J.evol.equ. 7 (2007), 471–495

1424-3199/07/030471-25, *published online* February 08, 2007
DOI 10.1007/s00028-006-0298-z

Journal of Evolution Equations

Asymptotic Complexity in Filtration Equations

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Abstract. We show that the solutions of nonlinear diffusion equations of the form $u_t = \Delta\Phi(u)$ appearing in filtration theory may present complicated asymptotics as $t \to \infty$ whenever we alternate infinitely many times in a suitable manner the behavior of the nonlinearity Φ. Oscillatory behaviour is demonstrated for finite-mass solutions defined in the whole space when they are renormalized at each time $t > 0$ with respect to their own second moment, as proposed in [Tos05, CDT05]; they are measured in the L^1 norm and also in the Euclidean Wasserstein distance W_2. This complicated asymptotic pattern formation can be constructed in such a way that even a chaotic behavior may arise depending on the form of Φ.

In the opposite direction, we prove that the assumption that the asymptotic normalized profile does not depend on time implies that Φ must be a power-law function on the appropriate range of values. In other words, the simplest asymptotic behavior implies a homogeneous nonlinearity.

1. Introduction

This paper is devoted to study the question of pattern formation for the family of filtration equations of the form

$$\frac{\partial u}{\partial t} = \Delta\Phi(u), \tag{1.1}$$

where Φ satisfies the basic set of hypotheses **(HB)**: Φ is a strictly increasing and continuous real function with $\Phi(0) = 0$, and smooth for $u > 0$, so that the equations are parabolic, at least in an extended way that is sufficient for the weak theories to apply. We consider finite mass nonnegative solutions $u(x, t)$ defined in $\mathbb{R}^N \times (0, \infty)$. We will make below precise assumptions that ensure that the Cauchy problem for such equation generates a continuous semigroup $u \mapsto T_t u_0 = u(t)$, cf. [BC81] that conserves the total mass of the solution. A further condition on Φ is assumed to avoid loss of mass for fast diffusion, and thus, to deduce mass-preservation together with a L^1–L^∞ smoothing effect (see [BB85, Ver79, Va05]) that we shall detail below. We impose the normalization of unit initial mass

$$\int_{\mathbb{R}^N} u_0(x)\, dx = 1 \tag{1.2}$$

for the rest of the paper. This condition does not restrict the generality of the results, since it can be obtained by a rescaling of the equation. We see the solution as the evolution in time of a probability distribution, $u(t) = u(\cdot, t)$. The total mass is the same for all times $t > 0$, but the mass distribution tends to spread in time. We want to discover the type of asymptotic patterns associated to the large-time behavior of this class of equations. Because of the spreading induced by diffusion, such patterns can only be seen after suitable normalization.

The standard cases. The question we address has been much studied for the most standard diffusion examples and admits a clear and simple answer in those cases. Thus, the asymptotic behavior of the linear heat equation, $u_t = \Delta u$, is determined by the heat kernel

$$B_1(x,t) = (4\pi t)^{-N/2}\, e^{-|x|^2/4t}. \tag{1.3}$$

A natural normalization in the case of the heat equation is to scale with the second moment of the solution. It is well-known that such renormalized orbits converge to the Gaussian profile

$$G(x) = (2\pi/N)^{-N/2}\, e^{-N|x|^2/2}, \tag{1.4}$$

corresponding to the heat kernel at time $t = 1/2N$ in which it has unit second moment. This is related to the Central Limit Theorem, see [Tos05].

In the case of the Porous Medium Equation (PME), $u_t = \Delta u^m$, $m > 1$, it is known that solutions converge to a Zel'dovich-Kompaneets-Barenblatt (ZKB) profile, i.e., source solutions of the self-similar form:

$$B_m(x,t) = t^{-N/\lambda} F_m(x/t^{1/\lambda}), \quad F_m(\xi) = (C_m - k\,|\xi|^2)_+^{\frac{1}{m-1}} \tag{1.5}$$

where

$$k = \frac{(m-1)}{2m\lambda} \quad \text{and} \quad \lambda = N(m-1)+2. \tag{1.6}$$

The constant $C_m > 0$ is determined by fixing the mass to 1. The result extends to some range $(N-2)_+/2 < m < 1$ where the ZKB solutions exist and have finite mass. Here is the precise statement, [Va03]:

THEOREM 1.1. [Self-Similar Asymptotic Convergence for PME] *Let $u(x,t)$ be the unique weak solution of problem $u_t = \Delta u^m$, $m > (N-2)_+/N$, with initial data $u_0 \in L^1_+(\mathbb{R}^N)$ and unit mass. Then as $t \to \infty$ we have*

$$\lim_{t\to\infty} \|u(t) - B_m(t)\|_{L^1(\mathbb{R}^N)} = 0. \tag{1.7}$$

Convergence holds also in L^∞-norm in the proper scale:

$$\lim_{t\to\infty} t^{N/\lambda} \|u(t) - B_m(t)\|_{L^\infty(\mathbb{R}^N)} = 0. \tag{1.8}$$

The second-moment normalization of the solutions for the homogeneous nonlinear diffusion equation, $u_t = \Delta u^m$, shows the convergence towards the ZKB profile $B_m(x, t_{o,m})$ at the time $t_{o,m}$ in which it has unit second moment; see [Tos05] and next section for details.

Recently, the existence of a time-dependent asymptotic profile for second-moment normalized solutions of general nonlinear filtration equations has been shown in [CDT05] under suitable hypotheses on the nonlinearity, see next section. These asymptotic profiles form a one parameter family of functions indexed by time to which normalized solutions resemble in Euclidean Wasserstein distance. However, an open problem remained in this work, namely, the convergence or not in time of the asymptotic profile, and thus of all normalized solutions, to a universal profile depending only on the nonlinearity Φ.

Complicated behaviour. Based on the results for the power-law case and their perturbations, there is the temptation to conjecture that the nonlinear diffusive equations of the family (1.1) give always rise to simple asymptotic patterns, even to a universal profile, when acting on free space, $x \in \mathbb{R}^N$. We refer to [CDG05, GT05] for numerical work related to this question.

Our main result disproves that claim: it says that, by choosing the behavior of the nonlinearity Φ near zero in a convenient way, we can obtain a quite complicated family of renormalized asymptotic patterns for the same equation and solution, even a chaotic situation. The simplest nontrivial example is an equation whose asymptotic renormalized profile oscillates infinitely many times between (increasingly accurate approaches to) the Gaussian and the Barenblatt profiles, passing of course through a continuous family of transition patterns. The constructions we make are quite explicit for the reader's convenience but the idea can be applied for quite general nonlinearities. We need a number of technical assumptions on Φ that we detail in Subsections 2.2 and 2.3.

Let us recall that complicated patterns may result from simple diffusion equations when these are posed in bounded domains with different boundary conditions. Vazquez and Zuazua [VZ02] find complicated behaviour for the heat and porous medium equations as well as some hyperbolic equations posed in the whole space when the initial data are merely bounded; in that analysis, the complicated time behaviour reproduces the complexity of the initial data as $|x| \to \infty$, something that does not happen here since the data that we treat can be assumed to have compact support, see also the works of Cazenave et al. [CDW03, CDW06]. Another way patterns are formed is by interaction of diffusion terms with other terms that are included in the equations, like convection (e.g., Burgers patterns), reaction (e.g., blow-up patterns) or other. Patterns are a very important issue in Mathematical Biology, associated mainly to reaction-diffusion systems of PDE's.

In Section 2 we review and improve slightly the results concerning long-time asymptotics for normalized solutions. We first rephrase the results in [Tos05] from this point of view. Later, we review the construction made in [CDT05] of asymptotic profiles for general nonlinear diffusion equations. Section 3 is the core of this paper and proves the claim concerning the complicated asymptotic pattern formation in nonlinear diffusion, see Theorem 3.2. We conclude the article by an argument in the opposite direction. Thus, in Section 4 we show that simple asymptotic behavior implies self-similarity and homogeneity.

2. Intermediate Asymptotic Profiles

Let us introduce the main tools in the present analysis of asymptotic pattern formation. We will recall the main asymptotic results in [CDT05] and we will improve slightly over the hypotheses on the nonlinearity $\Phi(u)$ leading to such profiles, over the properties of the asymptotic profile, and over the convergence of normalized solutions towards it. The main two ingredients are an intrinsic renormalization of the solutions by their own variance and the strict contractivity of the Euclidean Wasserstein distance between measures for such renormalized flow map.

2.1. *The Toscani map*

G. Toscani [Tos05] proposed to use the second moment of the probability distribution of the solutions

$$\theta_u(t) = \int_{\mathbb{R}^N} |x|^2 u(x,t)\, dx \tag{2.1}$$

to renormalize the flow of the porous medium equation, $\Phi(u) = u^m$, in an intrinsic way, i.e., to consider the normalized solution:

$$v(x,t) = \theta_u(t)^{N/2} u(\theta_u(t)^{1/2} x, t). \tag{2.2}$$

There are three intuitions about this nonlinear scaling. Firstly, the second moment corresponds to the kinetic energy of the probability distribution in kinetic theory, where x represents speed, u is the particle density and θ is the kinetic energy. In fact, solutions where the only dependence on time is through their temperature are of asymptotic importance in several kinetic models for which related scalings have been used, see for instance [Tos04, BCT05, BCT06]. On the other hand, this scaling corresponds to the usual normalization for the distribution function of the sum of n independent random variables with common distribution function of fixed variance to obtain the Central Limit Theorem (see [Tos05] for more comments about this analogy and relations to the heat equation). Moreover, since we will prove that the variance diverges in time, then, from a geometric perspective, scaling back with the variance will hopefully drive the solutions to a limit.

In order to define the Toscani map we take a mass distribution $\rho \in \mathcal{M}$ where

$$\mathcal{M} = \left\{ \rho \in L^1_+(\mathbb{R}^N),\ \int_{\mathbb{R}^N} \rho(x)\, dx = \int_{\mathbb{R}^N} |x|^2 \rho(x)\, dx = 1 \right\},$$

we find the unique solution $u(x,t)$ to equation (1.1) with initial datum ρ, and consider its projection $v(x,t)$ obtained by the scaling (2.2); we then set $\mathcal{T}_\Phi(t)(\rho) := v(t)$ by definition. In short, we have:

DEFINITION 2.1. The *Toscani map* (or the *normalized* flow map) $\mathcal{T}_\Phi(t) : \mathcal{M} \longrightarrow \mathcal{M}$ is defined by

$$\mathcal{T}_\Phi(t)(\rho) := \theta_u(t)^{N/2} u(\theta_u(t)^{1/2}\cdot, t), \tag{2.3}$$

where $u(x,t)$ is the unique solution to equation (1.1) with initial datum ρ.

In order for this map to be well-defined we need to show propagation of the second moment; we will discuss this point in the next subsection. Now, let us come back to the particular case of the porous medium equation $\Phi(u) = u^m$. We show next that the ZKB profiles are precisely the fixed points of the Toscani map, which we denote in this particular case by $\mathcal{T}_m(t)$.

LEMMA 2.2. *Given $m > \frac{N}{N+2}$, the ZKB profile $B_m(x, t_{o,m})$ at the time $t = t_{o,m}$ for which*

$$\int_{\mathbb{R}^N} |x|^2 B_m(x, t_{o,m})\, dx = 1$$

is a fixed point of the Toscani map $\mathcal{T}_m(t)$ for all $t > 0$.

Proof. Let us assume $m \neq 1$, we can write that

$$B_m^m = t^{-N(m-1)/\lambda} \left(C_m - k \frac{|x|^2}{t^{2/\lambda}} \right) B_m$$

and thus

$$kt^{-1} \int_{\mathbb{R}^N} |x|^2 B_m(x,t)\, dx = t^{-N(m-1)/\lambda} C_m - \int_{\mathbb{R}^N} B_m^m(x,t)\, dx.$$

Now, we use that $\lambda^{-1}\, t^{-1} |x|^2 B_m(x,t) = -x \cdot \nabla B_m^m$ to obtain

$$t^{-1} \int_{\mathbb{R}^N} |x|^2 B_m(x,t)\, dx = N\lambda \int_{\mathbb{R}^N} |x|^2 B_m^m(x,t)\, dx,$$

which combined with the previous identity, yields the second moment of the ZKB profile:

$$\theta_m(t) = \int_{\mathbb{R}^N} |x|^2 B_m(x,t)\, dx = \frac{C_m}{k + \frac{1}{N\lambda}} t^{2/\lambda},$$

hence, $t_{o,m}$ is given explicitly by

$$t_{o,m} = \left(\frac{k + \frac{1}{N\lambda}}{C_m} \right)^{\lambda/2}.$$

Therefore, we have that

$$\theta_m(t) = t_{o,m}^{-2/\lambda} t^{2/\lambda}$$

for all $t \geq 0$. Now, by a simple substitution into formula (1.5), we check that

$$\theta_m(t)^{N/2} B_m(\theta_m(t)^{1/2} x, t) = B_m(x, t_{o,m}) \tag{2.4}$$

and in particular

$$\theta_m(t + t_{o,m})^{N/2} B_m(\theta_m(t + t_{o,m})^{1/2} x, t + t_{o,m}) = B_m(x, t_{o,m})$$

for all $t \geq 0$, and thus, by definition

$$\mathcal{T}_m(t)(B_m(x, t_{o,m})) = B_m(x, t_{o,m})$$

for all $t \geq 0$, taking into account that the unique solution of the porous medium equation with initial data $B_m(x, t_{o,m})$ is $B_m(x, t + t_{o,m})$. We leave the proof to the reader in the heat equation case. □

It can be easily proved [Tos05, Va06] that the second moment for the solutions of the porous medium equation increases with time and $\theta_u(t) \simeq \theta_m(t) \simeq t^{2/\lambda}$ as $t \to \infty$. Moreover, it was proved in [Tos05], using the results of rates of convergence obtained in [CT00], that all solutions normalized by (2.2) converge in $L^1(\mathbb{R}^N)$ to the ZKB profile $B(x, t_{o,m})$. We rephrase here this result and we slightly improve it showing convergence in different spaces. It is worth pointing out that this observation was the starting point in [CDT05] to obtain the result on existence of asymptotic profiles of general nonlinear filtration equations.

THEOREM 2.3. [Asymptotic Convergence for normalized solutions] [Tos05, CT00] *Let $u(x, t)$ be the unique weak solution of problem $u_t = \Delta u^m$, $m > 1$, with initial data $u_0 \in L^1_+(\mathbb{R}^N)$ of unit mass such that $|x|^{2+\delta} u_0 \in L^1(\mathbb{R}^N)$ for some $\delta > 0$ small enough. Then as $t \to \infty$, we have*

i) **Temperature stabilization**:

$$\left| \frac{\theta_u(t)}{\theta_m(t)} - 1 \right| = 0(t^{-2/\lambda}) \qquad \text{as } t \to \infty. \tag{2.5}$$

ii) L^1**-stabilization**: *given $u_0 \in \mathcal{M}$*

$$\|\mathcal{T}_m(t)(u_0) - B_m(t_{o,m})\|_{L^1(\mathbb{R}^N)} = 0(t^{-\chi}) \qquad \text{as } t \to \infty \tag{2.6}$$

for certain $\chi > 0$ depending on N and m.

iii) L^∞**-stabilization**: *given* $u_0 \in \mathcal{M}$

$$\lim_{t\to\infty} \|\mathcal{T}_m(t)(u_0) - B_m(t_{o,m})\|_{L^\infty(\mathbb{R}^N)} = 0. \tag{2.7}$$

Proof. The first two points of the previous theorem are just an adaptation to our notation of [Tos05, Theorem 4.1, Theorem 4.3] and we refer to it for the precise value of χ. Moreover, one can use the decay of the temperature 2.5 and the general convergence result in Theorem 1.1. to deduce convergence without rate in L^∞. Let us just show this last point. We first just decompose it into

$$\begin{aligned}\|\mathcal{T}_m(t)(u(t)) - B_m(t_{o,m})\|_{L^\infty(\mathbb{R}^N)} &\le \|\mathcal{T}_m(t)(u(t)) - \theta_u(t)^{N/2} B_m(\theta_u(t)^{1/2}\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \\ &\quad + \|\theta_u(t)^{N/2} B_m(\theta_u(t)^{1/2}\cdot, t) - B_m(t_{o,m})\|_{L^\infty(\mathbb{R}^N)} \\ &= I_1 + I_2\end{aligned}$$

and by using (1.8) and (2.5), we immediately deduce that

$$I_1 \le \frac{\theta_u(t)^{N/2}}{t^{N/\lambda}} t^{N/\lambda} \|u(\cdot, t) - B_m(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \to 0$$

as $t \to \infty$. Regarding the second term, we use the scaling property of B_m in (2.4) to write

$$\begin{aligned}\theta_u(t)^{N/2} B_m(\theta_u(t)^{1/2} x, t) &= \left[\frac{\theta_u(t)}{\theta_m(t)}\right]^{N/2} B_m\left(\left[\frac{\theta_u(t)}{\theta_m(t)}\right]^{1/2} x, t_{o,m}\right) \\ &= B_m(x, t_{o,m} + \eta(t))\end{aligned}$$

where $\eta(t)$ verifies $\eta(t) \to 0$ since

$$\frac{\theta_u(t)}{\theta_m(t)} \to 1 \text{ as } t \to \infty$$

Therefore, the second term I_2 can be written as

$$I_2 = \|B_m(t_{o,m}) - B_m(t_{o,m} + \eta(t))\|_{L^\infty(\mathbb{R}^N)}$$

The continuity of the profile $B_m(x, t)$ in t and the convergence of temperatures (2.5) shows that $I_2 \to 0$ as $t \to \infty$. □

2.2. *Second Moment Propagation and Evolution*

As mentioned above, in order to define the Toscani map, we have to show propagation of the second moment for the filtration equation (1.1). This problem was considered in [CDT05] and additional conditions on Φ were given implying the propagation of the variance. While this propagation seems easier in the degenerate diffusion case, it is not obvious in the fast diffusion case. We will not describe here the different hypotheses on Φ leading to this variance propagation property. Instead, we assume the following hypothesis on the flow:

(HPSM) *Given any initial data* $u_o \in L^1_+(\mathbb{R}^N)$ *with unit mass and finite second moment, then the unique solution* $u(x,t)$ *of equation* (1.1) *has finite second moment for all times.*[1]

The next step is to control the evolution of the second moment. The variance in the power-law case diverges as $t \to \infty$ with a rate $t^{2/\lambda}$. In order to derive an estimate from below of the variance, we can make use of the L^1–L^∞ smoothing effect proved in [BB85, Ver79]. Those results give a quantitative estimate on how the solution diffuses as $t \to \infty$, and thus, the solution becomes instantly $L^\infty(\mathbb{R}^N)$ and decays towards zero as $t \to \infty$ with a bound depending only on the initial mass. We refer to [Va05] for a recent account of smoothing effects and its application in nonlinear diffusion equations.

Under the extra assumption on the nonlinearity Φ

(H1: L^1**–**L^∞**)** $\exists C > 0$ *and* $m > (N-2)_+/N$ *such that* $\Phi'(u) \geq Cu^{m-1}$ *for all* $u > 0$

the equation ((1.1)) enjoys the mass-preserving property and an L^1–L^∞ regularizing property. In fact, this assumption means that the nonlinearity $\Phi(u)$ is more diffusive than the power law u^m, and thus, the diffusive properties of the filtration equation (1.1) have to be at least the ones dictated by the power-law equation $u_t = \Delta u^m$ (see [Va05] for details). The following theorem is originally due to L. Véron [Ver79].

THEOREM 2.4. [$\mathbf{L^1}$–$\mathbf{L^\infty}$ regularizing effect 1] [Ver79] *Let* $u(x,t)$ *be the solution to* (1.1)*, with* $u_0 \in L^1_+(\mathbb{R}^N)$*. Let the nonlinearity* Φ *satisfy assumptions* **(HB)**-**(H1:**L^1**–**L^∞**)** *above. Then, at any* $t > 0$*,*

$$\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx = 1,$$

$u(x,t) \in L^\infty(\mathbb{R}^N)$ *for all* $t > 0$*, and the following estimate holds*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C(N,m)\, t^{-\frac{N}{N(m-1)+2}} \|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{2}{N(m-1)+2}}. \tag{2.8}$$

A more general regularizing effect was obtained in [BB85]. Under the assumption on the nonlinearity:

(H2:L^1**–**L^∞**)** *Let* $N \geq 3$ *and assume the function* $\eta(r)$ *given by*

$$\eta(r) = \int_r^\infty \Phi(s)^{-N/(N-2)}\, ds$$

for all $r > 0$ *is finite for all* $r > 0$*. Moreover, assume that*

$$\int_r^\infty s^{N-1}\Gamma(s^{-1/(N-2)})\, ds = \infty \tag{2.9}$$

for all $r > 0$ *where the function* Γ *is the inverse of* Φ*.*

[1] We refer to [CDT05] for sufficient conditions implying this property.

THEOREM 2.5. [$\mathbf{L^1}$–$\mathbf{L^\infty}$ regularizing effect 2] [BB85] *Let $u(x,t)$ be the solution to* (1.1)*, with $u_0 \in L^1_+(\mathbb{R}^N)$. Let the nonlinearity Φ satisfy assumptions* **(HB)**-**(H2:**L^1–L^∞**)** *above. Then, at any $t > 0$,*

$$\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx = 1,$$

$u(x,t) \in L^\infty(\mathbb{R}^N)$ *for all $t > 0$, and the following estimate holds*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \Phi^{-1}\left[2(\Phi \circ \eta^{-1})\left(C(N)\, t^{\frac{N}{N-2}}\, \|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{2}{N-2}}\right)\right] = \Lambda(t). \tag{2.10}$$

REMARK 2.6. This regularizing effect is more flexible than the one given above since it decouples the behavior of the diffusivity at ∞ (giving the smoothing effect) and at 0 (giving the mass conservation). Let us point out that both smoothing effects imply that the L^∞ norm of the solution decays as $t \to \infty$ uniformly in the set of positive integrable functions with unit mass. This fact was used in [CDT05] to derive a lower bound on the evolution of the second moment. In the following, we will denote by **(H**L^1–L^∞**)** any of the above hypotheses leading to mass conservation and the L^1–L^∞ smoothing effect. Let us point out that the second assumption in **(H2:**L^1–L^∞**)** is implied by a control at 0 of the nonlinearity, for instance, there exist $\delta, C > 0$ and $m > (N-2)_+/N$ such that $\Phi'(u) \le Cu^{m-1}$ for all $u \in (0,\delta)$.

LEMMA 2.7. [Divergence in time of Second Moment] [CDT05] *Assume the nonlinearity Φ satisfies* **(HB)**-**(H**L^1–L^∞**)**-**(HPSM)**. *Then,*

$$\lim_{t\to\infty} \theta_u(t) = +\infty \tag{2.11}$$

uniformly for initial data in the set $\mathcal{M}$. Under the hypothesis **(H1:**L^1–L^∞**)***, we have an explicit lower estimate:*

$$\theta_u(t) \ge A t^{\frac{2}{N(m-1)+2}}$$

with A constant in $\mathcal{M}$.

2.3. *Contractivity in Euclidean Wasserstein distance*

The third ingredient we need for obtaining the asymptotic results is a contraction property. The fact that the normalized asymptotic profile in the case of equation $u_t = \Delta u^m$ is given by a fixed point of the Toscani map, shown in Lemma 2.2, leads us to try to prove contractivity properties of the Toscani map for general nonlinearities Φ.

The well-known fact that the flow map of the filtration equation, $u_t = \Delta\Phi(u)$, is L^1-contractive does not help for obtaining contractivity of the Toscani map. In fact, the scaling defining the Toscani map is mass-preserving. Recently, nonlinear filtration equations of the form $u_t = \Delta\Phi(u)$ have been shown to have a formal gradient flow structure with respect to the Euclidean Wasserstein distance.

More precisely, let us define the function $\Psi(u)$ by the following relation $u\Psi''(u) = \Phi'(u)$ with $\Psi(0) = 0$ and $\Psi'(1) = 0$ where we need to assume that Ψ' is integrable at 0. Let us consider the functional

$$E(\rho) = \int_{\mathbb{R}^N} \Psi(\rho(x))\, dx \tag{2.12}$$

defined on the set $\rho \in \mathcal{P}_2^{ac}(\mathbb{R}^N)$ of probability measures with second moment bounded and absolutely continuous with respect to Lebesgue. Let us now introduce the definition of the Euclidean Wasserstein distance between probability measures with second moment bounded:

$$W_2(\rho_0, \rho_1) = \inf \left\{ \int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y|^2\, d\gamma(x, y); \quad \gamma \in \Gamma(\rho_0, \rho_1) \right\}^{1/2}; \tag{2.13}$$

here $\Gamma(\rho_0, \rho_1)$ is the set of probability measures on $\mathbb{R}^N \times \mathbb{R}^N$ having marginals ρ_0 and ρ_1. Let us remind that the convergence in W_2 is equivalent to the convergence weak-* as measures plus convergence of the second moments [Vil03, Theorem 7.12].

F. Otto [Ott01] showed heuristically that the porous medium equation $u_t = \Delta u^m$ could be seen as a gradient-flow on the space of probability measures $\mathcal{P}_2(\mathbb{R}^N)$, endowed with a manifold structure and local metric whose induced distance coincides with the Euclidean Wasserstein distance W_2. This point of view is easily generalized to the nonlinear filtration equation $u_t = \Delta\Phi(u)$ and thus, equations (1.1) can be considered at least formally as the gradient flow of the entropy function (2.12) on the space of probability measures $\mathcal{P}_2(\mathbb{R}^N)$.

These heuristics have recently been rendered rigorous in [AGS05, CMV05] and generalized to a general metric setting. The main outcome of this theory that is relevant to us concerns the contractivity of the Euclidean Wasserstein distance W_2. The contractivity of W_2 is natural assuming the entropy functional (2.12) is convex with the right notion of convexity. This notion of convexity, named displacement convexity, was introduced by R.J. McCann [McC97]. We need to assume the following additional property on the nonlinearity Φ:

(HDC) $\Phi(u)\, u^{-(N-1)/N}$ is nondecreasing on $u \in (0, \infty)$.

Assumption **(HDC)** implies that the entropy functional (2.12) associated to equation ((1.1)) is well-defined and displacement convex [McC97]. The main conclusion out of this gradient-flow point of view for the present discussion is that the flow map of the filtration equation (1.1) is a non-expansive contraction in time with respect to the Euclidean Wasserstein distance W_2 in probability measures [CMV05, Stu05, AGS05, OW05].

THEOREM 2.8. [Non-strict contraction of W_2 for filtration equations] [CMV05, Stu05, AGS05, OW05] *Let* $u_1(x,t)$ *and* $u_2(x,t)$ *be solutions to* (1.1) *with initial data* $u_1(0), u_2(0) \in \mathcal{P}_2^{ac}(\mathbb{R}^N)$ *and the nonlinearity* Φ *verifying* **(HB)–(HDC)***, then*

$$W_2(u_1(t), u_2(t)) \le W_2(u_1(0), u_2(0)), \quad \text{for all } t \ge 0. \tag{2.14}$$

2.4. *Asymptotic Profiles*

Finally, let us summarize the main result in [CDT05] and its proof since we will use it in the sequel. We need to remind the reader a geometric property of the Euclidean Wasserstein distance (see [CDT05, McC05] for a proof).

LEMMA 2.9. [Chordal Euclidean Wasserstein Distance inequality] [CDT05] *Given any two probability densities* $\rho_0, \rho_1 \in \mathcal{P}_2^{ac}(\mathbb{R}^N)$*, then*

$$W_2(\theta_0^{N/2}\rho_0(\theta_0^{1/2}x), \theta_1^{N/2}\rho_1(\theta_1^{1/2}x)) \le \theta^{-1/2} W_2(\rho_0, \rho_1),$$

where $\theta = \min(\theta_0, \theta_1)$ *with* θ_0 *and* θ_1 *being the second moment of* ρ_0 *and* ρ_1 *respectively.*

The previous lemma has a geometric interpretation: we can think of the normalized densities as the projected densities on the unit sphere of the set of probability densities $\mathcal{P}_2(\mathbb{R}^N)$ endowed with the Euclidean Wasserstein distance. With this interpretation, the previous result follows from a chordal Euclidean type inequality [McC05].

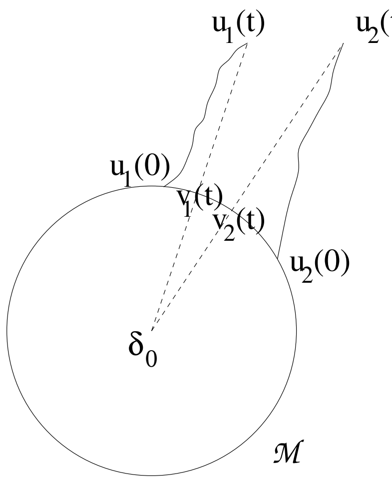

Figure 1 Geometric Interpretation of Proposition 2.10.

A direct consequence of Lemma 2.9 and Theorem 2.8 is an estimate on the contraction of the normalized solutions using the nonlinear scaling (2.2).

PROPOSITION 2.10. [Strict contraction of W_2 for normalized solutions] *Let* $u_1(x,t)$ *and* $u_2(x,t)$ *be solutions to* (1.1) *with initial data* $u_1(0), u_2(0) \in \mathcal{P}_2^{ac}(\mathbb{R}^N)$ *and the nonlinearity* Φ *verifying* **(HB)–(HPSM)–(HL^1–L^∞)–(HDC)**, *then*

$$W_2(v_1(t), v_2(t)) \leq \theta(t)^{-1/2} W_2(u_1(0), u_2(0)) \to 0 \quad as \quad t \to \infty, \tag{2.15}$$

where $\theta(t) = \min(\theta_1(t), \theta_2(t))$ *with* $\theta_1(t)$ *and* $\theta_2(t)$ *being the second moment of* $u_1(x,t)$ *and* $u_2(x,t)$ *respectively.*

Moreover, taking into account the uniform estimates on the decay of the variance in the set $\mathcal{M}$ obtained in Lemma 2.7, we obtain the contractivity for large times of the Toscani map $\mathcal{T}_\Phi(t) : \mathcal{M} \longrightarrow \mathcal{M}$. We refer to [CDT05] for all the details of the proof of the following result:

THEOREM 2.11. [Asymptotic profile for filtration equations] *Given* Φ *verifying the hypotheses* **(HB)–(HPSM)–(HL^1–L^∞)–(HDC)**, *there exists* $t_* > 0$ *and a one parameter curve of probability densities* $v_\infty(t) \in \mathcal{M}$ *defined for* $t \geq t_*$ *such that, for any solution of* (1.1) *with initial data* $u_0 \in \mathcal{M}$,

$$W_2\left(\mathcal{T}_\Phi(t)u_0, v_\infty(t)\right) \longrightarrow 0 \quad as \quad t \to \infty. \tag{2.16}$$

Moreover, the asymptotic profile $v_\infty(t)$ *is characterized as the unique fixed point of the Toscani map* $\mathcal{T}_\Phi(t)$ *in* $\mathcal{M}$ *and it is approached by iterative iteration on* $\mathcal{T}_\Phi(t)$ *starting with any initial seed in* $\mathcal{M}$. *Furthermore, the asymptotic profile* $v_\infty(t)$ *belongs to* $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ *and is a radially symmetric non increasing function. Moreover, if the nonlinear diffusion function* $\Phi(u)$ *is* C^∞ *for* $u > 0$, *the fixed point* $v_\infty(t)$ *is locally* C^∞ *wherever it is positive.*

Proof. The only properties stated in previous theorem not proven in [CDT05] concern the smoothness of the fixed points and the radial character of the asymptotic profile $v_\infty(t)$. Our first remark is that the fixed points are scaled solutions of the nonlinear diffusion equation just by definition of the Toscani map $\mathcal{T}_\Phi(t)$ and therefore, they enjoy the generic properties of solutions of the nonlinear diffusion equation. Thus, we conclude the continuity of the fixed points and the smoothness in the positive set whenever Φ is smooth for positive values of u.

Since the fixed points of the maps $\mathcal{T}_\Phi(t)$ are obtained through Banach fixed point theorem, the asymptotic profile $v_\infty(t)$ can be approximated by successive application of the Toscani map $\mathcal{T}_\Phi(t)$ to any initial data. Since continuous radially symmetric initial data produces continuous radially symmetric solutions, then we can assume that the approximation sequence $\{v_n\}$ in W_2 to the fixed point $v_\infty(t)$ consists of radially symmetric continuous probability densities.

Let us now show that the limit has to be a radially symmetric probability density. We know that $v_n \rightharpoonup v_\infty(t)$ as measures, thus

$$\lim_{n\to\infty} \int_{B(0,R)} dv_n = \int_{B(0,R)} dv_\infty(t) \tag{2.17}$$

for any $R > 0$. Since the limit $v_\infty(t)$ is continuous, thus the mass of the sequence of probability measures $\{v_n\}$ cannot concentrate at 0. We now make use the converse of the Prokhorov theorem [AGS05, Remark 5.1.6] and we have that the sequence $\{v_n\}$ is tight and thus, mass is not lost at ∞. Both previous arguments allow us to show that this approximating sequence seen as a sequence in the space of probability densities on the positive real line $r \in (0,\infty)$ with the weight $r^{N-1}dr$ is tight and therefore is relatively compact in this set, so any adherence point μ must be a radial probability measure. Therefore, we have for any continuous compactly supported function $\varphi(r)$ that

$$\lim_{n\to\infty} \int_0^\infty v_n(r)\varphi(r) r^{N-1}\, dr = \int_0^\infty \varphi(r) r^{N-1}\, d\mu$$

for a subsequence in n which is not relabelled. Now, by choosing a general continuous compactly supported function $\varphi(x)$, we have that

$$\lim_{n\to\infty} \int_0^\infty v_n(r) \int_{S^{N-1}} \varphi(r\omega)\, d\omega\, r^{N-1}\, dr = \int_0^\infty \int_{S^{N-1}} \varphi(r\omega)\, d\omega\, r^{N-1}\, d\mu.$$

Since v_n are radially symmetric, we get

$$\int_0^\infty v_n(r) \int_{S^{N-1}} \varphi(r\omega)\, d\omega\, r^{N-1}\, dr = \int_{\mathbb{R}^N} \varphi(x)\, v_n(x)\, dx$$

and as a consequence,

$$\int_0^\infty \int_{S^{N-1}} \varphi(r\omega)\, d\omega\, r^{N-1}\, d\mu = \int_{\mathbb{R}^N} \varphi(x)\, dv_\infty(t)$$

concluding that $v_\infty(t)$ is a radial continuous probability density.

Let us finally show that the asymptotic profile is in fact, radially non increasing. Since the approximation sequence is made out of smooth radially non increasing functions, we have that the radial distribution functions

$$F_n(r) = \int_{B(0,r)} dv_n$$

for all $r > 0$, are non decreasing, concave functions. Moreover, by (2.17), these functions converges pointwise to the radial distribution function $F_\infty(t)(r)$ of $v_\infty(t)$. Then, it is simple to check that $F_\infty(t)(r)$ is a non-decreasing concave function and thus, the asymptotic profile $v_\infty(t)$ is a radial non increasing distribution, since wherever is positive, is smooth and thus by concavity of the radial distribution function its derivative $v_\infty(t)$ is non increasing. □

REMARK 2.12. [Universal Asymptotic Profile for porous medium/fast diffusion equations] Let us remark that taking into account Lemma 2.2, the asymptotic profile for the porous medium type equation, $u_t = \Delta u^m$, is constant in time given by $v_\infty(t) = B_m(x, t_{o,m})$. Moreover, now we can improve Theorem 2.3. by showing the W_2**-stabilization**:

$$\lim_{t\to\infty} W_2(\mathcal{T}_m(t)(u_0), B_m(t_{o,m})) = 0. \tag{2.18}$$

REMARK 2.13. [Translation Invariance] It is clear from the translation invariance of equations (1.1) that the normalization with the second moment centered at 0 made in (2.2) does not play any particular role. In fact, it is easy to check due to the construction and the uniqueness of solution that if one decides to scale with the second moment centered at any other point the asymptotic profiles of the normalized solutions are just the translated profiles centered at that point.

REMARK 2.14. [Different Fixed Variance] Let us point out that the fixed value of the second moment chosen for the Toscani map, unit in this paper, can be arbitrarily chosen to be θ_o. Asymptotic profiles are then obtained for solutions with initial data with that given value of the second moment. These asymptotic profiles may depend on the second moment value θ_0. Furthermore, the relation between the asymptotic profiles for different second moment values is not explicit in general. Only in the particular case $\Phi(u) = u^m$, we can easily check they are related by the natural scaling due to homogeneity of the asymptotic ZKB profile.

3. Asymptotic Complexity

We first need to review a result of continuous dependence of solutions with respect the nonlinearity Φ.

THEOREM 3.1. [Continuous Dependence on Φ] *Consider a sequence of nonlinearities Φ_n converging to Φ uniformly in compact sets of* $[0,\infty)$*, all of them verifying the hypotheses* **(HB)–(HDC)** *and Φ satisfying* (2.9)*. Given an initial data $u_0 \in L^1_+(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ with unit mass and unit second moment such that $|x|^{2+\delta}u_0 \in L^1(\mathbb{R}^N)$ for some $\delta > 0$ small enough. Assume there exists a nondecreasing function $\Phi^*(u)$ such that $\Phi_n(u) \le \Phi^*(u)$ for all $u > 0$ and $\Phi^*(u)/u$ is bounded on $(0, R)$ for all $R > 0$. Then, the sequence of solutions u_n of the Cauchy problems:*

$$\frac{\partial u}{\partial t} = \Delta\Phi_n(u)$$

converges towards the solution u of the Cauchy problem (1.1) *verifying:*

i) $L^1\cap L^\infty$**-convergence***: $u_n \to u$ in $C([0,\infty), L^1(\mathbb{R}^N)) \cap C([\tau, T], C(\mathbb{R}^N))$ for all $0 < \tau < T$.*

ii) L^∞**-uniform bound***:*

$$\|u_n(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \|u_0\|_{L^\infty(\mathbb{R}^N)}. \tag{3.1}$$

iii) **Second-moment convergence***:*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^2 |u_n(x,t) - u(x,t)|\, dx = 0 \tag{3.2}$$

for all $t > 0$*.*

iv) W_2**-convergence***:*

$$\lim_{n\to\infty} W_2(u_n(t), u(t)) = 0 \tag{3.3}$$

for all $t > 0$*.*

Proof. The convergence result stated in L^1 is the main result of reference [BC81] and is where the assumption (2.9) over Φ is needed, the convergence in $C(\mathbb{R}^N)$ is consequence of the standard regularity theory of non linear diffusions [DiB83]. The uniform bound on the solutions follows from maximum principle. In order to deal with the second moment convergence we compute formally the evolution of the $|x|^{2+\delta}$ moment:

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^N} |x|^{2+\delta} u_n(x,t)\, dx &= (2+\delta)(N+\delta)\int_{\mathbb{R}^N} |x|^\delta \Phi_n(u_n)\, dx \\ &\le (2+\delta)(N+\delta)\int_{\mathbb{R}^N} |x|^\delta \Phi^*(u_n)\, dx. \end{aligned}$$

Now, since $\Phi^*(u)/u$ is bounded on the interval $(0, \|u_0\|_{L^\infty(\mathbb{R}^N)})$, say by $M > 0$ large enough, and $\Phi^*(u)$ is nondecreasing, then we estimate the right-hand side as

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^\delta \Phi^*(u_n)\, dx &\le M \int_{\mathbb{R}^N} |x|^\delta u_n\, dx \\ &\le M\left(1 + \int_{\mathbb{R}^N} |x|^{2+\delta} u_n\, dx\right) \end{aligned}$$

uniformly on n and $0 \le t \le T$ for all $T > 0$. Thus, we show that

$$\int_{\mathbb{R}^N} |x|^{2+\delta} u_n(x,t)\, dx \le C$$

is uniformly on n and $0 \le t \le T$ for all $T > 0$. Now, since $u_n(x,t) \to u(x,t)$ in $L^1(\mathbb{R}^N)$ for all $t > 0$ and the moment of order $2+\delta$ of u_n is uniformly bounded, we deduce the strong convergence of the second moments stated in iii). The rigorous derivation of the above inequalities goes through an approximation to render integration-by-parts rigorous. Finally, convergence in W_2 sense is just a consequence of the convergence in L^1 plus the convergence of the second moments [Vil03]. ☐

We proceed with the construction of a nonlinearity Φ with complicated asymptotics. Here is our main result:

THEOREM 3.2. [Asymptotic Oscillation] *There exists a nonlinearity* Φ *verifying hypothesis hypotheses* **(HB)–(HPSM)–(HL^1–L^∞)–(HDC)**, *and there exists a solution* $u(x,t)$ *of the Cauchy problem* (1.1) *such that it approaches two different asymptotic profiles in* L^1*: the heat kernel* $B_1(x,t)$ *and the ZKB-profile* $B_m(x,t)$ *with* $m > 1$, *along respective sequences* t_n', t_n'' *that go to infinity.*

Moreover, both the asymptotic profile $v_\infty(t)$ *associated to equation* (1.1) *and the normalized solutions* $\mathcal{T}_\Phi(t)(u_0)$ *oscillate at those time sequences between the two scaled asymptotic profiles: the Gaussian* $B_1(x, t_{o,1})$ *and* $B_m(x, t_{o,m})$*; in* W_2 *and in* L^1 *respectively.*

Proof. The construction produces at the same time the nonlinearity and a particular solution $u(t)$ of (1.1) with complicated asymptotics. It is inspired in the study of complexity of asymptotic behaviour (chaos) for bounded solutions of diffusion equations of [VZ02].

Step 0. We need to control the behavior at zero of the nonlinearity Φ but we need to satisfy certain control at infinity of the nonlinearity Φ in order to ensure the assumptions that imply L^1-L^∞ smoothing effects. This is of no importance since for bounded solutions the only part of the nonlinearity that counts is for $0 \le u \le \|u_0\|_{L^\infty(\mathbb{R}^N)} = R_o$. Therefore, we can always change the behaviour for u large enough in order to meet **(H1:L^1–L^∞)** or **(H2:L^1–L^∞)** for Φ. Therefore, we will restrict to the construction for $0 \le u \le R_o$.

On the other hand, since we need to meet condition **(HDC)**, the interpolations between different behaviors near zero should keep this property. Condition **(HDC)** is equivalent to verify that

$$\frac{u\Phi'(u)}{\Phi(u)} \ge \frac{N-1}{N}.$$

In fact, since we will swap between a linear behavior and a power behavior with exponent larger that 1 near 0, we are forced to do the regularizations in logarithmic variables, $(\log\Phi(u), \log u)$, and in fact our approximations will always verify for $u \le 1$ that

$$1 \le \frac{u\Phi'(u)}{\Phi(u)} \le m \tag{3.4}$$

or equivalently, $1 \le \frac{d\log\Phi(u)}{d\log u} \le m$ implying that for $u \le 1$, we have

$$u \ge \Phi(u) \ge u^m \qquad \text{and} \qquad m \ge \Phi'(u) \ge u^{m-1}$$

where we fixed $\Phi(1) = 1$. Although we will not write explicitly all regularisations/interpolations, we will graphically discuss this point later on, hoping to convince the reader.

Step 1. We start with the heat equation, $\Phi_1(s) = s$, for $0 < s < R_o$, and take as initial data any $u_0 \in L^1_+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with unit mass and unit second moment such that

$|x|^{2+\delta} u_0 \in L^1(\mathbb{R}^N)$ for some $\delta > 0$ small enough. Let us call this function $u_1(x,t)$, with $x \in \mathbb{R}^N$, $t > 0$. Let us fix $t_1 > 0$ such that $\theta_{u_1}(t_1) = \theta_1(t_1) = 1 + 2t_1 > 2$,

$$\|u_1(t_1) - B_1(t_1)\|_{L^1(\mathbb{R}^N)} \le \epsilon_1 \quad \text{and} \quad W_2(\mathcal{T}_{\Phi_1}(t_1)(u_0), B_1(t_{o,1})) \le \epsilon_1$$

for a given $0 < \epsilon_1 < 1$. The existence of t_1 is ensured by Theorems 1.1 and 2.11.

Step 2. We modify the nonlinearity in the interval $0 < s < \delta_1 < 1$ into the PME shape

$$\Phi_{2,\delta_1}(s) = c_2 s^m,$$

where c_2 is chosen to have agreement with $\Phi_1(s)$ at $s = \delta_1$, we now regularize in a tiny left neighborhood of $s = \delta_1$ to make it smooth while keeping (3.4). This is done by interpolating smoothly the corresponding straight lines in logarithmic variables. We keep the value $\Phi_{2,\delta_1}(s) = \Phi_1(s)$ for $s \ge \delta_1$. It is easy to check that $\Phi'_{2,\delta_1}(s) \ge s^{m-1}$ for $s \le 1$ from step 0, and thus, the nonlinearities $\Phi_{2,\delta_1}(s)$ verify all properties needed in Theorem 2.11.

We now recalculate the solution to find the new evolution starting from u_0, that we call $u_{2,\delta_1}(x,t)$, with $x \in \mathbb{R}^N$, $t > 0$. In order to compare u_1 and u_{2,δ_1} at $t = t_1$, we now are entitled to use the continuous dependence of the solution with respect to variations of the nonlinearity proved in Theorem 3.1. Let us choose in the sequel $\varepsilon_n = \varepsilon_1/2^{n-1}$. We conclude that for δ_1 small enough, we have the nonlinearity $\Phi_2 = \Phi_{2,\delta_1}$ such that its corresponding solution $u_2 = u_{2,\delta_1}$ satisfies

$$\|u_1(t_1) - u_2(t_1)\|_{L^1(\mathbb{R}^N)} \le \epsilon_2, \quad |\theta_{u_1}(t_1) - \theta_{u_2}(t_1)| \le \epsilon_2 \quad \text{and} \quad W_2(u_1(t_1), u_2(t_1)) \le \epsilon_2,$$

and thus,

$$\|u_1(t_1) - B_1(t_1)\|_{L^1(\mathbb{R}^N)} \le \epsilon_1 + \epsilon_2 \quad \text{and} \quad |\theta_1(t_1) - \theta_{u_2}(t_1)| \le \epsilon_2.$$

Moreover, using Lemma 2.9, we have

$$\begin{aligned} W_2(\mathcal{T}_{\Phi_1}(t_1)(u_0), \mathcal{T}_{\Phi_2}(t_1)(u_0)) &\le \frac{1}{\min(\theta_{u_1}(t_1), \theta_{u_2}(t_1))} W_2(u_1(t_1), u_2(t_1)) \\ &\le \frac{1}{4 - \varepsilon_1} W_2(u_1(t_1), u_2(t_1)) \le \varepsilon_2 \end{aligned}$$

and thus,

$$W_2(\mathcal{T}_{\Phi_2}(t_1)(u_0), B_1(t_{o,1})) \le \epsilon_1 + \epsilon_2.$$

We now let the evolution continue for another long time with nonlinearity Φ_2. Using (3.1), we are able to find a longer time $t_2 > t_1 > 0$, to be determined, where the maximum of the solution u_2 is as small as we want and therefore the temperature of the solution is as large as we want. Let us take t_2 such that $\theta_{u_2}(t_2) > 2^2$ and the maximum of the solution is less than δ_1. Now, we are basically dealing with a solution of the PME, let the time goes on even further to ensure that u_2 approaches the Barenblatt profile B_m:

$$\|u_2(t_2) - B_m(t_2)\|_{L^1(\mathbb{R}^N)} \leq \epsilon_2, \ \ W_2(\mathcal{T}_{\Phi_2}(t_2)(u_0), B_m(t_{o,m})) \\ \leq \epsilon_2 \text{ and } |\theta_m(t_2) - \theta_{u_2}(t_2)| \leq \epsilon_2.$$

Again, the existence of t_2 is ensured by Theorems 11, 2.3 and 2.11.

Step 3. We want to make a transition back into the heat equation for even smaller values of $u \leq \delta_2 < \delta_1$. Let us modify the nonlinear function $\Phi_2(s)$ into

$$\Phi_{3,\delta_2}(s) = c_3 s, \quad 0 < s \leq \delta_2,$$

while we keep $\Phi_{3,\delta_2}(s) = \Phi_2(s)$ for $s \geq \delta_2$. Here, c_3 is chosen to render the function $\Phi_{3,\delta_2}(s)$ continuous. We need to put convenient transition values in a tiny left interval at δ_2 to ensure smoothness while keeping the bounds (3.4). Again, one can check that $\Phi'_{3,\delta_2}(s) \geq s^{m-1}$ for $s \leq 1$ from step 0, and thus the nonlinearities $\Phi_{3,\delta_2}(s)$ verify all properties needed in Theorem 2.11.

We recalculate the solution starting from u_0 with Φ_{3,δ_2}, that we call $u_{3,\delta_2}(x,t)$. The continuous dependence of the solution proved in Theorem 3.1 allows us to conclude that for δ_2 small enough, we have the nonlinearity $\Phi_3 = \Phi_{3,\delta_2}$ such that its corresponding solution $u_3 = u_{3,\delta_2}$ satisfies

$$\|u_2(t_1) - u_3(t_1)\|_{L^1(\mathbb{R}^N)} \leq \epsilon_3, \ \ |\theta_{u_2}(t_1) - \theta_{u_3}(t_1)| \leq \epsilon_3 \ \text{ and } \ W_2(u_2(t_1), u_3(t_1)) \leq \epsilon_3,$$

and

$$\|u_2(t_2) - u_3(t_2)\|_{L^1(\mathbb{R}^N)} \leq \epsilon_3, \ \ |\theta_{u_2}(t_2) - \theta_{u_3}(t_2)| \leq \epsilon_3 \ \text{ and } \ W_2(u_2(t_2), u_3(t_2)) \leq \epsilon_3.$$

Therefore, it is easy to check that

$$\|u_3(t_1) - B_1(t_1)\|_{L^1(\mathbb{R}^N)} \leq \epsilon_1 + \epsilon_2 + \varepsilon_3 \ \text{ and } \ \|u_3(t_2) - B_m(t_2)\|_{L^1(\mathbb{R}^N)} \leq \epsilon_2 + \varepsilon_3,$$

and

$$|\theta_1(t_1) - \theta_{u_3}(t_1)| \leq \epsilon_2 + \varepsilon_3 \ \text{ and } \ |\theta_m(t_2) - \theta_{u_3}(t_2)| \leq \epsilon_2 + \varepsilon_3,$$

Moreover, using Lemma 2.9, we have

$$\begin{aligned} W_2(\mathcal{T}_{\Phi_2}(t_1)(u_0), \mathcal{T}_{\Phi_3}(t_1)(u_0)) &\leq \frac{1}{\min(\theta_{u_2}(t_1), \theta_{u_3}(t_1))} W_2(u_2(t_1), u_3(t_1)) \\ &\leq \frac{1}{(2-\varepsilon_2)(2-\varepsilon_2-\varepsilon_3)} W_2(u_2(t_1), u_3(t_1)) \leq \varepsilon_3 \end{aligned}$$

and thus,

$$W_2(\mathcal{T}_{\Phi_3}(t_1)(u_0), B_1(t_{o,1})) \leq \epsilon_1 + \epsilon_2 + \varepsilon_3.$$

Again, using Lemma 2.9, we have

$$\begin{aligned} W_2(\mathcal{T}_{\Phi_3}(t_2)(u_0), \mathcal{T}_{\Phi_2}(t_2)(u_0)) &\leq \frac{1}{\min(\theta_{u_3}(t_2), \theta_{u_2}(t_2))} W_2(u_2(t_2), u_3(t_2)) \\ &\leq \frac{1}{4(4-\varepsilon_3)} W_2(u_2(t_2), u_3(t_2)) \leq \varepsilon_3 \end{aligned}$$

and thus,

$$W_2(\mathcal{T}_{\Phi_3}(t_2)(u_0), B_m(t_{o,m})) \leq \epsilon_2 + \varepsilon_3.$$

Again, we let the evolution continue for another long time with Φ_3. We want to find a time $t_3 > t_2$ such that $\theta_{u_3}(t_3) > 2^3$ and the maximum of the solution is less than δ_2. Now, we are basically dealing with a solution of the heat equation, let the time goes on even further to ensure that u_3 approaches the Gaussian profile B_1:

$$\begin{aligned} &\|u_3(t_3) - B_1(t_3)\|_{L^1(\mathbb{R}^N)} \leq \epsilon_3, \; W_2(\mathcal{T}_{\Phi_3}(t_3)(u_0), B_1(t_{o,1})) \\ &\quad \leq \epsilon_3 \text{ and } |\theta_1(t_3) - \theta_{u_3}(t_3)| \leq \epsilon_3. \end{aligned}$$

Again, the existence of t_3 is ensured by Theorems 1.1, 2.3 and 2.11.

Step 4. We are back to the situation at the beginning with the only difference that a long time has been spent. But such a time is of low importance as for asymptotic behaviour. The rest of the construction of $\Phi(u)$ and its solution $u(x, t)$ follows by induction. The nonlinearity verifies (3.4) and therefore, taking into account step 0, it satisfies all the assumptions of Theorem 2.11.

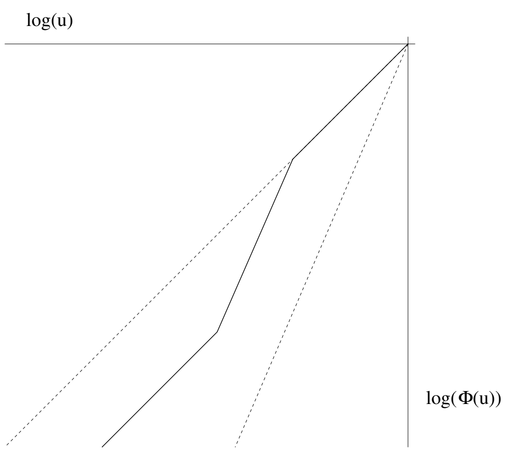

Figure 2 Sketch of the construction of the nonlinearity $\Phi(u)$ in logarithmic variables. The successive union of long-enough pieces of parallel straight lines to the ones with slope 1 and $m > 1$ respectively gives (after convenient and easy smoothing) the sought nonlinearity.

Step 5. Concerning the solution $u(x,t)$, there exist sequences of times $t_n' = t_{2n+1}$ and $t_n'' = t_{2n}$ for which the solution satisfies

$$\|u(t_n') - B_1(t_n')\|_{L^1(\mathbb{R}^N)} \le \epsilon,\ \ W_2(\mathcal{T}_\Phi(t_n')(u_0), B_1(t_{o,1})) \le \epsilon \text{ and } |\theta_u(t_n') - \theta_1(t_n')| \le \epsilon$$

and

$$\|u(t_n'') - B_m(t_n'')\|_{L^1(\mathbb{R}^N)} \le \epsilon,\ \ W_2(\mathcal{T}_\Phi(t_n'')(u_0), B_m(t_{o,m})) \le \epsilon \text{ and } |\theta_u(t_n'') - \theta_m(t_n'')| \le \epsilon$$

for all $\epsilon > 0$ small enough. Therefore, the oscillation in L^1 of the solution between the profiles B_1 and B_m is proved.

Let us show the oscillation of the asymptotic profile $v_\infty(t)$ for Φ given by Theorem 2.11. This theorem ensures that

$$W_2\left(\mathcal{T}_\Phi(t)(u_0), v_\infty(t)\right) \longrightarrow 0 \quad \text{as} \quad t \to \infty.$$

Therefore, for t large enough $d_2\left(\mathcal{T}_\Phi(t)(u_0), v_\infty(t)\right) \le \epsilon$ and thus,

$$W_2(v_\infty(t_n'), B_1(t_{o,1})) \le 2\epsilon \quad \text{and} \quad W_2(v_\infty(t_n''), B_m(t_{o,m})) \le 2\epsilon$$

for n large enough, that finishes this item of the proof.

Let us finally show the oscillation of the normalized solutions in L^1. In fact, it is already proved in [Tos05, Theorem 4.3], see also Theorem 2.3, that whenever we have

$$\|u(s) - B_m(s)\|_{L^1(\mathbb{R}^N)} \le \epsilon, \ \text{ and } |\theta_u(s) - \theta_m(s)| \le \epsilon,$$

with $m \ge 1$ and $s > 0$ then

$$\|\mathcal{T}_\Phi(t)(u_0) - B_m(t_{o,m})\|_{L^1(\mathbb{R}^N)} \le \eta_m(\epsilon)$$

where $\eta_m(\epsilon) \to 0$ as $\epsilon \to 0$. Therefore, applying the previous result to both sequences of times and the different exponent m, we conclude the proof. □

REMARK 3.3. [Generalizations] The same result holds true with any collection of profiles $\{B_{m_i} : m_i \ge 1, i{=}1, \cdots, l\}$ for every l. This alternating behaviour entails some curious geometrical effects. Thus, by oscillating between the Gaussian profile B_1 and the ZKB-profile B_m, we can keep the support of the profiles going to infinity and back all the time (at least in a highly approximated way).

By playing an expanding game over and over at every iteration of the first 3 steps of the proof of Theorem 3.2, we can show the following result:

COROLLARY 3.4. [Asymptotic Complexity] *There exists a nonlinearity Φ verifying hypothesis hypotheses* **(HB)**–**(HPSM)**–**(HL^1–L^∞)**–**(HDC)***, such that the adherence points of its asymptotic profile $v_\infty(t)$ contains the whole set of profiles $B_m(x, t_{o,m})$ with $m \ge 1$.*

Note that the adherence points of $v_\infty(t)$ is a closed set, hence we only need to prove for a dense numerable collection. We can also restrict the set of profiles $B_m(x, t_{o,m})$ to belong to a compact interval in the set of parameters m. We conjecture that the adherence points of $v_\infty(t)$ may contain a quite large set of other transition states. We are still not able to imagine well at this moment. It is possible that $v_\infty(t)$ does not approach any $B_m(x, t_{o,m})$ state, even if it is formed out of chunks of ZKB-profiles.

We can finally play the game of dealing with *chaotic* decisions defined by a general sequence in $\{0, 1\}^{\mathbb{N}}$ in which we perform 3-steps of oscillations between two profiles, say B_1 and B_{m_1}, $m_1 > 1$, whenever we have the 0 value and we perform 3-steps of oscillations between the first and another profile, say B_1 and B_{m_2}, $m_2 > 1$, whenever we have value 1. In the sense above, we can construct nonlinearities with chaotic behaviors:

COROLLARY 3.5. *There exists a nonlinearity Φ verifying hypotheses* **(HB)–(HPSM)–(HL^1–L^∞)–(HDC)***, such that the oscillations between the set of profiles $B_m(x, t_{o,m})$ follow a chaotic behavior.*

We are not using the word chaotic in the precise technical way. We recall that according to Devaney [Dev89], a dynamical system F_t is chaotic on a set S if the periodic points are dense, the flow is topologically transitive and it has sensitive dependence on the initial conditions. Checking these conditions for different constructions is an interesting direction for further research.

4. Simple asymptotics implies a power law

We now examine the case where the Toscani map has a single fixed point, $V(x)$, independent of t. We assume in the sequel that Φ is differentiable. We will prove the following theorem

THEOREM 4.1. [Self-Similarity Characterization] *If the Toscani map $\mathcal{T}_\Phi(t)$ for a Filtration Equation $u_t = \Delta\Phi(u)$ is constant in time for some open time interval I, then the function Φ is a power function on the range of V, the fixed point of $\mathcal{T}_\Phi(t)$. This means that $\Phi(u) = c\, u^m$ for some m and $c > 0$ and for all $u \in [0, A]$ for some A.*

Proof. Let $V(x)$ be the fixed point with unit mass and second moment.

Step 0. Let us fix a time $T \in I$ and let us consider the solution $u(x, t)$ of the PME in the time interval $I = (0, T)$ that gives rise to V. It follows from uniqueness that $u(x, t)$ is independent of the time T. This solution has initial value $u(x, 0) = V(x)$. This solution has a second moment $\theta_u(t) = \theta(t)$ that evolves in time from $\theta(0) = 1$ and $\theta(t) > 1$ for $t > 0$ and that is an increasing Lipschitz function in time since its time derivative is strictly positive and bounded. Moreover, we have

$$u(x,t) = \theta^{-N/2}(t)V(\theta^{-1/2}(t)x) \tag{4.5}$$

since $\mathcal{T}_\Phi(t)(V) = V$ for all $t \in I$. This means that the equation has a self-similar solution in the time interval $(0, T)$. We will show that this implies that the equation must be scale invariant, hence Φ is a power.

Step 1. By Theorem 2.11 we know that V is a continuous, radially symmetric, and decreasing probability density, $V = V(r)$, $r = |x|$. Moreover, the function is smooth wherever positive.

We also know that V and V' go to zero as $r \to \infty$ if positive everywhere. If the solution has compact support, then it has to be connected to zero by a decreasing curve and thus the support will be a ball of radius r_0 with $V = \Phi(V)_r = 0$ at the boundary of the support $r = r_0$. This last fact follows from general properties of radially decreasing solutions of nonlinear filtration equations since their pressure $p(r,t) = P(u(r,t))$ defined by the formula

$$P(u) = \int_0^u \frac{\Phi'(s)}{s}\, ds$$

has first derivative everywhere.

Next, we examine the consequences of the self-similarity shown above. If we substitute the form (4.5) into the equation and write $V = V(y)$ with $y = \theta^{-1/2}r$, we get

$$-\frac{1}{2}\theta'(NV + yV_y) = y^{1-N}(\Phi'(\theta^{-N/2}V)V_y)_y \tag{4.6}$$

in the strong sense in (y,t) whenever V is positive, plus boundedness and symmetry conditions at $y = 0$. We want to separate $\theta(t)$ from $V(y)$ as much as possible.

Integration in y gives

$$\Phi'(\theta^{-N/2}V)V_y = -\frac{1}{2}\theta' y^N V + C.$$

The constant can be eliminated using the conditions at infinity. Fix now $y > 0$ and take two different times in the interval I, t_0 and t. Write $\lambda(t) = (\theta(t_0)/\theta(t))^{N/2}$. Put $z = \theta^{-N/2}(t_0)V(y)$. Then,

$$\Phi'(\lambda(t)\,z) = -\theta'(t)\,\frac{y^N V}{2V_y}, \quad \Phi'(z) = -\theta'(t_0)\,\frac{y^N V}{2V_y},$$

hence, calling $F = \Phi'$ we get the functional equation

$$F(\lambda(t)\,z) = F(z)G(t) \tag{4.7}$$

where $G(t) = \theta'(t)/\theta'(t_0)$.

Step 2. Under conditions of differentiability for the solution $V(r)$ and the second moment $\theta(t)$, we get the conclusion that F is a power function by taking derivatives in z and t and comparing. Indeed,

$$\lambda F'(\lambda(t)\,z) = F'(z)G(t), \quad \lambda' z F'(\lambda(t)\,z) = F(z)G'(t),$$

hence, dividing and putting $t = t_0$ so that $\lambda(t_0) = 1$ we get

$$\frac{zF'(z)}{F(z)} = \frac{G'(t_0)}{\lambda'(t_0)G(t_0)} = \beta.$$

Step 3. In general, this argument cannot be performed since we cannot ensure that the second moment has second derivative everywhere. We can either justify the argument by approximation or else re-think the original argument which is essentially an algebraic problem. We choose this path. We first remark that $\theta(t)$ is strictly increasing and thus, the function $\lambda(t)$ is invertible. Then, the algebraic relation (4.7) is equivalent to say

$$F(\mu\,z) = F(z)H(\mu)$$

for all $0 < z < \bar{Z}$ and $0 < \mu \le 1$, with $H(s) = G(\lambda^{-1}(s))$. It is now a standard exercise in algebra to show that F is a power after taking logarithms: if $\eta = \log z$, $\zeta = \log \mu$, $f(\eta) = \log F(e^{\eta})$ and $h(\eta) = \log H(e^{\eta})$, we get

$$f(\eta\,\zeta) = f(\eta) + h(\zeta) \tag{4.8}$$

for η in some open interval $I \in \mathbb{R}$ and $\zeta \in (-\infty, 0]$. It is rather standard to prove that when f is continuous and satisfies the additive condition (4.8), then it must be a linear function in its interval on I. It follows that F is a power function in the interior of the range of the solution V.

Step 4. Once we know that F is a power, hence Φ' is a power, $\Phi'(u) = cu^{\beta}$. Moreover, coming back to (4.7), we deduce that

$$\left(\frac{\theta(t_0)}{\theta(t)}\right)^{\beta N/2} = \frac{\theta'(t)}{\theta'(t_0)}$$

and thus, we derive the growth of the moment for free:

$$\theta(t) = d_1(t + d_2)^p, \quad p = \frac{2}{2 + \beta N},$$

with $d_1, d_2 \in \mathbb{R}^+$ as expected. ☐

Acknowledgment.

This work arose out of the wish to better understand Toscani's proposal of intrinsic renormalization and the properties of the asymptotic profile. We thank P. L. Lions for putting to us the question whether simple behaviour implies a power law in the seminar of JAC at Collège de France in May 2005. JAC was partially supported by DGI-MEC (Spain) Project MTM2005–08024 and JLV by Project MTM2005–08760-C02-01 (Spain). Both authors also acknowledge the support from the Acc. Integ. Program HF2003–0121 for research stays at Université Paris-Dauphine, allowing us to follow this discussion during our commuting in the Paris subway system. Final discussion took place during the Workshop on "Nonlinear diffusions: entropies, asymptotic behavior and applications" at Banff International Research Station, whose hospitality we gratefully acknowledge.

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