $q \geq 0$ on $[0, T]$.

Define for $t \in [0, T]$ and $z \in H$

$$\varphi^t(z) := \int_\Omega q(t) j(z(x) - \psi(t, x)) dx.$$

To verify the condition (B), put

$$\tilde{z} := z - \psi(s) + \psi(t).$$

Then,

$$|\tilde{z} - z|_H = |\psi(t) - \psi(s)|_H \le \int_s^t |\psi'(\tau)|_H d\tau$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) = (q(t) - q(s))\varphi^s(z).$$

Therefore, we can choose

$$p_r(t) := \int_0^t |\psi'(\tau)|_H d\tau \quad \text{and} \quad q_r(t) := q(t).$$

To verify the condition (C)–(a), first note that for $\mu > 0$

$$\varphi^t_\mu(z) = \int_\Omega q(t) j_\mu(z(x) - \psi(t,x))dx,$$

where φ^t_μ and j_μ are the Yosida-approximations of φ^t and j, respectively. Next, put $z_\lambda := (I+\lambda A)^{-1}(z-\lambda\Delta\psi)$ and note that $\partial j_\mu = \beta_\mu$, where β_μ is the Yosida-approximation of β. Then, we have

$$\begin{aligned}
\varphi^t_\mu(z) - \varphi^t_\mu(z_\lambda) &\ge \int_\Omega q(t)\beta_\mu(z_\lambda(x) - \psi(t,x))(z(x) - z_\lambda(x))dx \\
&= \lambda \int_\Omega q(t)\beta_\mu(z_\lambda(x) - \psi(t,x))(-\Delta(z_\lambda(x) - \psi(t,x)))dx \\
&= \lambda \int_\Omega q(t)\beta'_\mu(z_\lambda(x) - \psi(t,x))|\nabla(z_\lambda - \psi(t))|^2 dx \\
&\ge 0,
\end{aligned}$$

where we have used the relation

$$z - \lambda\Delta\psi - z_\lambda = (I - (I+\lambda A)^{-1})(z - \lambda\Delta\psi) = \lambda A(I+\lambda A)^{-1}(z - \lambda\Delta\psi) = -\lambda\Delta z_\lambda$$

and

$$\beta_\lambda(z_\lambda(x) - \psi(t,x)) = \beta_\lambda(-\psi(t,x)) = 0 \qquad \text{on } \partial\Omega.$$

By letting $\mu \to 0$, we have $\varphi^t(z_\lambda) \le \varphi^t(z)$. Thus by Lemma 5.1, we have (C)–(a) with $g := -\Delta\psi$.

The condition (C)–(b) is easily verified by taking $h \equiv 0$.

5.2. *Bilateral constraints*

Here we define φ^t to be the indicator function of a convex set $K(t)$:

$$\varphi^t := I_{K(t)},$$

where

$$K(t) := \{z \in H;\quad \psi_1(t) \le z \le \psi_2(t) \quad \text{in } \Omega\}$$

with $\psi_i \in W^{1,2}(0,T;H) \cap L^2(0,T;H^1(\Omega))$, $\Delta\psi_i \in L^2(0,T;H)$, $i = 1, 2$, $\psi_1 \le \psi_2$ in $[0,T] \times \Omega$, $\psi_1 \le 0 \le \psi_2$ on $[0,T] \times \Omega$.

The condition (B) is verified by choosing

$$\tilde{z} := [z - \psi_2(s) + \psi_2(t) - \psi_1(t)]^+ + \psi_1(t),$$

since we then have (cf. Yamazaki [15, Lemma 2.1])

$$|\tilde{z} - z|_H \le \int_s^t (|\psi_1'(\tau)|_H + |\psi_2'(\tau)|_H) d\tau$$

and we can take

$$p_r(t) := \int_0^t (|\psi_1'(\tau)|_H + |\psi_2'(\tau)|_H) d\tau \quad \text{and} \quad q_r \equiv 0.$$

To verify (C)–(a), note first that

$$\varphi^t = \varphi_1^t + \varphi_2^t,$$

where $\varphi_i^t = I_{K_i(t)}$, $i = 1, 2$, with

$$K_1(t) := \{z \in H;\ z \ge \psi_1(t) \text{ in } \Omega\}, \qquad K_2(t) := \{z \in H;\ z \le \psi_2(t) \text{ in } \Omega\}.$$

By the maximal principle, we can show for $i = 1, 2$

$$\varphi_i^t((I + \lambda A)^{-1}(z - \lambda\Delta\psi_i(t))) \le \varphi_i^t(z)$$

and therefore by Lemma 5.1

$$(\partial\varphi_{i,\lambda}^t(z), A_\lambda z) \ge (\partial\varphi_{i,\lambda}^t(z), (I + \lambda A)^{-1}(-\Delta\psi_i(t))) \ge -|\Delta\psi_i(t)|_H |\partial\varphi_{i,\lambda}^t(z)|_H.$$

Here note that

$$\partial\varphi_\lambda^t(z) = -\frac{[z - \psi_1]^-}{\lambda} + \frac{[z - \psi_2]^+}{\lambda} = \partial\varphi_{1,\lambda}^t(z) + \partial\varphi_{2,\lambda}^t(z)$$

and

$$|\partial\varphi_\lambda^t(z)|_H = |\partial\varphi_{1,\lambda}^t(z)|_H + |\partial\varphi_{2,\lambda}^t(z)|_H.$$

Therefore we have

$$(\partial\varphi_\lambda^t(z), A_\lambda z) \ge -\left(|\Delta\psi_1(t)|_H + |\Delta\psi_1(t)|_H\right) |\partial\varphi_\lambda^t(z)|_H$$

and the condition (C)–(a) is satisfied.

The condition (C)–(b) is verified easily by taking $h = (\psi_1 \wedge 0) \vee \psi_2$.

5.3. *Constraints on the gradients*

Here we assume that the domain Ω is convex and put $\varphi^t := I_{K_3(t)}$ with

$$K_3(t) := \{z \in H_0^1(\Omega); \quad |\nabla z(x)| \le a(t) \quad \text{in } \Omega\},$$

where $a \in W^{1,2}(0,T)$ and $a(t) \ge c_0$ for some constant $c_0 > 0$. Then, the condition (B) is verified by choosing

$$\tilde{z} := \frac{a(t)}{a(s)} z, \quad p_r(t) := \frac{r}{c_0}\int_0^t |a'(\tau)| d\tau \quad \text{and} \quad q_r \equiv 0.$$

Since the domain Ω is convex, we have by [7, Lemma III.4] that the condition (5.1) is satisfied and hence (C)–(a) as well by Lemma 5.1.

Finally, the condition (C)–(b) is verified by choosing $h \equiv 0$.

6. Proof of Lemmas

6.1. *Proof of Lemma 3.4*

Since our condition (B) is the same as [10, Section 5, (E)], it implies the condition [10, (a.2)], and hence Φ is proper; that is, not identically equal to infinity. Also, its convexity and lower semi-continuity are consequences of those of φ^t, [10, (a.1)] and Fatou's lemma.

Now, to characterize the subdifferential of Φ as in Lemma 3.4, we define an operator $\mathcal{A} : L^2(0,T;H) \to L^2(0,T;H)$ as follows: for $v, v^* \in L^2(0,T;H)$, $v^* \in \mathcal{A}v$ if and only if

$$v^*(t) \in \partial\varphi^t(v(t)) \quad \text{a.e. } t \in (0,T).$$

Then, it is easy to see that $\mathcal{A}$ is monotone and $\mathcal{A} \subset \partial\Phi$. We can also verify that the inverse of $I + \mathcal{A}$ is defined by: $v^* = (I + \mathcal{A})^{-1} v$ if and only if

$$v^*(t) = (I + \partial\varphi^t)^{-1}(v(t)) \quad \text{a.e. } t \in (0,T).$$

By Lemma 3.1, this is well-defined and defined everywhere on $L^2(0,T;H)$. Therefore (cf. [6, Proposition 2.2]), $\mathcal{A}$ is maximal monotone and hence equal to $\partial\Phi$.

6.2. *Proof of Lemma 4.1*

As the function k we can take

$$k(t) := \int_0^t \tilde{k}(s) ds,$$

where $\tilde{k} \in W^{1,2}(0, T; H) \cap L^\infty(0, T; D(A^{1/2})) \cap L^2(0, T; D(A))$ is the solution of the following problem:

$$\tilde{k}'(t) + A\tilde{k}(t) + \partial\varphi^t(\tilde{k}(t)) \ni 0 \qquad \text{a.e. } t \in (0, T), \tag{4.1}$$
$$\tilde{k}(0) = u_1 \quad (\in D(A^{1/2}) \cap D(\varphi^0)).$$

We also have $\sup_{0 \le t \le T} |\varphi^t(\tilde{k}(t))| < \infty$. To show the existence of such a solution, we first solve the approximate problem for $\lambda > 0$

$$\tilde{k}_\lambda'(t) + A_\lambda \tilde{k}_\lambda(t) + \partial\varphi_\lambda^t(\tilde{k}_\lambda(t)) = 0 \qquad \text{a.e. } t \in (0, T),$$
$$\tilde{k}_\lambda(0) = u_1.$$

We can derive the uniform estimate of $\{\tilde{k}_\lambda\}_{0<\lambda\le 1}$ in a way similar to that in Section 4 as follows.

First, take the function h in (C)–(b) and multiply (4.1) by $\tilde{k}_\lambda - h$ to obtain as Proposition 4.2

$$|\tilde{k}_\lambda|_{L^\infty(0,T;H)} + |A_\lambda^{1/2}\tilde{k}_\lambda|_{L^2(0,T;H)} + |\varphi_\lambda^{(\cdot)}(\tilde{k}_\lambda)|_{L^1(0,T)} \le N_1,$$

where $N_1 > 0$ is independent of $\lambda \in (0, 1]$.

Next, multiply (4.1) by $A\tilde{k}_\lambda$ to obtain

$$\frac{1}{2}\frac{d}{dt}|A_\lambda^{1/2}\tilde{k}_\lambda|^2 + |A_\lambda\tilde{k}_\lambda|^2 + (\partial\varphi_\lambda^t(\tilde{k}_\lambda), A\tilde{k}_\lambda) = 0$$

and by $\tilde{k}_\lambda'$ to obtain

$$|\tilde{k}_\lambda'|^2 + \frac{1}{2}\frac{d}{dt}|A_\lambda^{1/2}\tilde{k}_\lambda|^2 + (\partial\varphi_\lambda^t(\tilde{k}_\lambda), \tilde{k}_\lambda') = 0.$$

Then, using the condition (C)–(a) and Lemma 4.3 to these equalities, respectively, we can derive a uniform estimate

$$|\tilde{k}_\lambda|_{W^{1,2}(0,T;H)} + |A_\lambda^{1/2}\tilde{k}_\lambda|_{L^\infty(0,T;H)} + |A_\lambda\tilde{k}_\lambda|_{L^2(0,T;H)} + \sup_{0\le t\le T}|\varphi_\lambda^t(k(t))| \le N_2,$$

where $N_2 > 0$ is independent of $\lambda \in (0, 1]$, in the same way as in Section 4.2. We note here that, in applying Lemma 4.3-(b), we have to choose the number $r > 0$ so that $r \ge N_1$.

Now, as in Section 3, we can take a limit to obtain the desired solution $\tilde{k}$. Therefore the function k defined above has the desired properties.

6.3. *Proof of Lemma 5.1*

By (5.1), we have for $z^* \in \partial\varphi(z)$

$$(z^*, \frac{z - (I + \lambda A)^{-1}(z + \lambda g)}{\lambda}) \ge \frac{1}{\lambda}\left\{\varphi(z) - \varphi((I + \lambda A)^{-1}(z + \lambda g))\right\} \ge 0.$$

Hence

$$(z^*, A_\lambda z) \geq (z^*, (I + \lambda A)^{-1} g).$$

Here, noting $\partial\varphi_\lambda(z) \in \partial\varphi((I + \lambda\partial\varphi)^{-1}z)$, we choose $\partial\varphi_\lambda(z)$ and $(I + \lambda\partial\varphi)^{-1}z$ for z^* and z respectively to obtain

$$(\partial\varphi_\lambda(z), A_\lambda(I + \lambda\partial\varphi)^{-1}z) \geq (\partial\varphi_\lambda(z), (I + \lambda A)^{-1} g).$$

Therefore, by the non-negativity of A_λ, we have

$$\begin{aligned}(\partial\varphi_\lambda(z), A_\lambda z) &= \frac{1}{\lambda}(z - (I + \lambda\partial\varphi)^{-1}z, A_\lambda z)\\ &\geq \frac{1}{\lambda}(z - (I + \lambda\partial\varphi)^{-1}z, A_\lambda(I + \lambda\partial\varphi)^{-1}z)\\ &= (\partial\varphi_\lambda(z), A_\lambda(I + \lambda\partial\varphi)^{-1}z)\\ &\geq (\partial\varphi_\lambda(z), (I + \lambda A)^{-1} g).\end{aligned}$$

Thus the lemma is proved.

Appendix: Convex functions and subdifferentials

Here we recall some basic notions and properties concerning convex functions and their subdifferentials. We refer to Brézis [6] for the details and proofs.

Let $\varphi : H \to \{\infty\}$ be a proper, i.e., not identically equal to ∞, l.s.c. (lower semi-continuous), and convex function. Its effective domain $D(\varphi)$ is defined by $D(\varphi) := \{z \in H; \varphi(z) < \infty\}$.

For a closed convex set K in H, the indicator function I_K of K is defined by:

$$I_K(z) := \begin{cases} 0 & \text{for } z \in K \\ \infty & \text{for } z \in H \setminus K. \end{cases}$$

I_K is a proper, l.s.c., and convex function.

The subdifferential $\partial\varphi$ of a proper l.s.c., and convex function φ is a (possibly multi-valued) operator in H defined by: $z^* \in \partial\varphi(z)$ if and only if $z \in D(\varphi)$ and

$$(z^*, y - z) \leq \varphi(y) - \varphi(z)$$

for all $y \in D(\varphi)$. The domain $D(\partial\varphi)$ of $\partial\varphi$ is defined by $D(\partial\varphi) := \{z \in H;\ \partial\varphi(z) \neq \emptyset\}$. It is easy to see that $\partial\varphi$ is monotone, that is, there holds

$$(z_1^* - z_2^*, z_1 - z_2) \geq 0$$

for any $z_i^* \in \partial\varphi(z_i)$, $z_i \in D(\partial\varphi)$, $i = 1, 2$. Moreover $\partial\varphi$ is maximal monotone, that is, its graph is maximal with respect to the inclusion relation in the family of graphs of monotone operators.

By the maximal monotonicity of $\partial\varphi$, the operator $J_\lambda := (I + \lambda\partial\varphi)^{-1}$ is defined everywhere on H and is a contraction for all $\lambda > 0$ (cf. [6, Proposition 2.2]). For $\lambda > 0$, the Yosida-approximation $\partial\varphi_\lambda$ of $\partial\varphi$ is defined by

$$\partial\varphi_\lambda := \frac{I - J_\lambda}{\lambda}.$$

It is known (cf. [6, Proposition 2.6]) that $\partial\varphi_\lambda$ is maximal monotone with $D(\partial\varphi) = H$ and is Lipschitz continuous with a Lipschitz constant $1/\lambda$.

For $\lambda > 0$ the Yosida-approximation φ_λ of φ is defined by

$$\varphi_\lambda(z) := \inf_{y\in H}\left\{\varphi(y) + \frac{1}{2\lambda}|z-y|^2\right\}.$$

Then, we have (cf. [6, Proposition 2.11]) that φ_λ is continuous and convex on H,

$$\varphi_\lambda(z) = \varphi(J_\lambda z) + \frac{1}{2\lambda}|z - J_\lambda z|^2.$$

and $\varphi_\lambda(z) \uparrow \varphi(z)$ as $\lambda \downarrow 0$ for all $z \in H$. From this, we can show (cf. [1, Definition 1.2]) that if $z_n \to z$ weakly in H and $\lambda_n \to 0$, then

$$\varphi(z) \le \liminf_{n\to 0} \varphi_{\lambda_n}(z_n). \qquad (A.1)$$

The subdifferential, in fact, the Fréchet derivative, of φ_λ is equal to the Yosida-approximation $\partial\varphi_\lambda$ of $\partial\varphi$.

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