J.evol.equ. 6 (2006) 113–144
1424–3199/06/010113–32
DOI 10.1007/s00028-005-0234-7

Published Online First 15 February 06

Journal of Evolution Equations

# Local estimates and global existence for strongly nonlinear parabolic equations with locally integrable data

FABIANA LEONI and BENEDETTA PELLACCI

*Abstract.* We obtain existence results for some strongly nonlinear Cauchy problems posed in $\mathbb{R}^N$ and having merely locally integrable data. The equations we deal with have as principal part a bounded, coercive and pseudo-monotone operator of Leray-Lions type acting on $L^p(0, T; W^{1,p}_{\text{loc}}(\mathbb{R}^N))$, they contain absorbing zero order terms and possibly include first order terms with natural growth. For any $p > 1$ and under optimal growth conditions on the zero order terms, we derive suitable local a-priori estimates and consequent global existence results.

## 1. Introduction

In this paper we prove the existence of distributional solutions for two classes of strongly nonlinear Cauchy problems, which include, as model examples,

$$\begin{cases} u_t - \operatorname{div}(|Du|^{p-2}Du) + g(u) = f(x,t) & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_t - \operatorname{div}(|Du|^{p-2}Du) + j(u) = |Du|^p + f(x,t) & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

In both problems (1.1) and (1.2), $p$ is a given exponent greater than 1 and the data $u_0(x)$ and $f(x,t)$ are allowed to be merely *locally* integrable functions. Precisely, for Problem (1.1) we address the case

$$u_0 \in L^1_{\text{loc}}(\mathbb{R}^N), \qquad f \in L^1(0,T; L^1_{\text{loc}}(\mathbb{R}^N)),$$

whereas for problem (1.2), for which we obtain locally bounded solutions, we assume that

$$u_0 \in L^\infty_{\text{loc}}(\mathbb{R}^N), \qquad f \in L^m(0,T; L^q_{\text{loc}}(\mathbb{R}^N)) \quad \text{with } \frac{N}{pq} + \frac{1}{m} < 1.$$

*Mathematics Subject Classification* (2000): 35K55, 35K65.
*Key words*: Nonlinear parabolic equations, $p$-laplacian with $p > 1$, $L^1_{\text{loc}}$ data, global existence, absorbing zero order terms, first order terms with natural growth.

Equations such as (1.1) and (1.2) arise in many physical applications, as they describe diffusion processes accompanied by absorption terms which are proportional to the concentration represented by the solution $u$. The mathematical pathology we are focusing on is the low and local summability of the data.

Generally speaking, when dealing with locally integrable data, suitably growing zero order terms are needed in the equation in order to obtain local a-priori estimates and, consequently, existence (and sometimes uniqueness) results (see [8, 9, 12, 16, 17]).

Actually, in [17] the elliptic versions both of problems (1.1) and (1.2) have been shown to have a globally defined distributional solution by assuming on the function $g : \mathbb{R} \to \mathbb{R}$, that

$$g \text{ is continuous and } g(s)\, s \geq 0, \tag{1.3}$$

$$g \text{ is odd and } s \in (0, +\infty) \mapsto \frac{g(s)}{s^{p-1}} \text{ is increasing,} \tag{1.4}$$

$$\int^{+\infty} \frac{ds}{(g(s)\, s)^{1/p}} < +\infty, \tag{1.5}$$

and, on the function $j : \mathbb{R} \to \mathbb{R}$, that

$$j \text{ is continuous, odd, increasing,} \tag{1.6}$$

$$\int^{+\infty} \frac{ds}{(j(s))^{1/p}} < +\infty. \tag{1.7}$$

Roughly speaking, these conditions require on the odd functions $g(s)$ and $j(s)$ a growth as $s \to +\infty$ faster than the powers $s^{p-1}$ and $s^p$ respectively. Moreover, they have been proved to be sharp in order to apply any natural method of approximation for solving the given problems.

Problem (1.1) has been studied in [9] by assuming that $g(s) = |s|^{\sigma-1}s$ with $\sigma > p - 1$ and $p > 2 - 1/(N + 1)$. We extend the basic result of [9] in different directions. We actually consider any $p > 1$ and any nonlinearity $g$ satisfying either (1.3) or (1.3)–(1.5) in dependence of $p$. Indeed, in parabolic equations the time derivative $u_t$ is an absorbing term analogous to a zero order term of the form $g_0(u) = u$. Hence, when studying Problem (1.1) for $p < 2$, we do not need assumptions (1.4) and (1.5), so that even the case $g \equiv 0$ is allowed for this range of $p$. Conversely, for $p \geq 2$ the time derivative term is not so absorbing as to yield a-priori estimates, and we need to assume the whole set of conditions (1.3)–(1.5) as in the elliptic framework.

We obtain a solution $u$ sharing the same regularity of the solutions of nonlinear parabolic equations posed in bounded domains and having $L^1$ or measures data (see [1, 5, 6]). Thus, $u$ is obtained in a suitable Marcinkiewicz space, as well as its spatial gradient $Du$, which will be not in general a locally integrable vector valued function if $p \leq 2 - 1/(N + 1)$ (see Theorem 2.2). In this case, we adapt the theory developed in [4] for nonlinear elliptic equations with $L^1$ data. This enables us to define the gradient $Du$ as a measurable function and , even if it has no more the usual distributional interpretation, we find out that $|Du|^{p-1}$ is locally integrable and that $u$ satisfies the equation in (1.1) in the usual sense of distributions.

When $p > 2-1/(N+1)$, we obtain a solution $u$ whose gradient $Du$ is locally integrable, and, more than that, we prove that $u$ belongs to the space $L^s(0, T; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N))$ for suitable exponents $s$ and $q$ as in [9] (see Theorem 2.4).

For problems such as (1.2), the presence of a first order term having natural growth requires stronger absorbing properties on the nonlinearity $j(s)$. Namely, conditions (1.6) and (1.7) are assumed in order to obtain a locally bounded solution (see Theorem 2.5). Note that, unlikely the previous case, we need (1.6) and (1.7) for *any* $p > 1$. To motivate our assumptions, let us heuristically argue as follows. Suppose that $v$ is a bounded positive solution of

$$v_t - \mathrm{div}(|Dv|^{p-2}Dv) + j(v) - |Dv|^p = f.$$

If we want to find a-priori estimates for $v$, then we can equvalently look for uniform bounds on the function $u := (p-1)(\exp(v/(p-1)) - 1)$. A direct computation shows that $u$ is a solution of

$$\begin{aligned}(p-1)(u+p-1)^{p-2}u_t &- (p-1)^{p-1}\mathrm{div}(|Du|^{p-2}Du)\\ &+ (u+p-1)^{p-1}j\left((p-1)\log\left(\frac{u}{p-1}+1\right)\right) = f(u+p-1)^{p-1}.\end{aligned}$$

Now, the term involving $u_t$ grows like $u^{p-1}$, so that it is of no help for any $p$. Thus, we have to impose conditions (1.3)–(1.5) on the function

$$g(s) = (s+p-1)^{p-1}j\left((p-1)\log\left(\frac{s}{p-1}+1\right)\right),$$

and this means to impose (1.6)–(1.7) on $j$.

Bounded solutions of strongly nonlinear equations including first order terms with natural growth and having globally integrable data have been obtained in [10, 11] in the elliptic case, and in [13, 19] for time dependent equations. As in the case of globally integrable data, in order to prove the existence of a distributional solution of problem (1.2), it is enough to have an a-priori estimate in $L^\infty(0, T; L^\infty_{\mathrm{loc}}(\mathbb{R}^N))$. This is precisely our achievement (see Lemma 3.12).

Let us finally mention that the techniques developed in the present paper, which rely on growth conditions on the equation's coefficients, are expected to extend to the case of doubly nonlinear principal part of the form $-\mathrm{div}(|u|^{m-1}u\,|Du|^{p-2}Du)$, even if we have not expanded in this direction.

The paper is organized as follows. In the following section we pose the general problems and state our main results. In Section 3 we prove some technical lemmas and we derive local a-priori estimates, which will be applied to prove the main results in Section 4.

## 2. Setting of the problems and statement of the main results

Our first existence results concern the following class of Cauchy problems

$$\begin{cases} u_t - \operatorname{div}(a(x,t,Du)) + h(x,t,u) = f(x,t) & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1}$$

with data $f(x,t)$ and $u_0(x)$ satisfying

$$f \in L^1(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^N)), \quad u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N). \tag{2.2}$$

We denote by $Du$ the (suitably defined) gradient with respect to the space variable $x$ of the function $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$, and by $u_t$ the derivative with respect to time $t$.

We assume that the function $a : \mathbb{R}^N \times (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ defines a bounded, coercive and pseudo-monotone operator of Leray-Lions type acting on $L^p(0,T; W^{1,p}(\mathbb{R}^N))$. More precisely, we require that $a(x,t,\xi)$ is measurable with respect to $(x,t) \in \mathbb{R}^N \times (0,T)$ for every $\xi \in \mathbb{R}^N$ and continuous with respect to $\xi$ for almost every $(x,t) \in \mathbb{R}^N \times (0,T)$, and that there exist constants $1 < p < N$ and $\Lambda \geq \lambda > 0$ such that, for every $\xi \in \mathbb{R}^N$ and for a. e. $(x,t) \in \mathbb{R}^N \times (0,T)$,

$$|a(x,t,\xi)| \leq \Lambda |\xi|^{p-1}, \tag{2.3}$$

$$a(x,t,\xi)\xi \geq \lambda |\xi|^p, \tag{2.4}$$

$$(a(x,t,\xi) - a(x,t,\eta))(\xi - \eta) > 0. \tag{2.5}$$

As far as the function $h : \mathbb{R}^N \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is concerned, we assume that it is measurable with respect to $(x,t) \in \mathbb{R}^N \times (0,T)$ for every $s \in \mathbb{R}$ and continuous with respect to $s \in \mathbb{R}$ for almost every $(x,t) \in \mathbb{R}^N \times (0,T)$. In addition, we assume that, for every $s_0 > 0$,

$$H_{s_0}(x,t) = \sup_{|s| \leq s_0} |h(x,t,s)| \in L^1(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^N)), \tag{2.6}$$

and that, for a.e. $(x,t) \in \mathbb{R}^N \times (0,T)$ and for all $s \in \mathbb{R}$,

$$h(x,t,s)\operatorname{sign}(s) \geq \begin{cases} 0 & \text{if } p < 2s, \\ |g(s)| & \text{if } p \geq 2, \end{cases} \tag{2.7}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a function satisfying (1.3), (1.4) and (1.5).

In order to state our first main result, let us introduce for every $k > 0$ the truncated functions $T_k : \mathbb{R} \to \mathbb{R}$ defined as

$$T_k(s) := \max\{-k, \min\{s, k\}\}, \tag{2.8}$$

and let us set

$$S_k(y) := \int_0^y T_k(s)\, ds. \tag{2.9}$$

The functions $T_k(s)$ play an essential role in the $L^1$-theory both of elliptic and parabolic equations (see [4], [21]). We recall here a simple but fundamental property proved in [4].

LEMMA 2.1. *If* $u : \mathbb{R}^N \times (0, T) \to \mathbb{R}$ *is a measurable function such that for every* $k > 0$ $T_k(u) \in L^1(0, T; W^{1,1}_{\text{loc}}(\mathbb{R}^N))$*, then there exists a measurable function* $v : \mathbb{R}^N \times (0, T) \to \mathbb{R}^N$*, which is unique in the a. e. sense, and such that, for every* $k > 0$ *and a.e. in* $\mathbb{R}^N$*,*

$$DT_k(u) = v\chi_{\{|u|\le k\}},$$

*where* $\chi_{\{|u|\le k\}}$ *is the characteristic function of the set* $\{(x, t) \in \mathbb{R}^N \times (0, T) : |u(x, t)| \le k\}$*. Moreover, if* $u \in L^1(0, T; W^{1,1}_{\text{loc}}(\mathbb{R}^N))$*, then* $v$ *coincides with the distributional spatial gradient of* $u$*.*

The above result is a direct consequence of Lemma 2.1 in [4] applied to the functions $x \mapsto u(x, t)$ for a.e. fixed $t \in (0, T)$. From now on, for every function $u$ as in Lemma 2.1 we set $Du = v$, and this vector valued function will be referred to as the weak gradient of $u$.

In the following we will also make use of the notion of Marcinkiewicz space $M^q(\Omega)$ (see e.g., [3]), that is the space of all measurable functions $v$ defined on the open set $\Omega \subset \mathbb{R}^N$ such that

$$\sup_{\sigma\in(0,+\infty)} [\sigma \operatorname{meas}(\{x \in \Omega : |v(x)| > \sigma\})^{1/q}] < +\infty, \tag{2.10}$$

where meas$(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$. Let us recall that, for $q > 1$, $M^q(\Omega)$ is a Banach space endowed with the norm defined by the sup appearing in (2.10), and, if $0 < \varepsilon < q - 1$ and $\Omega$ has finite measure, then the following continuous inclusions hold

$$L^q(\Omega) \hookrightarrow M^q(\Omega) \hookrightarrow L^{q-\varepsilon}(\Omega). \tag{2.11}$$

For $q > 0$, we use the notation $M^q_{\text{loc}}(\mathbb{R}^N \times [0, T])$ to denote the set of functions which belong to $M^q(\Omega \times (0, T))$ for every bounded set $\Omega \subset \mathbb{R}^N$.

We can now state our first existence result.

THEOREM 2.2. *Let* $1 < p < N$ *and assume that conditions from* (2.2) *to* (2.7) *hold true. Then, problem* (2.1) *has a distributional solution* $u \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$*, such that* $T_k(u) \in L^p((0, T); W^{1,p}_{\text{loc}}(\mathbb{R}^N))$ *for every* $k > 0$ *and*

$$\begin{cases} u \in M^{s_1}_{\text{loc}}(\mathbb{R}^N \times [0, T]) & \text{with } s_1 = \max\{1, p - 1 + \frac{p}{N}\}, \\ |Du| \in M^{q_1}_{\text{loc}}(\mathbb{R}^N \times [0, T]) & \text{with } q_1 = \max\{\frac{p}{2}, p - \frac{N}{N+1}\}. \end{cases} \tag{2.12}$$

REMARK 2.3. Theorem 2.2 asserts the existence of a function $u \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$, endowed with the weak gradient $Du$ in the sense of Lemma 2.1, such that $|a(x, t, Du)|$ and

$h(x, t, u)$ belong to $L^1(0, T; L^1_{\rm loc}(\mathbb{R}^N))$, and which satisfies problem (2.1) in the sense of distributions. Let us point out that the regularity (2.12) is the best possible for solutions of nonlinear parabolic equations with $L^1$ or measures data (see [1, 5, 6]). In particular, in [1] inclusions (2.12) have been derived for entropy solutions of initial-boundary value problems with source term $f \equiv 0$ and initial datum globally integrable. Thus, the weak gradient of the constructed solution is not in general expected to be locally integrable. Actually, from (2.12) we obtain $|Du| \in L^1(0, T; L^1_{\rm loc}(\mathbb{R}^N))$ iff $q_1 > 1$, that is iff $p > 2 - 1/(N+1)$.

In the case $p > 2 - 1/(N+1)$, we can further specify the regularity of the solution $u$ obtained in Theorem 2.2 accordingly to the following result.

THEOREM 2.4. *Let*

$$2 - \frac{1}{N+1} < p < N,$$

*and assume that conditions* (2.2)–(2.7) *hold true. Then, Problem* (2.1) *has a distributional solution* $u \in C([0, T]; L^1_{\rm loc}(\mathbb{R}^N)) \cap L^s(0, T; W^{1,q}_{\rm loc}(\mathbb{R}^N))$*, for all exponents* $s$ *and* $q$ *such that*

$$1 \le q < q(p) := \begin{cases} \frac{N}{3N+1-p(N+1)} & \text{if } p \le 2, \\ \frac{N(p-1)}{N-1} & \text{if } p > 2, \end{cases} \tag{2.13}$$

$$1 \le s < s(q) := \begin{cases} p & \text{if } q \le \frac{p}{2}, \\ q\,\frac{p(N+1)-2N}{q(N+1)-N} & \text{if } q > \frac{p}{2}. \end{cases} \tag{2.14}$$

We exploit the zero order terms in order to balance the local character of the summability of the data also for equations having first order terms with natural growth.

In fact, we give an existence result for the following class of Cauchy problems

$$\begin{cases} u_t - \operatorname{div}(a(x, t, Du)) + h(x, t, u) + F(x, t, Du) = f(x, t) & \text{in } \mathbb{R}^N \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.15}$$

with data $u_0$ and $f$ such that

$$u_0 \in L^\infty_{\rm loc}(\mathbb{R}^N), \qquad f \in L^m(0, T; L^q_{\rm loc}(\mathbb{R}^N)) \quad \text{with } \frac{N}{pq} + \frac{1}{m} < 1. \tag{2.16}$$

The function $h$ is now assumed to satisfy, besides (2.6), in place of (2.7) the stronger condition

$$h(x, t, s)\,\mathrm{sign}(s) \ge |j(s)|, \tag{2.17}$$

with $j : \mathbb{R} \to \mathbb{R}$ satisfying (1.6) and (1.7).

On the function $F(x, t, \xi)$ we assume, besides continuity with respect to $\xi \in \mathbb{R}^N$ for a.e. $(x, t) \in \mathbb{R}^N \times (0, T)$ and measurability with respect to $(x, t) \in \mathbb{R}^N \times (0, T)$ for every fixed $\xi \in \mathbb{R}^N$, the following growth condition

$$|F(x, t, \xi)| \le \gamma\, |\xi|^p \tag{2.18}$$

for a.e. $(x, t) \in \mathbb{R}^N \times (0, T)$ and for all $\xi \in \mathbb{R}^N$, for some given $\gamma > 0$.

THEOREM 2.5. *Let $1 < p < N$. Under assumptions* (2.16)*,* (2.3)–(2.5)*,* (2.6)*,* (2.17) *and* (2.18)*, Problem* (2.15) *has a distributional solution in*

$$L^p(0, T; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)) \cap L^\infty(0, T; L^\infty_{\mathrm{loc}}(\mathbb{R}^N)).$$

## 3. Local a–priori estimates

In this section, after some technical lemmas, we derive the a–priori estimates we need for our existence results.

Let us start with the following property of the function $G_k : \mathbb{R} \to \mathbb{R}$ defined as

$$G_k(s) = s - T_k(s), \tag{3.1}$$

where, for every $k > 0$, $T_k(s)$ is the truncated function defined in (2.8).

LEMMA 3.1. *Let $p > 1$, $\Omega \subset \mathbb{R}^N$ a bounded open set, and $u \in L^p(0, T; W^{1,p}_0(\Omega) \cap L^2(\Omega))$, with $u_t \in L^{p'}(0, T; W^{-1,p'}(\Omega) + L^2(\Omega))$. Let further $\psi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $\psi(0) = 0$. Then, for every $\eta \in C^1(\overline{\Omega})$ and for all $k > 0$, the function $v := \psi(G_k(u)\eta)$ belongs to $L^p(0, T; W^{1,p}_0(\Omega) \cap L^2(\Omega))$ and, for every $\tau \in [0, T]$, we have*

$$\int_0^\tau \langle u_t(t), v(t)\rangle\, dt = \int_\Omega \frac{\Psi\,(G_k(u(x, \tau))\eta(x)) - \Psi\,(G_k(u(x, 0))\eta(x))}{\eta(x)}\, dx, \tag{3.2}$$

*with $\Psi(s) := \int_0^s \psi(y)\, dy$.*

REMARK 3.2. Notice that the integral in the right-hand side of (3.2) is well defined. Indeed, since $u \in L^p(0, T; W^{1,p}_0(\Omega) \cap L^2(\Omega))$ and $u_t \in L^{p'}(0, T; W^{-1,p'}(\Omega) + L^2(\Omega))$, we have that $u \in C([0, T]; L^2(\Omega))$, so that the functions $x \mapsto u(x, t)$ are well defined for every $t \in [0, T]$. Moreover, since $\psi$ is Lipschitz continuous and $\psi(0) = 0$, the function $\Psi$ satisfies the growth condition

$$|\Psi(s)| \le c_0\, s^2,$$

whence

$$\begin{aligned}\left|\frac{\Psi\,(G_k(u(x, t))\eta(x))}{\eta(x)}\right| &\le c_0\, G_k^2(u(x, t))|\eta(x)| \\ &\le c_0\, u(x, t)^2|\eta(x)| \in L^1(\Omega) \quad \forall\, t \in [0, T].\end{aligned} \tag{3.3}$$

*Proof.* The regularity of $v$ immediately follows from the summability of $u$ and the properties of $\psi$.

In order to prove (3.2), let us use a density argument. We consider a sequence of functions $u_n \in \mathcal{D}([0,T]; W_0^{1,p}(\Omega)\cap L^2(\Omega))$, such that $u_n \to u$ in $L^p(0,T; W_0^{1,p}(\Omega)\cap L^2(\Omega))$ and $u_n' := u_{n,t} \to u_t$ in $L^{p'}(0,T; W^{-1,p'}(\Omega)+L^2(\Omega))$. We notice that for every $n$ the function

$$g_n(t) = \int_\Omega \frac{\Psi\,(G_k(u_n(x,t))\eta(x))}{\eta(x)}\,dx$$

is Frechét differentiable on $[0,T]$, and we have

$$g_n'(t) = \int_\Omega \psi\,(G_k(u_n(x,t))\eta(x))\;u_n'(x,t)\,dx = \langle u_n'(t), \psi\,(G_k(u_n)\eta)\rangle.$$

We integrate and we get

$$\int_0^\tau \langle u_n'(t), \psi\,(G_k(u_n)\eta)\rangle\,dt = \int_\Omega \frac{\Psi\,(G_k(u_n(x,\tau))\eta(x)) - \Psi\,(G_k(u_n(x,0))\eta(x))}{\eta(x)}\,dx.$$

Since $\psi(G_k(u_n)\eta) \to \psi(G_k(u)\eta)$ in $L^p(0,T; W_0^{1,p}(\Omega)\cap L^2(\Omega))$ and $u_n'$ strongly converges to $u_t$ in $L^{p'}(0,T; W^{-1,p'}(\Omega)+L^2(\Omega))$, we can pass to the limit in the left hand side of the above identity. Furthermore, since $u_n$ converges to $u$ in $C([0,T]; L^2(\Omega))$, by inequality (3.3) with $u$ replaced by $u_n$ and the dominated convergence theorem we can pass to the limit in the right hand side too, and we get the conclusion. □

In order to localize the test functions we need to use in our equations, the following result will be of crucial importance. For the proof we refer to [17].

LEMMA 3.3. *Let $\tilde{g} : [0,+\infty) \to [0,+\infty)$ be a continuous and increasing function, satisfying $\tilde{g}(0) = 0$ and (1.4)–(1.5) for some $p > 1$. Then, for every $C > 0$ and $\alpha \geq 0$, there exist a positive constant $\Gamma$ and an increasing function $\varphi : [0,1] \to [0,1]$ of class $C^1$, with $\varphi(0) = \varphi'(0) = 0$, $\varphi(1) = 1$, such that, for every $\sigma \in [0,1]$ and $t \geq 0$,*

$$t^{\alpha+p-1}\frac{\varphi'(\sigma)^p}{\varphi(\sigma)^{p-1}} \leq \frac{1}{C}\,t^\alpha \tilde{g}(t)\varphi(\sigma) + \Gamma. \tag{3.4}$$

REMARK 3.4. Lemma 3.3 may be regarded as a generalization of Young inequality. Indeed, if $\tilde{g}(s) = s^q$ for some $q > p-1$, then we can choose $\varphi(\sigma) = \sigma^{p(\alpha+q)/(q-p+1)}$ and inequality (3.4) is nothing but Young inequality.

We will also use the following compactness result (see [22]).

LEMMA 3.5. *Let $1 \leq p < \infty$ and $X, B, Y$ be Banach spaces, with $X \subset\subset B \subset Y$. Let $\{f_n\}$ be a sequence bounded in $L^p(0,T;X)$, such that $\{f_n'\}$ is bounded in $L^1(0,T;Y)$. Then $\{f_n\}$ is bounded in $L^p(0,T;B)$.*

We can now derive some a-priori estimates for every bounded solution of the following initial-boundary value problem

$$\begin{cases} u_t - \operatorname{div}(a(x,t,Du)) + h(x,t,u) = f(x,t) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3.5}$$

Here $\Omega$ is a bounded open set in $\mathbb{R}^N$ and $a$ and $h$ are assumed to satisfy respectively (2.3)–(2.5) and (2.6)–(2.7).

Assume that, for $f \in L^\infty(\Omega \times (0,T))$ and $u_0 \in L^\infty(\Omega)$, there exists a bounded weak solution of Problem (3.5), that is a function $u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(\Omega \times (0,T))$, such that $u_t \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, $u(x,0) = u_0(x)$ and

$$\int_0^T \langle u_t, v\rangle\, dt + \int_0^T\!\!\int_\Omega a(x,t,Du)Dv\, dx\, dt + \int_0^T\!\!\int_\Omega h(x,t,u)v\, dx\, dt$$
$$= \int_0^T\!\!\int_\Omega f v\, dx\, dt\,, \tag{3.6}$$

for all test functions $v \in L^p(0,T; W_0^{1,p}(\Omega))$.

LEMMA 3.6. *Under the above assumptions, let $u(x,t)$ satisfy* (3.6). *For $0 < r < R$ let further $B_r \subset B_R$ be concentric balls contained in $\Omega$. Then, there exists a positive constant $M$ depending on $N,\ p,\ \lambda,\ \Lambda,\ r,\ R,\ T$, on the function $g$ appearing in* (2.7)*, and on the norms $\|f\|_{L^1(B_R\times(0,T))}$, $\|u_0\|_{L^1(B_R)}$, such that*

$$\sup_{t\in[0,T]} \int_{B_r} |u(x,t)|\, dx \le M, \tag{3.7}$$

$$\int_0^T\!\!\int_{F_{k,r}(t)} |Du|^p\, dx\, dt \le M, \tag{3.8}$$

$$\int_0^T\!\!\int_{B_r} |DT_k(u)|^p\, dx\, dt \le M\,(k+1), \tag{3.9}$$

*where, for every $k \ge 0$ and a.e. $t \in (0,T)$, we have set*

$$F_{k,r}(t) := \{x \in B_r : k \le |u(x,t)| < k+1\}. \tag{3.10}$$

*Proof.* Let $\zeta \in \mathcal{D}(B_R)$ be a cut-off function, with $0 \le \zeta \le 1$ in $B_R$ and $\zeta \equiv 1$ in $B_r$. Let further $g(s)$ be the function appearing in assumption (2.7), and let us introduce the function

$$\tilde g(s) := \begin{cases} s & \text{if } p < 2\,, \\ g(s) & \text{if } p \ge 2. \end{cases}$$

By (1.3), (1.4) and (1.5), $\tilde{g}$ satisfies the assumptions of Lemma 3.3., so that we can construct a function $\varphi$ satisfying (3.4) with $\alpha = 1$ and with a constant $C > 0$ to be fixed in the sequel. We set

$$\eta := \varphi(\zeta),$$

and we notice that also $\eta$ is a cut-off function of class $C^1(B_R)$, with $\eta \equiv 1$ in $B_r$. Lemma 3.1 implies that, for every $\tau \in [0, T]$, $v = T_1(G_k(u)\eta)\,\chi_{(0,\tau)}$ is an admissible test function in (3.6). Therefore, by (3.2), we obtain

$$\begin{aligned}\int_{B_R} \frac{S_1(G_k(u(x,\tau))\eta)}{\eta} + \int_0^\tau\!\!\int_{B_R} a(x,t,Du)\, DT_1(G_k(u)\eta)\\ + \int_0^\tau\!\!\int_{B_R} h(x,t,u)\, T_1(G_k(u)\eta)\\ = \int_0^\tau\!\!\int_{B_R} f(x,t)\, T_1(G_k(u)\eta) + \int_{B_R} \frac{S_1(G_k(u_0(x))\eta)}{\eta},\end{aligned}$$

where $S_1(s)$ is the primitive of $T_1(s)$ defined in (2.9).

We further observe that, for a.e. $t \in (0,\tau)$, one has

$$\begin{aligned}&DT_1(G_k(u)\eta) = [Du\,\eta + G_k(u)\,D\eta]\chi_{E_{k,R}(t)},\\ &E_{k,R}(t) := \{x \in B_R : \eta k < |u|\eta < \eta k + 1\}.\end{aligned}$$

Hence, by assumptions (2.3) and (2.4), it easily follows that

$$\begin{aligned}&\int_{B_R} \frac{S_1(G_k(u(x,\tau))\eta)}{\eta} + \lambda \int_0^\tau \int_{E_{k,R}(t)} |Du|^p\eta + \int_0^\tau \int_{B_R} h(x,t,u) T_1(G_k(u)\eta)\\ &\le \|f\|_{L^1(B_R\times(0,T))} + \|u_0\|_{L^1(B_R)} + c\int_0^\tau \int_{E_{k,R}(t)} |Du|^{p-1}|G_k(u)|\varphi'(\zeta),\end{aligned}$$

for some positive constant $c$ depending on $\Lambda$, $R$ and $r$. From Young inequality we then obtain

$$\begin{aligned}&\int_{B_R} \frac{S_1(G_k(u(x,\tau))\eta)}{\eta} + \int_0^\tau\!\!\int_{E_{k,R}(t)} |Du|^p\eta + \int_0^\tau\!\!\int_{B_R} h(x,t,u)\, T_1(G_k(u)\eta)\\ &\le c\left(1 + \int_0^\tau \int_{E_{k,R}(t)} \frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\, |G_k(u)|^p\right),\end{aligned} \tag{3.11}$$

for some $c > 0$ which now depends on $\lambda$, $\Lambda$, $p$, $R$, $r$ and on the norms $\|f\|_{L^1(B_R\times(0,T))}$, $\|u_0\|_{L^1(B_R)}$.

Let us apply now Lemma 3.3 inequality (3.4) with $\alpha = 1$, $t = |G_k(u)|$ and $C = 4\,c\,\max\{1, T\}$ yields

$$
\begin{aligned}
&c \int_0^\tau \int_{E_{k,R}(t)} \frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}} |G_k(u)|^p \\
&\quad \le \frac{1}{4\max\{1, T\}} \int_0^\tau \int_{E_{k,R}(t)} \tilde{g}(G_k(u))\, G_k(u)\eta + \gamma,
\end{aligned} \tag{3.12}
$$

with $\gamma = c\,\Gamma\,T\,\text{meas}(B_R)$.

If $p < 2$, then $\tilde{g}(s) = s$, and from (3.12) it follows that

$$
\begin{aligned}
&c \int_0^\tau \int_{E_{k,R}(t)} \frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}} |G_k(u)|^p \le \frac{1}{4\,T} \int_0^\tau \int_{E_{k,R}(t)} (G_k(u))^2\eta \ + \gamma \\
&\quad \le \frac{1}{2\,T} \int_0^\tau \int_{B_R} \frac{S_1(G_k(u)\eta)}{\eta} \ + \gamma \\
&\quad \le \frac{1}{2} \sup_{t\in[0,T]} \int_{B_R} \frac{S_1(G_k(u(x,t))\eta(x))}{\eta(x)}\, dx + \gamma.
\end{aligned}
$$

In this case, we insert the above inequality into (3.11), use (2.7) and take the supremum with respect to $\tau \in [0, T]$. Thus, we deduce

$$
\begin{aligned}
&\sup_{t\in[0,T]} \int_{B_R} \frac{S_1(G_k(u(x,t))\eta(x))}{\eta(x)}\, dx \\
&\quad + \int_0^T \int_{E_{k,R}(t)} |Du(x,t)|^p \eta(x)\, dx\, dt \le 4\,(\gamma + c).
\end{aligned} \tag{3.13}
$$

If $p \ge 2$, then $\tilde{g}(s) = g(s)$ and from (3.12), (1.4), and (2.7) we obtain

$$
\begin{aligned}
c \int_0^\tau \int_{E_{k,R}(t)} \frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}} |G_k(u)|^p &\le \frac{1}{2} \int_0^\tau \int_{B_R} g(u) T_1(G_k(u)\eta) + \gamma \\
&\le \frac{1}{2} \int_0^\tau \int_{B_R} h(x,t,u)\, T_1(G_k(u)\eta) + \gamma.
\end{aligned}
$$

The above inequality, together with (3.11), by the arbitrariness of $\tau \in [0, T]$, still yields (3.13). Hence, estimate (3.13) is established for every $p > 1$.

We now observe that, for every $t \in [0, T]$,

$$
\begin{aligned}
&\int_{B_r \cap \{|u|\ge 1\}} |u(x,t)|\, dx \le \int_{B_r} S_1(G_1(u(x,t)))dx + c \\
&\quad \le \int_{B_R} \frac{S_1(G_k(u(x,t))\eta(x))}{\eta(x)} dx + c.
\end{aligned}
$$

This, toghether with (3.13) for $k = 1$, immediately gives (3.7).

Furthermore, since

$$F_{k,r}(t) \subseteq E_{k,R}(t),$$

inequality (3.8) is a direct consequence of (3.13) and the positivity of the function $S_1$.

Finally, we set

$$\Omega_{k,r}(t) := \{x \in B_r : |u(x,t)| < k\}, \tag{3.14}$$

we denote by $[k]$ the largest integer smaller than or equal to $k$, and we use (3.8). Then, from definition (3.10), we obtain

$$\begin{aligned}\int_0^T\int_{B_r} |DT_k(u)|^p\,dx\,dt &= \int_0^T\int_{\Omega_{k,r}(t)} |Du|^p\,dx\,dt \le \int_0^T\int_{\Omega_{([k]+1),r}(t)} |Du|^p\,dx\,dt \\ &= \sum_{i=0}^{[k]}\int_0^T\int_{F_{i,r}(t)} |Du|^p\,dx\,dt \le M\,(k+1),\end{aligned}$$

namely (3.9). □

As a consequence of the previous lemma, we now derive some a–priori estimates for $u$ and for its gradient $Du$ in appropriate Marcinkiewicz space. These estimates extend to the parabolic case the analogous bounds obtained in the $L^1$–theory for elliptic equations (see [1, 4]).

LEMMA 3.7. *Let $u \in L^p(0,T;W^{1,p}(B_r)) \cap L^\infty(0,T;L^1(B_r))$, with $1 < p < N$, be a function satisfying estimates (3.7) and (3.9) for every $k > 0$. Then, there exists a positive constants $C_1$, depending only on $r$, $N$, $p$ and $M$, such that*

$$\|u\|_{M^{s_1}(B_r\times(0,T))} \le C_1, \quad \||Du|\|_{M^{q_1}(B_r\times(0,T))} \le C_1, \tag{3.15}$$

*with $s_1$ and $q_1$ as in* (2.12).

*Proof.* Let $k > 0$ and $\sigma \in (1, p^*)$, where $p^* = Np/(N-p)$ is the Sobolev embedding exponent. Interpolation inequality applied to $T_k(u)$ for a.e. fixed $t \in (0,T)$ gives

$$\int_{B_r} |T_k(u)|^\sigma \le \left(\int_{B_r} |T_k(u)|\right)^{(1-\vartheta)\sigma} \left(\int_{B_r} |T_k(u)|^{p^*}\right)^{\sigma\vartheta/p^*},$$

with $\vartheta \in (0,1)$ such that

$$\frac{1}{\sigma} = 1 - \vartheta + \frac{\vartheta}{p^*}.$$

This and Sobolev embedding inequality with average yield

$$\int_{B_r} |T_k(u)|^\sigma \leq c \left( \int_{B_r} |T_k(u)| \right)^{(1-\vartheta)\,\sigma} \left[ \int_{B_r} |T_k(u)| + \left( \int_{B_r} |DT_k(u)|^p \right)^{1/p} \right]^{\sigma\,\vartheta} ,$$

with $c > 0$ depending on $N$, $p$, and $r$. From (3.7) it then follows

$$\int_{B_r} |T_k(u)|^\sigma \leq c \left[ 1 + \left( \int_{\Omega_{k,r}(t)} |Du|^p \right)^{\sigma\,\vartheta/p} \right],$$

where $\Omega_{k,r}(t)$ is defined in (3.14) and with $c$ depending now also on $M$ and $\sigma$. Let us choose $\sigma = p\,(N+1)/N$, so that $\sigma\,\vartheta/p = 1$. By integrating on $(0, T)$ and by using (3.9), we get

$$\int_0^T \int_{B_r} |T_k(u)|^\sigma \,dx\,dt \leq c\,(k+1)\,, \tag{3.16}$$

where $c > 0$ depends on $N$, $p$, $r$, $M$ and $T$. If we define the set

$$\mathcal{A}_k := \{(x, t) \in B_r \times (0, T) : |u(x, t)| > k\},$$

then from (3.16) we easily deduce, for all $k \geq 1$,

$$\text{meas}(\mathcal{A}_k) \leq \frac{1}{k^\sigma} \iint_{\mathcal{A}_k} |T_k(u)|^\sigma \leq \frac{1}{k^\sigma} \int_0^T \int_{B_r} |T_k(u)|^\sigma \leq c\,\frac{k+1}{k^\sigma} \leq \frac{2\,c}{k^{\sigma-1}}.$$

Since $\sigma - 1 = s_1$, the above inequality gives the estimate for $u$ in (3.15).

In order to estimate $|Du|$, let us consider for $h > 0$ the set

$$\mathcal{B}_h := \{(x, t) \in B_r \times (0, T) : |Du(x, t)| > h\}.$$

We have, for every $k > 0$,

$$\begin{aligned} \mathcal{B}_h &= \{(x, t) : |Du(x, t)| > h, |u(x, t)| \leq k\} \cup \{(x, t) : |Du(x, t)| > h, |u(x, t)| > k\} \\ &\subseteq \{(x, t) : |Du(x, t)| > h, |u(x, t)| \leq k\} \cup \mathcal{A}_k. \end{aligned}$$

Hence,

$$\begin{aligned} \text{meas}(\mathcal{B}_h) &\leq \text{meas}\{(x, t) : |Du(x, t)| > h |u(x, t)| \leq k\} + \text{meas}(\mathcal{A}_k) \\ &\leq \frac{1}{h^p} \int_0^T \int_{\Omega_{k,r}(t)} |Du|^p \,dx\,dt + \frac{c}{k^{s_1}} \leq c \left( \frac{k+1}{h^p} + \frac{1}{k^{s_1}} \right), \end{aligned}$$

where we have used (3.9) and the estimate of $u$ in $M^{s_1}(B_r \times (0, T))$. If we take the minimum with respect to $k > 0$ of the last term, then we obtain

$$\text{meas}(\mathcal{B}_h) \leq \frac{c}{h^{p\,s_1/(s_1+1)}},$$

for a not relabeled constant $c > 0$ and for all $h \geq 1$. Since $ps_1/(s_1+1) = q_1$, (3.15) is completely proved. □

In Lemma 3.7, the summability established for the functions $u(x,t)$ and $|Du(x,t)|$ is the same with respect to $x$ and with respect to $t$. In the following result, which is the local version of the estimates obtained in [5], we assume that the exponent $p$ is suitably large and we obtain different regularity properties with respect to space and time.

LEMMA 3.8. *Let $2 - 1/N < p < N$, $r > 0$ and let $u \in L^p(0,T;W^{1,p}(B_r)) \cap L^\infty(0,T;L^1(B_r))$ be a function satisfying* (3.7) *and* (3.8). *Then, there exists a positive constant $C_2$ depending only on $M$, $r$, $T$, $q$ and $s$ such that*

$$\int_0^T \|u(t)\|^s_{W^{1,q}(B_r)}\,dt \leq C_2\,, \tag{3.17}$$

*for all exponents $q$, $s$ satisfying*

$$1 \leq q < q(p) := \frac{N(p-1)}{N-1}, \tag{3.18}$$

$$0 < s < s(q) := \min\left\{p, q\,\frac{p(N+1)-2N}{q(N+1)-N}\right\}. \tag{3.19}$$

*Proof.* For $k \geq 0$ integer, let $F_{k,r}(t)$ be the set defined in (3.10). For every $\mu > 0$ and a.e. $t \in (0,T)$, we have

$$\text{meas}(F_{k,r}(t)) \leq \int_{F_{k,r}(t)} \left(\frac{|u(x,t)|+1}{k+1}\right)^\mu dx.$$

Hence, for every $q < p$, Hölder inequality with exponent $p/q$ yields

$$\begin{aligned}\int_{B_r} |Du|^q &= \sum_{k=0}^\infty \int_{F_{k,r}(t)} |Du|^q \leq \sum_{k=0}^\infty \left(\int_{F_{k,r}(t)} |Du|^p\right)^{q/p} \text{meas}(F_{k,r}(t))^{1-q/p} \\ &\leq \sum_{k=0}^\infty \frac{1}{(k+1)^{\mu(1-q/p)}} \left(\int_{F_{k,r}(t)} |Du|^p\right)^{q/p} \left(\int_{F_{k,r}(t)} (|u|+1)^\mu\right)^{1-q/p} \\ &\leq \left(\sum_{k=0}^\infty \frac{1}{(k+1)^{\mu(p/q-1)}} \int_{F_{k,r}(t)} |Du|^p\right)^{q/p} \left(\int_{B_r} (|u|+1)^\mu\right)^{1-q/p}.\end{aligned}$$

Then, integration on $(0, T)$ and again Hölder inequality yield, for all $s < p$,

$$\int_0^T \left(\int_{B_r} |Du|^q\right)^{s/q} \leq \int_0^T \left(\sum_{k=0}^{\infty} \frac{1}{(k+1)^{\mu(p/q-1)}} \int_{F_{k,r}(t)} |Du|^p\right)^{s/p} \left(\int_{B_r} (|u|+1)^{\mu}\right)^{s(1/q-1/p)}$$

$$\leq \left[\sum_{k=0}^{\infty} \frac{1}{(k+1)^{\mu(p/q-1)}} \int_0^T \int_{F_{k,r}(t)} |Du|^p\right]^{s/p} \left[\int_0^T \left[\int_{B_r} (|u|+1)^{\mu}\right]^{\frac{s(p-q)}{q(p-s)}}\right]^{1-s/p}.$$

We now choose

$$\mu := \frac{q\,(p(N+1)-N) - s\,(q(N+1)-N)}{N\,(p-q)}, \tag{3.20}$$

so that, by assumption (3.19) on $s$,

$$\mu\left(\frac{p}{q}-1\right) > 1.$$

Therefore, from the above and (3.8) we obtain

$$\int_0^T \left(\int_{B_r} |Du|^q\right)^{s/q} \leq c\left[\left(\int_0^T \left(\int_{B_r} |u|^{\mu}\right)^{\frac{s(p-q)}{q(p-s)}}\right)^{1-s/p} + 1\right],$$

for every $q < p$, with $s$ satisfying (3.19), $\mu$ defined by (3.20) and $c > 0$ depending on $M, r, N, T, p, q$ and $s$, as every constant appearing henceforth.

Next, we notice that, thanks to assumptions (3.18) and (3.19), we can assume without loss of generality that

$$s > q\,\frac{N-p}{N-q},$$

which implies, by definition (3.20),

$$1 < \mu < q^* := \frac{Nq}{N-q}.$$

Hence, by (3.7) and interpolation inequality for $u$ between $L^1(B_r)$ and $L^{q^*}(B_r)$, we end up with

$$\int_0^T \left(\int_{B_r} |Du|^q\right)^{s/q} \leq c\left[\left(\int_0^T \left(\int_{B_r} |u|^{q^*}\right)^{s/q^*}\right)^{1-s/p} + 1\right].$$

From this, by Sobolev embedding inequality with average and estimate (3.7), we obtain the following chain of inequalities

$$\int_0^T \left( \int_{B_r} |u|^{q^*} \right)^{s/q^*} \le c \left[ \int_0^T \left( \int_{B_r} |Du|^q \right)^{s/q} + 1 \right]$$
$$\le c \left[ \left( \int_0^T \left( \int_{B_r} |u|^{q^*} \right)^{s/q^*} \right)^{1-s/p} + 1 \right].$$

Hence, we first deduce that

$$\int_0^T \left( \int_{B_r} |u|^{q^*} \right)^{s/q^*} \le c,$$

and then we conclude that

$$\int_0^T \left( \int_{B_r} |Du|^q \right)^{s/q} \le c.$$

□

REMARK 3.9. If $p > 2 - 1/(N+1)$, we can have $s \ge 1$ in estimate (3.17) provided that we restrict the range for $q$ as in (2.13). In this case, (3.17) gives an a–priori estimate for $u$ in $L^s(0, T; W^{1,q}(B_r))$, for all exponent $s$ satisfying (2.14). If we require $s = q$, then we obtain the bound

$$\int_0^T \int_{B_r} |Du(x,t)|^q \, dx \, dt \le C_3 \qquad \text{for all } 1 \le q < p - \frac{N}{N+1}, \tag{3.21}$$

which, together with (3.7), gives also

$$\int_0^T \int_{B_r} |u(x,t)|^s \, dx \, dt \le C_4 \qquad \text{for all } 1 \le s < p + \frac{p}{N} - 1. \tag{3.22}$$

Thus, in virtue of (2.11), we obtain from Lemma 3.8 essentially the same information given by Lemma 3.7. Nevertheless, Lemma 3.7, besides providing slightly better estimates than (3.21) and (3.22), works for any $p > 1$.

Let us now turn to settle the tools we need to derive a–priori estimates for problem (2.15).

We will make use of the following version of the well known Gagliardo-Nirenberg embedding theorem. For the proof we refer to [2] for the case $p = 2$, and to [14] for $1 < p < N$.

LEMMA 3.10. *Let $\Omega \subset \mathbb{R}^N$ open and bounded, $T > 0$, $1 < p < N$ and let further $w \in L^\infty(0,T; L^p(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega))$. There exists a positive constant K depending only on N and p such that*

$$\left[\int_0^T \left(\int_\Omega |w|^\sigma\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}} \le K \left( \sup_{(0,T)} \int_\Omega |w|^p + \int_0^T \int_\Omega |Dw|^p \right),$$

*for every pair of exponent $\sigma$ and $\mu$ such that*

$$p \le \sigma \le p^*, \qquad p \le \mu \le \infty, \qquad \frac{N}{p\,\sigma} + \frac{1}{\mu} = \frac{N}{p^2}. \tag{3.23}$$

We will also need the following result (see [23]).

LEMMA 3.11. *Let $\omega(h,r)$ be a nondecreasing in r, and nonincreasing in h, function defined in $[0,+\infty) \times [0,1]$; suppose that there exist constants $k_0 \ge 0$, $D$, $\alpha$, $\gamma > 0$ and $\beta > 1$ such that*

$$\omega(h,r) \le \frac{D\,\omega(k,R)^\beta}{(h-k)^\alpha (R-r)^\gamma},$$

*for all $h > k \ge k_0$ and $0 \le r < R \le 1$. Then, for every $\rho$ in $(0,1)$, there exists $d > 0$, given by*

$$d^\alpha = \frac{D\, 2^{\frac{\beta(\alpha+\gamma)}{\beta-1}}\, \omega(k_0,1)^{\beta-1}}{(1-\rho)^\gamma},$$

*such that*

$$\omega(d,\rho) = 0.$$

Our final a–priori estimate concerns bounded solutions of the problem

$$\begin{cases} u_t - \operatorname{div}(a(x,t,Du)) + h(x,t,u) + F(x,t,Du) = f(x,t) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.24}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set and $a(x,t,\xi)$, $h(x,t,s)$ and $F(x,t,\xi)$ are Caratheodory functions satisfying respectively assumptions (2.3)–(2.5), (2.6) and (2.17), and (2.18). Assume that for $f \in L^\infty(\Omega \times (0,T))$ and $u_0 \in L^\infty(\Omega)$ there exists a weak solution $u \in L^\infty(\Omega \times (0,T)) \cap L^p(0,T; W_0^{1,p}(\Omega))$ of problem (3.24), that is a function such that $u_t \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, $u(x,0) = u_0(x)$ and

$$\int_0^T \langle u_t, v\rangle + \int_0^T \int_\Omega a(x,t,Du)\,Dv + \int_0^T \int_\Omega h(x,t,u)v + \int_0^T \int_\Omega F(x,t,Du)\,v = \int_0^T\int_\Omega f v \tag{3.25}$$

for every test function $v \in L^\infty(\Omega \times (0,T)) \cap L^p(0,T; W_0^{1,p}(\Omega))$.

The following result is a parabolic version of the estimate obtained in Theorem 9 of [17]. An analogous result for globally integrable data is proved in [13].

LEMMA 3.12. *Under the above assumptions with* $1 < p < N$*, let* $u(x,t)$ *satisfy* (3.25)*. Then, for every couple of concentric balls* $B_r \subset B_R \subseteq \Omega$*, the following estimate holds*

$$\|u\|_{L^\infty(B_r\times(0,T))} \le C_0\,,$$

*where* $C_0$ *is a positive constant depending on* $N,\ T,\ p,\ \lambda,\ \Lambda,\ \gamma,\ r$ *and on the norms* $\|u_0\|_{L^\infty(B_R)}$ *and* $\|f\|_{L^m(0,T;L^q(B_R))}$*, provided that the exponents* $m,\ q > 1$ *satisfy*

$$\frac{N}{pq} + \frac{1}{m} < 1. \tag{3.26}$$

*Proof.* We prove the estimate for $0 \le r < R \le 1$, and then get the full statement by a standard covering argument. For any fixed $\beta > 0$, let us consider the function

$$\tilde g(s) := |s|^{p-2} s j\left(\frac{\log(\beta|s|^{p-1}+1)}{\beta}\right), \qquad s \in \mathbb{R}\,,$$

where $j(s)$ is the function appearing in (2.17). Note that our assumptions on $j$ imply that the above defined $\tilde g(s)$ satisfies (1.3), (1.4) and (1.5). Thus, we can apply Lemma 3.3 and construct a function $\varphi$ satisfying (3.4) with such a $\tilde g$, $\alpha = 0$ and a constant $C$ to be chosen later on.

Let us further define the function

$$\psi_\beta(s) := \left(\frac{e^{\beta|s|}-1}{\beta}\right)\operatorname{sign}(s)\,, \qquad s \in \mathbb{R},$$

and let us fix a smooth function $\zeta \in \mathcal{D}(B_R)$, such that $0 \le \zeta \le 1$, $\zeta \equiv 1$ in $B_r$ and $|D\zeta| \le 2/(R-r)$. We divide the rest of the proof into three steps.

STEP 1. We are going to take as test function in (3.25) $v$ defined as

$$v = \psi_\beta(G_k(u))\eta(x),$$

where $\eta(x) = \varphi(\zeta(x))$, $G_k$ is defined by (3.1) and $k \ge \|u_0\|_{L^\infty(B_R)}$. Note that for a.e. $t \in (0,T)$, if we argue as in Lemma 3.1, then we obtain

$$\langle u_t(t), v(t)\rangle = \frac{d}{dt}\left(\int_\Omega \Psi_\beta(G_k(u))(x,t)\,\eta(x)\,dx\right),$$

with $\Psi_\beta(s) = \int_0^s \psi_\beta(\sigma)\,d\sigma$. Hence

$$\int_0^T \langle u_t(t), v(t)\rangle dt = \int_\Omega \Psi_\beta(G_k(u))(x,T)\,\eta(x)\,dx - \int_\Omega \Psi_\beta(G_k(u_0))(x)\,\eta(x)\,dx$$

$$= \int_\Omega \Psi_\beta(G_k(u))(x,T)\,\eta(x)\,dx \ge 0,$$

since, by our choice of $k$, we have $G_k(u_0) \equiv 0$, and, moreover, $\eta \geq 0$ and $\Psi_\beta(s) \geq 0$ for all $s \in \mathbb{R}$. We further observe that for a.e. $t \in (0, T)$ the test function $v(t)$ is supported on the set

$$A_{k,R}(t) = \{x \in B_R \, : |u(x,t)| \geq k\} \, .$$

From (3.25) it then follows

$$\begin{aligned}
&\int_0^T \int_{A_{k,R}(t)} a(x,t,Du) DG_k(u)\, \mathrm{e}^{\beta|G_k(u)|}\eta \\
&\quad + \int_0^T \int_{A_{k,R}(t)} a(x,t,Du) D\zeta\, \varphi'(\zeta)\, \psi_\beta(G_k(u)) \\
&\quad + \int_0^T \int_{A_{k,R}(t)} h(x,t,u)\, \psi_\beta(G_k(u))\, \eta + \int_0^T \int_{A_{k,R}(t)} F(x,t,Du)\, \psi_\beta(G_k(u))\, \eta \\
&\leq \int_0^T \int_{A_{k,R}(t)} f(x,t)\, \psi_\beta(G_k(u))\, \eta.
\end{aligned}$$

Since in $A_{k,R}(t)$ we have $Du = DG_k(u)$, by assumptions (2.3), (2.4), (2.17) and (2.18) we obtain

$$\begin{aligned}
&\lambda \int_0^T \int_{A_{k,R}(t)} |DG_k(u)|^p e^{\beta|G_k(u)|}\eta + \int_0^T \int_{A_{k,R}(t)} j(u)\, \psi_\beta(G_k(u))\, \eta \\
&\leq \frac{2\,\Lambda}{(R-r)} \int_0^T \int_{A_{k,R}(t)} |DG_k(u)|^{p-1}\varphi'(\zeta)\, \psi_\beta(|G_k(u)|) \\
&\quad + \gamma \int_0^T \int_{A_{k,R}(t)} |DG_k(u)|^p \psi_\beta(|G_k(u)|)\, \eta + \int_0^T \int_{A_{k,R}(t)} |f(x,t)|\, \psi_\beta(|G_k(u)|)\eta.
\end{aligned}$$

From the above inequality, by using Young inequality and Lemma 3.3. with $\alpha = 0$ and $t^{p-1} = |\psi_\beta(G_k(u))|$, exactly as in the elliptic case (see [17]), we obtain, for any $\beta > (\gamma+1)/\lambda$,

$$\begin{aligned}
&\int_0^T \int_{A_{k,R}(t)} \left| D\left(\psi_\beta(G_k(u)/p)\, \eta^{1/p}\right)\right|^p + j(k) \int_0^T \int_{A_{k,R}(t)} \psi_\beta(|G_k(u)|)\, \eta \\
&\leq c \left[ \int_0^T \int_{A_{k,R}(t)} |f(x,t)| \psi_\beta(|G_k(u)|)\, \eta + \frac{1}{(R-r)^p} \int_0^T \mathrm{meas}(A_{k,R}(t)) \right],
\end{aligned}$$

for a positive constant $c$ depending on $p$, $\lambda$, $\Lambda$ and $\gamma$.

Now, for every fixed constant $H > 0$, the first integral on the right hand side can be estimated as follows

$$\int_0^T \int_{A_{k,R}(t)} |f(x,t)|\, \psi_\beta(|G_k(u)|)\, \eta$$
$$\leq H \int_0^T \int_{A_{k,R}(t)} \psi_\beta(|G_k(u)|)\, \eta + \int_0^T \int_{A_{k,R}(t)\cap\{|f|>H\}} |f(x,t)|\, \psi_\beta(|G_k(u)|)\, \eta,$$

so that, for $k \geq j^{-1}(c\, H)$, it follows

$$\int_0^T \int_{A_{k,R}(t)} |D(\psi_\beta(G_k(u)/p)\, \eta^{1/p})|^p$$
$$\leq c \left[ \int_0^T \int_{A_{k,R}(t)\cap\{|f|>H\}} |f(x,t)|\, \psi_\beta(|G_k(u)|)\, \eta + \frac{1}{(R-r)^p} \int_0^T \operatorname{meas}(A_{k,R}(t)) \right].$$

Furthermore, an easy computation shows that, for all $s \in \mathbb{R}$,

$$\psi_\beta(|s|) \leq \beta^{p-1} \frac{(e^\beta - 1)}{(e^{\beta/p} - 1)^p} \psi_\beta(|s|/p)^p + \frac{(e^\beta - 1)}{\beta}.$$

Since $(R - r) < 1$, the above inequalities imply (for a not relabeled constant $c > 0$)

$$\int_0^T \int_{A_{k,R}(t)} \left| D(\psi_\beta(G_k(u)/p)\, \eta^{1/p}) \right|^p \leq \frac{c}{(R-r)^p} \int_0^T \int_{A_{k,R}(t)} (|f(x,t)| + 1)$$
$$+ c \int_0^T \int_{A_{k,R}(t)\cap\{|f|>H\}} |f(x,t)|\, (\psi_\beta(|G_k(u)|/p)\, \eta^{1/p})^p. \qquad (3.27)$$

STEP 2. In this step we use (3.25) with test function

$$v = e^{\beta/p\, |G_k(u)|}\, \psi_\beta(|G_k(u)|/p)^{p-1} \operatorname{sign}(u)\, \eta\, \chi_{(0,\tau)},$$

with $\tau \in (0, T]$ arbitrarily fixed and $k \geq \|u_0\|_{L^\infty(B_R)}$ as in the previous step. Note that such a function cannot be used directly in (3.25) because of the possible lackness of smoothness near 0 of the function $s \in [0, +\infty) \mapsto \psi_\beta(s/p)^{p-1}$. Nevertheless, by a standard regularizing argument and thanks to the increasing monotonicity of the function $s \in [0, +\infty) \mapsto \psi_\beta(s/p)^{p-1}$, it is not hard to check that the following inequality holds

$$
\begin{aligned}
&\int_\Omega |\psi_\beta(G_k(u)(\tau)/p)\,\eta^{1/p}|^p \\
&\qquad + \frac{\beta}{p}\int_0^\tau\!\!\int_\Omega a(x,t,Du)\,DG_k(u)\mathrm{e}^{\beta/p\,|G_k(u)|}\,\psi_\beta(|G_k(u)|/p)^{p-1}\,\eta \\
&\qquad + \int_0^\tau \int_\Omega h(x,t,u)\,\mathrm{e}^{\beta/p\,|G_k(u)|}\,\psi_\beta(|G_k(u)|/p)^{p-1}\,\mathrm{sign}(u)\,\eta \\
&\qquad + \int_0^\tau \int_\Omega F(x,t,Du)\,\mathrm{e}^{\beta/p\,|G_k(u)|}\,\psi_\beta(|G_k(u)|/p)^{p-1}\,\mathrm{sign}(u)\,\eta \\
&\quad \le -\int_0^\tau \int_\Omega a(x,t,Du)\,D\zeta\,\varphi'\mathrm{e}^{\beta/p\,|G_k(u)|}\,\psi_\beta(|G_k(u)|/p)^{p-1}\mathrm{sign}(u) \\
&\qquad + \int_0^\tau \int_\Omega f(x,t)\,\mathrm{e}^{\beta/p\,|G_k(u)|}\,\psi_\beta(|G_k(u)|/p)^{p-1}\,\mathrm{sign}(u)\,\eta.
\end{aligned}
$$

We now argue exactly as in Step 1, and we obtain, in particular,

$$
\begin{aligned}
&\int_\Omega |\psi_\beta(G_k(u)(\tau)/p)\,\eta^{1/p}|^p \\
&\quad \le c\Bigg[\int_0^\tau\!\!\int_{A_{k,R}(t)\cap\{|f|>H\}} |f(x,t)|\,\mathrm{e}^{\beta/p\,|G_k(u)|}\,\psi_\beta(|G_k(u)|/p)^{p-1}\,\eta \\
&\qquad + \frac{1}{(R-r)^p}\int_0^\tau \mathrm{meas}(A_{k,R}(t))\Bigg],
\end{aligned}
$$

for any fixed $H > 0$, with $k \ge j^{-1}(c\,H)$ and $c > 0$ depending only on $p$, $\lambda$, $\Lambda$ and $\gamma$.

We further observe that, for all $s \ge 0$,

$$
\mathrm{e}^{\frac{\beta}{p}s}\,\psi_\beta(s/p)^{p-1} \le \frac{\mathrm{e}^{\beta/p}}{(\mathrm{e}^{\beta/p}-1)}\left[\psi_\beta(s/p)^p + \beta\left(\frac{\mathrm{e}^{\beta/p}-1}{\beta}\right)^p\right],
$$

so that from the above we obtain

$$
\begin{aligned}
&\int_\Omega |\psi_\beta(G_k(u)(\tau)/p)\,\eta^{1/p}|^p \\
&\quad \le c\Bigg[\int_0^\tau \int_{A_{k,R}(t)\cap\{|f|>H\}} |f(x,t)|\,(\psi_\beta(|G_k(u)|/p)\,\eta^{1/p})^p \\
&\qquad + \frac{1}{(R-r)^p}\int_0^\tau\!\!\int_{A_{k,R}(t)} (|f(x,t)|+1)\Bigg].
\end{aligned}
$$

Hence, by taking the supremum with respect to $\tau \in (0,T)$, we finally get

$$\sup_{\tau\in(0,T)}\int_\Omega |\psi_\beta(G_k(u)(\tau)/p)\,\eta^{1/p}|^p \le \frac{c}{(R-r)^p}\int_0^T\!\!\int_{A_{k,R}(t)}(|f(x,t)|+1) + c\int_0^T\!\!\int_{A_{k,R}(t)\cap\{|f|>H\}}|f(x,t)|\,(\psi_\beta(|G_k(u)|/p)\,\eta^{1/p})^p. \tag{3.28}$$

STEP 3. In this step we derive the final estimate on $\|u\|_{L^\infty(B_r\times(0,T))}$ from (3.27), (3.28) and Lemma 3.10. For, if we set

$$w = \psi_\beta(G_k(u)/p)\,\eta^{1/p},$$

then Lemma 3.10 applied to $w$ and both inequalities (3.27) and (3.28) imply that

$$\left[\int_0^T\left(\int_\Omega |w|^\sigma\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}} \le c\left[\int_0^T\!\!\int_{A_{k,R}(t)\cap\{|f|>H\}}|f(x,t)|\,|w|^p + \frac{1}{(R-r)^p}\int_0^T\!\!\int_{A_{k,R}(t)}(|f(x,t)|+1)\right], \tag{3.29}$$

for all $\mu$ and $\sigma$ satisfying (3.23). The assumption (3.26) on the exponents $m$ and $q$ allows us to choose $\sigma$ and $\mu$ as folllows

$$\sigma = p\,\frac{(Nm'+pq')}{Nm'}\,,\qquad \mu = p\,\frac{(Nm'+pq')}{Nq'}\,. \tag{3.30}$$

With such a choice for $\sigma$ and $\mu$, since both of them are larger than $p$, we can apply Hölder inequality to estimate the first integral on the right hand side of (3.29). This yields

$$\begin{aligned}\int_0^T\int_{A_{k,R}(t)\cap\{|f|>H\}}&|f(x,t)|\,|w|^p\\ &\le \int_0^T\left(\int_\Omega|w|^\sigma\right)^{\frac{p}{\sigma}}\left(\int_{A_{k,R}(t)\cap\{|f|>H\}}|f(x,t)|^{\frac{\sigma}{(\sigma-p)}}\right)^{1-\frac{p}{\sigma}}\\ &\le\left[\int_0^T\left(\int_\Omega|w|^\sigma\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}}\left[\int_0^T\left(\int_{A_{k,R}(t)\cap\{|f|>H\}}|f(x,t)|^{\frac{\sigma}{(\sigma-p)}}\right)^{\frac{\mu(\sigma-p)}{\sigma(\mu-p)}}\right]^{1-\frac{p}{\mu}}.\end{aligned}$$

Observe now that by definitions (3.30) and by condition (3.26), we also have

$$\frac{\sigma}{\sigma-p} < q\,,\quad \frac{\mu}{\mu-p} < m.$$

Therefore, since

$$\text{meas}(A_{k,R}(t) \cap \{|f| > H\}) \leq \text{meas}(B_R) \leq \text{meas}(B_1)\,,$$

by Hölder inequality we get

$$\int_0^T \int_{A_{k,R}(t)\cap\{|f|>H\}} |f(x,t)|\,|w|^p$$
$$\leq c\left[\int_0^T \left(\int_\Omega |w|^\sigma\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}} \left[\int_0^T \left(\int_{A_{k,R}(t)\cap\{|f|>H\}} |f(x,t)|^q\right)^{\frac{m}{q}}\right]^{\frac{1}{m}},$$

for some positive constant $c$ depending on $N$ and $T$. If we use the above inequality into (3.29), and choose $H = H_0$ so large as to make the quantity $\|f\,\chi_{\{|f|>H_0\}}\|_{L^m(0,T;L^q(B_R))}$ small enough, then we obtain

$$\left[\int_0^T\left(\int_\Omega |w|^\sigma\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}} \leq \frac{c}{(R-r)^p}\int_0^T\int_{A_{k,R}(t)} (|f(x,t)|+1).$$

From the above estimate and again from Hölder inequality we immediately derive

$$\left[\int_0^T\left(\int_\Omega |w|^\sigma\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}} \leq \frac{c}{(R-r)^p}\left(\int_0^T \text{meas}(A_{k,R}(t))^{\frac{m'}{q'}}\right)^{\frac{1}{m'}}, \tag{3.31}$$

where $c$ now depends also on $\|f\|_{L^m(0,T;L^q(B_R))}$.

On the other hand, by definition of $w$, for a.e. $t \in (0,T)$ and for every $h > k$ we have

$$\int_\Omega |w|^\sigma \geq \int_{A_{k,r}(t)} |\psi_\beta(G_k(u)/p)|^\sigma \geq \int_{A_{h,r}(t)} |\psi_\beta((h-k)/p)|^\sigma$$
$$\geq \left(\frac{h-k}{p}\right)^\sigma \text{meas}(A_{h,r}(t)),$$

and thus, also by (3.30),

$$\left[\int_0^T\left(\int_\Omega |w|^\sigma\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}} \geq \left(\frac{h-k}{p}\right)^p \left(\int_0^T \text{meas}(A_{h,r}(t))^{\frac{\mu}{\sigma}}\right)^{\frac{p}{\mu}}$$
$$= \left(\frac{h-k}{p}\right)^p \left(\int_0^T \text{meas}(A_{h,r}(t))^{\frac{m'}{q'}}\right)^{\frac{p}{\mu}}.$$

From the above inequality and (3.31) we finally deduce

$$\int_0^T \text{meas}(A_{h,r}(t))^{\frac{m'}{q'}} \leq \frac{c}{(h-k)^\mu (R-r)^\mu}\left(\int_0^T \text{meas}(A_{k,R}(t))^{\frac{m'}{q'}}\right)^{\frac{\mu}{m'p}},$$

for every $h > k \geq k_0 = \max\{\|u_0\|_{L^\infty(B_1)},\ j^{-1}(c\,H_0)\}$ and $0 \leq r < R \leq 1$.

Since condition (3.26) implies that

$$\frac{\mu}{m'p} = \frac{1}{q'} + \frac{p}{m'N} = 1 + \frac{p}{N} - \frac{p}{N}\left(\frac{1}{m} + \frac{N}{pq}\right) > 1,$$

we can apply Lemma 3.11 to the function

$$\omega(h, r) := \int_0^T \operatorname{meas}(A_{h,r}(t))^{\frac{m'}{q'}}\, dt,$$

and the conclusion follows. □

## 4. Proofs of the theorems

This section is devoted to the proof of Theorems 2.2, 2.4 and 2.5.

The first proof relies on several arguments taken from [4, 7, 8, 17, 20]. We give the details for the sake of completeness.

*Proof of Theorem* 2.2. We proceed by approximation: we construct a sequence of penalized problems whose solutions are proved to converge to a solution of (2.1).

For $n \geq 1$, let $B_n$ denote the ball of $\mathbb{R}^N$ centered at 0 with radius $n$, and let us set

$$f_n(x, t) := T_n(f(x, t)), \qquad u_0^n(x) := T_n(u_0(x)),$$

and

$$h_n(x, t, s) := \begin{cases} T_n(h(x, t, s)) & \text{if } p \leq 2, \\ T_n\left(\frac{h(x,t,s)}{g(s)}\right) g(s) & \text{if } p > 2. \end{cases}$$

Note that

$$|f_n| \leq |f|,\ f_n \to f \text{ in } L^1(0, T; L^1_{\mathrm{loc}}(\mathbb{R}^N)), \tag{4.1}$$

$$|u_0^n| \leq |u_0|,\ u_0^n \to u_0 \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^N), \tag{4.2}$$

$$|h_n| \leq |h|,\ h_n \to h \text{ a.e. in } \mathbb{R}^N \times (0, T) \times \mathbb{R}. \tag{4.3}$$

Moreover, $h_n$ still satisfies (2.7) and, in place of (2.6), the stronger condition

$$\sup_{s \in (0, s_0)} h_n(x, t, s) \in L^\infty(\mathbb{R}^N \times (0, T)).$$

Standard existence theory for parabolic equations (see [18]) implies that the initial–boundary value problem

$$\begin{cases} u_n{}' - \operatorname{div}(a(x, t, Du_n)) + h_n(x, t, u_n) = f_n(x, t) & \text{in } B_n \times (0, T), \\ u_n(x, t) = 0 & \text{on } \partial B_n \times (0, T), \\ u_n(x, 0) = u_0^n(x) & \text{in } B_n, \end{cases} \tag{$P_n$}$$

has at least one weak solution $u_n \in L^p(0,T;W_0^{1,p}(B_n)) \cap L^\infty(B_n \times (0,T)) \cap C([0,T];L^2(B_n))$, with $u_n{}' := u_{n,t} \in L^{p'}(0,T;W^{-1,p'}(B_n))$.

Let $\rho > 0$ be fixed. For all $n \geq 5\rho$, Lemma 3.6 applied to $u_n$ with $R = 5\rho$ and $r = 4\rho$ gives that, for every fixed $k > 0$,

$$\{T_k(u_n)\}_{n\geq 5\rho} \text{ is bounded in } L^p(0,T;W^{1,p}(B_{4\rho})), \tag{4.4}$$

as well as, by Lemma 3.7,

$$\{u_n\}_{n\geq 5\rho} \text{ is bounded in } M^{s_1}(B_{4\rho} \times (0,T)), \tag{4.5}$$

$$\{|Du_n|\}_{n\geq 5\rho} \text{ is bounded in } M^{q_1}(B_{4\rho} \times (0,T)). \tag{4.6}$$

Note that, by definition of $q_1$ in (2.12) and by (2.11), property (4.6) implies that $\{|Du_n|^{p-1}\}_{n\geq 5\rho}$ is bounded in $L^q(B_{4\rho} \times (0,T))$ for every $1 \leq q < q_1/(p-1)$, so that, thanks to assumption (2.3),

$$\{|a(x,t,Du_n)|\}_{n\geq 5\rho} \text{ is bounded in } L^q(B_{4\rho} \times (0,T)), \quad \forall\, 1 \leq q < \frac{q_1}{p-1}. \tag{4.7}$$

Further consequences of (4.4), (4.5) and (4.6) are derived in the following claims.

CLAIM 1. $\{h_n(x,t,u_n)\}_{n\geq 5\rho}$ is bounded in $L^1(B_{3\rho} \times (0,T))$.

Indeed, let $\zeta \in \mathcal{D}(B_{4\rho})$ be a cut–off function, with $\zeta \equiv 1$ in $B_{3\rho}$, and let $\mathcal{T}: \mathbb{R} \to \mathbb{R}$ be an increasing Lipschitz continuous function, such that $\mathcal{T}(0) = 0$ and $|\mathcal{T}| \leq 1$. From the weak formulation of problem $(P_n)$ applied with test function $\mathcal{T}(u_n)\,\zeta$, we obtain, after crossing out some nonnegative terms and using (4.1)–(4.2),

$$\int_0^T \int_{B_{4\rho}} h_n(x,t,u_n)\,\mathcal{T}(u_n)\,\zeta \leq \|f\|_{L^1(B_{4\rho}\times(0,T))} + \|u_0\|_{L^1(B_{4\rho})} + \int_0^T \int_{B_{4\rho}} |a(x,t,Du_n)|\,|D\zeta|.$$

Hence, by (4.7) and (2.2), $\{h_n(x,t,u_n)\,\mathcal{T}(u_n)\}_{n\geq 5\rho}$ is bounded in $L^1(B_{3\rho} \times (0,T))$. From this and assumption (2.6) Claim 1 then follows.

CLAIM 2. There exists $u \in M^{s_1}(B_{3\rho} \times (0,T))$, with $T_k(u) \in L^p(0,T;W^{1,p}(B_{3\rho}))$ for all $k > 0$, such that, up to a subsequence,

$$u_n \to u \quad \text{a.e. in } B_{3\rho} \times (0,T), \quad T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } L^p(0,T;W^{1,p}(B_{3\rho})).$$

For, we borrow an argument from [20]. Fix $k > 0$ and take a regularized of $T_k(s)$, namely an odd, nondecreasing function $\mathcal{T}_k : \mathbb{R} \to \mathbb{R}$ of class $C^2$ such that $\mathcal{T}_k(s) = s$ for $|s| \leq k/2$, and $\mathcal{T}_k(s) = k$ for $|s| \geq k$. Note that, by (4.4), $\{\mathcal{T}_k(u_n)\}_{n\geq 5\rho}$ is a bounded sequence in $L^p(0,T;W^{1,p}(B_{4\rho}))$. Moreover, since $u_n$ solves $(P_n)$, we have the following identity in $L^1(0,T;W^{-1,p'}(B_n)+L^1(B_n))$

$$\begin{aligned}(\mathcal{T}_k(u_n))' &= \mathcal{T}_k'(u_n)\,u_n' \\ &= \mathrm{div}(a(x,t,Du_n)\mathcal{T}_k'(u_n)) - a(x,t,Du_n)Du_n\mathcal{T}_k''(u_n) \\ &\quad -h_n(x,t,u_n)\mathcal{T}_k'(u_n) + f_n\mathcal{T}_k'(u_n).\end{aligned}$$

From (2.2), (2.3), (4.1), (4.4), and Claim 1, it then follows that $\{\mathcal{T}_k'(u_n)\}_{n\geq 5\rho}$ is bounded in $L^{p'}(0,T;W^{-1,p'}(B_{3\rho}))+L^1(B_{3\rho}\times(0,T))$. Hence, by a classical compactness result (see [22]), $\{\mathcal{T}_k(u_n)\}_{n\geq 5\rho}$ is relatively compact in $L^p(B_{3\rho}\times(0,T))$. In particular, up to a (not relabeled) subsequence, $\{\mathcal{T}_k(u_n)\}_{n\geq 5\rho}$ converges in measure and a.e. in $B_{3\rho}\times(0,T)$, as $n\to\infty$ and for every fixed $k>0$. We can now show that $\{u_n\}_{n\geq 5\rho}$ converges in measure by arguing as in [4]. We observe that, for all $n,m\geq 5\rho$ and $\sigma,k>0$,

$$\begin{aligned}&\mathrm{meas}(\{(x,t)\in B_{3\rho}\times(0,T)\ :\ |(u_n-u_m)(x,t)|>\sigma\}) \\ &\quad\leq \mathrm{meas}(\{|\mathcal{T}_k(u_n)-\mathcal{T}_k(u_m)|>\sigma\}) + \mathrm{meas}\,(\{|u_n|>k/2\}) + \mathrm{meas}\,(\{|u_m|>k/2\})\,.\end{aligned}$$

Property (4.5) implies that the sets $\{|u_n|>k/2\}$, $\{|u_m|>k/2\}$ have small measure for large $k$ uniformly with respect to $n,m$. Therefore, for every small $\epsilon>0$ we can find $k_\epsilon>0$ not depending on $n$ nor on $m$, such that

$$\begin{aligned}&\mathrm{meas}(\{(x,t)\in B_{3\rho}\times(0,T) : |(u_n-u_m)(x,t)|>\sigma\}) \\ &\quad\leq \mathrm{meas}(\{|\mathcal{T}_{k_\epsilon}(u_n)-\mathcal{T}_{k_\epsilon}(u_m)|>\sigma\})+\epsilon.\end{aligned}$$

Since $\{\mathcal{T}_{k_\epsilon}(u_n)\}_{n\geq 5\rho}$ converges in measure in $B_{3\rho}\times(0,T)$, by the arbitrariness of $\epsilon>0$ we get the same for $\{u_n\}_{n\geq 5\rho}$. Hence, by going to a further subsequence if necessary, there exists $u$ measurable such that

$$u_n\to u\quad \text{a.e. in } B_{3\rho}\times(0,T), \tag{4.8}$$

and, by (4.5), we also have $u\in M^{s_1}(B_{3\rho}\times(0,T))$. Furthermore, (4.4) implies that $\{T_k(u_n)\}_{n\geq 5\rho}$ weakly converges in $L^p(0,T;W^{1,p}(B_{3\rho}))$. On the other hand, by the dominated convergence theorem and (4.8), we get that $\{T_k(u_n)\}_{n\geq 5\rho}$ strongly converges to $T_k(u)$ in $L^q(B_{3\rho}\times(0,T))$ for every $q$. Hence, the weak limit of $\{T_k(u_n)\}_{n\geq 5\rho}$ must be $T_k(u)$, and Claim 2 is completely proved. Note also that, since $T_k(u)\in L^p(0,T;W^{1,p}(B_{3\rho}))$ for every $k>0$, then $u$ is provided with its weak gradient $Du$ in the sense of Lemma 2.1.

CLAIM 3. $h_n(x,t,u_n)\to h(x,t,u)$ strongly in $L^1(B_{3\rho}\times(0,T))$, up to a subsequence.

By Claim 2 and the definition of $h_n$, we have that, up to a subsequence, $h_n(x,t,u_n)\to h(x,t,u)$ a.e. in $B_{3\rho}\times(0,T)$. Hence, in order to prove Claim 3, it is enough to show the equintegrability of $\{h_n(x,t,u_n)\}_{n\geq 5\rho}$ in $B_{3\rho}\times(0,T)$. In order to do this, we argue as in [8] and we apply the weak formulation of $(P_n)$ with test function $T_1(G_k(u_n))\,\zeta$, where $\zeta \in \mathcal{D}(B_{4\rho})$ is a cut–off function constantly equal to 1 in $B_{3\rho}$. As in the proof of Claim 1, we obtain

$$\iint_{\mathcal{A}^n_{k+1,3\rho}} |h_n(x,t,u_n)| \le \iint_{\mathcal{A}^n_{k,4\rho}} |a(x,t,Du_n)|\,|D\zeta| + \iint_{\mathcal{A}^n_{k,4\rho}} |f(x,t)| + \int_{B_{4\rho}\cap\{|u_0|>k\}} |u_0(x)|,$$

with $\mathcal{A}^n_{k,\rho} := \{(x,t) \in B_\rho \times (0,T) : |u_n(x,t)| > k\}$. By (4.5), meas$(\mathcal{A}^n_{k,4\rho}) \to 0$ as $k \to \infty$ uniformly in $n$, so that, by (4.7) and (2.2), for every small $\epsilon > 0$ we can find a large enough $k_\epsilon > 0$ such that, for all $n \ge 5\rho$,

$$\iint_{\mathcal{A}^n_{k_\epsilon+1,3\rho}} |h_n(x,t,u_n)| \le \epsilon.$$

Therefore, for any measurable set $E \subset B_{3\rho} \times (0,T)$, by using also (4.3) and (2.6), we get

$$\iint_E |h_n(x,t,u_n)| \le \epsilon + \iint_{E\cap(\mathcal{A}^n_{k_\epsilon+1,3\rho})^c} |h_n(x,t,u_n)| \le \epsilon + \iint_E H_{k_\epsilon+1}(x,t),$$

so that the left hand side integral is arbitrarily small for meas$(E)$ small enough, uniformly with respect to $n$. This is precisely equintegrability for $\{h_n(x,t,u_n)\}_{n\ge5\rho}$, whence Claim 3 follows.

CLAIM 4. $Du \in M^{q_1}(B_{2\rho} \times (0,T))$ and, up to a subsequence,

$$Du_n \to Du \quad \text{a.e. in } B_{2\rho} \times (0,T)\,,$$
$$a(x,t,Du_n) \to a(x,t,Du) \quad \text{strongly in } L^1(B_{2\rho} \times (0,T)).$$

Indeed, we can apply the technique developed in [7] in order to show that $\{Du_n\}_{n\ge5\rho}$ converges in measure in $B_{2\rho} \times (0,T)$. We fix $\sigma > 0$ and, for all $n, m \ge 5\rho$, we consider the set

$$\mathcal{B}_{n,m} := \{(x,t) \in B_{2\rho} \times (0,T) : |D(u_n - u_m)(x,t)| > \sigma\}.$$

Henceforth, unless otherwise stated, all the set we consider are contained in $B_{2\rho} \times (0,T)$ and we omit the dependence on $\sigma$, which is fixed. We have

$$\mathcal{B}_{n,m} \subseteq \{|Du_n| > \nu\} \cup \{|Du_m| > \nu\} \cup \{|u_n - u_m| > k\} \cup \{|Du_n| \le \nu,\ |Du_m| \le \nu, |u_n - u_m| \le k,\ |Du_n - Du_m| > \sigma\}\,.$$

Let us call $\mathcal{C}^{\nu,k}_{n,m}$ the last set above. Then, by (4.6), for every small $\epsilon > o$ we can select $\nu_\epsilon$ large enough to have, for all $n, m \ge 5\rho$,

$$\text{meas}(\mathcal{B}_{n,m}) \le \epsilon + \text{meas}(\{|u_n - u_m| > k\}) + \text{meas}(\mathcal{C}^{\nu_\epsilon,k}_{n,m}). \tag{4.9}$$

In order to estimate the measure of $\mathcal{C}_{n,m}^{\nu_\epsilon,k}$, we apply the weak formulations of $(P_n)$ and $(P_m)$ both with test function $T_k(u_n - u_m)\,\zeta$, where $\zeta \in \mathcal{D}(B_{3\rho})$ is a cut-off function satisfying $\zeta \equiv 1$ in $B_{2\rho}$. We subtract the two resulting identities, cross out the positive term given by the integration of $(u_n - u_m)'$, and we use (4.1), (4.2), (4.7) and Claim 1. We in particular obtain

$$\begin{aligned}\iint_{\mathcal{C}_{n,m}^{\nu_\epsilon,k}} &(a(x,t,Du_n) - a(x,t,Du_m))\,(Du_n - Du_m)\\ &\leq 2k[\|f\|_{L^1(B_{3\rho}\times(0,T))} + \|u_0\|_{L^1(B_{3\rho})}\\ &\quad + \sup_{n\geq 5\rho}(\|h_n\|_{L^1(B_{3\rho}\times(0,T))} + \|a(x,t,Du_n)D\zeta\|_{L^1(B_{3\rho}\times(0,T))})], \qquad (4.10)\end{aligned}$$

with $h_n = h_n(x,t,u_n)$. Next, we introduce the compact set

$$K_\epsilon := \{(\xi_1,\xi_2) \in \mathbb{R}^{2N} : |\xi_1| \leq \nu_\epsilon, |\xi_2| \leq \nu_\epsilon, |\xi_1 - \xi_2| \geq \sigma\},$$

and we notice that, by assumption (2.5), the function

$$\gamma_\epsilon(x,t) := \min_{(\xi_1,\xi_2)\in K_\epsilon} [(a(x,t,\xi_1) - a(x,t,\xi_2))\,(\xi_1 - \xi_2)]$$

is strictly positive for a.e. $(x,t) \in B_{2\rho}\times(0,T)$. Therefore, there exists $\delta_\epsilon > 0$ corresponding to $\epsilon$ such that, for any measurable set $E \subset B_{2\rho}\times(0,T)$,

$$\iint_E \gamma_\epsilon(x,t)\,dx\,dt < \delta_\epsilon \implies \mathrm{meas}(E) < \epsilon.$$

Now, by (4.10), we can choose $k = k_\epsilon$ so small as to have

$$\iint_{\mathcal{C}_{n,m}^{\nu_\epsilon,k_\epsilon}} \gamma_\epsilon(x,t) \leq \iint_{\mathcal{C}_{n,m}^{\nu_\epsilon,k_\epsilon}} (a(x,t,Du_n) - a(x,t,Du_m))\,(Du_n - Du_m) < \delta_\epsilon,$$

which implies

$$\mathrm{meas}(\mathcal{C}_{n,m}^{\nu_\epsilon,k_\epsilon}) < \epsilon.$$

Hence, from (4.9) it follows that

$$\mathrm{meas}(\mathcal{B}_{n,m}) \leq 2\,\epsilon + \mathrm{meas}(\{|u_n - u_m| > k_\epsilon\}).$$

Since by Claim 2 $\{u_n\}_{n\geq 5\rho}$ converges, up to a subsequence, in measure in $B_{2\rho}\times(0,T)$, from the above inequality we then deduce the same for $\{Du_n\}_{n\geq 5\rho}$.

Hence, up to a further subsequence, $\{Du_n\}_{n\geq 5\rho}$ converges a.e. to some vector $v(x,t)$, which belongs to $M^{q_1}(B_{2\rho}\times(0,T))^N$ by (4.6). Therefore, $\{DT_k(u_n)\}_{n\geq 5\rho}$ converges to $v\,\chi_{\{|u|<k\}}$ a.e. in $B_{2\rho}\times(0,T)$ and, by (4.4), strongly in $L^q(B_{2\rho}\times(0,T))^N$ for all $q < p$.

On the other hand, Claim 2 implies that $\{DT_k(u_n)\}_{n\geq 5\rho}$ converges to $DT_k(u)$ weakly in $L^q(B_{2\rho}\times(0,T))^N$ for all $q<p$. Thus, $DT_k(u)=v\,\chi_{\{|u|<k\}}$ for every $k>0$, that is $v=Du$ is the weak gradient of $u$ in the sense of Lemma 2.1 and $\{Du_n\}_{n\geq 5\rho}$ converges a.e. to $Du$, always up to a subsequence. The strong convergence of $\{a(x,t,Du_n)\}_{n\geq 5\rho}$ is then a consequence of (4.7), and Claim 4 is completely proved.

CLAIM 5. $u_n\to u$ strongly in $C([0,T];L^1(B_\rho))$, up to a subsequence.

For, we can argue in a similar way we have done for the previous claim, and we apply the weak formulations of $(P_n)$ and $(P_m)$ with test function $T_1(u_n-u_m)\,\zeta\,\chi_{(0,t)}$, for a fixed $t\in[0,T]$ and with $\zeta\in\mathcal{D}(B_{2\rho})$ identically 1 in $B_\rho$. If we subtract the two resulting identities and we now cross out the positive term involving the gradients, in place of (4.10) we obtain

$$\begin{aligned}\int_{B_\rho} S_1(u_n-u_m)(x,t)\le \|(a(x,t,Du_n)-a(x,t,Du_m))D\zeta\|_{L^1(B_{2\rho}\times(0,T))}\\ +\|h_n-h_m\|_{L^1(B_{2\rho}\times(0,T))}+\|u_0^n-u_0^m\|_{L^1(B_{2\rho})}\\ +\|f_n-f_m\|_{L^1(B_{2\rho}\times(0,T))}.\end{aligned}$$

Claim 5 then follows from the above inequality as a consequence of definition (2.9), (4.1), (4.2) and Claims 3 and 4.

By applying repeteadly Claims 2, 3, 4 and 5 and by following a diagonal procedure as in [17], we can extract from $\{u_n\}_{n\geq 1}$ a (not relabeled) subsequence for which there exists a globally defined function $u$ such that

$$\begin{aligned}&u_n\to\ u\in M^{s_1}_{\mathrm{loc}}(\mathbb{R}^N\times[0,T])\quad\text{a.e. in }\ \mathbb{R}^N\times(0,T)\,,\\ &Du_n\to\ Du\in M^{q_1}_{\mathrm{loc}}(\mathbb{R}^N\times[0,T])^N\quad\text{a.e. in }\ \mathbb{R}^N\times(0,T)\,,\\ &T_k(u)\rightharpoonup\ T_k(u)\quad\text{weakly in }\ L^p(0,T;W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N))\,,\\ &a(x,t,Du_n)\to\ a(x,t,Du)\quad\text{strongly in }L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^N))\,,\\ &h_n(x,t,u_n)\to\ h(x,t,u)\quad\text{strongly in }L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^N))\,,\\ &u_n\to\ u\quad\text{strongly in }C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N)).\end{aligned}$$

The last convergence implies that $u(x,0)=u_0(x)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, whereas the other limit equalities allow $n\to\infty$ in the weak formulation of $(P_n)$. This yields that $u$ satisfies problem (2.1) in the sense of distributions.

*Proof of Theorem* 2.4. The conclusion follows by arguing exactly as in the proof of Theorem 2.2 and by applying Lemma 3.8 together with Remark 3.9.

REMARK 4.13. It is not hard (see [21]) to show that the solution $u$ constructed both in Theorem 2.2 and in Theorem 2.4 enjoys the following property

$$\begin{aligned}
&\int_0^T \langle \varphi_t, T_k(u-\varphi)\,\vartheta\rangle\,dt + \int_{\mathbb{R}^N} [S_k((u-\varphi)(x,T)) - S_k(u_0(x)-\varphi(x,0))]\vartheta\,dx \\
&\quad + \int_0^T\int_{\mathbb{R}^N} a(x,t,Du)\,D\left(T_k(u-\varphi)\vartheta\right)\,dx\,dt \\
&\quad + \int_0^T\int_{\mathbb{R}^N} h(x,t,u)\,T_k(u-\varphi)\vartheta\,dx\,dt \\
&\leq \int_0^T\int_{\mathbb{R}^N} f(x,t)\,T_k(u-\varphi)\vartheta\,dx\,dt \qquad (4.11)
\end{aligned}$$

for all $k > 0$, $\varphi \in L^p(0,T;W^{1,p}_{\text{loc}}(\mathbb{R}^N)) \cap L^\infty(0,T;L^\infty_{\text{loc}}(\mathbb{R}^N))$, with $\varphi_t \in L^{p'}((0,T);W^{-1,p'}_{\text{loc}}(\mathbb{R}^N))$, and $\vartheta \in \mathcal{D}(\mathbb{R}^N)$, with $\vartheta \geq 0$.

This inequality is a local version of the *entropy inequality* introduced in [4] for elliptic equations, and in [21] for the time dependent case. It makes sense for functions $u \in C([0,T];L^1_{\text{loc}}(\mathbb{R}^N))$ such that $T_k(u) \in L^p(0,T;W^{1,p}_{\text{loc}}(\mathbb{R}^N))$ for every $k \geq 0$, and it may be used as a stronger definition of solution for problem (2.1). Indeed, if $u$ is such that $|a(x,t,Du)|$, $h(x,t,u) \in L^1(0,T;L^1_{\text{loc}}(\mathbb{R}^N))$ and (4.11) holds true, then $u$ is a distributional solution of (2.1). For globally integrable data, the entropy inequality is an extra condition giving uniqueness of solution (see [4, 21]). In case of locally integrable data, the uniqueness of solution is still an interesting open problem.

We conclude this section with the following

*Proof of Theorem* 2.5. We proceed by approximation as in the proof of Theorem 2.2, and we construct, by using e.g. a standard fixed point argument, a sequence of functions $u_n \in L^\infty(B_n \times (0,T)) \cap L^p(0,T;W^{1,p}_0(B_n))$ solving the problems

$$\begin{cases}
u_n{}' - \operatorname{div}(a(x,t,Du_n)) + h_n(x,t,u_n) & \\
\quad + F_n(x,t,Du_n) = f_n(x,t) & \text{in } B_n \times (0,T), \\
u_n(x,t) = 0 & \text{on } \partial B_n \times (0,T), \\
u(x,0) = u_0(x) & \text{in } B_n,
\end{cases} \qquad (P_n')$$

with $u_n' := u_{n,t}$, $f_n := T_n(f)$, $h_n := T_n(h/j)\,j$ and $F_n := T_n(F)$. Note that $h_n$ and $F_n$ still satisfy respectively (2.17) and (2.18). Thus, we can apply Lemma 3.12 and we obtain that

$$\{u_n\}_{n\geq 1} \quad \text{is bounded in } \; L^\infty(0,T;L^\infty_{\text{loc}}(\mathbb{R}^N)).$$

By following now exactly the argument of [19] and by applying a diagonal procedure as in the proof of Theorem 2.2, we obtain the existence of a function $u \in L^\infty(0, T; L^\infty_{\mathrm{loc}}(\mathbb{R}^N)) \cap L^p(0, T; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N))$ such that, up to a subsequence,

$$u_n \to u \quad \text{strongly in } L^p(0, T; W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)).$$

Then, we can let $n \to \infty$ in the weak formulation of $(P_n')$ and we obtain that $u$ is a distributional solution of (2.15).

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*Fabiana Leoni*
*Dipartimento di Matematica*
*Università di Roma La Sapienza*
*P.le A. Moro 2*
*I–00185 Roma*
*Italy*
*e-mail: leoni@mat.uniroma1.it*

*Benedetta Pellacci*
*Dipartimento di Scienze Applicate*
*Università di Napoli Pathenope*
*Via A. De Gasperi 5*
*I–80133 Napoli*
*Italy*
*e-mail: benedetta.pellacci@uniparthenope.it*