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The heat equation with generalized Wentzell boundary condition

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Abstract. Let Ω be a bounded subset of $\mathbf{R}^N$, $a \in C^1(\overline{\Omega})$ with $a > 0$ in Ω and A be the operator defined by $Au := \nabla \cdot (a\nabla u)$ with the generalized Wentzell boundary condition

$$Au + \beta\frac{\partial u}{\partial n} + \gamma u = 0 \qquad \text{on} \quad \partial\Omega.$$

If $\partial\Omega$ is in C^2, β and γ are nonnegative functions in $C^1(\partial\Omega)$, with $\beta > 0$, and $\Gamma := \{x \in \partial\Omega : a(x) > 0\} \neq \emptyset$, then we prove the existence of a (C_o) contraction semigroup generated by $\overline{A}$, the closure of A, on a suitable L^p space, $1 \leq p < \infty$ and on $C(\overline{\Omega})$. Moreover, this semigroup is analytic if $1 < p < \infty$.

1. Introduction

Of concern is the heat equation

$$\frac{\partial u}{\partial t} = Au \qquad \text{for} \quad t \geq 0, \quad \text{in} \quad \Omega \tag{1.1}$$

with the generalized Wentzell boundary condition

$$Au + \beta\frac{\partial u}{\partial n} + \gamma u = 0 \qquad \text{on} \quad \partial\Omega, \tag{1.2}$$

where Ω is a bounded subset of $\mathbf{R}^N$ with a sufficiently regular boundary ($\partial\Omega$ in C^2 will do), $Au = \nabla \cdot (a\nabla u)$, $a \in C^1(\overline{\Omega})$, $a > 0$ in Ω, β, γ are nonnegative functions in $C^1(\partial\Omega)$ with β strictly positive and $n(x)$ is the unit outer normal at x. Moreover, we assume that $\Gamma := \{x \in \partial\Omega : a(x) > 0\} \neq \emptyset$. We shall show that the corresponding Cauchy problem is governed by a strongly continuous contraction semigroup on a suitable L^p space, for $1 \leq p < \infty$ and on $C(\overline{\Omega})$; and the semigroup is analytic for $1 < p < \infty$.

In the one dimensional (linear and nonlinear) case in $C[0, 1]$ the problem of generation of a semigroup associated to (1.1), (1.2) was investigated for the first time in [9].

For the Robin boundary condition (i.e. $\beta\frac{\partial u}{\partial n} + \gamma u = 0$ on $\partial\Omega$), the relevant space is $L^p(\Omega, dx)$. Classical results along these lines go back to Agmon, Douglis and Nirenberg [1]; see also e.g. Lunardi [24] and Taira [32] and, more recently, Colombo and Vespri [5] and Daners [7]. We refer to [4] and [25] and to the references contained therein for the motivations and the study of general Wentzell boundary conditions in the theory of partial differential equations.

The surprising aspect of our results and proofs is that the natural L^p space is $L^p(\overline{\Omega}, d\mu)$, where

$$d\mu := dx\Big|_{\Omega} \oplus \frac{a\,dS}{\beta}\Big|_{\Gamma}, \tag{1.3}$$

dx denotes the Lebesgue measure on Ω and $\frac{a\,dS}{\beta}$ denotes the natural surface measure dS on $\Gamma := \{x \in \partial\Omega : a(x) > 0\}$ with weight $\frac{a}{\beta}$, and we assume $\Gamma \neq \emptyset$. It is easy to see that $L^p(\overline{\Omega}, d\mu)$ may be identified with

$$L^p(\Omega, dx) \times L^p\left(\Gamma, \frac{a\,dS}{\beta}\right), \quad 1 \leq p < \infty. \tag{1.4}$$

The identification is an isometric isomorphism if the product in (1.4) is given the "l^p-sum norm".

If we plug (1.1) into (1.2), the boundary condition becomes

$$\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial n} + \gamma u = 0.$$

Hence the term Au corresponds to introduce a dynamic condition on the boundary. At the resolvent level,

$$\lambda u - Au = h \quad \text{in} \quad \Omega,$$

$$Au + \beta\frac{\partial u}{\partial n} + \gamma u = 0 \quad \text{on} \quad \partial\Omega$$

becomes

$$\lambda u - Au = h \quad \text{in} \quad \Omega,$$

$$\beta\frac{\partial u}{\partial n} + (\gamma + \lambda)u = h \quad \text{on} \quad \partial\Omega,$$

which is an inhomogeneous elliptic equation with an inhomogeneous boundary condition. There is a large literature on solving such problems. But the semigroup approach requires a solution with appropriate estimates. This is where the space defined by (1.4) enters the picture.

We could replace A by a more general elliptic differential operator of order $2m$, in divergence form with corresponding suitable boundary conditions. This would introduce additional complications. Hence we keep the elliptic operator simple in order to concentrate on the effect of the nontrivial boundary condition (1.2).

Various authors have used spaces of the form $L^p(\Omega, d\mu_1) \times L^p(\partial\Omega, d\mu_2)$ in the study of elliptic or parabolic boundary value problems with nonhomogeneous boundary conditions.

The present paper is the first one, to our knowledge, to use spaces incorporating specifically function spaces on the boundary (i.e. the spaces X_p we shall introduce in Section 2, rather than $L^p(\Omega)$ or $W^{1,p}(\Omega)$) for problems with *homogeneous* boundary conditions.

2. The basic calculations and spaces

Let A represent the operator $\nabla \cdot (a\nabla)$ acting on functions defined on $\overline{\Omega}$, where Ω is a bounded subset of $\mathbf{R}^N$ with a smooth enough boundary so that the divergence theorem holds on Ω and $a \in C(\overline{\Omega})$ with $a > 0$ in Ω. We consider the elliptic equation

$$\lambda u - Au = h \quad \text{in} \quad \Omega, \tag{2.1}$$

with $Re\, \lambda > 0$ and $h : \overline{\Omega} \to \mathbf{C}$ given, with the generalized Wentzell boundary condition

$$Au + \beta \frac{\partial u}{\partial n} + \gamma u = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

Here β, γ, h are sufficiently smooth functions (β, γ on $\partial\Omega$, h on $\overline{\Omega}$) with γ nonnegative and $\beta \geq \epsilon_o > 0$. In the expression $\frac{\partial u}{\partial n}$, $n = n(x)$ refers to the unit outer normal to $\partial\Omega$ at x. From (2.1) we have $Au = \lambda u - h$; plugging this into (2.2) gives

$$\beta \frac{\partial u}{\partial n} + (\gamma + \lambda)u = h \quad \text{on} \quad \partial\Omega. \tag{2.3}$$

(To be more precise, we should replace h by $h|_{\partial\Omega}$ in (2.3), etc., but the meaning of (2.3) should be clear.)

To solve the elliptic boundary value problem (2.1), (2.3), we shall use a weak formulation and the Riesz representation theorem. The equation itself together with elliptic boundary value theory gives us a unique solution $u \in C^2(\Omega \cup \Gamma)$, if a, h, γ, β and $\partial\Omega$ are smooth enough, and $u \in C^2(\overline{\Omega})$ if $\Gamma = \partial\Omega$; see e.g. Gilbarg and Trudinger [14] or Lunardi [24] for elliptic theory in the uniformly elliptic case. But we wish to find explicit bounds describing how the solution depends on λ, and these bounds come naturally from energy methods.

Multiply (2.1) by $\overline{v} \in H^1(\Omega)$ and integrate over Ω. The result is

$$\lambda \int_\Omega u\overline{v}\, dx - \int_\Omega (\nabla \cdot (a\nabla u))\overline{v}\, dx = \int_\Omega h\overline{v}\, dx, \tag{2.4}$$

which by the divergence theorem gives

$$\lambda \int_\Omega u\overline{v}\, dx + \int_\Omega \nabla u \cdot \nabla\overline{v}\, a\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n}\overline{v}\, a\, dS = \int_\Omega h\overline{v}\, dx.$$

Using $\frac{\partial u}{\partial n} = \frac{(h-(\gamma+\lambda)u)}{\beta}$ (from (2.3)) and noting $\beta \geq \epsilon_o > 0$, we deduce

$$\begin{aligned} \lambda \int_\Omega u\overline{v}\, dx + \int_\Omega \nabla u \cdot \nabla\overline{v}\, a\, dx + \int_\Gamma (\lambda + \gamma)u\overline{v}\, a \frac{dS}{\beta} \\ = \int_\Omega h\overline{v}\, dx + \int_\Gamma h\overline{v}\, a\, \frac{dS}{\beta}, \end{aligned} \tag{2.5}$$

where $\Gamma = \{x \in \partial\Omega : a(x) > 0\}$. Let $L(u, v)$ be the left hand side of (2.5), and let $F(v)$ be the corresponding right hand side. (Here h is fixed.) Let $\mathcal{H}$ be the completion of $C^1(\overline{\Omega})$ in the norm

$$\|u\|_{\mathcal{H}} := \left(\int_\Omega |u|^2 \, dx + \int_\Omega |\nabla u|^2 \, a \, dx + \int_\Gamma |u|^2 \, \frac{a \, dS}{\beta} \right)^{\frac{1}{2}}.$$

Thus L is a bounded sesquilinear form on $\mathcal{H}$, and F is a bounded conjugate linear functional on $\mathcal{H}$:

$$\begin{aligned} |L(u, v)| &\le \max\{|\lambda|, 1\} \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \\ &\quad + (|\lambda| + \|\gamma\|_\infty) \|u\|_{L^2(\Gamma, \frac{adS}{\beta})} \|v\|_{L^2(\Gamma, \frac{adS}{\beta})} \\ &\le C_1(\lambda) \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \\ |F(v)| &\le \|h\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \end{aligned}$$

provided $h \in \mathcal{H}$. Also,

$$\begin{aligned} Re\, L(u, u) &\ge Re\, \lambda \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega, adx)} \\ &\quad + Re\, \lambda \|u\|^2_{L^2(\Gamma, \frac{adS}{\beta})} \\ &\ge C_2(\lambda) \|u\|^2_{\mathcal{H}}. \end{aligned}$$

By the Riesz representation theorem (or the Lax-Milgram lemma), for all $h \in \mathcal{H}$ there is a unique $u \in \mathcal{H}$ such that $L(u, v) = F(v)$ holds for all $v \in \mathcal{H}$. That is, (2.5) holds, and this u is our weak solution of (2.1), (2.2). As before, $u \in C^2(\Omega \cup \Gamma)$ if $a, \gamma, \beta, h, \partial\Omega$ are smooth enough. Let us also notice that, if $a > 0$ on $\overline{\Omega}$ and $\beta = a|_{\partial\Omega}$, then, according to Daners [7, p. 4213], the existence of weak solutions is assured even if a is only bounded and measurable on $\overline{\Omega}$ and $\partial\Omega$ is Lipschitz.

Let $U := (u, v)$ where $u : \Omega \to \mathbf{C}$ and $v : \partial\Omega \to \mathbf{C}$ are measurable functions such that

$$\int_\Omega |u|^p \, dx + \int_\Gamma |v|^p a \frac{dS}{\beta} < \infty.$$

Let us define the $\|\cdot\|_p^*$ norm of U as follows

$$\|U\|_p^* := \left[\int_\Omega |u|^p \, dx + \int_\Gamma |v|^p \, a \frac{dS}{\beta} \right]^{\frac{1}{p}}$$

for $1 \le p < \infty$, and observe that the $L^p(\overline{\Omega}, d\mu)$ norm and the $\|\cdot\|_p^*$ norm are the same. From now on denote this norm by $|||\cdot|||_p$. Moreover, if we identify every $u \in C(\overline{\Omega})$ with $U = (u|_\Omega, u|_{\partial\Omega}) \in C(\overline{\Omega}) \times C(\partial\Omega)$, we define X_p to be the completion of $C(\overline{\Omega})$ in the norm $|||\cdot|||_p$. But one can easily show that

$$X_p = L^p(\overline{\Omega}, d\mu)$$

(indeed $C(\overline{\Omega})$ is densely injected in $L^p(\Omega, dx) \times C(\partial\Omega)$ and $L^p(\Omega, dx) \times C(\partial\Omega)$ is densely injected in $L^p(\overline{\Omega}, \mu)$ and X_p contains elements of the form $(0, g)$ where $0 \neq g \in L^p(\Gamma, \frac{adS}{\beta})$, since $\Gamma \neq \emptyset$). Still, we expect all solutions u of the heat equation with generalized Wentzell boundary conditions to satisfy $u(\cdot, t) \in C(\overline{\Omega})$ for all $t > 0$, and so we can identify $u(\cdot, t)$ with $U(\cdot, t) = (u(\cdot, t)|_\Omega, u(\cdot, t)|_{\partial\Omega}) \in X_p$. We shall return to this point later.

Taking X_∞ to be the completion of $C(\overline{\Omega})$ in the $|||\cdot|||_\infty$ norm, which is defined in the obvious way, for every $U \in X_\infty$ we have that

$$|||U|||_\infty = \lim_{p\to\infty} |||U|||_p = \|U\|_{L^\infty(\Omega)},$$

and so X_∞ is $C(\overline{\Omega})$.

We next study (2.1), (2.2) in an X_p context with $1 < p < \infty$.

In the following we will be interested only to find estimates, hence we will not care if $p \geq 2$ or not. Multiply (2.1) by $\overline{v}$ where $v := |u|^{p-2}u\chi_{\{u\neq 0\}}$ (so that $v = 0$ wherever $u = 0$) and integrate over Ω. The result is

$$\lambda \int_\Omega |u|^p\, dx - \int_\Omega (\nabla \cdot (a\nabla u))\overline{v}\, dx = \int_\Omega h\overline{v}\, dx. \tag{2.6}$$

The right hand side of (2.6) satisfies

$$\left|\int_\Omega h\overline{v}\, dx\right| \leq \|h\|_{L^p(\Omega)}\|\overline{v}\|_{L^{p'}(\Omega)} = \|h\|_{L^p(\Omega)}\|u\|^{p-1}_{L^p(\Omega)} \tag{2.7}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Next,

$$-\int_\Omega (\nabla \cdot (a\nabla u))\overline{v}\, dx = \int_\Omega \nabla u \cdot \nabla\overline{v}\, a\, dx - \int_\Gamma \frac{\partial u}{\partial n}\overline{v}\, a\, dS := T - \widetilde{T}.$$

Note that

$$\frac{\partial}{\partial x_i}|u|^2 = 2|u|\frac{\partial|u|}{\partial x_i} = \frac{\partial}{\partial x_i}(u\overline{u}) = 2Re\left(u\frac{\partial\overline{u}}{\partial x_i}\right),$$

whence

$$\nabla|u| = \frac{Re(u\nabla\overline{u})}{|u|}.$$

This implies

$$\nabla|u|^q = q|u|^{q-1}\nabla|u| = q|u|^{q-2}Re(u\nabla\overline{u}). \tag{2.8}$$

For T we obtain:

$$\begin{aligned} T &= \int_\Omega \nabla u \cdot \nabla(|u|^{p-2}\overline{u})\, a\, dx \\ &= \int_\Omega \nabla u \cdot (\nabla\overline{u})|u|^{p-2}\, a\, dx + \int_\Omega (\nabla u)\overline{u}(p-2)|u|^{p-4}Re(u\nabla\overline{u})\, a\, dx \end{aligned}$$

by (2.8). Since

$$Re(\overline{u}\nabla u)Re(u\nabla\overline{u}) = [Re(u\nabla\overline{u})]^2 \geq 0,$$

we have

$$Re\, T \in [k_1\delta, k_2\delta],$$

where

$$k_1 := \min\{1, p-1\}, \quad k_2 := \max\{1, p-1\},$$
$$\delta := \int_\Omega |\nabla u|^2 |u|^{p-2}\, a\, dx.$$

By (2.6) and the above calculations, for $v := |u|^{p-2} u \chi_{\{u \neq 0\}}$,

$$\lambda \int_\Omega |u|^p\, dx + \int_\Omega \nabla u \cdot \nabla \overline{v}\, a\, dx - \int_\Gamma \frac{\partial u}{\partial n} \overline{v}\, a\, dS = \int_\Omega h\overline{v}\, dx,$$

or, using (2.3),

$$\lambda \int_\Omega |u|^p\, dx + T + \lambda \int_\Gamma |u|^p a \frac{dS}{\beta} + \int_\Gamma \gamma |u|^p a \frac{dS}{\beta}$$
$$= \int_\Omega h\overline{v}\, dx + \int_\Gamma h\overline{v}\, a \frac{dS}{\beta}. \tag{2.9}$$

Arguing as in (2.7),

$$\left| \int_\Gamma h\overline{v}\, \frac{adS}{\beta} \right| \leq \|h\|_{L^p(\Gamma, \frac{adS}{\beta})} \|u\|^{p-1}_{L^p(\Gamma, \frac{adS}{\beta})}. \tag{2.10}$$

Now take the real part of (2.9); this implies

$$Re\lambda\, |||u|||_p^p + [\geq 0] \leq (\|h\|_{L^p(\Omega)} + \|h\|_{L^p(\Gamma, \frac{adS}{\beta})}) |||u|||_p^{p-1}$$
$$\leq 2^{1-\frac{1}{p}} |||h|||_p |||u|||_p^{p-1}, \tag{2.11}$$

whence

$$(Re\, \lambda) |||u|||_p \leq 2^{1-\frac{1}{p}} |||h|||_p.$$

Here we have used

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

for $a, b > 0$ and $p > 1$. Rewrite (2.11) in the more precise form

$$Re\, \lambda\, |||u|||_p^p + Re\, T + \int_\Gamma \gamma |u|^p\, a \frac{dS}{\beta} \leq 2^{1-\frac{1}{p}} |||h|||_p |||u|||_p^{p-1}. \tag{2.12}$$

By [29, p. 216], if $a > 0$ in $\overline{\Omega}$ (for the uniformly elliptic case),

$$|Im\, T| \leq C_o(p) Re\, T, \tag{2.13}$$

where

$$C_o(p) := M \frac{|p-2|}{\sqrt{p-1}}, \tag{2.14}$$

and

$$M := \max \left\{ \|a\|_\infty, \left\| \frac{1}{a} \right\|_\infty \right\}.$$

Despite the assumption of a Dirichlet boundary condition in [29, pp. 215–216], the calculation given there allows the conclusion (2.13).

Now in (2.9) taking the imaginary part implies

$$|Im\, \lambda|\, |||u|||_p^p - |Im\, T| \leq 2^{1-\frac{1}{p}} |||h|||_p\, |||u|||_p^{p-1}. \tag{2.15}$$

Multiply (2.12) by $C_o(p)$ (see (2.14)) and add (2.15) to it; taking into account (2.13), the result is

$$(C_o(p)\, Re\, \lambda + |Im\, \lambda|)|||u|||_p + [\geq 0] \leq C_1(p)|||h|||_p$$

which implies

$$|||u|||_p \leq \frac{C_2(p)}{|\lambda|} |||h|||_p \tag{2.16}$$

whenever $Re\, \lambda > 0,\ h \in X_p,\ 1 < p < \infty$, and u is the weak solution of (2.1),(2.2). Our ultimate interpretation of this calculation will be that the closure of A (with generalized Wentzell boundary conditions) generates an analytic semigroup on X_p, for $1 < p < \infty$ (see Theorem 3.1 below).

Our "discovery" of the norm $||| \cdot |||_p$ is actually a rediscovery. Maz'ja [26], [27] introduced similar notions in 1960. His main result was

$$\|u\|^2_{L^{\frac{2n}{n-1}}(\Omega)} \leq C(n, Vol(\Omega))(\|\nabla u\|^2_{L^2(\Omega)} + \|u|_{\partial\Omega}\|^2_{L^2(\partial\Omega)})$$

(see [27, Corollary 4.11.1/2]), so that the norm on the right hand side is equivalent to the $H^1(\Omega)$ norm (see also [7, Section 2]) and to the $\mathcal{H}$ norm when $a > 0$ in $\overline{\Omega}$.

In the positive operator $\nabla \cdot (a\nabla)$ we could let a be a positive definite symmetric matrix rather than a positive function. The proofs would still work with some reasonably obvious modifications. In order to keep this exposition rather simple, we shall give the arguments only in the case of the function a, or from a different point of view, a times the identity matrix.

3. Generation and analyticity in X_p

In the degenerate case, when $\Gamma = \{x \in \partial\Omega : a(x) > 0\} \neq \partial\Omega$, the space X_p and its norm only contain boundary information on Γ, not on all of $\partial\Omega$, and so the boundary condition $Au + \beta\frac{\partial u}{\partial n} + \gamma u = 0$ on S needs only be stated for $S = \Gamma$ rather than for $S = \partial\Omega$. Let us define C^2_{BC} by

$$C^2_{BC} := \left\{ u \in C^2(\Omega \cup \Gamma) \cap C(\overline{\Omega}) : \nabla \cdot (a\nabla u) + \beta \frac{\partial u}{\partial n} + \gamma u = 0 \quad \text{on} \quad \Gamma \right\}$$

and observe that if we identify each element $u \in C^2_{BC}$ with $(u|_\Omega, u|_{\partial\Omega})$, where $u \in C^2(\overline{\Omega})$ and $u|_{\partial\Omega} \in C^2(\partial\Omega)$, then C^2_{BC} is densely injected in X_p, hence X_p coincides also with the completion of $C^2(\overline{\Omega})$ with respect to the norm $|||\cdot|||_p$, $1 \leq p < \infty$. Identify $u \in C(\overline{\Omega})$ with $U = (u|_\Omega, u|_{\partial\Omega}) = (w, v)$ in X_p. Then the problem (2.1), (2.3) becomes

$$\lambda \begin{pmatrix} w \\ v \end{pmatrix} - \begin{pmatrix} \nabla \cdot (a\nabla) & 0 \\ -\beta\frac{\partial}{\partial n} & -\gamma \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} k \\ l \end{pmatrix}, \tag{3.1}$$

where, for $h \in C(\overline{\Omega})$, $(h|_\Omega, h|_{\partial\Omega}) = (k, l) \in X_p$, for $Re\,\lambda > 0$. In (2.1) we had used A in place of $\nabla \cdot (a\nabla u)$. The generator G is given formally by the matrix

$$\begin{pmatrix} \nabla \cdot (a\nabla) & 0 \\ -\beta\frac{\partial}{\partial n} & -\gamma \end{pmatrix}.$$

This gives the action of G and

$$\begin{aligned} C^2_{BC} \subset D(G) \subset \Big\{ & u \in W^{2,p}_{loc}(\Omega) \cap X_p : Au \quad \text{exists in the sense of traces} \\ & \text{as well defined member of} \quad L^p\left(\Gamma, a\frac{dS}{\beta}\right), \\ & \text{and} \quad \nabla \cdot (a\nabla u) + \beta\frac{\partial u}{\partial n} + \gamma u = 0 \quad \text{on} \quad \Gamma \Big\}. \end{aligned} \tag{3.2}$$

Here every $u \in C^2(\overline{\Omega})$ is identified with $U = (u|_\Omega, u|_{\partial\Omega}) \in W^{2,p}(\Omega)$.

The main result of this paper is expressed in the following theorem.

THEOREM 3.1. *Let Ω be a bounded subset of $\mathbf{R}^N$ with boundary $\partial\Omega$ of class C^2. Let $a \in C^1(\overline{\Omega})$ with $a > 0$ in Ω, $Au := \nabla \cdot (a\nabla u)$, $\Gamma := \{x \in \partial\Omega : a(x) > 0\}$, and assume that $\Gamma \neq \emptyset$. If β, γ are nonnegative functions in $C^1(\partial\Omega)$ with $\beta > 0$, then $\overline{G}$, the closure of the operator*

$$G = \begin{pmatrix} A & 0 \\ -\beta\frac{\partial}{\partial n} & -\gamma \end{pmatrix}$$

with domain

$$D_p(G) \ := \ \Big\{ u \in C^2(\Omega \cup \Gamma) \cap C(\overline{\Omega}) : \ Au \in L^p(\Omega, dx),$$
$$Au + \beta \frac{\partial u}{\partial n} + \gamma u = 0, \ \text{on} \quad \Gamma \Big\},$$

generates a (C_o) *contraction semigroup on* X_p, *for* $1 \le p \le \infty$. *This semigroup is analytic for* $1 < p < \infty$.

Proof. Note that from $u \in W^{2,p}(\Omega)$ it follows that $\frac{\partial u}{\partial n} \in L^p(\Gamma, \frac{adS}{\beta})$ for $1 < p < \infty$. Let $1 < p < \infty$ and $q = \frac{p}{p-1}$. We denote the pairing between X_p and X_q by $\langle \cdot, \cdot \rangle$.

Here is the dissipative calculation of G on X_p, $1 < p < \infty$. Let $Ju := |u|^{p-2}\overline{u}\chi_{\{u \neq 0\}}$ be the duality map of X_p (modulo a constant multiple which depends on $|||u|||_p$). To show that G is dissipative, it suffices to show

$$Re \, \langle Gu, Ju \rangle \ \le 0$$

for $u \in D_p(G)$. We have

$$\begin{aligned} \langle Gu, Ju \rangle \ &= \ \int_{\overline{\Omega}} Gu(|u|^{p-2}\overline{u}) \, d\mu \\ &= \ \int_{\Omega} Au(|u|^{p-2}\overline{u}) \, dx - \int_{\Gamma} \beta \frac{\partial u}{\partial n} |u|^{p-2} \overline{u} \frac{adS}{\beta} - \int_{\Gamma} \gamma u |u|^{p-2} \overline{u} \frac{adS}{\beta} \\ &= \ - \int_{\Omega} \nabla u \cdot \nabla(|u|^{p-2}\overline{u}) \, a \, dx - \int_{\Gamma} \gamma |u|^p \, \frac{adS}{\beta} \end{aligned}$$

by the divergence theorem. Since $\beta, \gamma \ge 0$ and

$$- \int_{\Omega} \nabla u \cdot \nabla(|u|^{p-2}\overline{u}) \, a \, dx = -T$$

(see the lines following (2.8)) has real part nonpositive, it follows that

$$Re\langle Gu, Ju \rangle \ \le 0.$$

Note that G is closable since G is dissipative, and its closure $\overline{G}$ is dissipative. The solution of (3.1) satisfies (2.1) and (2.3), or, equivalently, (2.1) and (2.2). Given $h \in X_p \cap \mathcal{H}$, there is a solution $u \in \mathcal{H}$ of (3.1) by the Lax-Milgram lemma from Section 2. For $h \in C^{1,\epsilon}(\overline{\Omega})$, elliptic regularity shows that $u \in D(G)$. Under our assumptions, in the uniformly elliptic case (when $a > 0$ on $\overline{\Omega}$), this follows from Colombo and Vespri [5, Theorem 2.1] or directly from Gilbarg and Trudinger [14, Theorem 6.31, p. 128] when $\beta, \gamma \in C^{1,\epsilon}(\Omega)$ and $\partial\Omega \in C^{2,\epsilon}$. But in the degenerate case of $\Gamma \neq \partial\Omega$ an additional argument is needed, involving local regularity of the solutions to the problem $\lambda u - Au = h$. We omit the details.

In fact, $u \in \mathcal{H} \cap X_p$ follows from (2.11). Then the closure $\overline{G}$ of $(G, D_p(G))$ is densely defined and m-dissipative on X_p, $1 < p < \infty$. Thus by the Hille-Yosida theorem (cf. e.g. [8], [16], [29]), $\overline{G}$ generates a (C_o) contraction semigroup $(T_p(t))_{t\geq 0}$ on X_p.

Let G_p be the version of G acting on X_p, $1 \leq p \leq \infty$. Then $G_p^* = \overline{G_q}$ and $T_p(t)^* = T_q(t)$ for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Also $G_p^* \supset G_q$ and $T_p(t)^* \supset T_q(t)$ for $p \in \{1, \infty\}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Thus the problems associated to $\overline{G_p}$ are *selfadjoint* ones. It is sufficient to prove that $\overline{G_p}$ is m-dissipative on X_p for $2 \leq p < \infty$; the case of $1 < p < 2$ then follows by duality. Thus some of the above calculations could be restricted to the case of $p \geq 2$.

Observe that for $p = 2$, the operators $\overline{G_2}$ and $T_2(t)$ are selfadjoint on X_2, whence the operator $T_2(t) - I$ can be identified by $e^{tm} - 1$, where m is a suitable real measurable function, $m \leq 0$. Then $\|T_2(t) - I\| \leq 1$ for each $t > 0$. Hence, as a consequence of the M. Riesz interpolation theorem (or the Stein interpolation theorem), we have that

$$\limsup_{t\to 0^+} \|T_p(t) - I\| < 2,$$

for any p satisfying $1 < p < 2$ or $2 < p < \infty$. Using Neuberger's theorem (see e.g. [16, Exercise 5.10.5, p. 38] and [29, Corollary 5.7, p. 68]), exactly as in the paper [15], we have that $(T_p(t))_{t\geq 0}$ is an analytic semigroup for $1 < p < \infty$.

Notice that, if $a > 0$ on $\overline{\Omega}$, then the analyticity for $(T_p(t))_{t\geq 0}$, $1 < p < \infty$, follows directly from (2.16).

This completes the proof for $1 < p < \infty$. □

The cases $p = 1$ and $p = \infty$ will be treated in the next section.

4. The cases of X_1 and X_∞

In this section we deal with the proof of Theorem 3.1, when $p = 1$ or $p = \infty$. Let us remark that, in (2.16), $C_2(p) \to \infty$ as $p \to 1$ and as $p \to \infty$. Thus there is no easy proof that the semigroup $\mathcal{T}$ generated by G (or rather its closure) on X_1 or on $X_\infty = C(\overline{\Omega})$ is analytic. However, we show that $\overline{G}$ generates a (C_o) contraction semigroup on these spaces. Using quite different techniques, we are able to show, in a paper in preparation [13], that $\mathcal{T}$ is analytic on X_1 and is a differentiable semigroup on X_∞, provided that $a > 0$ on $\overline{\Omega}$. But we shall not give the (nontrivial) arguments here.

Note that $X_p \subset X_r$ if $1 \leq r \leq p \leq \infty$. If we solve $u' = Gu$ with $u(0) = f \in X_p$, $1 \leq p$, then $u(t) \in \bigcap_{m=1}^{\infty} D(G^m) \subset C^2(\overline{\Omega})$ for each $t > 0$. Thus there is no ambiguity in interpreting the meaning of the boundary condition for all positive times.

Let u be real and satisfy (2.1), (2.3) for $\lambda > 0$. Multiply (2.1) by sign u_+ and integrate over Ω. The result is

$$\lambda \|u_+\|_{L^1(\Omega)} - \int_{[u>0]\cap\Omega} \nabla \cdot (a\nabla)u \, dx = \int_{[u>0]\cap\Omega} h \, dx. \tag{4.1}$$

Next, by the divergence theorem,

$$\int_{[u>0]\cap\Omega} Au\,dx = \int_{\Gamma_o} \frac{\partial u}{\partial n}\,a\,dS$$

where $\Gamma_o := \partial[u>0] \cup \overline{(\partial\Omega \cap [u>0])}$. Thus $\Gamma_o = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 \cap \partial\Omega = \emptyset$ and $u = 0$ on Γ_1, so that $\frac{\partial u}{\partial n} \le 0$ on Γ_1 while $\Gamma_2 \subset \partial\Omega$. Hence

$$\begin{aligned}\int_{\Gamma_o} \frac{\partial u}{\partial n}\,a\,dS &= [\le 0] + \int_{\Gamma_2} \frac{\partial u}{\partial n}\,a\,dS \\ &\le -\int_{\Gamma_2} \left(\frac{\gamma+\lambda}{\beta}\right) u\,a\,dS + \int_{\Gamma_2} ha\frac{dS}{\beta} \\ &\le -\lambda \int_{\Gamma_2} ua\frac{dS}{\beta} + \int_{\Gamma_2} ha\frac{dS}{\beta},\end{aligned}$$

since $\gamma \ge 0$, $\beta > 0$ and $u \ge 0$ on Γ_2. Since $\Gamma_2 \subset \{u > 0\}$, the above calculations combine to give

$$\lambda \|u_+\|_{L^1(\Omega)} + \lambda \int_{\partial\Omega} u_+ a\frac{dS}{\beta} \le \int_{[u>0]\cap\Omega} h_+\,dx + \int_{[u>0]\cap\partial\Omega} h_+ \frac{adS}{\beta}. \tag{4.2}$$

The same reasoning applied to $u_- = (-u)_+$ gives

$$\lambda \|u_-\|_{L^1(\Omega)} + \lambda \int_{\partial\Omega} u_- \frac{adS}{\beta} \le \int_{[u<0]\cap\Omega} h_-\,dx + \int_{[u<0]\cap\partial\Omega} h_- \frac{adS}{\beta}. \tag{4.3}$$

Adding (4.2) and (4.3) yields

$$\lambda |||u|||_1 \le |||h|||_1. \tag{4.4}$$

Thus G is dissipative on X_1. The range condition follows as before (indeed $(I-\overline{G})(D(\overline{G})) \supset X_2$, which is dense in X_1). This leads to the definition of the semigroup $(T(t))_{t\ge 0}$ on real functions in X_1. For the complex case we simply take

$$T(t)(f+ig) = T(t)f + iT(t)g.$$

In order to prove the assertion for $p = \infty$, take $\lambda > 0$. Letting $p \to \infty$ in

$$\lambda |||u|||_p \le |||h|||_p$$

gives

$$\lambda \|u\|_\infty \le \|h\|_\infty,$$

since $\lim_{p\to\infty} |||v|||_p = \|v\|_{L^\infty(\Omega)}$. Since $C^2(\overline{\Omega}) \subset D(G)$, $D(G)$ is dense in $C(\overline{\Omega})$. If $h \in C(\overline{\Omega}), \lambda > 0$ and $\lambda u - \overline{G}u = h$, then $u \in \bigcap_{1<p<\infty}(X_p \cap D(\overline{G}))$. Thus $u \in$

$\bigcap_{1<p<\infty} W^{2,p}(\Omega)$ and $Au \in C(\overline{\Omega})$. This defines the natural domain for G on $C(\overline{\Omega})$, and no additional regularity is required of β and γ.

Let us remark that the range condition is essentially $p-$*independent*, since it suffices to solve (2.1), (2.2) for $h \in C^{\infty}(\overline{\Omega})$, and that the closure of G generates a Feller semigroup on $C(\overline{\Omega})$, by using standard approximation arguments.

5. The one-dimensional case

In one dimension, with Ω being the unit interval (0, 1), the normal derivative becomes

$$\frac{\partial u}{\partial n}(j) = (-1)^{j+1}u'(j) = (-1)^{j+1}\frac{du}{dx}(j)$$

for $j = 0, 1$. The X_p-norm becomes

$$|||f|||_p = \left(\int_0^1 |f(x)|^p\, dx + \frac{a_1|f(1)|^p}{\beta_1} + \frac{a_o|f(0)|^p}{\beta_o}\right)^{\frac{1}{p}}$$

for $1 \le p < \infty$, where $a_j := a(j) \ge 0$, $a_o^2 + a_1^2 > 0$ and $\beta_j := \beta(j) > 0$, $j = 0, 1$. The closure of the operator

$$G = \begin{pmatrix} \frac{d}{dx}(a\frac{d}{dx}) & 0 \\ -\beta\frac{\partial}{\partial n} & -\gamma \end{pmatrix}$$

generates a contraction semigroup on

$$X_p = \left(L^p[0,1],\ dx\Big|_{(0,1)} \oplus \frac{adS}{\beta}\Big|_{\{0,1\}} \right)$$

for $1 \le p < \infty$, and on $X_\infty = C[0, 1]$, where dS is the Dirac measure at both endpoints. Moreover, the semigroup is analytic when $1 < p < \infty$.

The Cauchy problem governed by this operator and the corresponding semigroup is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right), \qquad t \ge 0, \qquad x \in (0,1),$$

$$u(x, 0) = f(x), \qquad x \in (0,1),$$

$$(au_x)_x(j,t) + (-1)^{j+1}\beta_j u_x(j,t) + \gamma_j u(j,t) = 0, \quad j = 0,1, \quad t \ge 0,$$

where $\beta_j > 0$ and $\gamma_j := \gamma(j) \ge 0$ for $j = 0, 1$.

In our earlier paper [9], we established this result in $C[0, 1] = X_\infty$, using slightly different notation; we provided the $(-1)^{j+1}$ coefficient in the first order term at the endpoints, so that the coefficient of $u_x(j, t)$ had different signs for $j = 0, 1$. We had observed that the operator was not dissipative on $L^p(0,1)$ for $1 < p < \infty$ and therefore we focussed on the supremum norm. If dS denotes point evaluation, it is now clear that

$$X_p = L^p\left([0,1], dx\bigg|_{(0,1)} \oplus \frac{adS}{\beta}\bigg|_{\{0,1\}}\right),$$

in which $L^p(0,1)$ sits as a subspace of codimension 2, is the right space for this problem.

In the special case of $p = \infty$, the two norms (i.e. the X_∞-norm and the $C[0,1]$-norm) coincide.

We illustrate briefly why dissipativity fails in $L^2(0,1)$. Let $\epsilon > 0$, $u \in C^2[0,1] \cap D_2(G)$ be real and fixed on $[\epsilon, 1]$ and assume that a does not vanish at both endpoints. We modify u on $[0,\epsilon]$ so that $u(0) = 1$ is fixed and compute

$$\langle Gu, u\rangle = \int_0^1 (au')'(x)u(x)\,dx = -\int_0^1 a(x)(u'(x))^2\,dx + [au'u]_0^1.$$

By the boundary conditions

$$u'(j) = (-1)^j \frac{(au')'(j) + \gamma_j u(j)}{\beta_j}$$

at $j = 0, 1$, we deduce that

$$\langle Gu, u\rangle = -\|u'\|^2_{L^2((0,1),\,adx)} + \left[au\left(\frac{s((au')' + \gamma u)}{\beta}\right)\right]_0^1$$

where $s(j) = (-1)^j$. By suitable varying u on $[0,\epsilon]$, $\|u'\|^2_{L^2((0,1),\,adx)}$ will only change slightly, and all other terms will be constant except for

$$-\frac{a(0)u(0)s(0)(au')'(0)}{\beta(0)} = -\frac{a(0)(au')'(0)}{\beta(0)},$$

which we can make arbitrarily large by making $(au')'(0)$ negative with modulus large enough. Thus an inequality of the form $Re\,\langle Gu, u\rangle \le \omega\|u\|_2^2$ will not hold for any $\omega \in \mathbf{R}$.

6. Connections with representation formulas

Let X, Y be Banach spaces of functions on Ω to $\mathbf{C}$ such that $Y \hookrightarrow X$, and let Z be a Banach space of functions from $\partial\Omega$ to $\mathbf{C}$. Consider

$$\begin{cases} u'(t) = Lu(t) + h(t), \quad t \ge 0, \\ u(0) = f, \\ Bu(t) = \varphi(t), \end{cases} \tag{6.1}$$

where $L \in \mathcal{B}(Y, X)$ (= the bounded linear operators from Y to X), and $B \in \mathcal{B}(Y, Z)$. We also impose the compatibility condition

$$Bf = \varphi(0), \tag{6.2}$$

obtained by setting $t = 0$ in (6.1). We make the hypotheses:

(H1) $L_0 = L|_{N(B)}(= L|_{KerB})$ *generates a* (C_o) *semigroup* $(S(t))_{t\geq 0}$ *on* X,

(H2) *For some real number* λ *and each* $\psi \in Z$ *there exists a unique solution* $w = D_\lambda \psi$ *satisfying* $(L - \lambda I)w = 0$, $Bw = \psi$ *with* $D_\lambda \in \mathcal{B}(Z, Y)$.

Here D_λ is called the *Dirichlet operator*. Using (H1), (H2), we shall derive a representation formula for the solution u of (6.1). This is well-known; cf. [2], [17]–[19] and [23] and the many references contained therein. See also the references in [23] to the earlier work of Lasiecka and Triggiani.

Here we shall relate this representation formula (6.3) to the solution of (2.1), (2.2); this is a new result. For specific examples of the spaces X, Y, Z in a concrete problem see the example in [17]. Let the solution u of (6.1) be $u = v + w$ where $w(t) = D_\lambda \varphi(t)$, $t \geq 0$. Then v satisfies

$$\begin{aligned} v' &= u' - w' = Lu + h - D_\lambda \varphi', \\ v(0) &= f - D_\lambda \varphi(0), \qquad Bv = 0. \end{aligned}$$

Thus v satisfies

$$\begin{aligned} v' &= L_0 v + Lw + h - D_\lambda \varphi' = L_0 v + h + D_\lambda(\lambda\varphi - \varphi'), \\ v(0) &= f - D_\lambda \varphi(0). \end{aligned}$$

Consequently (using (H1), (H2) and the variation of parameters formula for v),

$$\begin{aligned} u(t) &= S(t)(f - D_\lambda \varphi(0)) + D_\lambda \varphi(t) \\ &\quad + \int_0^t S(t-s)[D_\lambda(\lambda\varphi(s) - \varphi'(s)) + h(s)]\, ds. \end{aligned} \tag{6.3}$$

Denote by $\hat{u}(\xi)$ the Laplace transform

$$\hat{u}(\xi) = \int_0^\infty e^{-\xi t} u(t)\, dt, \quad \text{at} \quad \xi > 0.$$

Then $\widehat{u}\,'(\xi) = \xi\hat{u}(\xi) - u(0)$, and so

$$\begin{cases} \xi\hat{u}(\xi) = L\hat{u}(\xi) + f + \hat{h}(\xi), \\ B\hat{u}(\xi) = \hat{\varphi}(\xi). \end{cases} \tag{6.4}$$

In (2.1), (2.3) write ξ in place of λ and write $\tilde{u}$ for u; the result is

$$\begin{cases} \xi\tilde{u}(\xi) = A\tilde{u}(\xi) + h, \\ \beta\frac{\partial\tilde{u}}{\partial n}(\xi) + (\gamma + \xi)\tilde{u}(\xi) = h. \end{cases} \tag{6.5}$$

We can identify (6.4) with (6.5) if we *fix* $\xi > 0$, identify h (from (2.1)) with $f + \hat{h}(\xi)$ (from (6.4)) as well as with $\hat{\varphi}(\xi)$ (on $\partial\Omega$), and we identify B with

$$v \mapsto Bv = \beta\frac{\partial v}{\partial n} + (\gamma + \xi)v =: B_\xi v. \tag{6.6}$$

Let us take $h \equiv 0$ in (6.1) (and hence in (6.3) also); then taking f (in (6.1)) equal to the h in (2.1) makes the first equations in (6.4), (6.5) agree when we identify $\hat{u}$ with $\tilde{u}$ and L with A (or G). We make the second equations in (6.4), (6.5) agree by also taking

$$\varphi(t) = h\delta_0(t);$$

but then we need B to depend on ξ as in (6.6). This B is a Robin type operator.

Now consider the parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= Au && \text{for} \quad (x,t) \in \Omega \times [0,\infty), \\ u(x,0) &= f(x) && \text{for} \quad x \in \Omega, \\ Au + \beta\frac{\partial u}{\partial n} + \gamma u &= 0 && \text{for} \quad (x,t) \in \partial\Omega \times [0,\infty). \end{aligned}$$

Let $\tilde{u}(\xi)$ be the Laplace transform of the solution u, evaluated at $\xi > 0$. By fixing ξ and taking $h = f$, then (6.5) holds. Since we identified this with (6.4), we take the Laplace transform of the representation formula (6.3) evaluated at ξ; this gives us a formula for $\tilde{u}(\xi)$ which is an explicit formula (involving $S(t)$ and D_λ). But its dependence on ξ is very complicated and it cannot be easily inverted to get a formula for $u(t)$.

Note that, for

$$\begin{aligned} u_t &= Au + h(t) \quad \text{in} \quad \Omega, \\ Au &+ \beta\frac{\partial u}{\partial n} + \gamma u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

the boundary condition becomes

$$u_t + \beta\frac{\partial u}{\partial n} + \gamma u = h(t) \quad \text{on} \quad \partial\Omega.$$

This is a *dynamic* boundary condition involving motion on the boundary. Such conditions arise in the applications and have been studied by Amann [2], Amann and Escher [3], Grobbelaar-Van Dalsen and Sauer [20]–[21], Sauer [30], and Hintermann [22].

7. Historical and other concluding remarks

In the 1930's John von Neumann determined all selfadjoint extensions of symmetric operators on a Hilbert space $\mathcal{H}$. He did this with the aid of the Cayley transform. This work was later extended by Mark Krein and Ralph Phillips who showed how to find all m-dissipative extensions of dissipative (linear) operators on $\mathcal{H}$, and this work was put into final form by M. Crandall and R. Phillips [6]. A more complicated space than $\mathcal{H}$ is $C(K)$, where K is a compact Hausdorff space. For instance, one may consider $K = [0,1]$. Let

A be a dissipative linear operator on $C(K)$ such that $A1 = 0$ and $(\lambda I - A)^{-1}$ is a positive operator for each $\lambda > 0$. (For instance, take $A := \frac{d^2}{dx^2}$ acting on the smooth functions with compact support in $(0, 1)$, plus the constants.) The problem is to find all extensions of A which generate positive contraction (C_o) semigroups on $C(K)$. This problem is still open. But it was in the context of this problem that Wentzell and generalized Wentzell boundary conditions first arose. These conditions make sense in $C(K)$ spaces. Our earlier paper [9] illustrates this in $K = [0, 1]$.

One of our main discoveries in the present paper is that, when one extends this theory properly to an L^p setting, one must replace $L^p(\Omega, dx)$ by $L^p(\overline{\Omega}, d\mu)$ which is a "bigger space", incorporating a weighted L^p space on (a portion of) the boundary. The weight function is important and plays a role in the dissipative calculation. Thus, for instance, consider the heat equation

$$u_t = u_{xx}, \qquad x \in [0, 1],$$

with boundary conditions

$$u_{xx}(j) + (-1)^{j+1}\beta_j u_x(j) + \gamma_j u(j) = 0, \qquad j = 0, 1,$$

where β and γ are defined on $\{0, 1\}$ with $\beta(j) := \beta_j > 0$ and $\gamma(j) := \gamma_j \geq 0$ for $j = 0, 1$. Let

$$G := \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ -\beta\frac{\partial}{\partial n} & -\gamma \end{pmatrix}$$

(where $\frac{\partial u(j)}{\partial n} = (-1)^{j+1}\frac{\partial u(j)}{\partial x}$ for $j = 0, 1$), on $X_2 := L^2((0, 1), dx) \times L^2(\{0, 1\}, \eta\, dS)$, where η is any weight function defined in $\{0, 1\}$ with $\eta(j) = \eta_j$ for $j = 0, 1$ and dS is point evaluation. For $u \in C^2[0, 1]$ we have

$$\begin{aligned} \langle Gu, Ju\rangle &= \left\langle \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ -\beta\frac{\partial}{\partial n} & -\gamma \end{pmatrix}\begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix}, \begin{pmatrix} u \\ u|_{\partial\Omega} \end{pmatrix}\right\rangle \\ &= \int_0^1 u''\overline{u}\, dx - \beta_1 u'(1)\overline{u}(1)\eta_1 \\ &\quad + \beta_o u'(0)\overline{u}(0)\eta_o - \gamma_1|u(1)|^2\eta_1 - \gamma_o|u(0)|^2\eta_o. \end{aligned}$$

But

$$\int_0^1 u''\overline{u}\, dx = -\int_0^1 |u'|^2\, dx + u'(1)\overline{u}(1) - u'(0)\overline{u}(0).$$

Thus,

$$Re\,\langle Gu, Ju\rangle \ \leq Re\,\{u'(1)\overline{u}(1)[1 - \beta_1\eta_1] + u'(0)\overline{u}(0)[-1 + \beta_o\eta_o]\},$$

which is clearly nonpositive if $\beta_j = \frac{1}{\eta_j}$, but not necessarily in general.

For example, consider

$$u(x) := \frac{x^4}{4}.$$

Clearly $u = u' = u'' = 0$ at $x = 0$, and $|u'(x)|^2 = x^6$. Then, if we choose $\eta \equiv 1$, $\beta(1) = \frac{1}{10}$ and $\gamma(1) = \frac{2}{5}$, we have

$$\begin{aligned}\langle Gu, Ju\rangle &= -\frac{1}{7} + \frac{1}{4}(1 - \beta_1) - \frac{\gamma_1}{16}\\ &= -\frac{1}{7} + \frac{9}{40} - \frac{1}{40}\\ &= \frac{2}{35} > 0.\end{aligned}$$

Hence, G cannot be dissipative on $L^2((0, 1), dx) \times L^2(\{0, 1\}, dS)$.

Other authors have worked in spaces similar to X_p. It is a natural space for studying inhomogeneous elliptic boundary value problems. During the Conference on Evolution Equations held at Levico (Italy) in November 2000, the paper of H. Amann and J. Escher [3], which contains some interesting results, was brought to our attention. They worked in $L^p(\Omega, dx) \times L^p(\Gamma, dS)$ where Γ is a portion of the boundary. Their paper is very general and technically complicated; it deals with a general class of inhomogeneous parabolic initial – boundary value problems with dynamic boundary conditions. There is some overlap between this paper and [3], but we have chosen to make our paper selfcontained because on the parts of overlap, our proofs are much simpler, and we cover results that are not covered by [3]. For example, they assume throughout uniform ellipticity, so that in their work the coefficient a cannot vanish at any boundary point. Nevertheless the ideas of this paper and [3] can be combined to get extensions of the union of the two papers' results. Finally the results of this paper can be extended to situations involving nonlinear boundary conditions (see [13]).

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