Ann. Comb.

https://doi.org/10.1007/s00026-024-00713-2

Annals of Combinatorics

A Multiparameter Refinement of Euler's Theorem

Andrew Y. Z. Wang and Lei Zhang

Abstract. Euler's partition theorem states that every integer has as many partitions into odd parts as into distinct parts. In this work, we reveal a new result behind this statement. On one hand, we study the partitions into odd parts according to the residue modulo 4 of the size of those parts occurring an odd number of times. On the other hand, we discuss the partitions into distinct parts with respect to the position of odd parts in the sequence. Some other statistics are also considered together, including the length, alternating sum and minimal odd excludant.

Mathematics Subject Classification. 05A17, 11P84.

Keywords. Euler's partition theorem, Minimal odd excludant, Refinement.

1. Introduction

A partition λ is a finite weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. The terms λ_i are called the parts of λ, and the number of parts of λ is called the length of λ, denoted $\ell(\lambda)$. The weight of λ is the sum of its parts, denoted $|\lambda|$. We say λ is a partition of n if $|\lambda| = n$.

An important part of the theory of partitions concerns partition identities, which have a long history starting with Euler's celebrated partition theorem [8].

Theorem 1.1. *The partitions of n into odd parts are equinumerous with the partitions of n into distinct parts.*

For convenience, a partition λ is called odd if each part of λ is odd, and distinct if every pair of parts differs by at least one. Let $\mathcal{O}$ and $\mathcal{D}$ denote the set of odd and distinct partitions respectively. We next discuss several statistics on partitions and study their joint distribution on $\mathcal{O}$ and $\mathcal{D}$.

Published online: 08 August 2024

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, the alternating sum of λ is defined to be

$$\ell_a(\lambda) := \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \cdots + (-1)^{l-1}\lambda_l.$$

Denote by $o(\lambda)$ the number of distinct odd parts occurring an odd number of times in λ, and denote by $n_o(\lambda)$ the number of odd parts in λ. Define the minimal odd excludant of λ, denoted $\mathrm{moe}(\lambda)$, to be the smallest positive odd integer that is not a part of λ. This newer statistic was introduced by Andrews and Newman [2] in 2019, and has many fruitful applications [3, 12, 13]. We now give an example to illustrate these statistics. Let $\lambda = (7, 7, 5, 5, 5, 3, 2, 1, 1)$. Then, $\ell_a(\lambda) = 4, o(\lambda) = 2, n_o(\lambda) = 8$, and $\mathrm{moe}(\lambda) = 9$.

Let $\mathcal{O}_{1,4}$ and $\mathcal{O}_{3,4}$ be the set of odd partitions with the minimal odd excludant congruent to 1 and 3 modulo 4, respectively. Denote by $\mathcal{D}_e$ and $\mathcal{D}_o$ the set of distinct partitions of even and odd length, respectively. The first author of the present paper and Xu [15] presented a strong refinement of Euler's identity, which reads as follows.

Theorem 1.2. *The number of odd partitions of n with length l, m parts occurring an odd number of times and the minimal odd excludant congruent to* 1 *(respectively,* 3*) modulo* 4 *is the same as the number of distinct partitions of n into an even (respectively, odd) number of parts with m odd parts and alternating sum l. Equivalently,*

$$\sum_{\lambda \in \mathcal{O}_{1,4}} x^{\ell(\lambda)} y^{o(\lambda)} q^{|\lambda|} = \sum_{\pi \in \mathcal{D}_e} x^{\ell_a(\pi)} y^{n_o(\pi)} q^{|\pi|},$$

$$\sum_{\lambda \in \mathcal{O}_{3,4}} x^{\ell(\lambda)} y^{o(\lambda)} q^{|\lambda|} = \sum_{\pi \in \mathcal{D}_o} x^{\ell_a(\pi)} y^{n_o(\pi)} q^{|\pi|}.$$

In this paper, we strengthen further Theorem 1.2 by taking into account the relationship between the statistics $o(\lambda)$ and $n_o(\pi)$. Given a partition λ, let $o_{1,4}(\lambda)$ and $o_{3,4}(\lambda)$ denote the number of distinct parts occurring an odd number of times in λ and congruent to 1 and 3 modulo 4, respectively. Clearly, $o(\lambda) = o_{1,4}(\lambda) + o_{3,4}(\lambda)$. For a partition $\pi = (\pi_1, \pi_2, \ldots, \pi_l)$, we call each π_{2i-1} an odd indexed part, and each π_{2i} an even indexed part. Namely, whether π_i is an odd or even indexed part depends on the parity of the subscript i. Let $n_{o,o}(\pi)$ and $n_{o,e}(\pi)$ denote the number of odd and even indexed odd parts of π, respectively. It is clear that $n_o(\pi) = n_{o,o}(\pi) + n_{o,e}(\pi)$.

We come to the main result of this work.

Theorem 1.3. *The number of odd partitions of n with length l, i parts congruent to* 1 *modulo* 4 *and occurring an odd number of times, j parts congruent to* 3 *modulo* 4 *and occurring an odd number of times, and the minimal odd excludant congruent to* 1 *(respectively,* 3*) modulo* 4 *is equal to the number of distinct partitions of n into an even (respectively, odd) number of parts with alternating sum l, i odd indexed odd parts and j even indexed odd parts. Equivalently,*

$$\sum_{\lambda \in \mathcal{O}_{1,4}} x^{\ell(\lambda)} y^{o_{1,4}(\lambda)} z^{o_{3,4}(\lambda)} q^{|\lambda|} = \sum_{\pi \in \mathcal{D}_e} x^{\ell_a(\pi)} y^{n_{o,o}(\pi)} z^{n_{o,e}(\pi)} q^{|\pi|},$$

$$\sum_{\lambda\in\mathcal{O}_{3,4}} x^{\ell(\lambda)}y^{o_{1,4}(\lambda)}z^{o_{3,4}(\lambda)}q^{|\lambda|} = \sum_{\pi\in\mathcal{D}_o} x^{\ell_a(\pi)}y^{n_{o,o}(\pi)}z^{n_{o,e}(\pi)}q^{|\pi|}.$$

We demonstrate Theorem 1.3 for $n = 10$ in Table 1.

Remark 1.4. Over the years, there have been many different refinements of Euler's theorem; see [5–7,9] and the references therein for more information. Theorem 1.3 is a new refinement of Euler's result.

The rest of the paper is organized as follows. In Sect. 2, we present some preliminary results, which are useful to our later proofs. The goal of Sect. 3 is to establish the generating function to keep track of the statistics on odd partitions. In Sect. 4, we find an expression for the joint distribution of the statistics on distinct partitions of fixed length, and finish the proof of Theorem 1.3.

2. Preliminaries

Throughout this paper, we use the following standard notation

$$\begin{aligned}(a;q)_0 &:= 1,\\ (a;q)_n &:= \prod_{i=0}^{n-1}(1-aq^i),\ n\geq 1,\\ (a;q)_\infty &:= \lim_{n\to\infty}(a;q)_n,\end{aligned}$$

and always assume that $|q| < 1$.

In this work, we employ some fundamental tools in the theory of q-series. Among the most useful summation formulas is the q-binomial theorem, one special case of which [1, p. 7, Eq. (1.2.4)] is stated as follows.

Lemma 2.1. *For* $|z| < \infty$,

$$\sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)/2}}{(q;q)_n} = (-z;q)_\infty. \tag{2.1}$$

We also need the second Heine transformation [1, p. 9, Corollary 1.2.4].

Lemma 2.2. *For* $|t| < 1$ *and* $|c/b| < 1$,

$${}_2\phi_1\left({a,b \atop c};q,t\right) = \frac{(c/b;q)_\infty(bt;q)_\infty}{(c;q)_\infty(t;q)_\infty}\,{}_2\phi_1\left({abt/c,b \atop bt};q,\frac{c}{b}\right), \tag{2.2}$$

where

$${}_2\phi_1\left({a,b \atop c};q,t\right) := \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}t^n.$$

Another important formula in q-series is the Rogers–Fine identity, which was first proved by Rogers [14] and rediscovered independently by Fine [10].

Table 1. An illustration of Theorem 1.3

$\lambda \in \mathcal{O}_{1,4}$	$\ell(\lambda)$	$o_{1,4}(\lambda)$	$o_{3,4}(\lambda)$	$\pi \in \mathcal{D}_e$	$\ell_a(\pi)$	$n_{o,o}(\pi)$	$n_{o,e}(\pi)$
$(7,3)$	2	0	2	$(4,3,2,1)$	2	0	2
(5^2)	2	0	0	$(6,4)$	2	0	0
$(3^3,1)$	4	1	1	$(7,3)$	4	1	1
$(3^2,1^4)$	6	0	0	$(8,2)$	6	0	0
$(3,1^7)$	8	1	1	$(9,1)$	8	1	1
$\lambda \in \mathcal{O}_{3,4}$	$\ell(\lambda)$	$o_{1,4}(\lambda)$	$o_{3,4}(\lambda)$	$\pi \in \mathcal{D}_o$	$\ell_a(\pi)$	$n_{o,o}(\pi)$	$n_{o,e}(\pi)$
$(9,1)$	2	2	0	$(5,4,1)$	2	2	0
$(7,1^3)$	4	1	1	$(5,3,2)$	4	1	1
$(5,3,1^2)$	4	1	1	$(6,3,1)$	4	1	1
$(5,1^5)$	6	2	0	$(7,2,1)$	6	2	0
(1^{10})	10	0	0	(10)	10	0	0

Lemma 2.3. *We have*

$$\sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(\beta;q)_n}\tau^n=\sum_{n=0}^{\infty}\frac{(\alpha;q)_n(\alpha\tau q/\beta;q)_n(\beta\tau)^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta;q)_n(\tau;q)_{n+1}}. \tag{2.3}$$

We now establish some preliminary results, which play a central role in our later proofs.

Lemma 2.4. *We have*

$$\sum_{k=0}^{\infty}\frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k}(-xyq)^k=\frac{1+xzq^{-1}}{1+xyq}\sum_{k=0}^{\infty}\frac{(-xz^{-1}q^3;q^4)_k}{(-xyq^5;q^4)_k}(-xzq^{-1})^k, \tag{2.4}$$

$$\sum_{k=0}^{\infty}\frac{(-xz^{-1}q^3;q^4)_k}{(-xyq^5;q^4)_k}(-xzq^3)^k=\frac{1+xyq}{1+xzq^3}\sum_{k=0}^{\infty}\frac{(-xy^{-1}q^5;q^4)_k}{(-xzq^7;q^4)_k}(-xyq)^k. \tag{2.5}$$

Proof. Replacing q by q^4 and setting $a=-xy^{-1}q, b=q^4, c=-xzq^3, t=-xyq$ in (2.2), we get

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k}(-xyq)^k\\&=\frac{(-xzq^{-1};q^4)_\infty(-xyq^5;q^4)_\infty}{(-xzq^3;q^4)_\infty(-xyq;q^4)_\infty}\sum_{k=0}^{\infty}\frac{(-xz^{-1}q^3;q^4)_k}{(-xyq^5;q^4)_k}(-xzq^{-1})^k\\&=\frac{1+xzq^{-1}}{1+xyq}\sum_{k=0}^{\infty}\frac{(-xz^{-1}q^3;q^4)_k}{(-xyq^5;q^4)_k}(-xzq^{-1})^k.\end{aligned}$$

Replacing x by xq^2 and interchanging y with z in (2.4) produces (2.5). □

Theorem 2.5. *We have*

$$\begin{aligned}&\sum_{i=0}^{\infty}\frac{(-xzq^{-1};q^4)_i x^i y^i q^{2i^2+3i}}{(x^2q^2;q^4)_i(q^4;q^4)_i}\sum_{k=0}^{\infty}\frac{(-xz^{-1}q^3;q^4)_k x^k z^k q^{2k^2+(4i+1)k}}{(x^2q^{4i+2};q^4)_k(q^4;q^4)_k}\\&=\frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty}\sum_{k=0}^{\infty}(-1)^k(xyq)^k\frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k}.\end{aligned}$$

Proof. Replacing q by q^4 and setting $a=-q^2/\tau, b=-xz^{-1}q^3, c=x^2q^{4i+2}, t=xz\tau q^{4i+1}$ in (2.2), and then letting $\tau\to 0$, we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-xz^{-1}q^3;q^4)_k x^k z^k q^{2k^2+(4i+1)k}}{(x^2q^{4i+2};q^4)_k(q^4;q^4)_k}\\&=\frac{(-xzq^{4i-1};q^4)_\infty}{(x^2q^{4i+2};q^4)_\infty}\sum_{k=0}^{\infty}(-xz^{-1}q^3;q^4)_k(-xzq^{4i-1})^k.\end{aligned}$$

Employing the above identity, we can derive that

$$\sum_{i=0}^{\infty} \frac{(-xzq^{-1};q^4)_i x^i y^i q^{2i^2+3i}}{(x^2q^2;q^4)_i(q^4;q^4)_i} \sum_{k=0}^{\infty} \frac{(-xz^{-1}q^3;q^4)_k x^k z^k q^{2k^2+(4i+1)k}}{(x^2q^{4i+2};q^4)_k(q^4;q^4)_k}$$
$$= \sum_{i=0}^{\infty} \frac{(-xzq^{-1};q^4)_i x^i y^i q^{2i^2+3i}}{(x^2q^2;q^4)_i(q^4;q^4)_i} \frac{(-xzq^{4i-1};q^4)_\infty}{(x^2q^{4i+2};q^4)_\infty} \sum_{k=0}^{\infty} (-xz^{-1}q^3;q^4)_k(-xzq^{4i-1})^k$$
$$= \frac{(-xzq^{-1};q^4)_\infty}{(x^2q^2;q^4)_\infty} \sum_{i=0}^{\infty} \frac{x^i y^i q^{2i^2+3i}}{(q^4;q^4)_i} \sum_{k=0}^{\infty} (-xz^{-1}q^3;q^4)_k(-xzq^{4i-1})^k$$
$$= \frac{(-xzq^{-1};q^4)_\infty}{(x^2q^2;q^4)_\infty} \sum_{k=0}^{\infty} (-xz^{-1}q^3;q^4)_k(-xzq^{-1})^k \sum_{i=0}^{\infty} \frac{(xyq^{4k+5})^i q^{2i^2-2i}}{(q^4;q^4)_i}$$
$$= \frac{(-xzq^{-1};q^4)_\infty}{(x^2q^2;q^4)_\infty} \sum_{k=0}^{\infty} (-xz^{-1}q^3;q^4)_k(-xyq^{4k+5};q^4)_\infty(-xzq^{-1})^k \qquad \text{(by (2.1))}$$
$$= \frac{(-xzq^{-1};q^4)_\infty(-xyq^5;q^4)_\infty}{(x^2q^2;q^4)_\infty} \sum_{k=0}^{\infty} \frac{(-xz^{-1}q^3;q^4)_k}{(-xyq^5;q^4)_k}(-xzq^{-1})^k$$
$$= \frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty} \sum_{k=0}^{\infty} \frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k}(-xyq)^k. \qquad \text{(by (2.4))}$$

The proof is complete. □

Corollary 2.6. *We have*

$$\sum_{i=0}^{\infty} \frac{(-xyq;q^4)_i x^i z^i q^{2i^2+5i}}{(x^2q^2;q^4)_{i+1}(q^4;q^4)_i} \sum_{k=0}^{\infty} \frac{(-xy^{-1}q;q^4)_{k+1} x^{k+1} y^{k+1} q^{2k^2+(4i+3)k+1}}{(x^2q^{4i+6};q^4)_k(q^4;q^4)_k}$$
$$= \frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty} \sum_{k=1}^{\infty} (-1)^{k-1}(xyq)^k \frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k}.$$

Proof. In Theorem 2.5, replacing x by xq^2 and interchanging y with z yields

$$\sum_{i=0}^{\infty} \frac{(-xyq;q^4)_i x^i z^i q^{2i^2+5i}}{(x^2q^6;q^4)_i(q^4;q^4)_i} \sum_{k=0}^{\infty} \frac{(-xy^{-1}q^5;q^4)_k x^k y^k q^{2k^2+(4i+3)k}}{(x^2q^{4i+6};q^4)_k(q^4;q^4)_k}$$
$$= \frac{(-xyq^5;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^6;q^4)_\infty} \sum_{k=0}^{\infty} (-xzq^3)^k \frac{(-xz^{-1}q^3;q^4)_k}{(-xyq^5;q^4)_k}$$
$$= \frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^6;q^4)_\infty} \sum_{k=0}^{\infty} (-xyq)^k \frac{(-xy^{-1}q^5;q^4)_k}{(-xzq^3;q^4)_{k+1}},$$

where we used (2.5) in the last step.

We then multiply both sides of the above identity by

$$\frac{xyq(1+xy^{-1}q)}{1-x^2q^2}$$

to get the desired result after $k+1$ replaced by k. □

3. Odd Partitions

Let $O_{1,4}(x,y,z;q)$ and $O_{3,4}(x,y,z;q)$ be the trivariate generating function defined by

$$O_{1,4}(x,y,z;q) := \sum_{\lambda\in\mathcal{O}_{1,4}} x^{\ell(\lambda)} y^{o_{1,4}(\lambda)} z^{o_{3,4}(\lambda)} q^{|\lambda|},$$
$$O_{3,4}(x,y,z;q) := \sum_{\lambda\in\mathcal{O}_{3,4}} x^{\ell(\lambda)} y^{o_{1,4}(\lambda)} z^{o_{3,4}(\lambda)} q^{|\lambda|}.$$

Theorem 3.1. *We have*

$$O_{1,4}(x,y,z;q) = \frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty}\sum_{k=0}^{\infty}(-1)^k(xyq)^k\frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k},$$
$$O_{3,4}(x,y,z;q) = \frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty}\sum_{k=1}^{\infty}(-1)^{k-1}(xyq)^k\frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k}.$$

Proof. The generating function for odd partitions with the minimal odd excludant $4k+1$ is

$$\begin{aligned}
&\prod_{i=1}^{k}(xyq^{4i-3}+x^2q^{2(4i-3)}+x^3yq^{3(4i-3)}+x^4q^{4(4i-3)}+\cdots)\\
&\quad\times\prod_{i=1}^{k}(xzq^{4i-1}+x^2q^{2(4i-1)}+x^3zq^{3(4i-1)}+x^4q^{4(4i-1)}+\cdots)\\
&\quad\times\prod_{i=k+1}^{\infty}(1+xyq^{4i+1}+x^2q^{2(4i+1)}+x^3yq^{3(4i+1)}+x^4q^{4(4i+1)}+\cdots)\\
&\quad\times\prod_{i=k}^{\infty}(1+xzq^{4i+3}+x^2q^{2(4i+3)}+x^3zq^{3(4i+3)}+x^4q^{4(4i+3)}+\cdots)\\
&=\prod_{i=1}^{k}\frac{xyq^{4i-3}+x^2q^{8i-6}}{1-x^2q^{8i-6}}\prod_{i=1}^{k}\frac{xzq^{4i-1}+x^2q^{8i-2}}{1-x^2q^{8i-2}}\prod_{i=k+1}^{\infty}\frac{1+xyq^{4i+1}}{1-x^2q^{8i+2}}\prod_{i=k}^{\infty}\frac{1+xzq^{4i+3}}{1-x^2q^{8i+6}}\\
&=x^{2k}y^kz^kq^{4k^2}\frac{(-xy^{-1}q;q^4)_k(-xz^{-1}q^3;q^4)_k(-xyq^{4k+5};q^4)_\infty(-xzq^{4k+3};q^4)_\infty}{(x^2q^2;q^8)_k(x^2q^6;q^8)_k(x^2q^{8k+10};q^8)_\infty(x^2q^{8k+6};q^8)_\infty}\\
&=(x^2yz)^kq^{4k^2}\\
&\quad\times\frac{(-xy^{-1}q;q^4)_k(-xz^{-1}q^3;q^4)_k(-xyq^{4k+5};q^4)_\infty(-xzq^{4k+3};q^4)_\infty(1-x^2q^{8k+2})}{(x^2q^2;q^4)_\infty}\\
&=\frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty}\frac{(-xy^{-1}q;q^4)_k(-xz^{-1}q^3;q^4)_k(1-x^2q^{8k+2})(x^2yz)^kq^{4k^2}}{(-xyq;q^4)_{k+1}(-xzq^3;q^4)_k}.
\end{aligned}$$

Summing over all k, we obtain

$$\begin{aligned}
&O_{1,4}(x,y,z;q)\\
&\quad=\sum_{k=0}^{\infty}\frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty}\\
&\qquad\times\frac{(-xy^{-1}q;q^4)_k(-xz^{-1}q^3;q^4)_k(1-x^2q^{8k+2})(x^2yz)^kq^{4k^2}}{(-xyq;q^4)_{k+1}(-xzq^3;q^4)_k}
\end{aligned}$$

$$= \frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty} \times \sum_{k=0}^{\infty} \frac{(-xy^{-1}q;q^4)_k(-xz^{-1}q^3;q^4)_k(1-x^2q^{8k+2})(x^2yz)^k q^{4k^2}}{(-xyq;q^4)_{k+1}(-xzq^3;q^4)_k}.$$

Replacing q by q^4, and letting $\alpha = -xy^{-1}q, \beta = -xzq^3, \tau = -xyq$ in the Rogers–Fine identity (2.3), we get

$$\sum_{k=0}^{\infty} \frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k}(-xyq)^k = \sum_{k=0}^{\infty} \frac{(-xy^{-1}q;q^4)_k(-xz^{-1}q^3;q^4)_k(1-x^2q^{8k+2})x^{2k}y^kz^kq^{4k^2}}{(-xyq;q^4)_{k+1}(-xzq^3;q^4)_k}.$$

We now arrive at

$$O_{1,4}(x,y,z;q) = \frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty}\sum_{k=0}^{\infty}(-xyq)^k\frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k}.$$

Similarly, we have

$$\begin{aligned} O_{3,4}(x,y,z;q) &= \frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty} \\ &\quad\times \sum_{k=0}^{\infty} \frac{(-xy^{-1}q;q^4)_{k+1}(-xz^{-1}q^3;q^4)_k(1-x^2q^{8k+6})x^{2k+1}y^{k+1}z^kq^{(2k+1)^2}}{(-xyq;q^4)_{k+1}(-xzq^3;q^4)_{k+1}} \\ &= \frac{(-xyq^5;q^4)_\infty(-xzq^3;q^4)_\infty(1+xy^{-1}q)xyq}{(x^2q^2;q^4)_\infty} \\ &\quad\times \sum_{k=0}^{\infty} \frac{(-xy^{-1}q^5;q^4)_k(-xz^{-1}q^3;q^4)_k(1-x^2q^{8k+6})x^{2k}y^kz^kq^{4k^2+4k}}{(-xyq^5;q^4)_k(-xzq^3;q^4)_{k+1}} \\ &= \frac{(-xyq^5;q^4)_\infty(-xzq^3;q^4)_\infty(1+xy^{-1}q)xyq}{(x^2q^2;q^4)_\infty}\sum_{k=0}^{\infty}\frac{(-xz^{-1}q^3;q^4)_k}{(-xyq^5;q^4)_k}(-xzq^3)^k, \end{aligned}$$

where the last equality was derived by replacing q by q^4 and setting

$$\alpha = -xz^{-1}q^3, \beta = -xyq^5, \tau = -xzq^3$$

in the Rogers–Fine identity (2.3).

Combining the above equation and (2.5) together, we get

$$\begin{aligned} O_{3,4}(x,y,z;q) &= \frac{(-xyq^5;q^4)_\infty(-xzq^3;q^4)_\infty(1+xy^{-1}q)xyq}{(x^2q^2;q^4)_\infty} \\ &\quad\times \frac{1+xyq}{1+xzq^3}\sum_{k=0}^{\infty}\frac{(-xy^{-1}q^5;q^4)_k}{(-xzq^7;q^4)_k}(-xyq)^k \\ &= \frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty}\sum_{k=0}^{\infty}\frac{(-xy^{-1}q;q^4)_{k+1}}{(-xzq^3;q^4)_{k+1}}(-1)^k(xyq)^{k+1} \\ &= \frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty}\sum_{k=1}^{\infty}(-1)^{k-1}(xyq)^k\frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k}. \end{aligned}$$

This completes the proof. □

4. Distinct Partitions

For a partition λ, let $n_{e,o}(\lambda)$ and $n_{e,e}(\lambda)$ be the number of odd and even indexed even parts of λ, respectively. Denote by $\mathcal{D}_k$ the set of distinct partitions of length k, and define

$$D_k^o(x,y,z;q) := \sum_{\lambda\in\mathcal{D}_k} x^{\ell_a(\lambda)} y^{n_{o,o}(\lambda)} z^{n_{o,e}(\lambda)} q^{|\lambda|},$$
$$D_k^e(x,y,z;q) := \sum_{\lambda\in\mathcal{D}_k} x^{\ell_a(\lambda)} y^{n_{e,o}(\lambda)} z^{n_{e,e}(\lambda)} q^{|\lambda|}.$$

It is straightforward to see that $D_0^o(x,y,z;q)=1$ and $D_0^e(x,y,z;q)=1$, and

$$\begin{aligned} D_1^o(x,y,z;q) &= \sum_{i=1}^{\infty} x^{2i-1} y q^{2i-1} + \sum_{i=1}^{\infty} x^{2i} q^{2i} = \frac{xyq}{1-x^2q^2} + \frac{x^2q^2}{1-x^2q^2} \\ &= \frac{xyq(1+xy^{-1}q)}{1-x^2q^2}, \\ D_1^e(x,y,z;q) &= \sum_{i=1}^{\infty} x^{2i-1} q^{2i-1} + \sum_{i=1}^{\infty} x^{2i} y q^{2i} = \frac{xq}{1-x^2q^2} + \frac{x^2yq^2}{1-x^2q^2} \\ &= \frac{xq(1+xyq)}{1-x^2q^2}. \end{aligned}$$

We next deduce the recurrence for $D_k^o(x,y,z;q)$ and $D_k^e(x,y,z;q)$.

Lemma 4.1. *For $k\geq 1$, we have*

$$D_{2k}^o(x,y,z;q) = \frac{zq^{2k}}{1-q^{4k}} D_{2k-1}^e(x,y,z;q) + \frac{q^{4k}}{1-q^{4k}} D_{2k-1}^o(x,y,z;q), \quad (4.1)$$

$$D_{2k}^e(x,y,z;q) = \frac{q^{2k}}{1-q^{4k}} D_{2k-1}^o(x,y,z;q) + \frac{zq^{4k}}{1-q^{4k}} D_{2k-1}^e(x,y,z;q), \quad (4.2)$$

$$\begin{aligned} D_{2k-1}^o(x,y,z;q) &= \frac{xyq^{2k-1}}{1-x^2q^{4k-2}} D_{2k-2}^e(x,y,z;q) \\ &\quad + \frac{x^2q^{4k-2}}{1-x^2q^{4k-2}} D_{2k-2}^o(x,y,z;q), \end{aligned} \quad (4.3)$$

$$\begin{aligned} D_{2k-1}^e(x,y,z;q) &= \frac{xq^{2k-1}}{1-x^2q^{4k-2}} D_{2k-2}^o(x,y,z;q) \\ &\quad + \frac{x^2yq^{4k-2}}{1-x^2q^{4k-2}} D_{2k-2}^e(x,y,z;q). \end{aligned} \quad (4.4)$$

Proof. Given a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{2k})\in\mathcal{D}_{2k}$, define π to be the partition

$$(\lambda_1-\lambda_{2k},\lambda_2-\lambda_{2k},\ldots,\lambda_{2k-1}-\lambda_{2k}).$$

Clearly, $\pi\in\mathcal{D}_{2k-1}$.

It is easy to see that $|\lambda| = |\pi| + 2k\lambda_{2k}$ and $\ell_a(\lambda) = \ell_a(\pi)$. Moreover, we have

$$n_{o,o}(\lambda) = \begin{cases} n_{e,o}(\pi), & \text{if } \lambda_{2k} \text{ is odd;} \\ n_{o,o}(\pi), & \text{if } \lambda_{2k} \text{ is even,} \end{cases}$$

and

$$n_{o,e}(\lambda) = \begin{cases} n_{e,e}(\pi) + 1, & \text{if } \lambda_{2k} \text{ is odd;} \\ n_{o,e}(\pi), & \text{if } \lambda_{2k} \text{ is even.} \end{cases}$$

We now can conclude that

$$\begin{aligned} \sum_{\lambda \in \mathcal{D}_{2k}} x^{\ell_a(\lambda)} y^{n_{o,o}(\lambda)} z^{n_{o,e}(\lambda)} q^{|\lambda|} &= \sum_{i=1}^{\infty} z q^{2k(2i-1)} \sum_{\pi \in \mathcal{D}_{2k-1}} x^{\ell_a(\pi)} y^{n_{e,o}(\pi)} z^{n_{e,e}(\pi)} q^{|\pi|} \\ &\quad + \sum_{i=1}^{\infty} q^{2k \times 2i} \sum_{\pi \in \mathcal{D}_{2k-1}} x^{\ell_a(\pi)} y^{n_{o,o}(\pi)} z^{n_{o,e}(\pi)} q^{|\pi|} \\ &= \frac{zq^{2k}}{1-q^{4k}} D^{e}_{2k-1}(x,y,z;q) \\ &\quad + \frac{q^{4k}}{1-q^{4k}} D^{o}_{2k-1}(x,y,z;q), \end{aligned}$$

which shows that (4.1) is true. Similarly, we can prove (4.2) and omit the details here.

Letting $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k-1}) \in \mathcal{D}_{2k-1}$, define π to be the partition

$$(\lambda_1 - \lambda_{2k-1}, \lambda_2 - \lambda_{2k-1}, \ldots, \lambda_{2k-2} - \lambda_{2k-1}).$$

Obviously, $\pi \in \mathcal{D}_{2k-2}$.

It is not hard to see that $|\lambda| = |\pi| + (2k-1)\lambda_{2k-1}$ and $\ell_a(\lambda) = \ell_a(\pi) + \lambda_{2k-1}$. Furthermore, we have

$$n_{o,o}(\lambda) = \begin{cases} n_{e,o}(\pi) + 1, & \text{if } \lambda_{2k-1} \text{ is odd;} \\ n_{o,o}(\pi), & \text{if } \lambda_{2k-1} \text{ is even,} \end{cases}$$

and

$$n_{o,e}(\lambda) = \begin{cases} n_{e,e}(\pi), & \text{if } \lambda_{2k-1} \text{ is odd;} \\ n_{o,e}(\pi), & \text{if } \lambda_{2k-1} \text{ is even.} \end{cases}$$

We now can derive that

$$\begin{aligned} \sum_{\lambda \in \mathcal{D}_{2k-1}} x^{\ell_a(\lambda)} y^{n_{o,o}(\lambda)} z^{n_{o,e}(\lambda)} q^{|\lambda|} &= \sum_{i=1}^{\infty} x^{2i-1} y q^{(2k-1)(2i-1)} \\ &\quad \times \sum_{\pi \in \mathcal{D}_{2k-2}} x^{\ell_a(\pi)} y^{n_{e,o}(\pi)} z^{n_{e,e}(\pi)} q^{|\pi|} \\ &\quad + \sum_{i=1}^{\infty} x^{2i} q^{(2k-1)\times 2i} \\ &\quad \times \sum_{\pi \in \mathcal{D}_{2k-2}} x^{\ell_a(\pi)} y^{n_{o,o}(\pi)} z^{n_{o,e}(\pi)} q^{|\pi|} \end{aligned}$$

$$= \frac{xyq^{2k-1}}{1-x^2q^{4k-2}} D^e_{2k-2}(x,y,z;q) + \frac{x^2q^{4k-2}}{1-x^2q^{4k-2}} D^o_{2k-2}(x,y,z;q),$$

which proves (4.3). Similarly, we can show that (4.4) is true, and we omit the details here. □

Theorem 4.2. *For $k \geq 0$, we have*

$$D^o_{2k}(x,y,z;q) = \frac{x^k q^{k(2k+1)}}{(x^2q^2;q^4)_k} \sum_{i=0}^{k} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}} y^i z^{k-i} q^{2i}, \tag{4.5}$$

$$D^e_{2k}(x,y,z;q) = \frac{x^k q^{k(2k+1)}}{(x^2q^2;q^4)_k} \sum_{i=0}^{k} \frac{(-xzq^{3};q^4)_i(-xz^{-1}q^{-1};q^4)_{k-i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}} y^i z^{k-i} q^{2(k-i)}, \tag{4.6}$$

$$D^o_{2k+1}(x,y,z;q) = \frac{x^{k+1} q^{(k+1)(2k+1)}}{(x^2q^2;q^4)_{k+1}} \sum_{i=0}^{k} \frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i+1}}{(q^4;q^4)_i(q^4;q^4)_{k-i}} y^{k-i+1} z^{i} q^{2i}, \tag{4.7}$$

$$D^e_{2k+1}(x,y,z;q) = \frac{x^{k+1} q^{(k+1)(2k+1)}}{(x^2q^2;q^4)_{k+1}} \sum_{i=0}^{k} \frac{(-xyq;q^4)_{i+1}(-xy^{-1}q;q^4)_{k-i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}} y^{k-i} z^{i} q^{2(k-i)}. \tag{4.8}$$

Proof. We proceed by induction on k. It is trivial for the case where $k = 0$.

We first prove (4.5) and (4.6). It follows from Lemma 4.1 that

$$\begin{bmatrix} D^o_{2k}(x,y,z;q) \\ D^e_{2k}(x,y,z;q) \end{bmatrix} = \frac{q^{2k}}{1-q^{4k}} \begin{bmatrix} q^{2k} & z \\ 1 & zq^{2k} \end{bmatrix} \begin{bmatrix} D^o_{2k-1}(x,y,z;q) \\ D^e_{2k-1}(x,y,z;q) \end{bmatrix} \tag{4.9}$$

and

$$\begin{bmatrix} D^o_{2k-1}(x,y,z;q) \\ D^e_{2k-1}(x,y,z;q) \end{bmatrix} = \frac{xq^{2k-1}}{1-x^2q^{4k-2}} \begin{bmatrix} xq^{2k-1} & y \\ 1 & xyq^{2k-1} \end{bmatrix} \begin{bmatrix} D^o_{2k-2}(x,y,z;q) \\ D^e_{2k-2}(x,y,z;q) \end{bmatrix},$$

where we used the matrix notation and matrix product. We now conclude that

$$\begin{bmatrix} D^o_{2k}(x,y,z;q) \\ D^e_{2k}(x,y,z;q) \end{bmatrix} = \begin{bmatrix} \frac{x^2q^{8k-2}+xzq^{4k-1}}{(1-q^{4k})(1-x^2q^{4k-2})} & \frac{xyq^{6k-1}+x^2yzq^{6k-2}}{(1-q^{4k})(1-x^2q^{4k-2})} \\ \frac{x^2q^{6k-2}+xzq^{6k-1}}{(1-q^{4k})(1-x^2q^{4k-2})} & \frac{xyq^{4k-1}+x^2yzq^{8k-2}}{(1-q^{4k})(1-x^2q^{4k-2})} \end{bmatrix} \begin{bmatrix} D^o_{2k-2}(x,y,z;q) \\ D^e_{2k-2}(x,y,z;q) \end{bmatrix}.$$

We now see that $D^o_{2k}(x,y,z;q)$ satisfies

$$\begin{aligned} D^o_{2k}(x,y,z;q) &= \frac{x^2q^{8k-2}+xzq^{4k-1}}{(1-q^{4k})(1-x^2q^{4k-2})} D^o_{2k-2}(x,y,z;q) \\ &\quad + \frac{xyq^{6k-1}+x^2yzq^{6k-2}}{(1-q^{4k})(1-x^2q^{4k-2})} D^e_{2k-2}(x,y,z;q) \\ &= \frac{(xq^{4k-1}+z)xq^{4k-1}}{(1-q^{4k})(1-x^2q^{4k-2})} \frac{x^{k-1}q^{2k^2-3k+1}}{(x^2q^2;q^4)_{k-1}} \\ &\quad \times \sum_{i=0}^{k-1} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i-1} y^i z^{k-i-1} q^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\ &\quad + \frac{(yq^{2k}+xyzq^{2k-1})xq^{4k-1}}{(1-q^{4k})(1-x^2q^{4k-2})} \frac{x^{k-1}q^{2k^2-3k+1}}{(x^2q^2;q^4)_{k-1}} \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{i=0}^{k-1} \frac{(-xzq^3;q^4)_i(-xz^{-1}q^{-1};q^4)_{k-i-1}y^iz^{k-i-1}q^{2(k-i-1)}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\
&= \frac{x^kq^{k(2k+1)}(1+xz^{-1}q^{4k-1})}{(x^2q^2;q^4)_k(1-q^{4k})} \\
&\quad \times \sum_{i=0}^{k-1} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i-1}y^iz^{k-i}q^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\
&\quad + \frac{x^kq^{k(2k+1)}}{(x^2q^2;q^4)_k(1-q^{4k})} \sum_{i=0}^{k-1} \frac{(-xzq^{-1};q^4)_{i+1}(-xz^{-1}q^{-1};q^4)_{k-i-1}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\
&\quad \times y^{i+1}z^{k-i-1}q^{4k-2i-2} \\
&= \frac{x^kq^{k(2k+1)}(1+xz^{-1}q^{4k-1})}{(x^2q^2;q^4)_k(1-q^{4k})} \\
&\quad \times \sum_{i=0}^{k-1} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i-1}y^iz^{k-i}q^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\
&\quad + \frac{x^kq^{k(2k+1)}}{(x^2q^2;q^4)_k(1-q^{4k})} \\
&\quad \times \sum_{i=1}^{k} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^{-1};q^4)_{k-i}y^iz^{k-i}q^{4k-2i}}{(q^4;q^4)_{i-1}(q^4;q^4)_{k-i}} \\
&= \frac{x^kq^{k(2k+1)}}{(x^2q^2;q^4)_k(1-q^{4k})} \Bigg(\frac{(-xz^{-1}q^3;q^4)_{k-1}(1+xz^{-1}q^{4k-1})z^k}{(q^4;q^4)_{k-1}} \\
&\quad + (1+xz^{-1}q^{4k-1}) \\
&\quad \times \sum_{i=1}^{k-1} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i-1}y^iz^{k-i}q^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\
&\quad + \sum_{i=1}^{k-1} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^{-1};q^4)_{k-i}y^iz^{k-i}q^{4k-2i}}{(q^4;q^4)_{i-1}(q^4;q^4)_{k-i}} \\
&\quad + \frac{(-xzq^{-1};q^4)_ky^kq^{2k}}{(q^4;q^4)_{k-1}} \Bigg) \\
&= \frac{x^kq^{k(2k+1)}}{(x^2q^2;q^4)_k(1-q^{4k})} \Bigg(\frac{(-xz^{-1}q^3;q^4)_kz^k}{(q^4;q^4)_{k-1}} + \frac{(-xzq^{-1};q^4)_ky^kq^{2k}}{(q^4;q^4)_{k-1}} \\
&\quad + (1+xz^{-1}q^{4k-1}) \sum_{i=1}^{k-1} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i-1}y^iz^{k-i}q^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\
&\quad + (1+xz^{-1}q^{-1}) \sum_{i=1}^{k-1} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i-1}y^iz^{k-i}q^{4k-2i}}{(q^4;q^4)_{i-1}(q^4;q^4)_{k-i}} \Bigg) \\
&= \frac{x^kq^{k(2k+1)}}{(x^2q^2;q^4)_k(1-q^{4k})} \Bigg(\frac{(-xz^{-1}q^3;q^4)_kz^k}{(q^4;q^4)_{k-1}} + \frac{(-xzq^{-1};q^4)_ky^kq^{2k}}{(q^4;q^4)_{k-1}} \\
&\quad + \sum_{i=1}^{k-1} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i-1}y^iz^{k-i}q^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}} \\
&\quad \times \Big((1+xz^{-1}q^{4k-1})(1-q^{4k-4i}) + (1+xz^{-1}q^{-1})(1-q^{4i})q^{4k-4i} \Big) \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{x^k q^{k(2k+1)}}{(x^2q^2;q^4)_k(1-q^{4k})}\Bigg(\frac{(-xz^{-1}q^3;q^4)_k z^k}{(q^4;q^4)_{k-1}} + \frac{(-xzq^{-1};q^4)_k y^k q^{2k}}{(q^4;q^4)_{k-1}} \\
&\quad + \sum_{i=1}^{k-1} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i-1} y^i z^{k-i} q^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}} \\
&\quad \times (1-q^{4k})(1+xz^{-1}q^{4k-4i-1})\Bigg) \\
&= \frac{x^k q^{k(2k+1)}}{(x^2q^2;q^4)_k(1-q^{4k})}\Bigg(\frac{(-xz^{-1}q^3;q^4)_k z^k}{(q^4;q^4)_{k-1}} + \frac{(-xzq^{-1};q^4)_k y^k q^{2k}}{(q^4;q^4)_{k-1}} \\
&\quad + \sum_{i=1}^{k-1} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i} y^i z^{k-i} q^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}}(1-q^{4k})\Bigg) \\
&= \frac{x^k q^{k(2k+1)}}{(x^2q^2;q^4)_k}\Bigg(\frac{(-xz^{-1}q^3;q^4)_k z^k}{(q^4;q^4)_k} + \frac{(-xzq^{-1};q^4)_k y^k q^{2k}}{(q^4;q^4)_k} \\
&\quad + \sum_{i=1}^{k-1} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i} y^i z^{k-i} q^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}}\Bigg) \\
&= \frac{x^k q^{k(2k+1)}}{(x^2q^2;q^4)_k} \sum_{i=0}^{k} \frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}} y^i z^{k-i} q^{2i}.
\end{aligned}
$$

Thus, (4.5) is true. We can prove (4.6) in a similar manner, and omit the details here.

We next turn to (4.7) and (4.8). It follows from (4.3) and (4.4) that

$$
\begin{bmatrix} D^o_{2k+1}(x,y,z;q) \\ D^e_{2k+1}(x,y,z;q) \end{bmatrix} = \frac{xq^{2k+1}}{1-x^2q^{4k+2}} \begin{bmatrix} xq^{2k+1} & y \\ 1 & xyq^{2k+1} \end{bmatrix} \begin{bmatrix} D^o_{2k}(x,y,z;q) \\ D^e_{2k}(x,y,z;q) \end{bmatrix}.
$$

Combining the above equation and (4.9) together, we obtain

$$
\begin{bmatrix} D^o_{2k+1}(x,y,z;q) \\ D^e_{2k+1}(x,y,z;q) \end{bmatrix} = \begin{bmatrix} \frac{xyq^{4k+1}(1+xy^{-1}q^{4k+1})}{(1-q^{4k})(1-x^2q^{4k+2})} & \frac{xyzq^{6k+1}(1+xy^{-1}q)}{(1-q^{4k})(1-x^2q^{4k+2})} \\ \frac{xq^{6k+1}(1+xyq)}{(1-q^{4k})(1-x^2q^{4k+2})} & \frac{xzq^{4k+1}(1+xyq^{4k+1})}{(1-q^{4k})(1-x^2q^{4k+2})} \end{bmatrix} \begin{bmatrix} D^o_{2k-1}(x,y,z;q) \\ D^e_{2k-1}(x,y,z;q) \end{bmatrix}.
$$

Now, we can verify that

$$
\begin{aligned}
D^o_{2k+1}(x,y,z;q) &= \frac{xyq^{4k+1}(1+xy^{-1}q^{4k+1})}{(1-q^{4k})(1-x^2q^{4k+2})} D^o_{2k-1}(x,y,z;q) \\
&\quad + \frac{xyzq^{6k+1}(1+xy^{-1}q)}{(1-q^{4k})(1-x^2q^{4k+2})} D^e_{2k-1}(x,y,z;q) \\
&= \frac{xyq^{4k+1}(1+xy^{-1}q^{4k+1})}{(1-q^{4k})(1-x^2q^{4k+2})} \frac{x^k q^{k(2k-1)}}{(x^2q^2;q^4)_k} \\
&\quad \times \sum_{i=0}^{k-1} \frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i} y^{k-i} z^i q^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\
&\quad + \frac{xyzq^{6k+1}(1+xy^{-1}q)}{(1-q^{4k})(1-x^2q^{4k+2})} \frac{x^k q^{k(2k-1)}}{(x^2q^2;q^4)_k} \\
&\quad \times \sum_{i=0}^{k-1} \frac{(-xyq;q^4)_{i+1}(-xy^{-1}q;q^4)_{k-i-1} y^{k-i-1} z^i q^{2(k-i-1)}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{x^{k+1}q^{2k^2+3k+1}(1+xy^{-1}q^{4k+1})}{(1-q^{4k})(x^2q^2;q^4)_{k+1}} \\
&\quad \times \sum_{i=0}^{k-1} \frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i}y^{k-i+1}z^iq^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\
&\quad + \frac{x^{k+1}q^{2k^2+3k+1}(1+xy^{-1}q)}{(1-q^{4k})(x^2q^2;q^4)_{k+1}} \\
&\quad \times \sum_{i=0}^{k-1} \frac{(-xyq;q^4)_{i+1}(-xy^{-1}q;q^4)_{k-i-1}y^{k-i}z^{i+1}q^{4k-2i-2}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\
&= \frac{x^{k+1}q^{2k^2+3k+1}(1+xy^{-1}q^{4k+1})}{(1-q^{4k})(x^2q^2;q^4)_{k+1}} \\
&\quad \times \sum_{i=0}^{k-1} \frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i}y^{k-i+1}z^iq^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\
&\quad + \frac{x^{k+1}q^{2k^2+3k+1}(1+xy^{-1}q)}{(1-q^{4k})(x^2q^2;q^4)_{k+1}} \\
&\quad \times \sum_{i=1}^{k} \frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i}y^{k-i+1}z^iq^{4k-2i}}{(q^4;q^4)_{i-1}(q^4;q^4)_{k-i}} \\
&= \frac{x^{k+1}q^{2k^2+3k+1}}{(1-q^{4k})(x^2q^2;q^4)_{k+1}} \left(\frac{(1+xy^{-1}q^{4k+1})(-xy^{-1}q;q^4)_k y^{k+1}}{(q^4;q^4)_{k-1}} \right. \\
&\quad + (1+xy^{-1}q^{4k+1}) \\
&\quad \times \sum_{i=1}^{k-1} \frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i}y^{k-i+1}z^iq^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i-1}} \\
&\quad + (1+xy^{-1}q) \\
&\quad \times \sum_{i=1}^{k-1} \frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i}y^{k-i+1}z^iq^{4k-2i}}{(q^4;q^4)_{i-1}(q^4;q^4)_{k-i}} \\
&\quad \left. + \frac{(1+xy^{-1}q)(-xyq;q^4)_k yz^kq^{2k}}{(q^4;q^4)_{k-1}} \right) \\
&= \frac{x^{k+1}q^{2k^2+3k+1}}{(1-q^{4k})(x^2q^2;q^4)_{k+1}} \left(\frac{(1+xy^{-1}q^{4k+1})(-xy^{-1}q;q^4)_k y^{k+1}}{(q^4;q^4)_{k-1}} \right. \\
&\quad + \sum_{i=1}^{k-1} \frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i}y^{k-i+1}z^iq^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}} \\
&\quad \times \left((1+xy^{-1}q^{4k+1})(1-q^{4k-4i}) + (1+xy^{-1}q)(1-q^{4i})q^{4k-4i}\right) \\
&\quad \left. + \frac{(1+xy^{-1}q)(-xyq;q^4)_k yz^kq^{2k}}{(q^4;q^4)_{k-1}} \right) \\
&= \frac{x^{k+1}q^{2k^2+3k+1}}{(1-q^{4k})(x^2q^2;q^4)_{k+1}} \left(\frac{(1+xy^{-1}q^{4k+1})(-xy^{-1}q;q^4)_k y^{k+1}}{(q^4;q^4)_{k-1}} \right. \\
&\quad + \sum_{i=1}^{k-1} \frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i}y^{k-i+1}z^iq^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}}
\end{aligned}
$$

$$
\begin{aligned}
&\times (1-q^{4k})(1+xy^{-1}q^{4k-4i+1}) \\
&+\frac{(1+xy^{-1}q)(-xyq;q^4)_k yz^k q^{2k}}{(q^4;q^4)_{k-1}}\Bigg) \\
=&\frac{x^{k+1}q^{2k^2+3k+1}}{(1-q^{4k})(x^2q^2;q^4)_{k+1}}\Bigg(\frac{(1+xy^{-1}q^{4k+1})(-xy^{-1}q;q^4)_k y^{k+1}}{(q^4;q^4)_{k-1}} \\
&+\sum_{i=1}^{k-1}\frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i+1}y^{k-i+1}z^iq^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}}(1-q^{4k}) \\
&+\frac{(1+xy^{-1}q)(-xyq;q^4)_k yz^k q^{2k}}{(q^4;q^4)_{k-1}}\Bigg) \\
=&\frac{x^{k+1}q^{2k^2+3k+1}}{(x^2q^2;q^4)_{k+1}}\Bigg(\frac{(-xy^{-1}q;q^4)_{k+1}}{(q^4;q^4)_k}y^{k+1} \\
&+\sum_{i=1}^{k-1}\frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i+1}}{(q^4;q^4)_i(q^4;q^4)_{k-i}}y^{k-i+1}z^iq^{2i} \\
&+\frac{(1+xy^{-1}q)(-xyq;q^4)_k}{(q^4;q^4)_k}yz^kq^{2k}\Bigg) \\
=&\frac{x^{k+1}q^{(k+1)(2k+1)}}{(x^2q^2;q^4)_{k+1}}\sum_{i=0}^{k}\frac{(-xyq;q^4)_i(-xy^{-1}q;q^4)_{k-i+1}}{(q^4;q^4)_i(q^4;q^4)_{k-i}}y^{k-i+1}z^iq^{2i}.
\end{aligned}
$$

Therefore, (4.7) is true. A similar argument yields (4.8). □

As a consequence of Theorem 4.2, we have the following result.

Corollary 4.3. *We have*

$$
\sum_{k=0}^{\infty}D^o_{2k}(x,y,z;q)=\frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty}\sum_{k=0}^{\infty}(-1)^k(xyq)^k\frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k},
$$

$$
\sum_{k=0}^{\infty}D^o_{2k+1}(x,y,z;q)=\frac{(-xyq;q^4)_\infty(-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty}\sum_{k=1}^{\infty}(-1)^{k-1}(xyq)^k\frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k}.
$$

Proof. Employing (4.5), it is routine to check that

$$
\begin{aligned}
\sum_{k=0}^{\infty}D^o_{2k}(x,y,z;q)&=\sum_{k=0}^{\infty}\frac{x^kq^{k(2k+1)}}{(x^2q^2;q^4)_k}\sum_{i=0}^{k}\frac{(-xzq^{-1};q^4)_i(-xz^{-1}q^3;q^4)_{k-i}y^iz^{k-i}q^{2i}}{(q^4;q^4)_i(q^4;q^4)_{k-i}} \\
&=\sum_{i=0}^{\infty}\frac{(-xzq^{-1};q^4)_iy^iz^{-i}q^{2i}}{(q^4;q^4)_i} \\
&\quad\times\sum_{k=i}^{\infty}\frac{(-xz^{-1}q^3;q^4)_{k-i}x^kz^kq^{k(2k+1)}}{(x^2q^2;q^4)_k(q^4;q^4)_{k-i}} \\
&=\sum_{i=0}^{\infty}\frac{(-xzq^{-1};q^4)_iy^iz^{-i}q^{2i}}{(q^4;q^4)_i} \\
&\quad\times\sum_{k=0}^{\infty}\frac{(-xz^{-1}q^3;q^4)_kx^{k+i}z^{k+i}q^{(k+i)(2k+2i+1)}}{(x^2q^2;q^4)_{k+i}(q^4;q^4)_k}
\end{aligned}
$$

$$= \sum_{i=0}^{\infty} \frac{(-xzq^{-1};q^4)_i x^i y^i q^{2i^2+3i}}{(x^2q^2;q^4)_i (q^4;q^4)_i}$$
$$\times \sum_{k=0}^{\infty} \frac{(-xz^{-1}q^3;q^4)_k x^k z^k q^{2k^2+(4i+1)k}}{(x^2q^{4i+2};q^4)_k (q^4;q^4)_k}$$
$$= \frac{(-xyq;q^4)_\infty (-xzq^3;q^4)_\infty}{(x^2q^2;q^4)_\infty} \sum_{k=0}^{\infty} (-1)^k (xyq)^k \frac{(-xy^{-1}q;q^4)_k}{(-xzq^3;q^4)_k},$$

where the last equality follows from Theorem 2.5.

Similarly, we can show that

$$\sum_{k=0}^{\infty} D^o_{2k+1}(x,y,z;q) = \sum_{i=0}^{\infty} \frac{(-xyq;q^4)_i x^i z^i q^{2i^2+5i}}{(x^2q^2;q^4)_{i+1} (q^4;q^4)_i}$$
$$\times \sum_{k=0}^{\infty} \frac{(-xy^{-1}q;q^4)_{k+1} (xy)^{k+1} q^{2k^2+(4i+3)k+1}}{(x^2q^{4i+6};q^4)_k (q^4;q^4)_k}.$$

Applying Corollary 2.6 to the above equality yields the second equation. □

We conclude this section with the following result, which shows Theorem 1.3 is true.

Corollary 4.4. *We have*

$$O_{1,4}(x,y,z;q) = \sum_{k=0}^{\infty} D^o_{2k}(x,y,z;q),$$
$$O_{3,4}(x,y,z;q) = \sum_{k=0}^{\infty} D^o_{2k+1}(x,y,z;q).$$

Proof. The desired result follows immediately from Theorem 3.1 and Corollary 4.3. □

5. Concluding Remarks

It was pointed out by one anonymous referee that Berkovich and Uncu [4] have discussed the location of odd parts in distinct partitions. They established the following elegant result.

Theorem 5.1. *The number of distinct partitions of n with i odd indexed odd parts and j even indexed odd parts is the same as the number of distinct partitions of n with i parts congruent to* 1 *modulo* 4 *and j parts congruent to* 3 *modulo* 4*.*

As an immediate consequence of Theorem 1.3 and Theorem 5.1, we get a simple refinement of Euler's partition theorem.

Corollary 5.2. *The set of odd partitions of n with i different parts congruent to* 1 *modulo* 4 *and occurring an odd number of times, and j different parts congruent to* 3 *modulo* 4 *and occurring an odd number of times is equinumerous with the set of distinct partitions of n with i parts that are congruent to* 1 *modulo* 4*, and j parts that are congruent to* 3 *modulo* 4*.*

In fact, Corollary 5.2 can be easily shown by applying Glaisher's merging/splitting proof of Theorem 1.1. See [11] for details. If we take other pairs of statistics into account, such as the length of odd partitions and the alternating sum of distinct partitions or the minimal odd excludant of odd partitions and the length of distinct partitions, Corollary 5.2 is no longer true. Namely, Corollary 5.2 cannot be strengthened to be similar in flavor to Theorem 1.3.

Acknowledgements

The authors would like to thank the anonymous referees for reading the manuscript carefully and providing helpful comments.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

References

[1] G.E. Andrews and B.C. Berndt, Ramanujan's Lost Notebook, Part II, Springer, 2009.

[2] G.E. Andrews and D. Newman, Partitions and the minimal excludant, Ann. Comb., 23 (2019), 249–254.

[3] G.E. Andrews and D. Newman, The minimal excludant in integer partitions, J. Integer Seq., 23 (2020), Article 20.2.3.

[4] A. Berkovich and A.K. Uncu, On partitions with fixed number of even-indexed and odd-indexed odd parts, J. Number Theory, 167 (2016), 7–30.

[5] C. Bessenrodt, A bijection for Lebesgue's partition identity in the spirit of Sylvester, Discrete Math., 132 (1994), 1–10.

[6] M. Bousquet-Mélou and K. Eriksson, Lecture hall partitions, Ramanujan J., 1 (1997), 101–110.

[7] W.Y.C. Chen, H.Y. Gao, K.Q. Ji and M.Y.X. Li, A unification of two refinements of Euler's partition theorem, Ramanujan J., 23 (2010), 137–149.

[8] L. Euler, Introductio in Analysin Infinitorum, Vol. 1, Marcum-Michaelum Bousquet, 1748.

[9] N.J. Fine, Some new results on partitions, Proc. Natl. Acad. Sci. USA, 34 (1948), 616–618.

[10] N.J. Fine, Basic Hypergeometric Series and Applications, American Mathematical Society, 1988.

[11] J.W.L. Glaisher, A theorem in partitions, Messenger Math., 12 (1883), 158–170.

[12] J.Y. Kang, R.Q. Li and A.Y.Z. Wang, A new refinement of Fine's partition theorem, Bull. Aust. Math. Soc., 104 (2021), 353–361.

[13] J.Y. Kang, R.Q. Li and A.Y.Z. Wang, Partition identities related to the minimal excludant, Discrete Math., 346 (2023), Article 113302.

[14] L.J. Rogers, On two theorems of combinatory analysis and some allied identities, Proc. Lond. Math. Soc., S2-16 (1917), 315–336.

[15] A.Y.Z. Wang and Z. Xu, The minimal odd excludant and Euler's partition theorem, Int. J. Number Theory, 20 (2024), 1445–1461.

Andrew Y. Z. Wang and Lei Zhang
School of Mathematical Sciences
University of Electronic Science and Technology of China
Chengdu 611731
People's Republic of China
e-mail: yzwang@uestc.edu.cn

Lei Zhang
e-mail: zhlmath@outlook.com

Communicated by Jeremy Lovejoy
Received: 16 January 2024.
Accepted: 24 July 2024.