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Weak solutions to stochastic porous media equations

GIUSEPPE DA PRATO and MICHAEL RÖCKNER

Abstract. A stochastic version of the porous medium equation is studied. The corresponding Kolmogorov equation is solved in a space $L^2(H, \nu)$ where ν is an invariant measure. Then a weak solution, that is a solution in the sense of the corresponding martingale problem, is constructed.

1. Introduction

The porous medium equation

$$\frac{\partial X}{\partial t} = \Delta(X^m), \quad m \in \mathbb{N}, \tag{1.1}$$

on a bounded open set $D \subset \mathbb{R}^d$ has been studied extensively. We refer to [1] for both the mathematical treatment and the physical background and also to [2, Section 4.3] for the general theory of equations of such type.

In this paper we are interested in a stochastic version of (1.1). Throughout this paper we assume

(H1) *m is odd*, $\quad m \geq 3$.

Furthermore, we consider Dirichlet boundary conditions for the Laplacian Δ. So, the stochastic partial differential equation we would like to solve for suitable initial conditions is the following:

$$dX(t) = (\alpha \Delta X(t) + \Delta(X^m(t)))dt + \sqrt{C}\, dW(t), \quad t \geq 0, \tag{1.2}$$

where $\alpha \geq 0$. As in [3], where similar equations were studied (but with $x \to x^m$ replaced by some $\beta : \mathbb{R} \to \mathbb{R}$ of linear growth, satisfying, in particular, $\beta' \geq c > 0$), it turns out that the appropriate state space is $H^{-1}(D)$, i.e. the dual of the Sobolev space $H_0^1 := H_0^1(D)$. Below we shall use the standard $L^2(D)$ dualization $\langle \cdot, \cdot \rangle$ between $H_0^1(D)$ and $H = H^{-1}(D)$ induced by the embeddings

$$H_0^1(D) \subset L^2(D)' = L^2(D) \subset H^{-1}(D) = H$$

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without further notice. Then for $x \in H$

$$|x|_H^2 = \int_D ((-\Delta)^{-1}x)(\xi)\, x(\xi) d\xi$$

and for the dual H' of H we have $H' = H_0^1$. We equip H with its Borel σ-algebra $\mathscr{B}(H)$.

$(W_t)_{t\geq 0}$ is a cylindrical Brownian motion in H and C is a positive definite bounded operator on H of trace class. To be more concrete below we assume:

(H2) *There exists λ_k, $k \in \mathbb{N}$, such that for the eigenbasis $\{e_k | k \in \mathbb{N}\}$ in H of Δ (with Dirichlet boundary conditions) we have*

$$Ce_k = \sqrt{\lambda_k}\, e_k \ \text{ for all } \ k \in \mathbb{N}.$$

(H3) *For $\alpha_k := \sup_{\xi \in D} |e_k(\xi)|^2$, $k \in \mathbb{N}$, we have*

$$K := \sum_{k=1}^{\infty} \alpha_k \lambda_k < +\infty.$$

Our aim is to construct a strong Markov weak solution for (1.2), i.e. a solution in the sense of the corresponding martingale problem (see [13] for the finite dimensional case), at least for a large set $\overline{H}$ of starting points in H which is left invariant by the process, that is with probability one $X_t \in \overline{H}$ for all $t \geq 0$. We follow the strategy first presented in [10] (and already carried out in the more dissipative cases in [5]). That is, first we construct a solution to the corresponding Kolmogorov equations and then a strong Markov process with continuous sample paths having transition probabilities given by that solution to the Kolmogorov equations. As in [5] we also prove that this process is for μ-a.e. starting point $x \in \overline{H}$ the (in distribution) unique continuous Markov process whose transition semigroup consists of continuous operators on $L^2(H, \mu)$ which is e.g. the case if μ is a sub-invariant measure.

Applying Itô's formula (on a heuristic level) to (1.2) one finds what the corresponding Kolmogorov operator, let us call it N_0, should be, namely

$$N_0\varphi(x) = \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k D^2\varphi(e_k, e_k) + D\varphi(x)(\Delta(\alpha x + x^m)), \quad x \in H, \tag{1.3}$$

where $D\varphi$, $D^2\varphi$ denote the first and second Fréchet derivatives of $\varphi : H \to \mathbb{R}$. So, we take $\varphi \in C_b^2(H)$.

In order to make sense of (1.3) one needs that $\Delta(x^m) \in H$ at least for "relevant" $x \in H$. Here one clearly sees the difficulties since x^m is, of course, not defined for any Schwartz distribution in $H = H^{-1}$, not to mention that it will not be in $H_0^1(D)$. So, a way out of this is to think about "relevant" $x \in H$. Our approach to this is first to look for an invariant measure for the solution to equation (1.2) which can now be defined "infinitesimally" (cf. [4]) without having a solution to (1.2) as the solution to the equation

$$N_0^*\mu = 0 \tag{1.4}$$

with the property that μ is supported by those $x \in H$ for which x^m makes sense and $\Delta(x^m) \in H$. 1.4 is a short form for

$$N_0\varphi \in L^1(H,\mu) \text{ and } \int_H N_0\varphi d\mu = 0 \text{ for all } \varphi \in C_b^2(H). \tag{1.5}$$

Any invariant measure for any solution of (1.2) in the classical sense will satisfy (1.4). Then we can analyze N_0, with domain $C_b^2(H)$ in $L^2(H,\mu)$, i.e. solve the Kolmogorov equation

$$\frac{dv}{dt} = \overline{N_0}v \tag{1.6}$$

for the closure $\overline{N_0}$ of N_0 on $L^2(H,\mu)$. This means, we have to prove that $\overline{N_0}$ generates a C_0-semigroup $T_t = e^{t\overline{N_0}}$ on $L^2(H,\mu)$. Subsequently, we have to show that $(T_t)_{t\geq 0}$ is given by a semigroup of probability kernels $(p_t)_{t\geq 0}$ (i.e. $p_t f$ is a μ-version of $T_t f \in L^2(H,\mu)$ for all $t \geq 0$, $f: H \to \mathbb{R}$, bounded, measurable) and such that there exists a strong Markov process with continuous sample paths in H whose transition function is $(p_t)_{t\geq 0}$. By definition this Markov process then will solve the martingale problem corresponding to (1.2).

The organization of this paper is as follows. In §2 we construct a solution μ to (1.4) and prove the necessary support properties of μ, more precisely, that for all $M \in \mathbb{N}$, $M \geq 2$

$$\mu(\{x \in L^2(D) | x^M \in H_0^1\}) = 1,$$

so that N_0 in (1.3) is μ-a.e. well defined for all $\varphi \in C_b^2(H)$. In §3 we prove that N_0, which is automatically closable in $L^2(H,\mu)$, is essentially maximal dissipative in $L^2(H,\mu)$, i.e. its closure $N := \overline{N_0}$ generates a C_0-semigroup in $L^2(H,\mu)$. In both §2 and §3 we rely on results in [3] in essential way, which we apply to suitable approximations, i.e. the function $x \mapsto x^m$ is replaced by

$$\beta_\varepsilon(x) := \frac{x^m}{1+\varepsilon x^{m-1}} + (\alpha+\varepsilon)x, \quad \varepsilon \in (0,1]$$

to which the results in [3] apply.

In §4 we construct the semigroup $(p_t)_{t\geq 0}$ of probability kernels and the corresponding Markov process. The technique to this is to prove that the capacity determined by N (defined in §2.1 below) is tight. So, since $C_b^2(H)$ is a core of N which is an algebra, a general result from [12] implies the existence of $(p_t)_{t\geq 0}$ and the Markov process.

In the recent paper [6] we already constructed solutions to (1.4) in the case $\alpha = 0$ and $m = 3$. In this paper we extend this result to $\alpha \in (0,+\infty)$ and arbitrary odd $m \in \mathbb{N}$. We emphasize that all further steps described above we can only perform if $\alpha > 0$. For the convenience of the reader we include the case $\alpha = 0$ in §2, thus recalling all relevant results from [6]. Starting from §3, however, we need $\alpha > 0$. We shall point out in detail why this is needed in the proof of Theorem 3.2.

2. Existence of an infinitesimal invariant measure

Throghout this section (H1)-(H3) are still in force. So, we first consider the following approximations for the Kolmogorov operator N_0. For $\varepsilon \in (0, 1]$ we define for $\varphi \in C_b^2(H)$, $x \in L^2(D)$ such that $\beta_\varepsilon(x) \in H_0^1$

$$N_\varepsilon\varphi(x) := \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k D^2\varphi(x)(e_k, e_k) + D\varphi(x)(\Delta\beta_\varepsilon(x)), \tag{2.1}$$

where

$$\beta_\varepsilon(r) := \frac{r^m}{1+\varepsilon r^{m-1}} + (\alpha+\varepsilon)r, \quad r \in \mathbb{R}. \tag{2.2}$$

We note that β_ε is Lipschitz continuous and recall the following result from [3] which is crucial for our further analysis, see [3, Theorems (3.1), (3.9), Remark 3.1]. To avoid confusion and for the reader's convenience we note that for $\varphi \in C_b^2(H)$, $x \in H$, what is denoted $D\varphi(x)$ in [3] is the image in H of our $D\varphi(x) \in H_0^1$ via the embedding $H_0^1 \subset L^2(D) \subset H$, i.e. corresponds to $\Delta(D\varphi(x))$.

THEOREM 2.1. *Let $\varepsilon \in (0, 1]$. Then there exists a probability measure μ_ε on H such that*

$$\mu_\varepsilon(H_0^1) = 1, \tag{2.3}$$

$$\int_H |x|_{H_0^1}^2 \,\mu_\varepsilon(dx) < +\infty, \tag{2.4}$$

$$\int_H |\beta_\varepsilon|_{H_0^1}^2 \,d\mu_\varepsilon = \int_H |\Delta\beta_\varepsilon|_H^2 \,d\mu_\varepsilon < +\infty \tag{2.5}$$

and

$$\int_H N_\varepsilon\varphi d\mu_\varepsilon = 0 \text{ for all } \varphi \in C_b^2(H). \tag{2.6}$$

REMARK 2.2. (i). In [3] only

$$\mu_\varepsilon(\{x \in L^2(D) | \beta_\varepsilon(x) \in H_0^1\}) = 1$$

was proved. But since $\beta_\varepsilon(0) = 0$, $\beta_\varepsilon(\mathbb{R}) = \mathbb{R}$, and

$$\beta_\varepsilon'(r) = r^{m-1}\ \frac{m + \varepsilon r^{m-1}}{(1 + \varepsilon r^{m-1})^2}\ + \alpha + \varepsilon \geq \alpha + \varepsilon \ \text{ for all }\ r \in \mathbb{R}, \tag{2.7}$$

it follows that the inverse β_ε^{-1} of β_ε is Lipschitz with $\beta_\varepsilon^{-1}(0) = 0$, so $\beta_\varepsilon(x) \in H_0^1$ is equivalent to $x \in H_0^1$ and (2.4) follows from (2.5), since

$$|\nabla x| = |\nabla \beta_\varepsilon^{-1}(\beta_\varepsilon(x))| \leq (\alpha + \varepsilon)^{-1} |\nabla \beta_\varepsilon(x)|.$$

We thank V. Barbu for pointing this out to us.

(ii) By Theorem 2.1 we have that $N_\varepsilon \varphi(x)$ is well defined for μ_ε-a.e. $x \in H$.

For $N \in \mathbb{N}$ we define

$$P_N x = \sum_{k=1}^{N} \langle x, e_k \rangle_k e_k, \quad x \in H.$$

Note that, since $\{e_k | k \in \mathbb{N}\}$ is the eigenbasis of the Laplacian we have that the respective restriction P_N is also an orthogonal projection on $L^2(D)$ and H_0^1 and on both spaces $(P_N)_{N\in\mathbb{N}}$ also converges strongly to the identity.

The following result was proved for $\alpha = 0$ in [6]. The proof for $\alpha \in [0, +\infty)$ is almost the same. To make this paper self-contained we include the proof in this general case.

PROPOSITION 2.3. $\{\mu_\varepsilon,\ \varepsilon \in (0, 1]\}$ *is tight on* H. *For any weak limit point* μ

$$\int_H |x|^2_{L^2(D)} \mu(dx) \leq \int_D (\alpha + 1)\, d\xi + \frac{1}{2} \operatorname{Tr} C.$$

In particular, $\mu(L^2(D)) = 1$.

Proof. For $n \in \mathbb{N}$ let $\chi_n \in C^\infty(\mathbb{R})$, $\chi_n(x) = x$ on $[-n, n]$, $\chi_n(x) = (n + 1)\operatorname{sign} x$, for $x \in \mathbb{R}\backslash[-(n + 2), n + 2]$, $0 \leq \chi_n' \leq 1$ and $\sup_{n\in\mathbb{N}} |\chi_n''| < +\infty$. Define for $n, N \in \mathbb{N}$

$$\varphi_{N,n}(x) := \frac{1}{2}\ \chi_n(|P_N x|_H^2).$$

Then $\varphi_{N,n} \in C_b^2(H)$ and for $x \in H$

$$\begin{aligned} N_\varepsilon \varphi_{N,n}(x) = \frac{1}{2}\ &\sum_{k=1}^{N} \lambda_k [2\chi_n''(|P_N x|_H^2) \langle P_N x, e_k \rangle_H^2 + \chi_n'(|P_N x|_H^2)] \\ &+ \chi_n'(|P_N x|_H^2) \langle P_N x, \Delta \beta_\varepsilon(x) \rangle_H. \end{aligned}$$

Hence integrating with respect to μ_ε, by (2.6) we find

$$\begin{aligned}
&\int_H \chi_n'(|P_N x|_H^2)\langle P_N x, \beta_\varepsilon(x)\rangle_{L^2(D)}\mu_\varepsilon(dx)\\
&\quad= \frac{1}{2}\sum_{k=1}^N \lambda_k \int_H [2\chi_n''(|P_N x|_H^2)\langle P_N x, e_k\rangle_H^2 + \chi_n'(|P_N x|_H^2)]\mu_\varepsilon(dx)\\
&\quad\le \frac{1}{2}\sum_{k=1}^N \lambda_k + \sup_{k\in\mathbb{N}}\lambda_k \int_H |\chi_n''(|P_N x|_H^2)|\ |P_N x|_H^2 \mu_\varepsilon(dx).
\end{aligned}$$

For all $n \in \mathbb{N}$ the integrand in the left hand side is bounded by

$$1_{\{|P_N x|_H^2 \le n+2\}}\ |P_N x|_H\ |\beta_\varepsilon(x)|_{H_0^1},$$

and similar bounds for the integrand in the right hand side hold. Therefore, (2.5) and Lebesgue's dominated convergence theorem allow us to take $N \to \infty$ and obtain

$$\begin{aligned}
&\int_H \chi_n'(|x|_H^2)\langle x, \beta_\varepsilon(x)\rangle_{L^2(D)}\mu_\varepsilon(dx)\\
&\quad\le \frac{1}{2}\sum_{k=1}^\infty \lambda_k + \sup_{k\in\mathbb{N}}\lambda_k \int_H |\chi_n''(|x|_H^2)|\ |x|_H^2\ \mu_\varepsilon(dx).\\
&\quad\le \frac{1}{2}\sum_{k=1}^\infty \lambda_k + \sup_{k\in\mathbb{N}}\lambda_k \int_{\{|x|_H^2 \ge n\}} |x|_H^2\ \mu_\varepsilon(dx).
\end{aligned}$$

Hence taking $n \to \infty$ by (2.4) and using the definition (2.2) of β_ε we arrive at

$$\int_H \int_D \left(\frac{x^{m+1}(\xi)}{1+\varepsilon x^{m-1}(\xi)} + (\alpha+\varepsilon)x^2(\xi)\right) d\xi\, \mu_\varepsilon(dx) \le \frac{1}{2}\,\mathrm{Tr}\, C.$$

Since m is odd and $\varepsilon \in (0, 1]$, this implies

$$\begin{aligned}
\int_H |x|_{L^2(D)}^2 \mu_\varepsilon(dx) &\le \int_H \int_D \left(\alpha + 1 + \frac{x^{m+1}(\xi)}{1+x^{m-1}(\xi)}\right) d\xi\, \mu_\varepsilon(dx)\\
&\le \int_D (\alpha+1)\, d\xi + \frac{1}{2}\,\mathrm{Tr}\, C.
\end{aligned}\tag{2.8}$$

Since $L^2(D) \subset H$ is compact, this implies that $\{\mu_\varepsilon | \varepsilon \in (0, 1]\}$ is tight on H. Since the map $x \to |x|_{L^2(D)}^2$ is lower semicontinuous and nonnegative in H all assertions follow. □

Later we need better support properties of μ. Therefore, our next aim is to prove the following:

THEOREM 2.4. *Let* (H1)-(H3) *hold and assume that either* $\alpha = 0, m = 3$ *or* $\alpha > 0$, $m \ge 3$, *m odd. Then:*

(i) *For all $M \in \mathbb{N}$, $M \geq 2$, there exists a constant $C_M = C_M(D, K) > 0$ such that*

$$\sup_{\varepsilon \in (0,1]} \int_H \int_D x^{2(M-1)}(\xi) |\nabla x(\xi)|^2 d\xi \, \mu_\varepsilon(dx) \leq C_M.$$

If $\alpha > 0$ this also holds for $M = 1$.

(ii) *For all $M \in \mathbb{N}$, $M \geq 2$, and any limit point μ as in Proposition* (2.3)

$$\int_H \int_D |\nabla(x^M)(\xi)|^2 d\xi \, \mu(dx) \leq C_M.$$

In particular, setting

$$H^1_{0,M} := \{x \in L^2(D) | x^M \in H^1_0\}$$

we have

$$\mu(H^1_{0,M}) = 1 \quad \text{for all } M \geq 2.$$

If $\alpha > 0$, this also holds for $M = 1$.

In order to prove Theorem 2.4 we need some preparation, i.e. more precise information about the μ_ε, $\varepsilon \in (0, 1]$. This can be deduced from (2.6), i.e. from the fact that μ_ε is an infinitesimally invariant measure for N_ε. So, we fix $\varepsilon \in (0, 1]$ and for the rest of this section we assume that (H1)-(H3) hold.

We need to apply (2.6) with φ replaced by $\varphi_M : L^{2M}(D) \to [0, +\infty)$, $M \in \mathbb{N}$, given by

$$\varphi_M(x) = \int_D x^{2M}(\xi) \, d\xi, \quad x \in L^{2M}(D).$$

Clearly, such functions are not in $C^2_b(H)$ so we have to construct proper approximations. So, define for $\delta \in (0, 1]$

$$f_{M,\delta}(r) := \frac{r^{2M}}{1 + \delta r^2}, \quad r \in \mathbb{R}. \tag{2.9}$$

Then for $r \in \mathbb{R}$

$$f'_{M,\delta}(r) = (1 + \delta r^2)^{-2}[2Mr^{2M-1} + 2\delta(M - 1)r^{2M+1}] \tag{2.10}$$

and

$$\begin{aligned} f''_{M,\delta}(r) = {} & 2(1 + \delta r^2)^{-3}[M(2M - 1)r^{2M-2} + \delta(4M^2 - 6M - 1)r^{2M} \\ & + \delta^2(M - 1)(2M - 3)r^{2M+2}]. \end{aligned} \tag{2.11}$$

We have chosen this approximation since below (cf. Lemma 2.7) it will be crucial that $f''_{M,\delta}$ is nonnegative if $M \geq 2$. More precisely we have

$$\begin{aligned}
&0 \leq f_{M,\delta}(r) \leq \frac{1}{\delta}\,|r|^{2M-2}\\
&0 \leq f'_{M,\delta}(r) \leq \frac{2M}{\delta}\,|r|^{2M-3}\\
&0 \leq f''_{M,\delta}(r) \leq 16M^2\,|r|^{2M-4}\inf\{r^2, 1/\delta\}.
\end{aligned} \tag{2.12}$$

REMARK 2.5. The following will be used below: if $x \in H_0^1$ is such that for $M \in \mathbb{N}$

$$\int_H x^{2(M-1)}(\xi)|\nabla x(\xi)|^2 d\xi < \infty, \tag{2.13}$$

then $x^M \in H_0^1$ and $x^{M-1}\nabla x = \frac{1}{M}\nabla x^M$, or using the notation introduced in Theorem 2.4.-(ii) equivalently $x \in H^1_{0,M}$. The proof is standard by approximation. So, we omit it. We also note that by Poincaré's inequality, $H^1_{0,M} \subset L^{2M}(D)$. More precisely, there exists $C(D) \in (0,\infty)$ such that

$$C(D)\int_D x^{2M}(\xi)d\xi \leq \int_D |\nabla x^M(\xi)|^2 d\xi = M^2\int_D x^{2(M-1)}(\xi)|\nabla x(\xi)|^2 d\xi, \tag{2.14}$$

for all x as above.

The following lemma is a consequence of (2.6) and crucial for our analysis of $\{\mu_\varepsilon,\ \varepsilon \in (0,1]\}$ and their limit points. For $\alpha = 0, m = 3$ its proof can be found in [6]. We include the general case here for the reader's convenience.

LEMMA 2.6. *Let $M \in \mathbb{N}$, $\delta \in (0,1]$. Assume that*

$$\int_H\int_D x^{2(M-2)}(\xi)|\nabla x(\xi)|^2 d\xi\,\mu_\varepsilon(dx) < \infty \quad \text{if } M \geq 3. \tag{2.15}$$

Then

$$\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{\infty}\lambda_k\int_H\int_D f''_{M,\delta}(x(\xi))e_k^2(\xi)d\xi\,\mu_\varepsilon(dx)\\
&\quad = \int_H\int_D f''_{M,\delta}(x(\xi))\beta'_\varepsilon(x(\xi))|\nabla x(\xi)|^2 d\xi\,\mu_\varepsilon(dx).
\end{aligned} \tag{2.16}$$

Proof. We first note that (2.15) holds for $M = 2$ by (2.3). For $\kappa \in (0,1]$ we define

$$f_{M,\delta,\kappa}(r) := f_{M,\delta}(r)e^{-\frac{1}{2}\kappa r^2}, \quad r \in \mathbb{R} \quad \text{if } M \geq 2$$

and $f_{1,\delta,\kappa} = f_{1,\delta}$. Then (2.10) and (2.11) imply that $f_{M,\delta,\kappa} \in C_b^2(\mathbb{R})$. Define

$$\varphi_{M,\delta,\kappa}(x) := \int_D f_{M,\delta,\kappa}(x(\xi))d\xi, \quad x \in L^2(D).$$

Then it is easy to check that $\varphi_{M,\delta,\kappa}$ is twice Gateaux differentiable on $L^2(D)$ and that for all $y, z \in L^2(D)$

$$\varphi'_{M,\delta,\kappa}(x)(y) = \int_D f'_{M,\delta,\kappa}(x(\xi))y(\xi)d\xi, \tag{2.17}$$

$$\varphi''_{M,\delta,\kappa}(x)(y,z) = \int_D f''_{M,\delta,\kappa}(x(\xi))y(\xi)z(\xi)d\xi. \tag{2.18}$$

Hence

$$\varphi_{M,\delta,\kappa} \circ P_N \in C_b^2(H)$$

and for all $x \in H_0^1$ (hence $\beta_\varepsilon(x) \in H_0^1$),

$$\begin{aligned} N_\varepsilon(\varphi_{M,\delta,\kappa} \circ P_N)(x) = \frac{1}{2} \sum_{k=1}^N \lambda_k \int_D f''_{M,\delta,\kappa}(P_N x(\xi)) e_k^2(\xi)d\xi \\ + \int_D f'_{M,\delta,\kappa}(P_N x(\xi)) P_N(\Delta\beta_\varepsilon(x))(\xi)d\xi. \end{aligned}$$

Since $P_N\Delta = \Delta P_N$, integrating by parts we obtain

$$\begin{aligned} N_\varepsilon(\varphi_{M,\delta,\kappa} \circ P_N)(x) = \frac{1}{2} \sum_{k=1}^N \lambda_k \int_D f''_{M,\delta,\kappa}(P_N x(\xi)) e_k^2(\xi)d\xi \\ - \int_D f''_{M,\delta,\kappa}(P_N x(\xi)) \langle \nabla(P_N x)(\xi), \nabla(P_N\beta_\varepsilon(x))(\xi)\rangle_{\mathbb{R}^d} d\xi. \end{aligned}$$

Since $(P_N)_{N\in\mathbb{N}}$ strongly converges to the identity in H_0^1, we conclude by (H3) that

$$\begin{aligned} \lim_{N\to\infty} N_\varepsilon(\varphi_{M,\delta,\kappa} \circ P_N)(x) = \frac{1}{2} \sum_{k=1}^\infty \lambda_k \int_D f''_{M,\delta,\kappa}(x(\xi)) e_k^2(\xi)d\xi \\ - \int_D f''_{M,\delta,\kappa}(x(\xi)) \beta'_\varepsilon(x)(\xi) |\nabla x(\xi)|^2 d\xi. \end{aligned}$$

Since β_ε is Lipschitz, by (2.3)–(2.5) and (H3) this convergence also holds in $L^1(H, \mu_\varepsilon)$. Hence (2.6) implies that

$$\begin{aligned} &\frac{1}{2} \sum_{k=1}^\infty \lambda_k \int_H \int_D f''_{M,\delta,\kappa}(x(\xi)) e_k^2(\xi)d\xi \mu_\varepsilon(dx) \\ &\quad = \int_H \int_D f''_{M,\delta,\kappa}(x(\xi)) \beta'_\varepsilon(x)(\xi) |\nabla x(\xi)|^2 d\xi \mu_\varepsilon(dx). \end{aligned} \tag{2.19}$$

So, for $M = 1$ the assertion is proved. If $M \geq 2$, an elementary calculation shows that by (2.12) there exists a constant $C(M, \delta) > 0$ (only depending on M and δ) such that

$$|f''_{M,\delta,\kappa}(x)| \leq C(M, \delta) r^{2(M-2)}, \quad r \in \mathbb{R}. \tag{2.20}$$

Hence by (H3), Remark 2.5 and assumption (2.15) we can apply Lebesgue's dominated convergence theorem to (2.19) and letting $\kappa \to 0$ we obtain the assertion. □

LEMMA 2.7. *Let $M \in \mathbb{N}$ and assume that* (2.15) *holds if* $M \geq 3$.

(i) *We have*

$$\begin{aligned} &\frac{K}{2} \int_H \int_D x^{2(M-1)}(\xi) d\xi \mu_\varepsilon(dx) \\ &\quad \geq \int_H \int_D x^{2(M-1)}(\xi) \left(\frac{x^{m-1}(\xi)}{1 + x^{m-1}(\xi)} + \alpha + \varepsilon \right) |\nabla x(\xi)|^2 d\xi \mu_\varepsilon(dx). \end{aligned} \tag{2.21}$$

(ii) *If $\alpha = 0$ and $m = 3$ then for $M \geq 2$*

$$\begin{aligned} &\frac{K}{2} \int_H \int_D \left(x^{2(M-1)}(\xi) + x^{2(M-2)}(\xi) \right) d\xi \mu_\varepsilon(dx) \\ &\quad \geq \int_H \int_D x^{2(M-1)}(\xi)\ |\nabla x(\xi)|^2 d\xi \mu_\varepsilon(dx) \\ &\quad = \frac{1}{M^2} \int_H \int_D |\nabla x^M(\xi)|^2 d\xi \mu_\varepsilon(dx), \end{aligned}$$

and

$$\int_H \int_D |\nabla x(\xi)|^2 d\xi \mu_\varepsilon(dx) \leq \frac{K}{2\varepsilon}. \tag{2.22}$$

(iii) *If $\alpha > 0$, then*

$$\begin{aligned} &\frac{K}{2} \int_H \int_D x^{2(M-1)}(\xi) d\xi \mu_\varepsilon(dx) \\ &\quad \geq \alpha \int_H \int_D x^{2(M-1)}(\xi)\ |\nabla x(\xi)|^2 d\xi \mu_\varepsilon(dx). \end{aligned} \tag{2.23}$$

Proof. (i) By (H3) the left hand side of (2.16) is dominated by

$$\frac{K}{2} \int_H \int_D f''_{M,\delta}(x(\xi)) d\xi \mu_\varepsilon(dx).$$

If $M \geq 2$, by assumption (2.15) and Remark 2.5 we know that

$$\int_H \int_D x^{2(M-1)}(\xi) d\xi \mu_\varepsilon(dx) < \infty$$

which trivially also holds for $M = 1$. So, by (2.11), (2.12) and Lebesgue's dominated convergence theorem we obtain that for $M \geq 2$

$$\begin{aligned}\frac{K}{2} &\int_H \int_D 2M(2M-1)x^{2(M-1)}(\xi)d\xi\mu_\varepsilon(dx)\\ &\geq \liminf_{\delta\to 0} \int_H \int_D f''_{M,\delta}(x(\xi))\beta'_\varepsilon(x(\xi))|\nabla x(\xi)|^2 d\xi\mu_\varepsilon(dx).\end{aligned}$$

Since $f''_{M,\delta} \geq 0$ for $M \geq 2$ and

$$\beta'_\varepsilon(r) \geq \frac{r^{m-1}}{1+r^{m-1}} + \alpha + \varepsilon \geq 0 \quad \text{for all} \quad r \in \mathbb{R},$$

we can apply Fatou's lemma to prove the assertion. If $M = 1$ we conclude in the same way by (2.3) and Lebesgue's dominated convergence theorem which applies since β'_ε is bounded and $|f''_{1,\delta}| \leq 3/2$ (as follows from (2.12)) for all $\delta \in (0, 1]$.

(ii) See [6, Lemma 2.7-(ii) and (iii)].

(iii) Since $m - 1$ is even, the assertion follows by (i).

□

By an induction argument we shall now prove that the integrals in (2.22) are all finite and at the same time prove the bounds claimed in Theorem 2.4.

Proof of Theorem 2.4. For the case $\alpha = 0, m = 3$ we refer to [6]. We only give the proof for $\alpha > 0, m \geq 3$. If $M = 1$ then the assertion holds by Lemma 2.7-(iii). Furthermore, by Remark (2.5)

$$\begin{aligned}\int_H \int_D x^{2(M-1)}(\xi)|\nabla(x(\xi))|^2 d\xi\mu_\varepsilon(dx) &= \frac{1}{M^2}\int_H \int_D |\nabla(x^M(\xi))|^2 d\xi\mu_\varepsilon(dx)\\ &\geq \frac{C(D)^2}{M^2}\int_H \int_D x^{2M}(\xi)d\xi\mu_\varepsilon(dx). \qquad (2.24)\end{aligned}$$

Now assertion (i) follows from Lemma 2.7-(iii) by induction.

To prove (ii) we start with the following

CLAIM. *For all* $M \in \mathbb{N}$

$$\Theta_M(x) = 1_{H^1_{0,M}}(x)\int_D |\nabla x^M(\xi)|^2 d\xi + \infty \cdot 1_{H\setminus H^1_{0,M}}(x), \quad x \in H \qquad (2.25)$$

is a lower semi-continuous function on H.

Since μ is a weak limit point of $\{\mu_\varepsilon |\ \varepsilon \in (0, 1]\}$ and $\Theta_M \geq 0$, the claim immediately implies assertion (ii).

To prove the claim let $\alpha > 0$ and $x_n \in \{\Theta_M \leq \alpha\}$, $n \in \mathbb{N}$ such that $x_n \to x$ in H as $n \to \infty$. By Poincaré's inequality $\{x_n | n \in \mathbb{N}\}$ is a bounded set in $L^{2M}(D)$. So $x_n \to x$ in H as $n \to \infty$ also weakly in $L^2(D)$, in particular $x \in L^2(D)$. Since $\{x_n^M | n \in \mathbb{N}\}$ is bounded in H_0^1, there exists a subsequence $(x_{n_k}^M)_{k \in \mathbb{N}}$ and $y \in H_0^1$ such that $x_{n_k}^M \to y$ in H as $k \to \infty$ weakly in H_0^1 and

$$\int_D |\nabla y(\xi)|^2 d\xi \leq \alpha.$$

Since the embedding $H_0^1 \subset L^2(D)$ is compact, $x_{n_k}^M \to y$ in H as $k \to \infty$ in $L^2(D)$. Selecting another subsequence if necessary, this convergence is $d\xi$-a.e., hence

$$x_{n_k} \to y^{\frac{1}{M}} \quad d\xi\text{-a.e.}$$

Since (selecting another subsequence if necessary) we also know that the Cesaro mean of $(x_{n_k})_{k \in \mathbb{N}}$ has a subsequence which converges $d\xi$-a.e. to x, hence $x^M = y$, so $x \in \{\Theta_M \leq \alpha\}$.

As a consequence from the previous proof we obtain:

COROLLARY 2.8. *Let $M \in \mathbb{N}$. Then Θ_M has compact level sets in H.*

Proof. We already know from the previous proof that Θ_M is lower semicontinuous. The relative compactness of their level sets is, however, clear by Poincaré's inequality since $L^{2M}(D) \subset H$ is compact. □

Since for $M \in \mathbb{N}$ and $x \in H_{0,M}^1$

$$|\Delta x^M|_H = \int_D |\nabla x^M(\xi)|^2 d\xi, \tag{2.26}$$

so $\Delta x^M \in H$, we can define the Kolmogorov operator in (1.3) rigorously for $x \in H_0^1 \cap H_{0,m}^1$. So, for $\varphi \in C_b^2(H)$, $\alpha \in [0, \infty)$

$$N_0\varphi(x) := \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k D^2\varphi(x)(e_k, e_k) + D\varphi(x)(\Delta(\alpha x + x^m)), \tag{2.27}$$

where we assume $m = 3$ if $\alpha = 0$. We note that by Theorem 2.4.-(ii) and (2.26), $N_0\varphi \in L^2(H, \mu)$ for any weak limit point μ of $\{\mu_\varepsilon | \varepsilon \in (0, 1]\}$ on H. Now we can prove our main result, namely that any such μ is an infinitesimally invariant measure for N_0 in the sense of [4], i.e. satisfies (1.4).

THEOREM 2.9. *Assume that $(H1)$-$(H3)$ hold and that either $\alpha = 0$, $m = 3$ or $\alpha > 0$, $m \geq 3$, m odd. Let μ as in Proposition 2.3. Then*

$$\int_H N_0\varphi d\mu = 0 \quad \text{for all} \quad \varphi \in C_b^2(H).$$

Proof. For $\alpha = 0, m = 3$ the assertion was proved in [6]. So, we only prove the case $\alpha > 0, m \geq 3, m$ odd. Let $\varphi \in C_b^2(H)$. For $N \in \mathbb{N}$ define $\varphi_N := \varphi \circ P_N$. Then for $x \in H^1_{0,M}$

$$\begin{aligned} N_0\varphi_N(x) &= \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k D^2\varphi(P_Nx)(e_k, P_Ne_k) + D\varphi_N(x)(\Delta(\alpha x + x^m)) \\ &= \frac{1}{2}\sum_{k=1}^{N}\lambda_k D^2\varphi(P_Nx)(e_k, e_k) + D\varphi(P_Nx)(P_N(\Delta(\alpha x + x^m))). \end{aligned}$$

If we can prove that

$$\int_H N_0\varphi_N d\mu = 0 \quad \text{for all} \quad N \in \mathbb{N}, \tag{2.28}$$

the same is true for φ by Lebesgue's dominated convergence theorem. So, fix $N \in \mathbb{N}$. Then by (2.6)

$$\begin{aligned} \int_H N_0\varphi_N d\mu &= \lim_{\varepsilon\to 0}\int_H \frac{1}{2}\sum_{k=1}^{N}\lambda_k D^2\varphi_N(x)(e_k, e_k)\mu_\varepsilon(dx) \\ &\quad + \int_H D\varphi_N(x)(\Delta(\alpha x + x^m))\mu(dx) \\ &= -\lim_{\varepsilon\to 0}\int_H D\varphi_N(x)(\Delta\beta_\varepsilon(x))\mu_\varepsilon(dx) \\ &\quad + \int_H D\varphi(P_Nx)(P_N(\Delta(\alpha x + x^m)))\mu(dx) \\ &= \lim_{\varepsilon\to 0}\sum_{i=1}^{N}\int_H [D\varphi(P_Nx)(e_i)\langle e_i, \Delta(\alpha x + x^m)\rangle_H\mu(dx) \\ &\quad - D\varphi(P_Nx)(e_i)\langle e_i, \Delta\beta_\varepsilon(x)\rangle_H\mu_\varepsilon(dx)]. \end{aligned} \tag{2.29}$$

For $i \in \{1, \ldots, N\}$ fixed we have

$$\begin{aligned} &\Big|\int_H D\varphi(P_Nx)(e_i)\langle e_i, \Delta(\alpha x + x^m)\rangle_H\mu(dx) \\ &\qquad - \int_H D\varphi(P_Nx)(e_i)\langle e_i, \Delta\beta_\varepsilon(x)\rangle_H\mu_\varepsilon(dx)\Big| \\ &\leq \Big|\int_H D\varphi(P_Nx)(e_i)\langle e_i, \Delta(\alpha x + x^m)\rangle_H(\mu - \mu_\varepsilon)(dx)\Big| \\ &\quad + \Big|\int_H D\varphi(P_Nx)(e_i)\langle e_i, \Delta(\alpha x + x^m - \beta_\varepsilon(x))\rangle_H\mu_\varepsilon(dx)\Big|. \end{aligned} \tag{2.30}$$

The right hand side's second summand is bounded by

$$|e_i|_{L^2(D)} \sup_{x\in H} |D\varphi(x)|_{H_0^1} \int_H \left(\int_D |\alpha x(\xi) + x^m(\xi) - \beta_\varepsilon(x(\xi))|^2\, d\xi\right)^{1/2} \mu_\varepsilon(dx). \quad (2.31)$$

We have

$$|\alpha r + r^m - \beta_\varepsilon(r)| = \left|\frac{\varepsilon r^{2m-1}}{1+\varepsilon r^{m-1}} - \varepsilon r\right| \le \varepsilon(|r|^{2m-1} + |r|), \quad r \in \mathbb{R}.$$

So, the term in (2.31) is dominated by

$$\varepsilon |e_i|_{L^2(D)} \sup_{x\in H} |D\varphi(x)|_{H_0^1} \int_H (||x|^{2m-1}|_{L^2(D)} + |x|_{L^2(D)})\mu_\varepsilon(dx),$$

which by Theorem 2.4-(i) and Remark 2.5 converges to 0 as $\varepsilon \to 0$.

Now we estimate the first summand in the right hand side of (2.30). So, we define

$$f(x) := D\varphi(P_N x)(e_i)\langle e_i, \Delta(\alpha x + x^m)\rangle_H.$$

Then since $\langle e_i, \Delta(\alpha x + x^m)\rangle_H = \langle e_i, \alpha x + x^m\rangle_{L^2(D)}$, it follows by the proof of the lower semicontinuity of Θ_m that f is continuous on the level sets of Θ_m (with Θ_m defined as in (2.25)). Furthermore, since

$$|f(x)| \le \sup_{x\in H} |D\varphi(x)|_{H_0^1}\, |\alpha x + x^m|_{L^2(D)},$$

it follows that

$$\lim_{R\to\infty} \sup_{\Theta_m \ge R} \frac{|f(x)|}{1+\Theta_m(x)} = 0.$$

Furthermore, by Corollary 2.8 the function $1 + \Theta_m$ has compact level sets. Hence by [11, Lemma 2.2], there exists $f_n \in C_b(H)$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \sup_{x\in H} \frac{|f(x) - f_n(x)|}{1+\Theta_m(x)} = 0. \quad (2.32)$$

But

$$\left|\int_H D\varphi(P_N x)(e_i)\langle e_i, \Delta(\alpha x + x^m)\rangle_H (\mu - \mu_\varepsilon)(dx)\right| \le \int_H |f(x) - f_n(x)|(\mu + \mu_\varepsilon)(dx) + \left|\int_H f_n(x)(\mu - \mu_\varepsilon)(dx)\right|.$$

For fixed n the second summand tends to 0 as $\varepsilon \to 0$ and the first one is dominated by

$$\sup_{x\in H} \frac{|f(x) - f_n(x)|}{1+\Theta_m(x)} \sup_{\varepsilon>0} \int_H (1+\Theta_m)d(\mu + \mu_\varepsilon),$$

which in turn by Theorem 2.4 and (2.32) tends to zero as $n \to \infty$. So, also the first summand in (2.30) tends to 0 as $\varepsilon \to 0$. Hence the right hand side of (2.29) is zero and (2.28) follows which completes the proof. □

3. Essential m-dissipativity of N_0

In this section we assume that $\alpha > 0$ and $m \geq 3$ is odd. We still assume (H1)-(H3) to hold. Let μ be a weak limit point of $\{\mu_\varepsilon | \varepsilon \in (0, 1]\}$ (cf. Proposition 2.3).

We already know that $N_0\varphi \in L^2(H, \mu)$ for all $\varphi \in C_b^2(H)$. We would like to consider $(N_0, C_b^2(H))$ as an operator on $L^2(H, \mu)$. For this we need to check that N_0 respects μ-classes.

LEMMA 3.1. *Let $\varphi \in C_b^2(H)$ such that $\varphi = 0$ μ-a.e.. Then $N_0\varphi = 0$ μ-a.e..*

Before we prove this lemma, we emphasize that we do not know whether $\mu(U) > 0$ for any non-empty open set $U \subset H$, so two functions in $C_b^2(H)$ may be not identically equal if they are equal μ_ε-a.e. So, Lemma 3.1. is really essential. Its proof is due to Z. Sobol. Then we have for all $\varphi, \psi \in C_b^2(H)$, $x \in H_0^1 \cap H_{0,m}^1$

$$N_0(\varphi\psi)(x) = \varphi(x)\ N_0\psi(x) + \psi(x)\ N_0\varphi(x) + \sum_{k=1}^{\infty} \lambda_k D\varphi(x)(e_k) D\psi(x)(e_k). \tag{3.1}$$

Proof of Lemma 3.1. Since $\mu(H_0^1 \cap H_{0,m}^1) = 1$, by (3.1) applied with $\psi = \varphi$ it follows by Theorem 2.9 that

$$\sum_{k=1}^{\infty} \lambda_k (D\varphi(x)(e_k))^2 = 0 \quad \mu\text{-a.e..}$$

Hence for all $\psi \in C_b^2(H)$ again by (3.1) and Theorem 2.9

$$\int_H \psi\ N_0\varphi\ d\mu = 0,$$

since $\varphi = 0$ μ-a.e.. But $C_b^2(H)$ is dense in $L^2(H, \mu)$, so $N_0\varphi = 0$ μ-a.e.

So, we can consider $(N_0, \widetilde{C_b^2(H)})$ as an operator on $L^2(H, \mu)$ where $(N_0, \widetilde{C_b^2(H)})$ denotes the μ-classes determined by $C_b^2(H)$. For notational convenience we shall also write $C_b^2(H)$ for the set of these classes if there is no confusion possible. It is well known and easy to see that (3.1) implies that $(N_0, C_b^2(H))$ is dissipative, so in particular closable, on $L^2(H, \mu)$. Let $(N_2, D(N_2))$ denote its closure.

THEOREM 3.2. *Assume that* (H1)-(H3) *hold and that $\alpha > 0, m \geq 3$, m odd. Let μ be a weak limit point of $\{\mu_\varepsilon | \varepsilon \in (0, 1]\}$. Then $(N_0, C_b^2(H))$ is essentially m-dissipative* (*i.e. $(N_2, D(N_2))$ is m-dissipative*) *on $L^2(H, \mu)$. Hence $(N_2, D(N_2))$ generates a C_0-semigroup $(e^{tN_2},\ t \geq 0)$ of linear contractions on $L^2(H, \mu)$. Furthermore, $(e^{tN_2})_{t\geq 0}$ is Markovian, i.e. $e^{tN_2}1 = 1$ and $e^{tN_2} f \geq 0$ for all nonnegative $f \in L^2(H, \mu)$ and all $t > 0$.*

Proof. Let $\lambda > 0$. We have to show that

$(\lambda - N_0)C_b^2(H)$ is dense in $L^2(H, \mu)$.

Let $\varepsilon \in (0, 1]$, $f \in C_b^2(H)$. Define for $\delta, \varepsilon \in (0, 1]$, $x \in H$

$$\beta_{\varepsilon,\delta}(x) := (\beta_\varepsilon^{-1} + \delta\Delta)^{-1}x + \delta x$$

It was proved in [3, Proof of Theorem 4.1] that there exists $\varphi_{\varepsilon,\delta} \in C_b^2(H)$ such that for all $x \in H$

$$\lambda\varphi_{\varepsilon,\delta}(x) - \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k D^2\varphi_{\varepsilon,\delta}(x)(e_k, e_k) + D\varphi_{\varepsilon,\delta}(x)(\Delta\beta_{\varepsilon,\delta}(x)) = f(x) \tag{3.2}$$

and

$$|D\varphi_{\varepsilon,\delta}(x)|_{H_0^1} \leq \frac{1}{\lambda}\, \|f\|_{C_b^1(H)} \ \text{ for all } \ x \in H. \tag{3.3}$$

(We note that β in [3] corresponds to our smooth β_ε. Therefore, an additional regularization of β as done in [3] is not necessary). Furthermore, it follows from [3, estimate (4.13) and the estimate preceding (4.12)] that for all $x \in H_0^1$

$$|D\varphi_{\varepsilon,\delta}(x)(\Delta(\beta_\varepsilon(x) - \beta_{\varepsilon,\delta}(x)))| \leq \frac{2}{\lambda}\, \|f\|_{C_b^1(H)}\,(|\Delta(\beta_\varepsilon(x))|_H + |\Delta x|_H) \tag{3.4}$$

and by [3, (4.12)] that for all $x \in H_0^1$

$$\lim_{\varepsilon\to 0} |D\varphi_{\varepsilon,\delta}(x)(\Delta(\beta_\varepsilon(x) - \beta_{\varepsilon,\delta}(x)))| = 0. \tag{3.5}$$

We note that by (3.2) for all $x \in H_0^1 \cap H_{0,m}^1$

$$\begin{aligned}
&\lambda\varphi_{\varepsilon,\delta}(x) - N_0\varphi_{\varepsilon,\delta}(x) = f(x) + D\varphi_{\varepsilon,\delta}(x)(\Delta(\beta_{\varepsilon,\delta}(x) - \alpha x - x^m))\\
&\quad = \ f(x) - D\varphi_{\varepsilon,\delta}(x)(\Delta(\beta_\varepsilon(x) - \beta_{\varepsilon,\delta}(x)))\\
&\qquad - \varepsilon D\varphi_{\varepsilon,\delta}(x)\left(\Delta\left(\frac{x^{2m-1}}{1 + \varepsilon x^{m-1}} \ - x\right)\right).
\end{aligned} \tag{3.6}$$

Here we emphasize that this equality only holds μ-a.e. if $\alpha > 0$, because only in this case we know that in addition to $\mu(H_{0,m}^1) = 1$, we also have that $\mu(H_0^1) = 1$. So, the following only makes sense if $\alpha > 0$. □

CLAIM.

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}(\lambda\varphi_{\varepsilon,\delta} - N_0\varphi_{\varepsilon,\delta}) = f \ \text{ in } \ L^2(H, \mu). \tag{3.7}$$

This will imply the assertion, by the Lumer-Phillips theorem since $C_b^2(H)$ is dense in $L^2(H,\mu)$. To prove (3.7) *in view of* (3.3)–(3.5) *it is enough to show that*

$$\int_H (|\Delta(\beta_\varepsilon(x))|_H^2 + |\Delta x|_H^2)\mu(dx) < \infty \quad \text{for all } \varepsilon \in (0,1] \tag{3.8}$$

and that

$$\int_H \left| \Delta \left(\frac{x^{2m-1}}{1+\varepsilon x^{m-1}} - x \right) \right|_H^2 \mu(dx) < \infty. \tag{3.9}$$

To prove (3.8) *we note that by* (2.7) *that for every $x \in H_0^1$*

$$\begin{aligned} &|\Delta(\beta_\varepsilon(x))|_H^2 = |\nabla \beta_\varepsilon(x)|_{L^2(D)}^2 = |\beta_\varepsilon'(x)|\nabla x||_{L^2(D)}^2 \\ &\leq 2\int_D (m^2 x^{2(m-1)}(\xi) + (\alpha+\varepsilon)^2)|\nabla x(\xi)|^2 d\xi \end{aligned}$$

which is in $L^2(H,\mu)$ by Theorem 2.4-(ii). *Since $|\Delta x|_H^2 = |\nabla x|_{L^2(D)}^2$,* (3.8) *follows.*

To prove (3.9) *note that*

$$\begin{aligned} &\left| \Delta \left(\frac{x^{2m-1}}{1+\varepsilon x^{m-1}} - x \right) \right|_H^2 = \int_D \left| \nabla \left(\frac{x^{2m-1}(\xi)}{1+\varepsilon x^{m-1}(\xi)} - x(\xi) \right) \right|^2 d\xi \\ &= \int_D \left(\frac{(2m-1)x^{2m-2}(\xi) - m\varepsilon x^{3m-3}(\xi)}{(1+\varepsilon x^{m-1}(\xi))^2} - 1 \right)^2 |\nabla x(\xi)|^2 d\xi. \end{aligned}$$

Since for $r \in \mathbb{R}$

$$\frac{(2m-1)r^{2m-2} - m\varepsilon r^{3m-3}}{(1+\varepsilon r^{m-1})^2} \leq \frac{(2m-1)r^{2m-2}}{1+\varepsilon r^{m-1}} \leq (2m-1)r^{2m-2},$$

we obtain that

$$\begin{aligned} &\left| \Delta \left(\frac{x^{2m-1}}{1+\varepsilon x^{m-1}} - x \right) \right|_H^2 \leq 2(2m-1)^2 \int_D x^{4m-4}(\xi)|\nabla x(\xi)|^2 d\xi \\ &\quad + 2\int_D |\nabla x(\xi)|^2 d\xi. \end{aligned}$$

Hence (3.9) *follows by Theorem* 2.4-(ii) (*which as stressed above now also holds for $M = 1$*). *The last assertion follows from* [7 *Appendix* B, *Lemma* 1.9]

4. Existence of weak solutions

In this section we assume that $\alpha > 0$ and $m \geq 3$, m odd. We shall construct weak solutions to the stochastic porous medium equation (1.2) in the sense that they solve the corresponding martingale problem. More precisely, we shall prove the following.

THEOREM 4.1 (Existence). (i) *There exists a conservative strong Markov process* $\mathbb{M} = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in H})$ *on* H *with continuous sample paths and with invariant measure* μ *such that for its transition semigroup* $(p_t)_{t\geq 0}$ *defined by*

$$p_t f(x) := \int_H f(X_t) d\mathbb{P}_x, \ t \geq 0, \ x \in H,$$

$f : H \to \mathbb{R}$, *bounded* $\mathscr{B}(H)$*-measurable, we have that* $p_t f$ *is a* μ*-version of* $e^{tN_2} f, \ t > 0$.

(ii) *There exists* $\overline{H} \in \mathscr{B}(H)$ *such that* $\mu(\overline{H}) = 1$, *for all* $x \in \overline{H}$

$$\mathbb{P}_x[X_t \in \overline{H} \quad \forall\, t \geq 0] = 1,$$

and for all probability measures ν *on* $(H, \mathscr{B}(H))$ *with* $\nu(\overline{H}) = 1$

$$\varphi(X_t) - \int_0^t N_0 \varphi(X_s) ds, \quad t \geq 0,$$

is an $(\mathscr{F}_t)$*-martingale under* $\mathbb{P}_\nu := \int_{\overline{H}} \mathbb{P}_x \nu(dx)$ *for all* $\varphi \in C_b^2(H)$ *and* $\mathbb{P}_\nu \circ X_0^{-1} = \nu$.

THEOREM 4.2 (Uniqueness). *Suppose that*

$$\mathbb{M}' = (\Omega', \mathscr{F}', (\mathscr{F}'_t)_{t\geq 0}, (X'_t)_{t\geq 0}, (\mathbb{P}'_x)_{x\in H})$$

is a continuous Markov process on H *whose transition semigroup* $(p'_t)_{t\geq 0}$ *consists of continuous operators on* $L^2(H, \mu)$ *(which is e.g. the case if* μ *is sub-invariant for* $(p'_t)_{t\geq 0}$*). If* $\mathbb{M}'$ *satisfies assertion* (ii) *of Theorem* 4.1 *for* $\nu := \mu$, *then for* μ*-a.e.* $x \in H$ $p'_t(x, dy) = p_t(x, dy)$ *for all* $t \geq 0$ *(where* p_t *is as in Theorem* 4.1-(i)*), i.e.* $\mathbb{M}'$ *has the same finite dimensional distributions as* $\mathbb{M}$ *for* μ*-a.e. starting point*.

We shall only prove Theorem 4.1-(i). The remaining parts are proved in exactly the same way as Theorem 7.4-(ii), Proposition 8.2 and Theorem 8.3 in [5] with the only exception that because we do not know whether $(p_t)_{t\geq 0}$ is Feller, all statements can only be proved μ-a.e.. So we do not want to repeat them here.

Our proof of Theorem 4.1-(i) is based on the theory of generalized Dirichlet forms developed in [12]. Indeed, by the last part of Theorem 3.2 $(N_2, D(N_2))$ is a Dirichlet operator in the sense of [9], [12]. Hence by [12, Proposition I. 4.6]

$$\mathscr{E}(u, v) := \begin{cases} (u, v)_{L^2(H,\mu)} - (N_2 u, v)_{L^2(H,\mu)}, & u \in D(N_2), \ v \in L^2(H, \mu), \\ (u, v)_{L^2(H,\mu)} - (N_2^* v, u)_{L^2(H,\mu)}, & u \in L^2(H, \mu), \ v \in D(N_2^*), \end{cases}$$

is a generalized Dirichlet form on $L^2(H, \mu)$ in the sense of [12, Definition I. 4.8] with

$$\mathscr{F} := (D(N_2), \|N_2 \cdot\|_{L^2(H,\mu)} + \|\cdot\|_{L^2(H,\mu)})$$

and with coercive part $\mathscr{A}$ identically equal to 0.

We emphasize here that the theory of generalized Dirichlet forms, in contrast to earlier versions (cf. e.g. [8], [9]), does not require any symmetry or sectoriality of the underlying operators. We refer to [12] for a beautiful exposition. As is well known to experts on potential theory on L^2-spaces (and as is clearly presented in [12]) the following two main ingredients are needed.

(a) There exists a core C of $(N_2, D(N_2))$ which is an algebra consisting of functions having (quasi) continuous μ-versions.
(b) The capacity determined by $(N_2, D(N_2))$ is tight.

(a) follows fron the essential m-dissipativity of N_0 on $C_b^2(H)$ proved in the previous section, so we can take $C := C_b^2(H)$. This is exactly why essential m-dissipativity is so important for probability theory, in particular, Markov processes. Before we prove (b) we recall the necessary definitions.

Let

$$G_\lambda^{(2)} := (\lambda - N_2)^{-1}, \quad \lambda > 0,$$

be the resolvent corresponding to N_2. A function $u \in L^2(H, \mu)$ is called 1-*excessive* if $u \geq 0$ and $\lambda G_{1+\lambda} u \leq u$ for all $\lambda > 0$. For an open set $U \subset H$ define

$$e_U := \inf\{u \in L^2(H, \mu) | u \text{ 1-excessive}, \ u \geq 1_U, \ \mu\text{-a.e.}\},$$

(cf. [12, Proposition III 1.7 (ii)]), and the 1-*capacity* of U by

$$\text{Cap } U := \int_H e_U d\mu.$$

(cf. [12, Definition III 2.5 with $\varphi \equiv 1$]). Cap is called *tight* if there exist increasing compact sets K_n, $n \in \mathbb{N}$, such that for $K_n^c := H \setminus K_n$

$$\lim_{n\to\infty} \text{Cap}(K_n^c) = 0.$$

Once we have proved this, i.e. have proved (b), Theorem 4.1-(i) follows from one of the main results of ([12, Theorem IV 2.2]). Indeed, in our situation the requirement in ([12, Theorem IV 2.2]) that quasi-regularity holds is equivalent to (b) and condition $D3$ in ([12, Theorem IV 2.2]) by ([12, Proposition IV 2.1]) follows from (a).

REMARK 4.3. We mention here that in Theorem 4.1 we do not state all facts known about $\mathbb{M}$; e.g. it is also proved in [12, Theorem IV 2.2, see also Definition IV 1.4] that all "μ-a.e." statements can be replaced by "quasi everywhere" (w.r.t. Cap) statements and that

$$x \mapsto \int_0^{+\infty} e^{-\lambda t} p_t f(x) dt$$

is Cap-quasi-continuous. Furthermore, [12, Theorem IV 2.2] only claims that $\mathbb{M}$ has cadlag paths, but a similar proof as that in [9, Chapter V, Sect. 1] gives indeed continuous paths because N_2 is a local operator.

To prove (b) it is enough to find a 1-excessive function $u : H \to \mathbb{R}^+$ with compact level sets

$$K_n := \{u \leq n\}, \quad n \in \mathbb{N},$$

because then $e_{K_n^c} \leq \frac{1}{n}\, u$, hence

$$\text{Cap } (K_n^c) \leq \frac{1}{n} \int_H u d\mu \to 0, \quad \text{as } n \to \infty.$$

So, the proof of Theorem 4.1-(i) is completed by the following proposition, since closed balls in $L^2(D)$ are compact in H.

PROPOSITION 4.4. *There exists $C \in (0, +\infty)$ such that*

$$u(x) := |x|^2_{L^2(D)} + C, \quad x \in H, \tag{4.1}$$

(with $|x|^2_{L^2(D)} := +\infty$ for $x \in H \backslash L^2(D)$) is 1-excessive with respect to the resolvent of N_2.

The idea to prove Proposition 4.4 is to show that for some $C \in (0, +\infty)$ and $\varphi(x) := |x|^2_{L^2(D)}$ we have

$$N_2 \varphi \leq C$$

and take $u := \varphi + C$.

Though we know by Theorem 2.4. and Poincaré's inequality that $\varphi \in L^p(H, \mu)$ for all $p \in [1, +\infty)$, it is not clear whether $\varphi \in D(N_2)$. But below we shall prove the following

LEMMA 4.5. *Let $(N_1, D(N_1))$ be the closure of $(N_0, C_b^2(H))$ on $L^1(H, \mu)$. Then $\varphi \in D(N_1)$ and*

$$N_1 \varphi \leq K \int_D 1 d\xi. \tag{4.2}$$

We note that since $N_0^* \mu = 0$, $(N_0, C_b^2(H))$ is also dissipative on $L^1(H, \mu)$ (cf. e.g. [7, Appendix B, Lemma 1.8]), hence closable. We recall that $(\lambda - N_0)(C_b^2(H))$ is dense in $L^2(H, \mu)$ (by the proof of Theorem 3.2 above), hence also dense in $L^1(H, \mu)$, so analogously $(N_1, D(N_1))$ generates a C_0 semigroup $(e^{tN_1})_{t \geq 0}$ of contractions on $L^1(H, \mu)$ and we can consider the corresponding resolvent

$$G_\lambda^{(1)} := (\lambda - N_1)^{-1}, \quad \lambda > 0,$$

Clearly, $G_\lambda^{(1)} = G_\lambda^{(2)}$ on $(\lambda - N_0)(C_b^2(H))$, hence

$$G_\lambda^{(1)} f = G_\lambda^{(2)} f \quad \text{for all } \lambda > 0,\ f \in L^2(H,\mu).$$

Therefore, for $\lambda > 0$ and $u := \varphi + K\int_D 1 d\xi$

$$\begin{aligned}\lambda G_{1+\lambda}^{(2)} u &= \lambda G_{1+\lambda}^{(1)} u = \lambda G_{1+\lambda}^{(1)} G_1^{(1)}(1-N_1)u\\ &= G_1^{(1)}(1-N_1)u - G_{1+\lambda}^{(1)}(1-N_1)u\\ &= u - G_{1+\lambda}^{(1)}(1-N_1)u \le u,\end{aligned}$$

since $(1-N_1)u \ge 0$ by (4.2). So, u is 1-excessive with respect to the resolvent of N_2. Therefore, to prove Proposition 4.4 we only have to prove Lemma 4.5.

Proof of Lemma 4.5. Define for $\delta \in (0,1)$

$$f_\delta(r) := f_{1,\delta}(r) = \frac{r^2}{1+\delta r^2}, \quad r \in \mathbb{R},$$

$$\varphi_\delta(r) := \int_D f_\delta(x(\xi))d\xi, \quad x \in L^2(D).$$

(cf. (2.9)). Then as in the proof of Lemma 2.6 we see that $\varphi_\delta \circ P_N \in C_b^2(H)$ and for $x \in H_0^1 \cap H_{0,m}^1$

$$\begin{aligned}N_0(\varphi_\delta \circ P_N)(x) = \frac{1}{2}\sum_{k=1}^N \lambda_k \int_D f_\delta''(P_N x(\xi)) e_k^2(\xi) d\xi\\ - \int_D f_\delta''(P_N x(\xi))\langle \nabla(P_N x)(\xi), \nabla(P_N(\alpha x + x^m))(\xi)\rangle_{\mathbb{R}^d} d\xi.\end{aligned}$$

Since $(P_N)_{N\in\mathbb{N}}$ strongly converges to the identity in H_0^1 and since $|f_\delta''| \le 3/2$, we conclude by (H3) that

$$\begin{aligned}\lim_{N\to\infty} N_0(\varphi_\delta \circ P_N)(x) = \frac{1}{2}\sum_{k=1}^\infty \lambda_k \int_D f_\delta''(x(\xi)) e_k^2(\xi) d\xi\\ - \int_D f_\delta''(x(\xi))\langle \nabla x(\xi), \nabla(\alpha x + x^m)(\xi)\rangle_{\mathbb{R}^d} d\xi\end{aligned} \tag{4.3}$$

and by Theorem 2.4 this convergence also holds in $L^1(H,\mu)$. By exactly the same arguments we can take $\delta \downarrow 0$ and conclude that the right hand side of (4.3) converges in $L^1(H,\mu)$ and for all $x \in H_0^1 \cap H_{0,m}^1$ to

$$\sum_{k=1}^\infty \lambda_k \int_D e_k^2(\xi) d\xi - 2(m-1)\int_D x^{m-1}(\xi)|\nabla x(\xi)|_{\mathbb{R}^d}^2 d\xi - \alpha \int_D |\nabla x(\xi)|_{\mathbb{R}^d}^2 d\xi,$$

which for $x \in H^1_0 \cap H^1_{0,m} \cap H^1_{0,\frac{m+1}{2}}$ is equal to

$$\sum_{k=1}^{\infty} \lambda_k \int_D e_k^2(\xi)d\xi - \frac{8(m-1)}{(m+1)^2} \int_D |\nabla x^{\frac{m+1}{2}}(\xi)|^2_{\mathbb{R}^d} d\xi - \alpha \int_D |\nabla x(\xi)|^2_{\mathbb{R}^d} d\xi. \tag{4.4}$$

Since $\varphi_\delta \to \varphi$ as $\delta \downarrow 0$ in $L^1(H, \mu)$, we conclude that $\varphi \in D(N_1)$ and that $N_1\varphi$ equals the expression in (4.4), since $\mu(H^1_0 \cap H^1_{0,m} \cap H^1_{0,\frac{m+1}{2}}) = 1$ by Theorem 2.4. Hence by (H3)

$$N_1\varphi \le K \int_D 1 d\xi.$$

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Giuseppe Da Prato
Scuola Normale Superiore
Piazza dei Cavalieri 7
56126 Pisa
Italy
e-mail:daprato@sns.it

Michael Röckner
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
Germany
e-mail: roeckner@mathematik.uni-bielefeld.de