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An Edge-Isoperimetric Problem for Powers of the Petersen Graph

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Abstract. In this paper we introduce a new order on the set of n-dimensional tuples and prove that this order preserves nestedness in the edge isoperimetric problem for the graph P^n, defined as the nth cartesian power of the well-known Petersen graph. The cutwidth and wirelength of P^n are also derived. These results are then generalized for the cartesian product of P^n and the m-dimensional binary hypercube.

Keywords: isoperimetric problem, Petersen graph, wirelength, cutwidth

1. Introduction

Various families of regular graphs have been studied for important practical applications in computer science. For example, they appear naturally as the interconnection topology (such as grids, tori, hypercube, de Bruijn graphs) for multiprocessor architectures and also in the context of communication networks design. The symmetries are provided by various notions of regularity simplify algorithms for different network related problems, such as message routing and information exchange among node-pairs.

From the theoretical point of view, the regular graphs also play a significant role in *edge isoperimetric problems*, among others, which is the subject of investigation in this paper. Mostly two versions of the edge isoperimetric problems have been considered in the literature.

Problem 1.1. Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality.

Problem 1.2. Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices.

Clearly, if a subset of vertices is *optimal* with respect to Problem 1.1, then its complement is also an optimal set. However, it is not true for Problem 1.2 in general, although this is indeed the case if the graph is regular. Moreover, for regular graphs the above two problems are equivalent in the sense that a solution for one also becomes a solution for the other.

In this paper, we focus our attention on Problem 1.2. Let $G = (V_G, E_G)$ be a graph and $A \subseteq V_G$. Denote

$$I_G(A) \; = \; \{(u,v) \in E_G \mid u,v \in A\}.$$
$$I_G(t) \; = \; \max_{|A|=t} |I_G(A)|.$$

Thus, for a given t, where $t = 1,\ldots,|V_G|$, we consider the problem of finding a subset A of vertices of G such that $|A| = t$ and $|I_G(A)| = I_G(t)$. Such subsets are called *optimal*. We say that optimal subsets are *nested* if there exists a total order $\mathcal{O}$ on the set V_G such that, for any $t = 1,\ldots,|V_G|$, the collection of the first t vertices in this order is an optimal subset. In this case we call the order $\mathcal{O}$ an *optimal order*.

Let us now concentrate on the graphs representable as cartesian products. Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, their *cartesian product* is defined as a graph $G \times H$ with the vertex-set $V_G \times V_H$ and the edge-set

$$\{((x,y),(u,v)) \mid x = u \text{ and } (y,v) \in E_H \text{ or } (x,u) \in E_G \text{ and } y = v\}.$$

A graph $G^n = G \times G \times \cdots \times G$ is called the nth *cartesian power* of G. Examples of product graphs include hypercubes, grids and tori.

Consider the edge isoperimetric problems for the cartesian powers G^n of a regular graph G. Such problems have been well studied for cliques, i.e., $G = K_p$. Representing the vertices of G^n as n-dimensional tuples, the results of Harper [11] and Lindsey [12] imply that the lexicographic order is an optimal order. Here, by the *lexicographic order* we mean the following: We say that an n-tuple $(x_1,\ldots,x_n)$ is lexicographically greater than $(y_1,\ldots,y_n)$ iff there exists an index i such that $x_j = y_j$ for $1 \le j < i$ and $x_i > y_i$.

These old classical results can be extended to various directions. For instance, taking a path instead of a clique leads to a grid. In this case, Problems 1.1 and 1.2 above are essentially different. The first problem does not have nested solutions, while the second does [1] (see also [7]). It is further shown in [3] that the results of [1,7] can be extended to the products of arbitrary trees and the function $I(t)$ depends just on t and the number of vertices in the trees, but not of the shape of the trees in the product. The *order*, $\mathcal{G}$, providing the nestedness in Problem 1.2 in this case, is much more complicated with respect to the lexicographic order. (For the definition of the order $\mathcal{G}$ and further details, readers are referred to [1,4,7].)

To summarize, the order $\mathcal{G}$ and the lexicographic order are the only known orders, which provide nestedness for products of some graphs in the edge isoperimetric problems. However, as shown in [4], the order $\mathcal{G}$ works for products of trees only. Therefore, two natural questions arise: (i) For products of which other graphs is the lexicographic order optimal with respect to the edge isoperimetric problems; and (ii) which other optimal orders can one expect?

In [5], Bezrukov and Elässer considered the cartesian powers of k-regular graphs with an even number of vertices $2p$ such that $k \geq 3p/2$. They have shown that, for the nth power of any such graph, the size of the edge cut separating a set from its complement is at least as large as that for the corresponding power of the graph H_p^k obtained as follows: Consider a clique K_{2p} and split its vertices into two disjoint cliques K_p' and K_p'' of order p each. Now construct the bipartite subgraph of K_{2p}, formed by the vertex sets of K_p' and K_p'', as the independent sets and remove from it some $p-k-1$ perfect matchings. Although the resulting graphs are non-isomorphic in general, they all (as well as their cartesian powers) have the same function $I(\cdot)$ [5]. Considering one of these graphs as H_p^k, it turns out that, for $k \geq 3p/2$, the edge isoperimetric problem for cartesian powers of H_p^k has nested solutions provided by the lexicographic order. Similar results can be derived for powers of complete bipartite graphs with deleted perfect matchings. It is interesting to note that violating the condition $k \geq 3p/2$ leads to the absence of nested solutions. Bezrukov and Elässer [5] also studied the powers of complete p-partite graphs and showed that the lexicographic order is still the optimal order. Thus, this extends a result of Ahlswede and Cai [2] concerning the powers of complete bipartite graphs.

It is worth mentioning that the lexicographic order yields the so-called *local-global principle* discovered by Ahlswede and Cai [2]. Following the main result of [2], if the lexicographic order is optimal for G^2, then so it is for G^n for any $n \geq 3$. The main difficulty in applying this powerful theorem is to establish that the lexicographic order is optimal for the second power of considered graphs. For this, no general methods are known yet. (See [4] for the local-global principles for some other orders.)

In all of the preceding results, the degree of the underlying regular graph is relatively large which is intuitively necessary for the lexicographic order to work. Now the question is what happens if the powers of regular graphs have smaller degree. For instance, considering the regular graphs of degree 1, we get the hypercubes for which the lexicographic order is still optimal. For powers of regular graphs of degree 2, e.g., a torus, there is no nested solutions in general [10]. Tori are well studied and some isoperimetric inequalities are known for them [6], which are sharp enough for most practical applications.

The next step is to consider the powers of regular graphs of degree 3, for which a huge collection of non-isomorphic graphs exists. However, to the best of our knowledge, none of them (excluding the 3-dimensional cube) has been studied with respect to the edge isoperimetric problems. This motivates our work.

We concentrate on the cartesian powers P^n of the Petersen graph P, which is a regular graph of degree 3 and diameter 2 as shown in Figure 1(a). Note that P is a vertex-symmetric as well as an edge-symmetric graph. The graphs P^n, known as *folded Petersen networks*, were proposed and extensively studied by Öhring and Das [9,13–16] as a communication-efficient interconnection network topology for multiprocessors. By definition, P^n is also regular, vertex- and edge-symmetric with 10^n vertices, degree $3n$ and diameter $2n$. Interestingly, the size of a minimum cut separating P^n into two equal parts is known exactly [4]. This fact also stimulates the cut problem having two parts of different cardinalities. It is an important property of a graph from the viewpoint of its minimum layout area in VLSI.

In this paper we answer several questions raised above. More precisely, we introduce a new order $\mathcal{P}^n$ on the set of n-dimensional tuples (which we call the *Petersen order*) and show that this order provides nestedness in the edge isoperimetric problem for P^n, the powers of the Petersen graph. This result allows us to compute exactly the cutwidth and wirelength of P^n, which are respectively defined as the maximum and the mean value of the minimum cut separating the graph into two parts. We extend these results to the product graph $P^n_m = P^n \times Q^m$, where Q^m is the m-dimensional hypercube. The graphs P^n_m, called the *folded Petersen cubes*, were first studied by Öhring and Das [16, 17]. It is interesting that, in this case, a lexicographic-type order is optimal.

The paper is organized as follows. In the next section we introduce the Petersen order, $\mathcal{P}^n$, on the vertex set of the graph P^n. Section 3 shows that, for $t = 1, \ldots, 10^n$, the set $\mathcal{F}^n(t)$ represented by the initial segment of the order $\mathcal{P}^n$ of length t is an optimal subset. Section 4 is devoted to computing the cutwidth and the wirelength of P^n. Section 5 presents the extensions of these results to the graphs P^n_m.

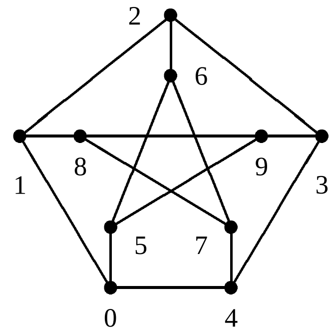
2. The Petersen Order $\mathcal{P}^n$ and Its Properties

The order $\mathcal{P}^1$ is shown in Figure 1(a) and it is an easy exercise to convince that it is optimal for the Petersen graph P.

Now, by induction on n, we define the total order $\mathcal{P}^n$ on the vertex set of P^n for $n \geq 2$. For this purpose let us first define the successor for any vector $(a_1, \ldots, a_n) \in V_{P^n}$ in the order $\mathcal{P}^n$ as follows. Denoting $(a'_2, \ldots, a'_n) = succ(a_2, \ldots, a_n)$ in the order $\mathcal{P}^{n-1}$, we define

$$succ(a_1, \ldots, a_n) = \begin{cases} (a_1 + 1, a_2, \ldots, a_n), & \text{if } a_1 \in \{0, 3, 5, 8\}, \\ (a_1 - 1, a'_2, \ldots, a'_n), & \text{if } a_1 \in \{1, 4, 6, 9\} \text{ and} \\ & \quad (a_2, \ldots, a_n) \neq (9, \ldots, 9), \\ (a_1, a'_2, \ldots, a'_n), & \text{if } a_1 \in \{2, 7\} \text{ and} \\ & \quad (a_2, \ldots, a_n) \neq (9, \ldots, 9), \\ (a_1 + 1, 0, \ldots, 0), & \text{if } a_1 \in \{1, 2, 4, 6, 7\} \text{ and} \\ & \quad (a_2, \ldots, a_n) = (9, \ldots, 9). \end{cases}$$

The order $\mathcal{P}^2$ is illustrated in Figure 1(b). The vertices of the graph $P^2 = P \times P$ are represented as the entries of a 10×10 matrix $\{a_{i,j}\}$, where $i, j = 0, \ldots, 9$. We assume in this figure that the entry $a_{0,0}$ is at the bottom left corner of the matrix. Furthermore, we assume that the elements $a_{0,0}, \ldots, a_{9,0}$ of the bottom row and the elements $a_{0,0}, \ldots, a_{0,9}$ of the leftmost column represent the vertices of the multiplicands of the product (i.e., vertices of P) taken in the order $\mathcal{P}^1$. The value of the matrix element $a_{i,j}$ is the number of the corresponding vertex of the graph P^2 in the order $\mathcal{P}^2$, as shown in Figure 1(b). With the help of a computer we have verified that the set $\mathcal{F}^2(t)$, which is represented by the initial segment of the order $\mathcal{P}^2$ of length t, is optimal for any t in the range $1 \leq t \leq 100$.



(a)

j/i	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
9	18	19	29	48	49	68	69	79	98	99
8	16	17	28	46	47	66	67	78	96	97
7	14	15	27	44	45	64	65	77	94	95
6	12	13	26	42	43	62	63	76	92	93
5	10	11	25	40	41	60	61	75	90	91
4	8	9	24	38	39	58	59	74	88	89
3	6	7	23	36	37	56	57	73	86	87
2	4	5	22	34	35	54	55	72	84	85
1	2	3	21	32	33	52	53	71	82	83
0	0	1	20	30	31	50	51	70	80	81

(b)

Figure 1: (a) The order $\mathcal{P}^1$; (b) the order $\mathcal{P}^2$.

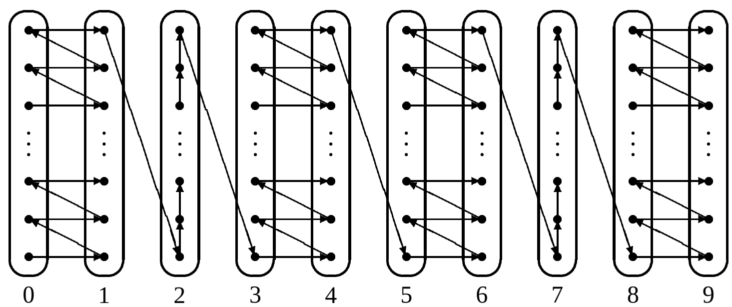


Figure 2: The structure of the order $\mathcal{P}^n$.

Using induction on n, it is easy to show that starting with the vector $(0,\dots,0)$ and following the successors, one can reach any other vector $(a_1,\dots,a_n) \in V_{P^n}$. This idea is schematically depicted in Figure 2 (cf. Figure 1 for $n = 2$), in which the ovals represent the vectors of the form $(i, a_2,\dots,a_n)$ ordered bottom-up in the order $\mathcal{P}^{n-1}$ (here i is shown under the ovals). Thus, the Petersen order $\mathcal{P}^n$ is well defined.

For $\mathbf{a}, \mathbf{b} \in V_{P^n}$ we write $\mathbf{a} > \mathbf{b}$ if the vertex $\mathbf{a}$ is greater than $\mathbf{b}$ in the order $\mathcal{P}^n$. By analyzing the inductive definition of the order $\mathcal{P}^n$, the following property, called *consistency* in [8], can be verified.

Lemma 2.1. *Let* $\mathbf{a} = (a_1,\dots,a_n)$, $\mathbf{b} = (b_1,\dots,b_n)$, *and* $a_i = b_i$ *for some i. Furthermore, let vectors* $\tilde{\mathbf{a}}$ *and* $\tilde{\mathbf{b}}$ *be obtained from* $\mathbf{a}$ *and* $\mathbf{b}$, *respectively, by omitting their ith entries. Then* $\mathbf{a} > \mathbf{b}$ *iff* $\tilde{\mathbf{a}} > \tilde{\mathbf{b}}$.

For $A \subseteq V_{P^n}$, $i = 1,\dots,n$, and $j = 0,\dots,9$, let us denote

$$P_i^n(j) \;=\; \{(\xi_1,\dots,\xi_n) \in V_{P^n} \mid \xi_i = j\},$$
$$A_i(j) \;=\; A \cap P_i^n(j).$$

We say that A is i-compressed if, for any $j = 0,\dots,9$, the subset $A_i(j)$ is an initial segment of the set $P_i^n(j)$ in the order $\mathcal{P}^{n-1}$. We call the set A *compressed* if A is i-compressed for $i = 1,\dots,n$. Standard arguments provide that there is no loss of generality in assuming that an optimal set A is compressed. We use the compression to verify the 2-dimensional solution of our problem with the help of a computer. Larry Harper noted that in this case there are $\binom{20}{10} = 352,716$ compressed sets. The complete choice of such a size is doable by computer but, without compression, there are $2^{100} \approx 1.3 \times 10^{30}$ possibilities, a prohibitively large number.

3. Proof of the Main Result

Let $\mathbf{a} = (a_1,\dots,a_n)$ be the largest vertex of the subset $A \subseteq V_{P^n}$ in the order $\mathcal{P}^n$ and let $\mathbf{b} = (b_1,\dots,b_n)$ be the smallest vertex of the complement $V_{P^n} \setminus A$ in this order. If $A \neq \mathcal{F}^n(t)$, then $\mathbf{a} > \mathbf{b}$. Since A is compressed, then so are the sets $A \setminus \mathbf{a}$ and $(A \setminus \mathbf{a}) \cup \mathbf{b}$.

For a graph G and $i = 1,\dots,|V_G|$, let us denote $\delta_G(i) = I_G(i) - I_G(i-1)$ and assume $\delta_G(0) = 0$. By analyzing Figure 1(a) the entries of Table 1 can be easily verified for the Petersen graph P.

Lemma 3.1. *With the above notations one has*

$$|I_{P^n}(A)| - |I_{P^n}((A \setminus \mathbf{a}) \cup \mathbf{b})| = \sum_{i=1}^{n} (\delta_P(b_i) - \delta_P(a_i)).$$

The lemma follows from the observation [2, 4] that for a compressed set A, it holds:

$$|I_{P^n}(A)| = \sum_{(x_1,\dots,x_n)\in A} \; \sum_{i=1}^{n} \delta_P(x_i).$$

Now we are ready to prove the main result.

Theorem 3.1. *For any $n \geq 1$ and t, $t = 1,\dots,10^n$, the set $\mathcal{F}^n(t)$ is optimal, where $\mathcal{F}^n(t)$ is represented by the initial segment of the order $\mathcal{P}^n$ of length t.*

Proof. We prove the theorem by induction on n. The case $n = 1$ is trivial and the case $n = 2$ follows from the mentioned results based on a computer search. Therefore, let us proceed with $n \geq 3$.

From the definition of the order $\mathcal{P}^n$, it can be concluded that if $\mathbf{a} > \mathbf{b}$, then one of the following five (disjoint) cases occurs:

Table 1: The values of $I_P(i)$ and $\delta_P(i)$.

i	0	1	2	3	4	5	6	7	8	9	10
$I_P(i)$	0	0	1	2	3	5	6	8	10	12	15
$\delta_P(i)$	0	0	1	1	1	2	1	2	2	2	3

(a) $a_1 - 1 > b_1$;
(b) $a_1 - 1 = b_1$ and $b_1 \in \{1, 2, 4, 6, 7\}$;
(c) $a_1 - 1 = b_1$, $b_1 \in \{0, 3, 5, 8\}$ and $(a_2, \ldots, a_n) \geq (b_2, \ldots, b_n)$;
(d) $a_1 = b_1$ and $(a_2, \ldots, a_n) > (b_2, \ldots, b_n)$;
(e) $a_1 + 1 = b_1$, $b_1 \in \{1, 4, 6, 9\}$ and $(a_2, \ldots, a_n) > (b_2, \ldots, b_n)$.

Let A be an optimal compressed set. Following the cases above we show that, in many of them, the condition $\mathbf{a} \in A$ implies $\mathbf{b} \in A$ due to the compression. The general strategy to show this is to find a vertex $\mathbf{c}$ satisfying $\mathbf{a} > \mathbf{c} > \mathbf{b}$ such that the vectors $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{c}$, $\mathbf{b}$ have an equal entry.

If such a vector $\mathbf{c}$ does exist, then, using Lemma 2.1, the condition $\mathbf{a} \in A$ implies $\mathbf{c} \in A$ which in turn implies $\mathbf{b} \in A$ because of the compression. On the other hand, if such a vector $\mathbf{c}$ does not exist, using Lemma 3.1, we show that replacing the vertex $\mathbf{a}$ with $\mathbf{b}$ yields a set B satisfying $|I_{P^n}(B)| \geq |I_{P^n}(A)|$. Clearly, after a finite number of such replacements one can transform A into $\mathcal{F}^n(|A|)$.

In the following we rigorously consider each one of the above five cases.

Case a. Assume $a_1 - 1 > b_1$.

(a1) Assume $a_1 - b_1 \geq 4$. Then $\mathbf{b} \in A$. Indeed, taking into account that $b_1 \leq 5$ and using the definition of the order $\mathcal{P}^n$ and Lemma 2.1, we get

$$\begin{aligned}\mathbf{a} = (a_1, \ldots, a_n) &\geq (b_1 + 4, a_2, \ldots, a_n) > (b_1 + 2, a_2, b_3, \ldots, b_n) \\ &> (b_1, b_2, \ldots, b_n) = \mathbf{b}.\end{aligned}$$

Note that any two consecutive vectors in this chain have an equal entry. Therefore, since $\mathbf{a} \in A$, and since A is compressed, then all the mentioned vectors are in A according to Lemma 2.1.

A similar approach will be used in analysis of all the remaining cases. We will just provide chains of appropriate vectors of P^n ordered in decreasing order $\mathcal{P}^n$. From now on we assume that $a_1 - b_1 \in \{2, 3\}$.

(a2) Assume $a_i > 1$ for some i where $2 \leq i \leq n-1$. Then $(a_i, a_{i+1}, \ldots, a_n) > (1, b_{i+1}, \ldots, b_n)$. On the other hand, $a_1 > b_1 + 1$ implies $(a_1, \ldots, a_{i-1}, 1) > (b_1, \ldots, b_i)$. Thus

$$\mathbf{a} = (a_1, \ldots, a_i, \ldots, a_n) > (a_1, \ldots, a_{i-1}, 1, b_{i+1}, \ldots, b_n) > (b_1, \ldots, b_n) = \mathbf{b}.$$

This implies $\mathbf{b} \in A$ since A is compressed.

(a3) Assume $b_i < 8$ for some i with $2 \leq i \leq n-1$. Then $(9, b_{i+1}, \ldots, b_n) > (b_i, \ldots, b_n)$. Similarly to case *a2*, it holds $(a_1, \ldots, a_i) > (b_1, b_2, \ldots, b_{i-1}, 9)$. Hence,

$$\mathbf{a} = (a_1, \ldots, a_i, \ldots, a_n) > (b_1, \ldots, b_{i-1}, 9, a_{i+1}, \ldots, a_n) > (b_1, \ldots, b_n) = \mathbf{b}.$$

Thus, $\mathbf{b} \in A$ since A is compressed.

(a4) Assume $a_i \le 1$ and $b_i \ge 8$ for $2 \le i \le n-1$. If $a_n > b_n$, then (cf. Lemma 2.1)

$$\mathbf{a} = (a_1,\dots,a_{n-1},a_n) > (a_1,\dots,a_{n-1},b_n) > (b_1,\dots,b_n) = \mathbf{b}.$$

Hence, $\mathbf{b} \in A$ since A is compressed.

If $a_n < b_n$, then replacing $\mathbf{a}$ with $\mathbf{b}$ we obtain a set B such that (cf. Lemma 3.1)

$$\begin{aligned} &|I_{P^n}(B)| - |I_{P^n}(A)| \\ &\quad = (\delta_P(b_1) - \delta_P(a_1)) + \sum_{i=2}^{n-1} (\delta_P(b_i) - \delta_P(a_i)) + (\delta_P(b_n) - \delta_P(a_n)) \\ &\quad \ge n-4, \end{aligned}$$

because $\delta_P(b_1) - \delta_P(a_1) \ge -1$ for $a_1 - b_1 \in \{2,3\}$, $\delta_P(b_i) - \delta_P(a_i) \ge \delta_P(8) - \delta_P(1) = 1$ for $2 \le i \le n-1$, and finally $\delta_P(b_n) - \delta_P(a_n) \ge -1$, since $b_n > a_n$ and equality takes place for $b_n = 5$, $a_n = 4$ only.

(a5) It remains to consider only the case $n = 3$, $\mathbf{a} = (a_1, a_2, 4)$, $\mathbf{b} = (b_1, b_2, 5)$, where $a_1 - b_1 \in \{2,3\}$, $a_2 \le 1$, and $b_2 \ge 8$. Now if $a_2 = 0$ or $b_2 = 9$, then $\delta_P(b_2) - \delta_P(a_2) \ge 2$ and, for the set B constructed in case (a4), one has $|I_{P^n}(B)| - |I_{P^n}(A)| \ge 0$.

Thus, we can assume that $\mathbf{a} = (a_1, 1, 4)$ and $\mathbf{b} = (b_1, 8, 5)$. Let us denote

$$\begin{aligned} X &= \{(a_1, x', x'') \mid 0 \le x' \le 1,\ 0 \le x'' \le 4\}, \\ Y &= \{(b_1, y', y'') \mid 9 \ge y' \ge 8,\ 9 \ge y'' \ge 5\}. \end{aligned}$$

Note that $|X| = |Y| = 10$, $X \subseteq A$ (since $\mathbf{a} \in A$) and $Y \cap A = \emptyset$ (since $\mathbf{b} \notin A$). Now consider a set $B = (A \setminus X) \cup Y$. It is easy to show that the set B is compressed. Taking into account that $\delta_P(b_1) - \delta_P(a_1) \ge -1$,

$$\begin{aligned} |I_{P^n}(B)| - |I_{P^n}(A)| &= \sum_{(a_1,y',y'') \in Y} (\delta_P(y') + \delta_P(y'')) - \sum_{(b_1,x',x'') \in X} (\delta_P(x') + \delta_P(x'')) \\ &\quad + 10 \cdot (\delta_P(b_1) - \delta_P(a_1)) \ge 45 - 15 - 10 = 20. \end{aligned}$$

Case b. Assume $a_1 - 1 = b_1$ and $b_1 \in \{1,2,4,6,7\}$.

The analysis of this case is quite similar to cases (a2)–(a4). The only difference is that now we can guarantee $\delta_P(b_1) - \delta_P(a_1) \ge 0$. Thus, for the set B constructed in case (a4), it holds $|I_{P^n}(B)| - |I_{P^n}(A)| \ge n - 3 \ge 0$.

Case c. Assume $a_1 - 1 = b_1$ and $b_1 \in \{0,3,5,8\}$. In this case $(a_2,\dots,a_n) \ge (b_2,\dots,b_n)$. Lemma 2.1 implies

$$\mathbf{a} = (a_1, a_2, \dots, a_n) \ge (a_1, b_2, \dots, b_n) > (b_1, b_2, \dots, b_n) = \mathbf{b}.$$

Hence, $\mathbf{b} \in A$ which is a contradiction.

Case d. Assume $a_1 = b_1$. In this case $\mathbf{b} \in A$ since A is 1-compressed.

Case e. Assume $a_1 + 1 = b_1$ and $b_1 \in \{1,4,6,9\}$. Denoting $(c_2,\dots,c_n) = succ(b_2,\dots,b_n)$, we have $(a_1, c_2, \dots, c_n) \in A$.

(e1) Assume $b_2 \in \{0,3,5,8\}$. Then $succ(b_2,\dots,b_n) = (b_2+1, b_3,\dots,b_n)$. Since $(a_1, b_2+1) > (a_1+1, b_2)$, we get

$$\mathbf{a} = (a_1,\dots,a_n) \geq (a_1, b_2+1, b_3,\dots,b_n) > (a_1+1, b_2, b_3,\dots,b_n) = \mathbf{b}.$$

Therefore, $\mathbf{b} \in A$, a contradiction.

(e2) Assume $b_2 \in \{2,7\}$. If $(b_3,\dots,b_n) \neq (9,\dots,9)$, then let $(d_3,\dots,d_n) = succ(b_3,\dots,b_n)$. One has $succ(b_2,\dots,b_n) = (b_2, d_3,\dots,d_n)$. Since $(a_1, d_3,\dots,d_n) > (a_1+1, b_3,\dots,b_n)$, using Lemma 2.1, we write

$$\mathbf{a} = (a_1,\dots,a_n) \geq (a_1, b_2, d_3,\dots,d_n) > (a_1+1, b_2, b_3,\dots,b_n) = \mathbf{b}.$$

This implies $\mathbf{b} \in A$.

Assume $(b_3,\dots,b_n) = (9,\dots,9)$. Then $succ(b_2,\dots,b_n) = (b_2+1, 0,\dots,0) \in A$. First assume that $(a_2,\dots,a_n) \geq succ(succ(b_2,\dots,b_n)) = (b_2+2, 0,\dots,0)$. Then $(a_1, b_2+2, 0,\dots,0) \in A$ since A is 1-compressed. Thus

$$\mathbf{a} \geq (a_1, b_2+2, 0\dots,0) > (a_1+1, b_2+1, 0,\dots,0) > (a_1+1, b_2, 9,\dots,9) = \mathbf{b}.$$

This implies $\mathbf{b} \in A$. If $(a_2,\dots,a_n) = succ(b_2,\dots,b_n) = (b_2+1, 0\dots,0)$. Then $\mathbf{a} = (b_1-1, b_2+1, 0,\dots,0)$ and $\mathbf{b} = (b_1, b_2, 9,\dots,9)$. Replacing $\mathbf{a}$ with $\mathbf{b}$ yields a set B such that

$$\begin{aligned}
&|I_{P^n}(B)| - |I_{P^n}(A)| \\
&\quad = (\delta_P(b_1) + \delta_P(b_2) + (n-2)\cdot\delta_P(9)) - (\delta_P(b_1-1) + \delta_P(b_2+1)) \\
&\quad = (\delta_P(b_1) - \delta_P(b_1-1)) + (\delta_P(b_2) - \delta_P(b_2+1)) + 3(n-2) \\
&\quad = 3(n-2)+1,
\end{aligned}$$

since $\delta_P(b_1) - \delta_P(b_1-1) = 1$ for $b_1 \in \{1,4,6,9\}$ and $\delta_P(b_2) - \delta_P(b_2+1) = 0$ for $b_2 \in \{2,7\}$.

(e3) Assume $b_2 \in \{1,4,6,9\}$ and $(b_3,\dots,b_n) = (9,\dots,9)$. Now $b_2 \neq 9$ since $(a_2,\dots,a_n) > (b_2,\dots,b_n)$. Then $succ(b_2,\dots,b_n) = (b_2+1, 0,\dots,0)$. First assume $(a_2,\dots,a_n) \geq (e_2,\dots,e_n) = succ(succ(b_2,\dots,b_n))$. Lemma 2.1 and the fact that A is 1-compressed imply $(a_1, e_2,\dots,e_n) \in A$.

Now if $b_2 \in \{1,6\}$, then $(e_2,\dots,e_n) = (b_2+1, 0,\dots,0,1)$. Hence,

$$\mathbf{a} \geq (a_1, b_2+1, 0,\dots,0,1) > (a_1+1, b_2+1, 0,\dots,0) > (a_1+1, b_2, 9,\dots,9) = \mathbf{b}.$$

If $b_2 = 4$, then $(e_2,\dots,e_n) = (6, 0,\dots,0)$ and

$$\mathbf{a} \geq (a_1, 6, 0,\dots,0) > (a_1+1, 5, 0,\dots,0) > (a_1+1, 4, 9,\dots,9) = \mathbf{b}.$$

In both cases $\mathbf{b} \in A$ since A is compressed.

If $(a_2,\dots,a_n) = succ(b_2,\dots,b_n)$, then $\mathbf{a} = (b_1-1, b_2+1, 0,\dots,0)$ and $\mathbf{b} = (b_1, b_2, 9,\dots,9)$. Replacing $\mathbf{a}$ with $\mathbf{b}$, we obtain a set B such that

$$\begin{aligned}
&|I_{P^n}(B)| - |I_{P^n}(A)| \\
&\quad = (\delta_P(b_1) + \delta_P(b_2) + (n-2)\cdot\delta_P(9)) - (\delta_P(b_1-1) + \delta_P(b_2+1)) \\
&\quad \geq 3(n-2)+1,
\end{aligned}$$

since $\delta_P(b_1) - \delta_P(b_1-1) = 1$ for $b_1 \in \{1,4,6,9\}$ and $\delta_P(b_2) \geq \delta_P(b_2+1)$ for $b_2 \in \{1,4,6\}$.

If $(b_3,\dots,b_n) \neq (9,\dots,9)$, then let us denote $(d_3,\dots,d_n) = succ(b_3,\dots,b_n)$. One has $succ(b_2,\dots,b_n) = (b_2-1, d_3,\dots,d_n)$. Now assume additionally that $(a_2,\dots,a_n) \geq (e_2,\dots,e_n) = succ(succ(b_2,\dots,b_n))$. Similarly to the above, $(a_1, e_2,\dots,e_n) \in A$ and $(e_2,\dots,e_n) = (b_2, d_3,\dots,d_n)$. Consequently,

$$\mathbf{a} \geq (a_1, b_2, d_3,\dots,d_n) > (a_1+1, b_2-1, d_3,\dots,d_n) > (a_1+1, b_2, b_3,\dots,b_n) = \mathbf{b}.$$

Therefore, $\mathbf{b} \in A$.

(e4) Lastly, we consider the case $b_2 \in \{1,4,6,9\}$, $(a_2,\dots,a_n) = succ(b_2,\dots,b_n)$ and assume $(b_3,\dots,b_n) \neq (9,\dots,9)$. Therefore, $\mathbf{a} = (b_1-1, b_2-1, a_3,\dots,a_n)$ and we can apply to the vectors $(a_2,\dots,a_n)$ and $(b_2,\dots,b_n)$ the same analysis as those for cases (e1)–(e3). If in the course of this analysis, we are be able to guarantee $\mathbf{b} \in A$ due to the compression, or to replace $\mathbf{a}$ with $\mathbf{b}$ without decreasing the function $I_{P^n}(\cdot)$, then we are done. Otherwise, just one case will give rise to problem again, namely when $b_3 \in \{1,4,6,9\}$, $(a_3,\dots,a_n) = succ(b_3,\dots,b_n)$ and $(b_4,\dots,b_n) \neq (9,\dots,9)$.

Continuing this way, the only remaining case left open is the case $\mathbf{a} = succ(\mathbf{b})$ such that

$$\mathbf{a} = (b_1-1, b_2-1,\dots,b_{n-1}-1, b_n+1), \quad \mathbf{b} = (b_1,\dots,b_n),$$

where $b_1,\dots,b_{n-1} \in \{1,4,6,9\}$ and $b_n \neq 9$. However, in this case the replacement of $\mathbf{a}$ with $\mathbf{b}$ leads to a set B with

$$|I_{P^n}(B)| - |I_{P^n}(A)| = \sum_{i=1}^{n-1} (\delta_P(b_i) - \delta_P(b_i-1)) + (\delta_P(b_n) - \delta_P(b_n+1)) \geq n-2,$$

since $\delta_P(b_i) - \delta_P(b_i-1) = 1$ for $b_i \in \{1,4,6,9\}$ and $\delta_P(b_n) - \delta_P(b_n+1) \geq -1$.

∎

4. Cutwidth and Wirelength of P^n

For $A \subseteq V_{P^n}$, denote

$$\partial(A) \;=\; \{(u,v) \in E_{P^n} \mid u \in A,\, v \notin A\},$$
$$g_n(t) \;=\; \min_{|A|=t} |\partial(A)|.$$

Since the graph P^n is regular of degree $3n$, for any $t = 1,\dots,10^n$, it holds:

$$I_{P^n}(t) + 2 \cdot g_n(t) = 3nt. \tag{4.1}$$

Therefore, for any initial segment A of the Petersen order $\mathcal{P}^n$, we have $|\partial(A)| = g_n(|A|)$.

For a graph G, let f be a bijective mapping $f : V_G \mapsto \{1,\dots,|V_G|\}$. Let

$$con_f(i) = |\{(u,v) \in E_G \mid f(u) \le i,\ f(v) \ge i+1\}|, \text{ for } 1 \le i < |V_G|,$$
$$cw(G) = \min_f \max_i \{con_f(i)\},$$
$$wl(G) = \min_f \left\{ \sum_{i=1}^{|V_G|-1} con_f(i) \right\}.$$

The parameters $cw(G)$ and $wl(G)$ are respectively called the *cutwidth* and *wire-length* of the graph G. For $G = P^n$, since the subsets with minimum value of the function $|\partial(\cdot)|$ are nested, it holds (cf. [8]):

$$cw_n = cw(P^n) = \max_t \{g_n(t)\},$$
$$wl_n = wl(P^n) = \sum_{t=0}^{10^n} g_n(t).$$

It can be easily shown that $cw_1 = 6$ and $wl_1 = 41$.

Theorem 4.1. *The cutwidth of the graph P^n is given by*

$$cw_n = \begin{cases} (6.25)\cdot 10^{n-1} + (2^{n-1}-4)/12, & \text{if } n \text{ is odd,} \\ (6.25)\cdot 10^{n-1} + (2^{n-1}-8)/12, & \text{if } n \text{ is even.} \end{cases}$$

Proof. First we show that, for $n \ge 2$, the maximum of $g_n(t)$ is attained when $t > 3\cdot 10^{n-1}$. Using the inductive structure of the order $\mathcal{P}^n$ (cf. Figure 2), for $1 \le t \le 2\cdot 10^{n-1}$, one has

$$g_n(t) \le 2t + 2\cdot cw_{n-1} + 1 \le 4\cdot 10^{n-1} + 2\cdot cw_{n-1} + 1.$$

Indeed, for $A = \mathcal{F}^n(t)$, the set $\partial(A)$ consists of the edges connecting $A_1(0)$ with $P_1^n(1) \cup P_1^n(4) \cup P_1^n(5)$, the edges connecting $A_1(1)$ with $P_1^n(2) \cup P_1^n(8)$, and of $\partial(A_1(0)) \cap P_1^n(0)$ and $\partial(A_1(1)) \cap P_1^n(1)$. Since $|A_1(0)| + |A_1(1)| = |A| = t$, the number of the edges of the first two types is

$$2\cdot(|A_1(0)| + |A_1(1)|) + (|A_1(0)| - |A_1(1)|) \le 2t + 1.$$

The number of edges of the third type does not exceed $2\cdot cw_{n-1}$.

Now, for t of the form $t = 2\cdot 10^{n-1} + t'$, where $0 < t' \le 10^{n-1}$, it holds

$$g_n(t) = 4\cdot 10^{n-1} + t' + g_{n-1}(t') \le 5\cdot 10^{n-1} + cw_{n-1}.$$

Indeed, for $A = \mathcal{F}^n(t)$, the set $\partial(A)$ consists of $2\cdot 10^{n-1}$ edges connecting $A_1(0)$ with $P_1^n(4) \cup P_1^n(5)$, of $10^{n-1} + (10^{n-1} - t')$ edges connecting $A_1(1)$ with $P_1^n(2) \cup P_1^n(8)$, of $2t'$ edges connecting $A_1(2)$ with $P_1^n(3) \cup P_1^n(6)$, and of $\partial(A_1(2)) \cap P_1^n(2)$. The size of the last set does not exceed cw_{n-1}.

Similarly one can show that, for $t = 3 \cdot 10^{n-1} + t'$, where $0 < t' \le 2 \cdot 10^{n-1}$, one has

$$g_n(t) = \begin{cases} 5 \cdot 10^{n-1} + 2 \cdot g_{n-1}(k), & \text{if } t' = 2k, \\ 5 \cdot 10^{n-1} + g_{n-1}(k+1) + g_{n-1}(k) + 1, & \text{if } t' = 2k+1. \end{cases} \tag{4.2}$$

Claim. Let $x < 5 \cdot 10^{n-1}$ be a number such that $g_n(x-1) < g_n(x) = cw_n$. Then $g_n(x+1) = g_n(x-1) = g_n(x) - 1$ for n odd, and $g_n(x+1) = g_n(x) = g_n(x-1) + 2$ for n even.

Proof of the claim. By induction on n. Clearly, the maximum of $g_1(x)$ is attained for $x = 4$ only, given by $g_1(4) = 6$. With the help of Table 1, it can be easily shown that the maximum of $g_2(x)$ is attained for all $x \in [37, 43]$ and equals 62. For $n \ge 3$, we proceed by induction. Let k be a number such that $g_{n-1}(k-1) < g_{n-1}(k) = cw_{n-1}$.

If n is odd, then $n-1$ is even and thus $g_{n-1}(k+1) = g_{n-1}(k) = cw_{n-1}$ by induction. Hence $g_{n-1}(k+1) + g_{n-1}(k) + 1 = 2 \cdot g_{n-1}(k) + 1$ and $cw_n = 2 \cdot cw_{n-1} + 1$ by Equation (4.2). Thus, for $x = 3 \cdot 10^{n-1} + 2k + 1$, we have $g_n(x) = cw_n$ by the choice of k. On the other hand, using Equation (4.2), $g_n(x+1) - g_n(x-1) = 2(g_{n-1}(k+1) - g_n(k)) = 0$. Thus $g_n(x+1) = g_n(x-1) = g_n(x) - 1$.

Similarly, if n is even, then $n-1$ is odd and thus $g_{n-1}(k+1) + g_{n-1}(k) + 1 = 2 \cdot g_{n-1}(k)$ by induction. Hence, in this case, $cw_n = 2 \cdot cw_{n-1}$ by Equation (4.2). Thus, for $x = 3 \cdot 10^{n-1} + 2k - 1$, it holds $g_n(x) = cw_n$ by the choice of k. On the other hand, $g_n(x+1) - g_n(x-1) = 2(g_{n-1}(k) - g_{n-1}(k-1)) = 2$. Thus $g_n(x+1) = g_n(x) = g_n(x-1) + 2$ and the claim follows. ∎

The proof of the claim implies

$$cw_n = \begin{cases} 5 \cdot 10^{n-1} + 2 \cdot cw_{n-1} + 1, & \text{if } n \text{ is odd}, \\ 5 \cdot 10^{n-1} + 2 \cdot cw_{n-1}, & \text{if } n \text{ is even}. \end{cases}$$

The solution of this recursion with $cw_1 = 6$ and $cw_2 = 62$ leads to the expression

$$cw_n = \begin{cases} 6 \cdot 10^{n-1} \cdot \sum_{i=0}^{(n-1)/2} (1/5)^{2i} + (2^{n-1} - 1)/3, & \text{if } n \text{ is odd}, \\ 6 \cdot 10^{n-1} \cdot \sum_{i=0}^{(n-2)/2} (1/5)^{2i} + (2^{n-1} + 1)/3, & \text{if } n \text{ is even}. \end{cases}$$

Hence the theorem. ∎

Theorem 4.2. *The wirelength of P^n is given by*

$$wl_n = (37/82) \cdot 100^n + (72/82) \cdot 18^{n-1} - (1/2) \cdot 10^n.$$

Proof. Denote $wl_n(t) = \sum_{i=1}^{t} g_n(i)$. We use the inductive structure of the order $\mathcal{P}^n$ again (cf. Figure 2). First consider the case $t \le 2 \cdot 10^{n-1}$. Now, for $t = 2k - 1$ and $A = \mathcal{F}^n(t)$ (cf. the proof of Theorem 4.1), one has

$$\begin{aligned} g_n(2k-1) &= |\partial(A_1(0)) \cap P_1^n(0)| + |\partial(A_1(1)) \cap P_1^n(1)| + 2|A| + 1 \\ &= g_{n-1}(k) + g_{n-1}(k-1) + 2(2k-1) + 1. \end{aligned}$$

Similarly, for $t = 2k$, one has $g_n(2k) = 2g_{n-1}(k) + 4k$. Therefore,

$$wl_n(2k) = 4 \cdot wl_{n-1}(k) - g_{n-1}(k) + 8\sum_{i=1}^{k} i - k.$$

This implies $wl_n(2 \cdot 10^{n-1}) = 4 \cdot wl_{n-1} + 8S - 10^{n-1}$, where $S = \sum_{i=0}^{10^{n-1}} i = (100^{n-1} + 10^{n-1})/2$.

Furthermore, for $t = 2 \cdot 10^{n-1} + k$, where $1 \le k \le 10^{n-1}$, a similar technique provides

$$wl_n(t) = wl_n(2 \cdot 10^{n-1}) + wl_{n-1}(k) + \sum_{i=1}^{k} i + 4k \cdot 10^{n-1}.$$

Therefore, $wl_n(3 \cdot 10^{n-1}) = 5 \cdot wl_{n-1} + 9S + 4 \cdot 10^{n-1} \cdot 10^{n-1} - 10^{n-1}$.

Finally, for $t = 3 \cdot 10^{n-1} + 2k$, where $1 \le k \le 10^{n-1}$, it holds

$$wl_n(t) = wl_n(3 \cdot 10^{n-1}) + 5 \cdot wl_{n-1} + 4 \cdot wl_{n-1}(k) - g_{n-1}(k) + k + 10k \cdot 10^{n-1}.$$

Therefore, $wl_n(5 \cdot 10^{n-1}) = 9 \cdot wl_{n-1} + 4 \cdot 10^{n-1} \cdot 10^{n-1} + 9S$.

Since $wl_n = 2 \cdot wl_n(5 \cdot 10^{n-1}) - g_n(5 \cdot 10^{n-1}) = 2 \cdot wl_n(5 \cdot 10^{n-1}) - 5 \cdot 10^{n-1}$, using the above formulas, we obtain

$$wl_n = 18 \cdot wl_{n-1} + 37 \cdot 100^{n-1} + 4 \cdot 10^{n-1} \quad \text{with} \quad wl_1 = 41,$$

and the theorem follows. ∎

5. Products of Petersen Powers with Hypercubes

Let Q^m be the graph of the m-dimensional hypercube which is the mth cartesian power of the clique with two vertices. Consider the edge-isoperimetric problem on the product graph $P_m^n = P^n \times Q^m$. These families of graphs are called folded Petersen cubes [16, 17] and are extensively studied to model multiprocessor interconnection networks.

5.1. Edge-Isoperimetric Problem on P_m^n

We show that the edge-isoperimetric problem on the graph P_m^n has nested solutions provided by the new order, $\mathcal{Q}_m^n$, presented below. We represent the vertices of P_m^n as $(n+m)$-dimensional vectors $(a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_m)$, where $(a_1, \ldots, a_n) \in P^n$ and $(\alpha_1, \ldots, \alpha_m) \in Q^m$. For vertices $\mathbf{a} = (a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_m)$ and $\mathbf{b} = (b_1, \ldots, b_n, \beta_1, \ldots, \beta_m)$ of $P^n \times Q^m$, we write $\mathbf{a} \succ \mathbf{b}$ in the order $\mathcal{Q}_m^n$ iff

(i) $(a_1, \ldots, a_n) > (b_1, \ldots, b_n)$ in the order $\mathcal{P}^n$, or
(ii) $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ and $(\alpha_1, \ldots, \alpha_m)$ is greater than $(\beta_1, \ldots, \beta_m)$ in the lexicographic order.

It is an easy exercise to ensure oneself that the order $\mathcal{Q}_m^n$ satisfies the consistency property [8] similar to Lemma 2.1.

We introduce compressions similar to that in Section 4. The initial segments of the order $\mathcal{Q}_m^n$ of length t will be denoted by the set $\mathcal{F}_m^n(t)$.

Theorem 5.1. *For any $n \geq 1$, $m \geq 1$ and $t = 1, \ldots, 10^n \cdot 2^m$, the set $\mathcal{F}_m^n(t)$ is optimal, where $\mathcal{F}_m^n(t)$ is represented by the initial segment of the order $\mathcal{Q}_m^n$ of length t.*

Proof. We prove by induction on $n+m$. The induction starts with $n+m = 2$. If $n = 2$, then the theorem is true by Theorem 3.1. If $m = 2$, then the theorem is obviously true as well. Let $n = m = 1$. Note that, for a compressed set $A \subseteq P_1^1$, it holds

$$|I_{P_1^1}(A)| = I_P(|\{(x,0) \in A\}|) + I_P(|\{(x,1) \in A\}|) + |\{(x,1) \in A\}|.$$

In this case the theorem is easy to verify by using Table 1.

Let us now proceed with $n+m \geq 3$. If $m = 0$, the proof follows from Theorem 3.1. If $n = 0$, then the theorem follows from the corresponding result of Harper for the hypercube Q^m [11]. So let us assume $n \geq 1$ and $m \geq 1$. Let A be a compressed optimal set, and let $\mathbf{a}$ and $\mathbf{b}$ be respectively the largest vector of A and the smallest vector of $P_m^n \setminus A$ in the order $\mathcal{Q}_m^n$. Furthermore, let these vectors be of the form

$$\mathbf{a} = (a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_m) \quad \text{and} \quad \mathbf{b} = (b_1, \ldots, b_n, \beta_1, \ldots, \beta_m).$$

If $A \neq \mathcal{F}_m^n(t)$, then $\mathbf{a} \succ \mathbf{b}$. In this case we can also assume that $a_i \neq b_i$ and $\alpha_j \neq \beta_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, otherwise $\mathbf{b} \in A$ because A is compressed. Therefore, $(\alpha_1, \ldots, \alpha_m)$ is the binary negation of $(\beta_1, \ldots, \beta_m)$.

Assume that $(\alpha_1, \ldots, \alpha_m)$ is lexicographically greater than $(\beta_1, \ldots, \beta_m)$. Then

$$\mathbf{a} = (a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_m) \succ (a_1, \ldots, a_n, \beta_1, \ldots, \beta_m) \succ (b_1, \ldots, b_n, \beta_1, \ldots, \beta_m) = \mathbf{b}.$$

This, along with the fact that A is compressed, implies $\mathbf{b} \in A$. This is a contradiction.

Now if $(\alpha_1, \ldots, \alpha_m) \neq (0, \ldots, 0)$, then let $(\alpha'_1, \ldots, \alpha'_m)$ be its predecessor in the lexicographic order. One has

$$\mathbf{a} = (a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_m) \succ (a_1, \ldots, a_n, \alpha'_1, \ldots, \alpha'_m) \succ (b_1, \ldots, b_n, \beta_1, \ldots, \beta_m) = \mathbf{b},$$

and thus $b \in A$ follows since $(\alpha'_1, \ldots, \alpha'_m)$ is not the negation of $(\beta_1, \ldots, \beta_m)$ and A is compressed. Similarly, if $(\beta_1, \ldots, \beta_m) \neq (1, \ldots, 1)$, then let $(\beta'_1, \ldots, \beta'_m)$ be its successor in the lexicographic order. One has

$$\mathbf{a} = (a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_m) \succ (b_1, \ldots, b_n, \beta'_1, \ldots, \beta'_m) \succ (b_1, \ldots, b_n, \beta_1, \ldots, \beta_m) = \mathbf{b},$$

which again implies $\mathbf{b} \in A$.

Hence, we can assume that $(\alpha_1, \ldots, \alpha_m) = (0, \ldots, 0)$ and $(\beta_1, \ldots, \beta_m) = (1, \ldots, 1)$. Consider the case when the vectors $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are not consecutive in the order $\mathcal{P}^n$. Recall that $a_1 \neq b_1$. We show that, for $n > 1$, there exists a vector $\mathbf{c} = (c_1, \ldots, c_n) \in P^n$ satisfying

$$(a_1, \ldots, a_n) > (c_1, \ldots, c_n) > (b_1, \ldots, b_n),$$

such that $a_i = c_i$ or $c_i = b_i$ for some i, $1 \leq i \leq n$. To show this, let us examine the definition of the order $\mathcal{P}^n$. If $a_1 - 1 > b_1$ then the vector $(a_1 - 1, a_2, \ldots, a_n)$ has the required property. Assume $a_1 - 1 = b_1$. If $(b_2, \ldots, b_n) \neq (9, \ldots, 9)$, then denote by $(b'_2, \ldots, b'_n)$ its successor in the order $\mathcal{P}^{n-1}$ and let $\mathbf{c} = (b_1, b'_2, \ldots, b'_n)$. If $(b_2, \ldots, b_n) =$

$(9,\dots,9)$ then denote by $(a'_2,\dots,a'_n)$ the predecessor of $(a_2,\dots,a_n)$ in the order $\mathcal{P}^{n-1}$ and assign $\mathbf{c}=(a_1,a'_2,\dots,a'_n)$. Finally, if $a_1+1=b_1$, then the vector $(b_1,b'_2,\dots,b'_n)$ leads to the solution.

Now if $n>1$ and $a_i=c_i$, then

$$\mathbf{a}=(a_1,\dots,a_n,0,\dots,0)\succ(c_1,\dots,c_n,1,\dots,1)\succ(b_1,\dots,b_n,1,\dots,1)=\mathbf{b}.$$

Since A is compressed, then $(c_1,\dots,c_n,1,\dots,1)\in A$ and thus $\mathbf{b}\in A$ which leads to a contradiction. Similarly, if $c_i=b_i$, then

$$\mathbf{a}=(a_1,\dots,a_n,0,\dots,0)\succ(c_1,\dots,c_n,0,\dots,0)\succ(b_1,\dots,b_n,1,\dots,1)=\mathbf{b}$$

and we have a contradiction too. Finally, if $n=1$ and hence $m\geq 2$, then

$$\mathbf{a}=(a_1,0,\dots,0)\succ(a_1-1,0,\dots,0,1)\succ(b_1,1,\dots,1)=\mathbf{b},$$

which implies a contradiction that $\mathbf{b}\in A$.

It remains to consider the case when the vectors $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$ are consecutive in the order $\mathcal{P}^n$. In this case, the replacement of $\mathbf{a}$ with $\mathbf{b}$ leads to a compressed set B such that

$$|I_{P_m^n}(B)|-|I_{P_m^n}(A)|=\sum_{i=1}^{n}\delta_P(b_i)-\sum_{i=1}^{n}\delta_P(a_i)+m.$$

We prove that the set B is optimal. This is obvious if $\sum_{i=1}^n\delta_P(b_i)\geq\sum_{i=1}^n\delta_P(a_i)$. However, if $\sum_{i=1}^n\delta_P(b_i)<\sum_{i=1}^n\delta_P(a_i)$, we show that these sums can differ by 1 only, which is a consequence of the following general fact:

If there exists a total order O of the vertex set of a graph G such that each of its initial segments forms an optimal subset, and if $\delta_G(i)<\delta_G(i+1)$ for some i, then $\delta_G(i)=\delta_G(i+1)-1$.

Indeed, assume the contrary, i.e., $\delta_G(i)=k$ and $\delta_G(i+1)\geq k+2$ for some k, and consider the sets S_i and S_{i+1} consisting of the first i and $i+1$ vertices of G in the order O. These sets are optimal and $|I_G(S_i)|=|I_G(S_i\setminus v_i)|+k$, where v_i is the ith vertex of G in the order O. Now

$$|I_G((S_i\setminus v_i)\cup v_{i+1})|\geq|I_G(S_i\setminus v_i)|+k+1>|I_G(S_i)|,$$

which contradicts the fact that the set S_i is optimal. Hence the proof of the theorem. ∎

5.2. Cutwidth and Wirelength of P_m^n

The simple structure of the order Q_m^n immediately derives the formulas for the cutwidth, $cw_{n,m}$, and the wirelength, $wl_{n,m}$, of the graph P_m^n. We assume that $n\geq 1$ and $m\geq 1$. Furthermore, let

$$q_m(r) \;=\; \min_{|A|=r}\big|\{(u,v)\in E_{Q^m}\;\big|\;u\in A,\;v\notin A\}\big|,$$

$$G_{n,m}(s) \;=\; \min_{|A|=s}\big|\{(u,v)\in E_{P_m^n}\;\big|\;u\in A,\;v\notin A\}\big|,$$

where the minima run over all corresponding subsets of size r. It follows from Theorem 5.1 that

$$G_{n,m}(s) = r \cdot g_n(t+1) + (2^m - r) \cdot g_n(t) + q_m(r),$$

where $s = 2^m \cdot t + r$ and $0 \le r < 2^m$. In these terms,

$$cw_{n,m} = \max_s \{G_{n,m}(s)\} \quad \text{and} \quad wl_{n,m} = \sum_{s=1}^{10^n \cdot 2^m} G_{n,m}(s).$$

Let $cw(Q^m) = \max_r\{q_m(r)\}$ be the cutwidth of the hypercube Q^m.

Now if n is even, then $g_n(t+1) = g_n(t) = cw_n$ for some t (cf. the claim in the proof of Theorem 4.2). Hence, $cw_{n,m} = 2^m \cdot cw_n + cw(Q^m)$. If n is odd, then, using the claim again, at most one of $g_n(t+1)$ and $g_n(t)$ equals cw_n. Therefore, we get

$$G_{n,m}(s) \le \begin{cases} 2^m(cw_n - 1) + q_m(t), & \text{if } g_n(t+1) < cw_n,\ g_n(t) < cw_n, \\ 2^m \cdot cw_n - t + q_m(t), & \text{if } g_n(t+1) < cw_n,\ g_n(t) = cw_n, \\ 2^m(cw_n - 1) + t + q_m(t), & \text{if } g_n(t+1) = cw_n,\ g_n(t) < cw_n. \end{cases}$$

It can be shown that the last expression is the largest if $t \ge 2^{m-1}$.

Note that $\max_{2^{m-1} \le t \le 2^m}\{t + q_m(t)\} = cw(Q^{m+1})$. Therefore, the cutwidth of P_m^n is given by

$$cw_{n,m} = \begin{cases} 2^m \cdot cw_n + cw(Q^m), & \text{if } n \text{ is even}, \\ 2^m(cw_n - 1) + cw(Q^{m+1}), & \text{if } n \text{ is odd}. \end{cases}$$

Similar arguments provide the wirelength of P_m^n as

$$wl_{n,m} = 2^m \cdot wl_n + 10^n \cdot wl(Q^m).$$

From these results one can derive formulas for $cw_{n,m}$ and $wl_{n,m}$ in terms of n and m by using Theorems 4.1 and 4.2, and known results $cw(Q^m) = (2^{m+1} - 2 + m(\bmod 2))/3$ and $wl(Q^m) = 2^{m-1}(2^m - 1)$ (cf., e.g., [4, 11]).

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