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Symmetry Parameters of Two-Generator Circulant Graphs

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Abstract. The derived graph of a voltage graph consisting of a single vertex and two loops of different voltages is a circulant graph with two generators. We characterize the automorphism groups of connected, two-generator circulant graphs, and give their determining and distinguishing number, and when relevant, their cost of 2-distinguishing. We do the same for the subdivisions of connected, two-generator circulant graphs obtained by replacing one loop in the voltage graph with a directed cycle.

Mathematics Subject Classification. 05C25, 05C25, 05C69.

Keywords. Circulant graphs, Determining number, Distinguishing number, Cost of 2-distinguishing.

1. Introduction

A voltage graph consists of a base directed graph $D = (V, E)$, a group Γ, and a voltage function $\phi : E \to \Gamma$. The associated derived directed graph D^ϕ has vertex set $\{u_a \mid u \in V, a \in \Gamma\}$ and arc set $\{e_a \mid e \in E, a \in \Gamma\}$; if $e = (u, v)$ and $\phi(e) = b$, then $e_a = (u_a, v_{ab})$. For more background on the voltage graph construction, see [12]. A particularly simple example has a base directed graph consisting of a single vertex u and two directed loops (sometimes called a bouquet), denoted B_2, and group $\Gamma = \mathbb{Z}_n$. Because the base graph has only one vertex, we can denote vertices in the derived graph simply as elements of $\mathbb{Z}_n$. We will denote an element of $\mathbb{Z}_n$ with an integer representative; for $a, b \in \mathbb{Z}$, we use the notation $a \equiv b$ to denote equality of the equivalence classes in $\mathbb{Z}_n$. The voltages on the two loops are denoted i and j. Then, the underlying undirected graph of the associated derived graph has vertex set $\mathbb{Z}_n$, with $a, b \in \mathbb{Z}_n$ adjacent if and only if $a - b \equiv \pm i$ or $a - b \equiv \pm j$. This is the circulant graph with two generators, commonly denoted $C_n(i, j)$. Figure 1 shows an example with $n = 10$, where the two loops have voltages $i = 1$ and $j = 4$.

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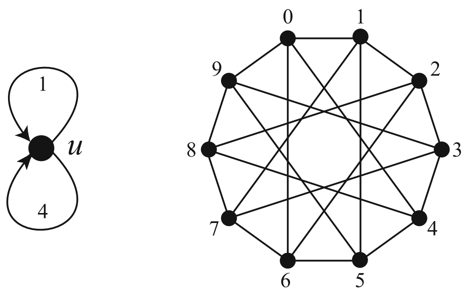

FIGURE 1. Voltage graph $(B_2, \mathbb{Z}_{10})$ and derived graph $C_{10}(1,4)$

We present some facts about two-generator circulant graphs. It is clear from the definition that $C_n(i,j) = C_n(n-i,j) = C_n(i,n-j) = C_n(n-i,n-j)$, so throughout this paper, we will assume that $0 < i < j \leq n/2$. If k is a unit in $\mathbb{Z}_n$, then multiplying vertices by k is a graph isomorphism $C_n(i,j) \cong C_n(ik,jk)$. In particular, if i is a unit in $\mathbb{Z}_n$, then $C_n(i,j) \cong C_n(1,i^{-1}j)$ (and if j is a unit, then $C_n(i,j) \cong C_n(1,ij^{-1})$). In what follows, we will assume that either $i = 1$ or neither i nor j is a unit in $\mathbb{Z}_n$. As noted in [6], $C_n(i,j)$ is connected if and only if $\gcd(n,i,j) = 1$. More generally, if $\gcd(n,i,j) = g$, then $C_n(i,j)$ consists of g components, all isomorphic to $C_{n/g}(i/g, j/g)$.

As the drawing of $C_{10}(1,4)$ in Fig. 1 illustrates, two-generator circulant graphs can be drawn symmetrically. More precisely, they are always vertex-transitive. For any $s \in \mathbb{Z}_n$, let σ_s be translation by s; that is, $\sigma_s(a) = s + a$ for all $a \in \mathbb{Z}_n$. This is the natural left action of the voltage group on the derived graph and it is easily verified to be a graph automorphism of $C_n(i,j)$. If a, b are vertices of $C_n(i,j)$, then $\sigma_{b-a}(a) = b$. Note that σ_{b-a} maps $\{a, a+i\}$ to $\{b, b+i\}$ and $\{a, a+j\}$ to $\{b, b+j\}$. Thus, $C_n(i,j)$ is edge-transitive if and only if there is an automorphism mapping an edge of the form $\{a, a+i\}$ to an edge of the form $\{b, b+j\}$. Since the reflection given by $\tau_{-1}(a) = -a$ is also an automorphism of $C_n(i,j)$, circulant graphs are edge-transitive if and only if they are arc-transitive. The classification of all arc-transitive circulant graphs was found independently by Kovacs [14] and Li [16]. Based on this work, Potočnik and Wilson recently noted in [18] that 4-regular, two-generator circulant graphs are edge-transitive if and only if they are either isomorphic to $C_n(1,j)$ for $j^2 \equiv \pm 1$ or isomorphic to $C_{2m}(1, m-1)$ for $m \geq 3$. Thus, for example, $C_{10}(1,4)$ in Fig. 1 is edge-transitive. The only 3-regular edge-transitive circulant graphs are $C_4(1,2) \cong K_4$ and $C_6(1,3) \cong K_{3,3}$; see [8].

Another way to characterize the symmetry of a graph G is to compute parameters that measure how easy it is to 'break' any nontrivial automorphisms of G. As one example of this, a determining set of graph G is a vertex subset W, such that the only graph automorphism that fixes each vertex in W is the identity. The size of a minimum determining set is the determining number of G, denoted by $\mathrm{Det}(G)$. (Some authors refer to this as the fixing number of the graph.) Another example is to assign d colors to the vertices in such a way that the only automorphism that preserves the color classes (setwise) is the identity. Such a coloring is called a d-distinguishing coloring; the minimum number of colors required for a distinguishing coloring is called the distinguishing number of the graph, denoted by $\mathrm{Dist}(G)$. For more background on determining and distinguishing number and the relationships between them; see [1]. It has been shown that many infinite families of graphs have distinguishing number 2; for such graphs, a further refinement is to determine the minimum size of a color class in a 2-distinguishing coloring. This parameter, introduced in [5], is called the cost of 2-distinguishing G and is denoted $\rho(G)$.

If a graph G is disconnected with components $C_1, C_2, \dots C_k$, then it is possible to calculate its symmetry parameters from those of its components. In particular, if all components have positive determining number, then $\mathrm{Det}(G) = \mathrm{Det}(C_1) + \cdots + \mathrm{Det}(C_k)$. However, the situation for distinguishing number is more complicated. If multiple components are isomorphic, then there are nontrivial automorphisms that permute components. We therefore need to know the number of nonisomorphic distinguishing colorings for each such component. In this paper, we focus on finding the symmetry parameters only for two-generator circulant graphs that are connected.

Partial results on the determining and distinguishing number of circulant graphs have been obtained. Recently, Brooks et al. [6] studied the determining number of powers of cycles. This motivated their study of general circulant graphs of the form $C_n(A)$, where $A \subseteq \mathbb{Z}_n$ and vertices u and v are adjacent if and only if $\pm(u - v) \in A$. They identify the determining number of circulant graphs with two generators $\{i, j\}$ with $i + j = \frac{n}{2}$, with $i = 1$ and $4 \leq j \leq \frac{n}{2}$ and, for even n, for $i = 2$ and $j > 1$ odd. Brooks et al. conjecture that if $C_n(i, j)$ is connected, then $\mathrm{Det}(C_n(i, j)) = 2$ if and only if $C_n(i, j)$ is twin-free. We prove that this is true except for $C_n(1, 3)$.

Gravier, Meslem, and Souad [11] investigated the distinguishing number of circulant graphs $C_n(A)$ where $n = mp \geq 3$, for some $m \geq 1$ and $p \geq 2$, and $A = \{kp + 1 \mid 0 \leq k \leq m - 1\}$. Restricted to two-generator circulant graphs, their results are $\mathrm{Dist}(C_{2p}(1, p - 1)) = 3$, if $p \geq 2$ and $p \neq 4$, and 5, if $p = 4$.

The presence of *twin* vertices, which are vertices having the same neighborhood, understandably affects symmetry parameters. For example, Gonzales and Puertas [10] looked at quotient graphs with respect to the twin relation to find upper and lower bounds on the determining number of an arbitrary graph. Brooks et al. [6] prove that if every vertex in $C_n(A)$ is in a set of k mutual twins, then $\mathrm{Det}(C_n(A)) = n - (n/k)$.

In this paper, we give complete results on the symmetry parameters of connected, two-generator circulant graphs. We begin by characterizing the automorphism group of such graphs. There are results on the automorphisms of special cases of $C_n(A)$, such as when it is arc-transitive or when n is prime, a prime power, or square-free, and/or the elements of A are divisors of n, or when the circulant graph has a rational spectrum; see [2,13,16,17]. We find the automorphism group of all connected, two-generator circulant graphs, with no restrictions on arc-transitivity or the prime factorization of n. As in [2], our proofs make extensive use of possible sets of common neighbors. Let H be

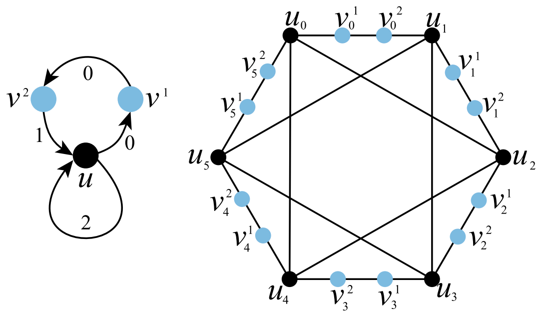

FIGURE 2. Voltage graph subdivided $(B_2, \mathbb{Z}_6)$ and derived graph $C_6(1\div 2, 2)$

the set of units in $\mathbb{Z}_n$ that preserve $\{\pm i, \pm j\}$ under multiplication, commonly denoted by $\mathrm{Aut}(\mathbb{Z}_n, \{\pm i, \pm j\})$. We show that if $C_n(i,j)$ is connected, twin-free, and not $C_n(1,3)$, $\mathrm{Aut}(C_n(i,j)) = \mathbb{Z}_n \rtimes H$. If $C_n(i,j)$ has twins, then every automorphism can be expressed as the composition of an element of $\mathbb{Z}_n \rtimes H$ and an automorphism that permutes sets of mutually twin vertices.

We also consider the derived graphs associated with voltage graphs obtained by subdividing one loop in B_2 with ℓ vertices of degree 2. Equivalently, the voltage graph is a directed cycle of length ℓ with a loop at one vertex. Using the fact that without loss of generality, we can assign a voltage of 0 to the arcs in a spanning tree of the base directed graph, we assign the subdivided loop's original voltage to the arc of the cycle directed to vertex u [12]. The associated derived graph is a subdivision of $C_n(i,j)$, which we denote by $C_n(i\div\ell, j)$ if the arc of voltage i is the one that has been subdivided; $C_n(i, j\div\ell)$ is defined analogously. See Fig. 2, in which the arc of voltage i in B_2 has been subdivided with $\ell = 2$ vertices of degree 2, producing derived graph $C_6(1\div 2, 2)$.

We find the automorphism groups of connected, two-generator circulant graphs that have been subdivided in this way, and use them to determine their symmetry parameters. Our results, summarized in Table 1, confirm and extend those in [6,11]. Note that subdividing an arc of voltage less than $n/2$ sometimes reduces the determining number and always reduces the cost of 2-distinguishing, indicating that the overall symmetry has been reduced. On the other hand, when an arc of voltage $n/2$ is subdivided, the derived graph changes from trivalent to tetravalent. As a result, the derived graph has additional symmetries and both the determining number and the cost of 2-distinguishing increase.

Throughout the entire paper, we assume $0 < i < j \leq n/2$ and $\gcd(n,i,j) = 1$. Thus, $C_n(i,j)$ is a connected, two-generator circulant graph. In Sect. 2, we characterize which $C_n(i,j)$ have twin vertices. In Sect. 3, we compute the symmetry parameters of such graphs. Section 4 exhibits the possible sets of common neighbors in twin-free $C_n(i,j)$. In Sect. 5, we characterize the automorphisms of $C_n(i,j)$, both for those that are twin-free and, because we use this information in the subdivided case, for those with twins. Section 6 gives

TABLE 1. Summary of symmetry parameters

	Det	Dist	ρ	Condition(s)
$C_n(i,j)$	$n-1$	n	n/a	$n \in \{4,5\}$
	4	4	n/a	$(n,i,j)=(6,1,3)$
	6	5	n/a	$(n,i,j)=(8,1,3)$
	4	3	n/a	$(n,i,j)=(10,1,3)$
	$n/2$	3	n/a	$i+j=n/2$, but $n \neq 8$
	2	2	3	Otherwise
$C_n(i \div \ell, j)$, or	1	2	1	$\ell \geq 2$ and $H=\{1,-1\}$
$C_n(i, j \div \ell)$, $j<n/2$	2	2	2	Otherwise
$C_n(i, j \div \ell)$, $j=n/2$	4	3	n/a	$\ell=1$, $j=2$
	$j+1$	2	$j+3$	$\ell=1$, $j \geq 3$
	j	2	$j+1$	$\ell=2$, $j \in \{2,3,4,5\}$ or $\ell=j=3$
	j	2	j	Otherwise

the symmetry parameters of twin-free $C_n(i,j)$. Finally, in Sect. 7, we find the automorphism group and symmetry parameters of subdivided $C_n(i,j)$. Section 7.1 deals with the case in which the subdivided loop in the voltage graph has voltage less than $n/2$, and Sect. 7.2 considers the case where this voltage equals $n/2$. We close with some ideas for future research in Sect. 8.

2. Twins in Two-Generator Circulant Graphs

The open neighborhood of a vertex v in a graph G, $N(v)$, is defined to be $\{u \in V(G) \mid uv \in E(G)\}$ and the closed neighborhood of v is $N[v]=\{v\} \cup N(v)$. Two vertices v and w are nonadjacent twins if $N(v)=N(w)$, and they are adjacent twins if $N[v]=N[w]$.

Note that if a vertex has an adjacent twin, it cannot also have a nonadjacent twin. We say a graph has twins if it has adjacent or nonadjacent twins. Finally, we define vertices u and v in a graph Gu to be *co-twins* if $N[us]$ and $N[v]$ are complementary sets in $V(G)$. If u and v are twins, the vertex map that interchanges u and v while leaving all other vertices fixed is a graph automorphism. In circulant graphs, exchanging a pair of co-twins also leads to additional automorphisms. Substantial effect on the symmetry parameters of the graph. The presence of twin and co-twin vertices therefore has a substantial effect on the symmetry parameters of the graph.

It is easy to verify that the only connected, two-generator circulant graph with co-twins is $C10(1,3)$; in this case, a and a+5 are co-twins for all $a \in \mathbb{Z}_{10}$. More generally, if $n=4k+2$ for some $2 \leq k \in \mathbb{Z}$, then any two vertices of the form a, a+2k+1 are co-twins in $C2k(1,3,\ldots,2kk-1)$.

In [6], Brooks et al. show that if $i+j=n/2$, then $N(a)=N(a+n/2)$. Lemma 1 strengthens this result by fully characterizing all twin vertices in connected, two-generator circulant graphs.

Lemma 1. (1) *If $n \in \{4,5\}$, then $C_n(i,j) = C_n(1,2) = K_n$, and so, any two distinct vertices are adjacent twins.*

(2) *If $n \geq 6$ and $j < n/2$, then $C_n(i,j)$ is twin-free if and only if $i+j \neq n/2$. If $i+j = n/2$, then for distinct a, b, $N(a) = N(b)$ if and only if either $b = a \pm n/2$ or $(n,i,j) = (8,1,3)$ and $b = a \pm 2$.*

(3) *If $n \geq 6$ and $j = n/2$, then $C_n(i,j)$ is twin-free except if $(n,i,j) = (6,1,3)$, in which case for distinct a, b, $N(a) = N(b)$ if and only if $b = a \pm 2$.*

Proof. Recalling $0 < i < j \leq n/2$, statement (1) is easy to verify. Therefore, assume $n \geq 6$. Since $C_n(i,j)$ is vertex-transitive, $C_n(i,j)$ has twin vertices if and only if 0 has a twin. Suppose vertex $0 \not\equiv a \in \mathbb{Z}_n$ is a twin of 0, so that $N(0) = N(a)$. In what follows, we use the fact that $0 \not\equiv a$ implies $a + k \not\equiv k$ for $k \in \mathbb{Z}_n$.

Because the proof is simpler, we first consider (3). If $j = n/2$, then $j \equiv -j$ and so $N(0) = \{\pm i, j\} = \{a \pm i, a + j\} = N(a)$. There are only two possible values of $a + i$.

Case 3a. If $a+i \equiv -i$, then $a+j \equiv i$ and $a-i \equiv j$. Then, $a \equiv -2i \equiv i+j$, so $3i \equiv -j \equiv n/2$. Thus, $\gcd(n,i,j) = i$, which by assumption implies $i = 1$, and hence, $(n,i,j) = (6,1,3)$.

Case 3b. If $a+i \equiv j$, then $a-i \equiv i$ and $a+j \equiv -i$. Then, $a \equiv 2i \equiv j-i$, so again $3i \equiv j \equiv n/2$, which forces $(n,i,j) = (6,1,3)$.

Thus, if $N(0) = N(a)$, then $(n,i,j) = (6,1,3)$. Moreover, $a \equiv \pm 2i \equiv \pm 2$.

Returning to (2), assume $j < n/2$. Then assuming $N(0) = N(a)$ gives $\{\pm 1, \pm j\} = \{a \pm i, a \pm j\}$. We again consider all possible values of $a + i$.

Case 2a. If $a + i \equiv -j$, then $a + j \equiv -i$. There are only two possibilities for $a - i$ and $a - j$. If $a - i \equiv i$ and $a - j \equiv j$, then $a \equiv 2i \equiv 2j$. Since $0 < i < j < n/2$, $0 < 2i < 2j < n$, so this is a contradiction. Thus, $a - i \equiv j$ and $a - j \equiv i$. Then, $a \equiv i + j \equiv -i - j$, which implies that $2i + 2j \equiv 0$, and so, $i + j = n/2$.

Case 2b. Next, suppose $a + i \equiv -i$. If $a - i \equiv i$, then $a + j \equiv -j$ and $a - j \equiv j$. These imply $a \equiv 2i \equiv 2j$, a contradiction. If $a - i \equiv j$, then $a - j \equiv i$ and $a + j \equiv -j$. These imply $a \equiv -2i \equiv -2j$, which again leads to the contradiction $2i \equiv 2j$. The remaining possibility is $a - i \equiv -j$, which implies $a + j \equiv i$ and $a - j \equiv j$. Then, $a \equiv -2i \equiv 2j$, which in turn implies $i + j = n/2$.

Case 2c. Finally, suppose $a+i \equiv j$, which implies $a-j \equiv -i$. In this case, there are only two possibilities for $a-i$ and $a+j$. If $a-i \equiv -j$ and $a+j \equiv i$, then $a \equiv i-j \equiv -i+j$, leading to the contradiction $2i \equiv 2j$. Hence, $a-i \equiv i$ and $a+j \equiv -j$, so $a \equiv 2i \equiv -2j \equiv -i+j$, and thus, $i+j = n/2$.

We conclude that if $N(0) = N(a)$, then $i+j = n/2$, so $N(0) = N(i+j) = \{\pm i, \pm j\}$. Thus, for every b, $N(b) = N(b + n/2)$. The calculations also show that either $a \equiv i+j$, or $a \equiv -2i \equiv 2j$ or $a \equiv 2i \equiv -2j$. In the latter two cases, $\gcd(n,i,j) = 1$ forces $(n,i,j) = (8,1,3)$, $N(0) = N(\pm 2)$, and more generally, for every b, $N(b) = N(b \pm 2)$. $\square$

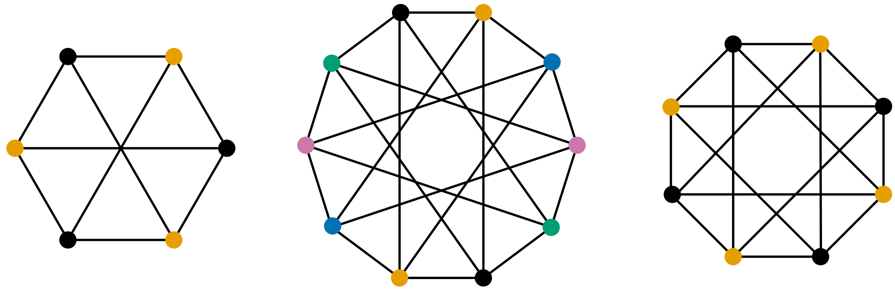

Figure 3. Circulant graphs with twins: $C_6(1,3)$, $C_{10}(1,4)$ and $C_8(1,3)$

Corollary 2. *Every connected, two-generator circulant graph with twins is edge-transitive.*

Proof. Clearly, $C_4(1,2) \cong K_4$, $C_5(1,2) \cong K_5$ and $C_6(1,3) \cong K_{3,3}$ are edge-transitive. By Lemma 1, it suffices to consider $C_n(i,j)$ where $i+j = n/2$. Let $m = n/2$, so that $2i+2j = 2\,m$. Since $\gcd(2\,m, i, j) = 1$, we have $\gcd(i,j) = 1$. It follows that either i or j is a unit. Without loss of generality, assume i is a unit, so $C_n(i,j) \cong C_{2\,m}(1, i^{-1}j)$. Since $2i + 2j \equiv 0$, $2 + 2i^{-1}j \equiv 0$, implying $i^{-1}j \equiv m-1$. □

The converse of Corollary 2 is false; for example, $C_{15}(1,4)$ is edge-transitive and twin-free. Figure 3 shows some examples of circulant graphs with twins.

3. Symmetry Parameters for $C_n(i,j)$ with Twins

Permuting a set of twin vertices and fixing all other vertices is nontrivial graph automorphism. This implies that a determining set for G must contain at all but one vertex from any set of mutually twin vertices. Additionally, in any distinguishing coloring, the vertices in a set of mutual twins must be assigned distinct colors.

For vertices x, y of a graph G, define the relation $x \sim y$ if x and y are twin vertices. It is easy to verify that $\sim$ is an equivalence relation on $V(G)$ and so we can create a quotient graph $\widetilde{G}$ with respect to the relation $\sim$, where the set of vertices is the set equivalence classes $[x] = \{y \in V(G) \mid x \sim y\}$ with $[x]$ adjacent to $[z]$ in $\widetilde{G}$ if and only if there exist $u \in [x]$ and $v \in [z]$, such that u and v are adjacent in G. We refer to $\widetilde{G}$ as the *twin quotient graph*. It can be verified that $\widetilde{G}$ is twin-free; see [4].

Using terminology from [4], a *minimum twin cover* T of a graph G is a subset of vertices that contains all but one vertex from each set of mutual twins. The following is a corollary of Theorem 19 of [4]:

Corollary 3. *Assume every vertex of G has at least one twin. Let T be minimum twin cover of G. Then, T is a minimum size determining set for G.*

The following result, based on an approach used by Boutin and Cockburn in [3] to find the symmetry parameters of orthogonality graphs, gives a relationship between the distinguishing numbers of a graph G and $\widetilde{G}$.

Theorem 4. *Let G be a graph in which every vertex is in a set of exactly k mutual twins and let $\widetilde{G}$ be the corresponding twin quotient graph. If* $\mathrm{Dist}(\widetilde{G}) = \widetilde{d}$*, then* $\mathrm{Dist}(G) = d$*, where d is the smallest positive integer, such that $\binom{d}{k} \geq \widetilde{d}$.*

Proof. Assume $\mathrm{Dist}(\widetilde{G}) = \widetilde{d}$. If $d \in \mathbb{N}$ satisfies $\binom{d}{k} \geq \widetilde{d}$, then from a palette of d colors, we can create $\widetilde{d}$ distinct subsets of k colors, which we can think of as $\widetilde{d}$ distinct k-color packets. By assumption, we can then color each vertex of $\widetilde{G}$ with a k-color packet in a distinguishing fashion. For each $x \in V(G)$, the equivalence class $[x] \in V(\widetilde{G})$ has been assigned a k-color packet; we randomly assign these k colors to the k vertices in $[x]$ a bijective fashion.

Automorphisms of G induce automorphisms in $\widetilde{G}$, so it is straightforward to verify that this coloring produces a distinguishing coloring of G and $\mathrm{Dist}(G) \leq d$. For any $d' < d$, the number of different k-color packets is less than $\widetilde{d}$. Thus, any d'-coloring of G induces a coloring of $\widetilde{G}$ by k-color packets that is not distinguishing. We conclude such a coloring is not distinguishing for G, so $\mathrm{Dist}(G) \geq d$. □

Theorem 5. *Assume $C_n(i,j)$ has twins.*

(1) *If $n \in \{4,5\}$, then* $\mathrm{Det}(C_n(i,j)) = n-1$ *and* $\mathrm{Dist}(C_n(i,j)) = n$.

(2) *If $(n,i,j) = (6,1,3)$, then* $\mathrm{Det}(C_n(i,j)) = \mathrm{Dist}(C_n(i,j)) = 4$.

(3) *If $n \geq 6$ and $i+j = n/2$, then* $\mathrm{Det}(C_8(1,3)) = 6$ *and* $\mathrm{Dist}(C_8(1,3)) = 5$ *and for all other values,* $\mathrm{Det}(C_n(i,j)) = n/2$ *and* $\mathrm{Dist}(C_n(i,j)) = 3$.

Proof. The three statements in this theorem align with the three cases in Lemma 1.

Statement (1) handles the cases in which $C_n(i,j) = K_n$.

For statement (2), a minimum twin cover of $C_6(1,3)$ is $T = \{0,1,2,3\}$, so by Corollary 3, the determining number is 4. The twin quotient graph is K_2, so, by Theorem 4, the distinguishing number is 4.

For statement (3), we first consider the case $(n,i,j) = (8,1,3)$. A minimum twin cover is $T = \{0,1,2,3,4,5\}$ and the twin quotient graph is again K_2. Thus, by Corollary 3 and Theorem 4, the determining number and distinguishing number are both 5.

Finally, we consider the case $n \geq 6$, $i+j = n/2$ but $n \neq 8$. From Lemma 1, a and $a+i+j = a+n/2$ are twins for all $a \in \mathbb{Z}_n$. A minimum twin cover is $T = \{0,1,2,\ldots,n/2-1\}$, so the determining number is $n/2$. The twin quotient graph has order $n/2$ and can be easily seen to be a cycle. Thus, it is $C_{n/2}$, which has distinguishing number $\tilde{d} = 3$ if $n/2 \in \{3,4,5\}$ and distinguishing number $\tilde{d} = 2$ if $n/2 \geq 6$. In either case, the smallest d, such that $\binom{d}{2} \geq \tilde{d}$ is $d = 3$. □

Since there is only one two- generator graph with cotwins, namely $C_{10}(1,3)$, we use direct computation to find the symmetry parameters, $\det(C_{10}(1,3)) = 4$ and $\mathrm{Dist}(C_{10}(1,3)) = 3$.

4. Common Neighbors in Twin-Free $C_n(i,j)$

To find the determining and distinguishing numbers of $C_n(i,j)$ in the twin-free case, we first find its automorphism group. A key tool in our investigation involves possible sets of common neighbors. Since automorphisms respect adjacency and nonadjacency, if $\alpha \in \mathrm{Aut}(G)$ and u, v are vertices in a graph G with $N(u) \cap N(v) = \{w_1, \ldots w_\ell\}$, then $N(\alpha(u)) \cap N(\alpha(v)) = \{\alpha(w_1), \ldots \alpha(w_\ell)\}$. To use this fact, we must determine the possible sets of common neighbors.

Two vertices have common neighbors if and only if there is a path of length 2 between them. In $C_n(i,j)$, we find a and b have common neighbors if and only if $b \in \{a \pm 2i, a \pm 2j, a+i \pm j, a-i \pm j\}$. Exchanging the roles of a and b as needed, we need only consider the cases $b \in \{a+2i, a+2j, a+i \pm j\}$. Additionally, if $j = n/2$, we can further restrict to $b \in \{a+2i, a+i+j\}$.

Lemma 6. *Assume $C_n(i,j)$ is twin-free. Distinct vertices a and b have common neighbors if and only if we can label them, so that $b \in \{a+2i, a+2j, a+i \pm j\}$. Furthermore, the set of common neighbors of each such pair of vertices (a,b) is given by the following table, where we take the union of the applicable rows.*

	$(a, a+2i)$	$(a, a+2j)$	$(a, a+i+j)$	$(a, a+i-j)$
Always	$\{a+i\}$	$\{a+j\}$	$\{a+i, a+j\}$	$\{a+i, a-j\}$
$4i \equiv 0$	$\{a-i\}$	–	–	–
$4j \equiv 0$	–	$\{a-j\}$	–	–
$3i \equiv -j$	$\{a-i, a-j\}$	–	$\{a-i\}$	–
$3i \equiv j$	$\{a-i, a+j\}$	–	–	$\{a-i\}$
$3j \equiv -i$	–	$\{a-i, a-j\}$	$\{a-j\}$	–
$3j \equiv i$	–	$\{a+i, a-j\}$	–	$\{a+j\}$

Proof. By Lemma 1, the assumption that $C_n(i,j)$ is twin-free implies $n \geq 6$, $(n,i,j) \neq (6,1,3)$ and $i+j \neq n/2$. When $j < n/2$, our assumptions yield $2i \not\equiv 0$, $2j \not\equiv 0$, $2i \not\equiv 2j$, and $2i \not\equiv -2j$.

It is enough to assume $a = 0$ and to consider which elements of $N(a)$ and $N(b)$ may be shared. In each case, there are obvious members of $N(a) \cap N(b)$. All other possibilities lead to contradictions, except in the special conditions listed in the left column. □

When $j = n/2$, none of the special conditions can hold. However, in general, it is possible for two special conditions to hold. In this case, both affect the set of common neighbors. For example, in $C_{12}(3,5)$, both conditions $4i \equiv 0$ and $3j \equiv i$ hold. In this case, $N(0) \cap N(2i) = \{\pm i\}$, $N(0) \cap N(i+j) = \{i, j\}$ and $N(0) \cap N(2j) = N(0) \cap N(i-j) = \{i, \pm j\}$.

Our assumption that $0 < i < j < n/2$ implies that each condition corresponds to exactly one linear equation in $\mathbb{Z}$. Given that we are interested only in intersection points corresponding to integral values of n, i and j, that also satisfy $\gcd(n,i,j)=1$ and $i+j\neq n/2$, we find only three cases where multiple special conditions are satisfied: $C_{12}(3,5)$ satisfies both $4i\equiv 0$ and $3j\equiv i$, $C_{10}(1,3)$ satisfies both $3i\equiv j$ and $3j\equiv -i$ and $C_{12}(1,3)$ satisfies both $4j\equiv 0$ and $3i\equiv j$. Because $5\equiv 5^{-1}$ in $\mathbb{Z}_{12}$, $C_{12}(3,5)\simeq C_{12}(1,3)$.

5. Automorphisms of Circulant Graphs

As noted in the introduction, for any $s\in\mathbb{Z}_n$, translation by s, given by $\sigma_s(a)=s+a$, is an automorphism of $C_n(i,j)$. Thus $\mathrm{Aut}(C_n(i,j))$ has a subgroup isomorphic to $\mathbb{Z}_n$. Additionally, let H be the set of automorphisms $\mathbb{Z}_n$ that preserve $\{\pm i,\pm j\}$; that is, $H=\mathrm{Aut}(\mathbb{Z}_n,\{\pm i,\pm j\})$. It is easy to verify that H is a subgroup of $\mathrm{Aut}\,(\mathbb{Z}_n)=U(n)$. For $t\in H$ and $a\in\mathbb{Z}_n$, let $\tau_t(a)=ta$. It is routine to verify the following special case of Godsil's Lemma 2.1 in [9].

Proposition 7. *For all $t\in H$, we have $\tau_t\in\mathrm{Aut}(C_n(i,j))$.*

It will always be the case that $\{\pm 1\}\subseteq H$. In fact, in many cases, these are the only two elements of H. Lemma 8 gives H in the edge-transitive cases.

Lemma 8. *For $C_4(1,2)$, $C_6(1,3)$ and $C_{2m}(1,m-1)$ where $m\geq 3$ is odd, $H=\{\pm 1\}$. For $C_{2m}(1,m-1)$ where $m\geq 3$ is even and $j=m-1$ and $C_n(1,j)$ where $j^2\equiv\pm 1$, $H=\{\pm 1,\pm j\}$.*

Proof. The results for $C_4(1,2)$ and $C_6(1,3)$ follow from $U(4)=U(6)=\{\pm 1\}$. Since $t\in H$ preserves $\{\pm 1,\pm j\}$, $t=t\cdot 1\in\{\pm 1,\pm j\}$, so $H\subseteq\{\pm 1,\pm j\}$. If m is odd, then $j=m-1$ is even, and so, $\pm j\notin U(2m)=U(n)$. If m is even, then $2m$ divides m^2, and so, $j^2=(m-1)^2=m^2-2m+1\equiv 1$. In particular, $j\in U(n)$. If $j^2\equiv\pm 1$, then clearly j and $-j$ preserve $\{\pm 1,\pm j\}$. $\square$

If $C_n(i,j)$ is not edge-transitive, then as noted in the introduction, no automorphism of $C_n(i,j)$ takes an edge of the form $\{a,a+i\}$ to an edge of the form $\{b,b+j\}$. Hence, in this case, $H=\mathrm{Aut}(\mathbb{Z}_n,\{\pm i\},\{\pm j\})$.

Lemma 9. *Let $0<i<j\leq n/2$ and $\gcd(n,i,j)=1$. If $t\in U(n)$ satisfies $ti\equiv i$ and $tj\equiv j$, then $t\equiv 1$. If t satisfies $ti\equiv -i$ and $tj\equiv -j$, then $t\equiv -1$.*

Proof. Since $\gcd(n,i,j)=1$, there exist $x,y,z\in\mathbb{Z}$, such that $xi+yj+zn=1$, which means $xi+yj\equiv 1$. Hence

$$t\equiv t(xi+yj)\equiv x(ti)+y(tj)\equiv\begin{cases}xi+yj\equiv 1, & \text{if } ti\equiv i \text{ and } tj\equiv j,\\ -(xi+yj)\equiv -1, & \text{if } ti\equiv -i \text{ and } tj\equiv -j.\end{cases}$$

$\square$

Corollary 10. *Assume $C_n(i,j)$ is not edge-transitive. If there exists $1\neq t\in U(n)$, such that $ti\equiv i$ and $tj\equiv -j$, then $H=\{\pm 1,\pm t\}$. Otherwise $H=\{\pm 1\}$.*

Proof. If there exists such a t, then $\pm t\in H=\mathrm{Aut}(\mathbb{Z}_n,\{\pm i\},\{\pm j\})$. Suppose $t^*\in H\backslash\{\pm 1,\pm t\}$. By Lemma 9, we can assume without loss of generality that $t^*i\equiv i$ and $t^*j\equiv -j$. Then, tt^* and t^2 fix both i and j, and so, by Lemma 9, $tt^*\equiv t^2\equiv 1$. Since t is a unit, this implies $t\equiv t^*$. $\square$

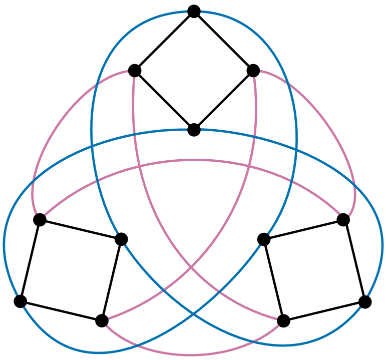

FIGURE 4. $C_{12}(2, 3)$

Example 1. For a non-edge-transitive example where $H \neq \{\pm 1\}$, let $(n, i, j) = (12, 2, 3)$. Then, $U(12) = \{1, 5, 7, 11\}$. Note that $t = 7$ satisfies $ti \equiv +i$ and $tj \equiv -j$. The (twin-free) circulant graph $C_{12}(2, 3)$ is shown in Fig. 4.

The possibilities for H are summarized in Table 2.

For any $s \in \mathbb{Z}_n$ and $t \in H$, we can compose the automorphisms σ_s and τ_t; for any $a \in \mathbb{Z}_n$, $(\sigma_s \circ \tau_t) \cdot (a) = s + ta$. do not commute, but $(\tau_t)^{-1} \circ \sigma_s \circ \tau_t = \sigma_{st^{-1}}$. In the next subsection, we will show that these are the only automorphisms of connected, twinfree, twogenerator circulant graphs, except $C_{10}(1, 3)$.

5.1. Automorphisms of Twin-Free $C_n(i, j)$

Proposition 11. *If $C_n(i, j)$ is not $C10(1, 3)$, is connected, and twin-free and $\alpha \in \mathrm{Aut}(C_n(i, j))$ satisfies $\alpha(0) = 0$, then α is an automorphism of the additive group $\mathbb{Z}_n$.*

Proof. Assume $\alpha \in \mathrm{Aut}(C_n(i, j))$ fixes 0. Since $\gcd(n, i, j) = 1$, there exist $x, y \in \mathbb{Z}$, such that $xi + yj \equiv 1$. It follows that for any $a \in \mathbb{Z}_n$, there exist $c, d \in \mathbb{Z}$, such that $ci + dj \equiv a$. It suffices to show that for all $0 \leq c, d \in \mathbb{Z}$

$$\alpha(ci + dj) \equiv c\alpha(i) + d\alpha(j). \tag{1}$$

This proof involves multiple cases, but the underlying strategy is to use induction on $m = c + d$. As indicated at the beginning of Sect. 3, the main tool is to apply α to an equation expressing a set of common neighbors. Here, we will assume that none of the special conditions of Lemma 6 holds; see [7] for these cases.

Because α preserves adjacency, for any $a \in \mathbb{Z}_n$, $\alpha(a + i) = \alpha(a) + x$ for some $x \in \{\pm i, \pm j\}$. Applying α to the equation $\{a + i\} = N(a) \cap N(a + i)$ gives $\{\alpha(a + i)\} = \{\alpha(a) + x\} = N(\alpha(a)) \cap N(\alpha(a + 2i))$. By Lemma 6, $\alpha(a+2i) = \alpha(a)+2x$. Similarly, if $\alpha(a+j) = \alpha(a)+y$, then $\alpha(a+2j) = \alpha(a)+2y$ and $\alpha(a + i + j) = \alpha(a) + x + y$.

It is clear that Eq. 1 holds for $m = 1$. Applying the results of the preceding paragraph when $a \equiv 0$, we get $\alpha(2i) = 2\alpha(i)$, $\alpha(2j) = 2\alpha(j)$ and $\alpha(i + j) = \alpha(i) + \alpha(j)$. Hence, Eq. 1 also holds for $m = 2$.

TABLE 2. Possibilities for H

	H	Conditions
	$\{\pm 1, \pm j\}$	$C_{2m}(1, m-1)$, $m \geq 3$ even ($j = m-1$)
Edge-transitive		$C_n(1, j)$, $j^2 \equiv \pm 1$
$H = \mathrm{Aut}(\mathbb{Z}_n, \{\pm i, \pm j\})$	$\{\pm 1\}$	$C_{2m}(1, m-1)$, $m \geq 3$ odd
		$C_4(1,2)$, $C_6(1,3)$
Not edge-transitive	$\{\pm 1, \pm t\}$	$1 \neq t \in U(n)$ satisfies $ti \equiv i$, $tj \equiv -j$
$H = \mathrm{Aut}(\mathbb{Z}_n, \{\pm i\}, \{\pm j\})$	$\{\pm 1\}$	Otherwise

If $j = n/2$, we only need to show $\alpha(ci + dj) \equiv c\alpha(i) + dj$ for all $c \geq 0$ and $d \in \{0, 1\}$. Let $m \geq 3$ and assume $\alpha(ci + dj) \equiv c\alpha(i) + d\alpha(j)$ for all $c+d \in \{m-1, m-2\}$, with $c, d \geq 0$; if $j = n/2$, further assume that $d \in \{0, 1\}$. We have either $c \geq 2$ or $d \geq 2$. Assume $c \geq 2$. For $a = (c-2)i + dj$

$$\alpha(a + i) = \alpha((c-1)i + dj) = (c-1)\alpha(i) + d\alpha(j) = \alpha(a) + \alpha(i),$$

so $\alpha(a + 2i) = \alpha(a) + 2\alpha(i)$. Thus, $\alpha(ci + dj) = \alpha((c-2)i + dj) + 2\alpha(i) = c\alpha(i) + d\alpha(j)$. Suppose instead $d \geq 2$. Then, letting $a = ci + (d-2)j$ and applying a similar argument completes the proof when no special conditions hold. □

Theorem 12. *If $C_n(i, j)$ is not $C_{10}(1, 3)$, is connected, and twin-free, then $\mathrm{Aut}(C_n(i, j)) = \mathbb{Z}_n \rtimes H$, where the action of $(s, t) \in \mathbb{Z}_n \rtimes H$ on a vertex of $C_n(i, j)$ is $(s, t) \cdot (a) = s + ta$.*

Proof. Let $\gamma \in \mathrm{Aut}(C_n(i, j))$. If γ fixes 0, then $\gamma \in \mathrm{Aut}(\mathbb{Z}_n) = U(n)$ by Proposition 11. Then, there exists $t \in U(n)$, such that $\gamma(a) = ta$. Since γ preserves adjacency, we find $t \in H$. If $\gamma(0) = s$, let σ_{-s} be the translation defined by $\sigma_{-a}(a) = -s + a$. Then, $\sigma_{-s} \circ \gamma$ is an automorphism of $C_n(i, j)$ that fixes 0, and hence, $(\sigma_{-s} \circ \gamma)(a) = ta$ for some $t \in H$. Then, $\gamma(a) = s + ta$; equivalently, $\gamma = \sigma_s \circ \tau_t$. We can represent γ with the ordered pair (s, t).

The argument above shows $\mathrm{Aut}(C_n(i, j)) = \mathbb{Z}_n H$. As noted earlier, for all $s \in \mathbb{Z}_n$ and $t \in H$, $(\tau_t)^{-1} \circ \sigma_s \circ \tau_t = \sigma_{st^{-1}}$, so $\mathbb{Z}_n$ is a normal subgroup of $\mathrm{Aut}(C_n(i, j))$. Clearly, $\mathbb{Z}_n \cap H$ contains only the identity automorphism. Hence, $\mathrm{Aut}(C_n(i, j))$ is the semidirect product $\mathbb{Z}_n \rtimes H$. □

This result aligns with Godsil's Lemma 2.2 in [9], because in the twin-free case, except $C_{10}(1, 3)$, the $\mathbb{Z}_n$ is a normal subgroup of $\mathrm{Aut}(C_n(i, j))$, so its normalizer is the entire automorphism group.

5.2. Automorphisms of $C_n(i, j)$ with Twins

If $C_n(i, j)$ has twins, then by Corollary 2, $C_n(i, j)$ is edge-transitive and hence arc-transitive. In [16], Li provides a description of the automorphism group of any arc-transitive circulant graph, based on its tensor-lexicographic decomposition into a normal circulant graph, some complete graphs, and an empty graph. Here, we take an more elementary approach for the special case of two-generator circulant graphs.

If $C_n(i, j)$ has twins, then $\mathbb{Z}_n \rtimes H$ is still a subgroup of the automorphism group. Additionally, $C_n(i, j)$ has automorphisms that permute mutual twin vertices. Recall from Sect. 3 that for any graph G, we can collapse sets of mutually twin vertices to define the twin quotient graph $\widetilde{G}$. Define $\pi : \mathrm{Aut}(G) \to \mathrm{Aut}(\widetilde{G})$ by $[\pi(\alpha)]\cdot[x] = [\alpha(x)]$ for all $\alpha \in \mathrm{Aut}(G)$ and $x \in V(G)$. Properties of automorphisms guarantee that $\pi(\alpha)$ is a bijection that respects adjacency and nonadjacency in $\widetilde{G}$. Note that $\ker(\pi)$ is a normal subgroup of $\mathrm{Aut}(G)$ consisting of automorphisms that simply permute the vertices within each equivalence class.

Lemma 13. *Let G be a graph of order n where every vertex is in a set of k mutual twins. Then,* $\ker(\pi) = (S_k)^{n/k}$ *and* $\mathrm{Aut}(\widetilde{G}) = \mathrm{Aut}(G)/\ker(\pi)$. *Hence*

$$|\,\mathrm{Aut}(G)| = |\ker(\pi)||\,\mathrm{Aut}(\widetilde{G})| = (k!)^{n/k}|\,\mathrm{Aut}(\widetilde{G})|.$$

Proof. It suffices to show that π is surjective and then apply the First Isomorphism Theorem for groups (see [15]). For each equivalence class of vertices in G, we select a class representative and label the vertices with subscripts in some order, $[x] = \{x_1, x_2, \ldots, x_k\}$. Suppose $\widetilde{\beta} \in \mathrm{Aut}(\widetilde{G})$. If $\widetilde{\beta}([x]) = [y]$, then let $\beta(x_m) = y_m$ for each $m \in \{1, 2, \ldots, k\}$. It is easy to verify that $\beta \in \mathrm{Aut}(G)$ and $\pi(\beta) = \widetilde{\beta}$. □

We can apply Lemma 13 to connected, two-generator circulant graphs with twins. The relevant twin quotient graphs are K_1, K_2 and $C_{n/2}$. We find $|\,\mathrm{Aut}(C_n(1, 2))| = (n!)^{n/n} \cdot 1$ for $n \in \{4, 5\}$, $|\,\mathrm{Aut}(C_6(1, 3))| = (3!)^{6/3} \cdot 2 = 72$, $|\,\mathrm{Aut}(C_8(1, 3))| = (4!)^{8/4} \cdot 2 = 1152$ and for $i + j = n/2$, but $n \neq 8$, $|\,\mathrm{Aut}(C_n(i, j))| = 2^{n/2}n$.

Using the Second Isomorphism Theorem for groups (see [15]) and a cardinality argument, we get the following.

Theorem 14. *If $C_n(i, j)$ is connected and has twins, then any automorphism of $C_n(i, j)$ is a composition of some $(s, t) \in \mathbb{Z}_n \rtimes H$ and an automorphism that permutes twins.*

It can be verified computationally that the automorphism group of the one connected, twogenerator circulant graph with cotwins, $C_{10}(1, 3)$, is $\mathbb{Z}_2 \times S_5$.

6. Symmetry Parameters for Twin-Free $C_n(i, j)$

After establishing $\mathrm{Aut}(C_n(i, j)) = \mathbb{Z}_n \rtimes H$ in the twin-free case, we can find the symmetry parameters with relative ease. The result below proves in the affirmative a conjecture on the determining number of connected, twin-free, two-generator circulant graphs of Brooks et al.[6].

Theorem 15. *If $C_n(i, j)$ is not $C_{10}(1, 3)$, is connected, and twin-free, then*

$$\mathrm{Det}(C_n(i, j)) = 2,\ \mathrm{Dist}(C_n(i, j)) = 2,\ \textit{and}\ \rho(C_n(i, j)) = 3.$$

Proof. For determining, first let $a \in \mathbb{Z}_n$. Since $a = (2a) + (-1) \cdot a$, we find $\{a\}$ is fixed by the nontrivial automorphism $(2a, -1)$. Thus, $\mathrm{Det}(C_n(i,j)) > 1$. Next, let $W = \{0, 1\}$ and assume $\alpha = (s,t) \in \mathrm{Aut}(C_n(i,j))$ fixes both vertices in W. Then, $0 \equiv \alpha(0) \equiv s + t \cdot 0 \equiv s$. Next, $1 \equiv \alpha(1) \equiv s + t \cdot 1 \equiv 0 + t \equiv t$. Hence, $\alpha = (0,1)$ which is the identity and W is a determining set. Thus, $\mathrm{Det}(C_n(i,j)) = 2$.

Next, we find a 2-coloring that is distinguishing and that has one color class of size 3. There are two cases to consider.

Case 1. Assume vertices i and j are not adjacent in $C_n(i,j)$. If $C_n(i,j)$ is not edge-transitive, color the vertices in $\{0,i,j\}$ red and all other vertices blue. Assume $\alpha = (s,t) \in \mathrm{Aut}(C_n(i,j))$ is an automorphism that preserves the color classes. Since $\{0,i,j\}$ induces a path, $\{\alpha(i), \alpha(j)\} = \{i,j\}$ and $\alpha(0) = 0$. Hence, $s \equiv 0$. Considering the possibilities for t from Corollary 10, we find $t \equiv 1$, so α is trivial.

In the case $C_n(1,j)$, $j^2 \equiv \pm 1$, we have $H = \{\pm 1, \pm j\}$. When $j^2 \equiv -1$, $\{0,i,j\}$ is still a color class in a 2-distinguishing coloring. However, when $j^2 \equiv 1$, we have that $t = j$ gives a nontrivial automorphism that preserves this set. In this case, $\{0,-i,j\}$ is instead a color class in a 2-distinguishing set.

Case 2. If i and j are adjacent in $C_n(i,j)$, then $i \in N(j) = \{0, j \pm i, 2j\}$. The assumptions on i and j require $i \equiv -i + j$, so $2i \equiv j$. The only edge-transitive two-generator circulant graphs satisfying this condition are $C_5(1,2)$ and $C_6(1,2)$, both of which have twins. Hence, in this case, $C_n(i,j)$ is not edge-transitive.

Using our assumptions, we find that $-j \notin N(i) = \{0, 2i, i \pm j\} = \{0, j, i \pm j\}$. Thus, we instead color the vertices in $\{-j, 0, i\}$ red and all other vertices blue. Arguing as above, we find that any automorphism α preserving these color classes must be trivial.

In both cases, $\mathrm{Dist}(C_n(i,j)) = 2$ and $\rho(C_n(i,j)) \leq 3$. A color class in any 2-distinguishing coloring cannot be a singleton set because $\mathrm{Det}(C_n(i,j)) = 2$. Furthermore, if $a \neq b$ in $\mathbb{Z}_n$, then the nontrivial automorphism $(a+b, -1)$ interchanges them, so a color class in a 2-distinguishing coloring cannot consist of just two vertices. Thus, $\rho(C_n(i,j)) = 3$. □

7. Subdivided Circulant Graphs

We next consider the symmetry parameters of the subdivided circulant graphs $C_n(i \div \ell, j)$ and $C_n(i, j \div \ell)$. Recall that these are the derived graphs associated to a bouquet voltage graph B_2 in which one of the arcs is subdivided by $\ell \geq 1$ vertices of degree 2. Because the voltage graph has order at least 2, we can no longer label vertices of the derived graph simply with elements of $\mathbb{Z}_n$. As shown in Fig. 2, we label them u_a and v_a^r where $a \in \mathbb{Z}_n$ and $r \in \{1, \ldots, \ell\}$.

7.1. $C_n(i \div \ell, j)$ and $C_n(i, j \div \ell)$, $j < n/2$

We first consider the case in which the arc with voltage i has been subdivided by ℓ vertices. In the derived graph, u_a is no longer adjacent to the (distinct) vertices u_{a+i} and u_{a-i}. Instead, for each $a \in \mathbb{Z}_n$, there is a path

$(u_a, v_a^1, v_a^2, \ldots, v_a^\ell, u_{a+i})$. of length $\ell+1$, all of whose interior vertices have degree 2. Note that there is no ambiguity regarding the subscripts on the degree-2 vertices: if $b \equiv a+i$, it cannot also be the case that $a \equiv b+i$, because $0 < 2i < n$. By definition

$$N(u_a) = \begin{cases} \{v_a^1, v_{a-i}^\ell, u_{a-j}, u_{a+j}\}, & j < n/2, \\ \{v_a^1, v_{a-i}^\ell, u_{a+j}\}, & j = n/2. \end{cases}$$

Thus, each u_a has a distinct pair of degree-2 neighbors, meaning that no two vertices of this type are twins. Additionally, each v_a^r, $r \in \{1, \ldots \ell\}$, is uniquely identified by its distances from u_a and u_{a+i}. Hence, $C_n(i \div \ell, j)$ is twin-free.

Neighborhood sizes dictate that the only candidates for subdivided graphs with co-twins are small and direct inspection shows none exist.

Clearly, $C_n(i \div \ell, j)$ is neither vertex-transitive nor edge-transitive. However, automorphisms of $C_n(i,j)$ extend uniquely to automorphisms of $C_n(i \div \ell, j)$, provided that they do not interchange edges of the form $\{a, a+i\}$ and $\{b, b+j\}$. We let

$$H' = H \cap \operatorname{Aut}(\mathbb{Z}_n, \{\pm i\}, \{\pm j\}) = \begin{cases} \{\pm 1\}, & C_n(i,j) \text{ is edge-transitive}, \\ H, & \text{otherwise}. \end{cases}$$

Lemma 16. *For any $\alpha' = (s,t) \in \mathbb{Z}_n \rtimes H'$, there is a unique $\alpha \in \operatorname{Aut}(C_n(i \div \ell, j))$, such that $\alpha(u_a) = \alpha'(u_a)$ for all $a \in \mathbb{Z}_n$. The action of the unique extension α is defined as*

$$\alpha(x) = \begin{cases} u_{s+ta}, & x = u_a, \\ v_{s+at}^r, & x = v_a^r \text{ and } ti \equiv i, \\ v_{s+ta-i}^{(\ell+1)-r}, & x = v_a^r \text{ and } ti \equiv -i. \end{cases} \tag{2}$$

Proof. First, suppose $\gamma, \lambda \in \operatorname{Aut}(C_n(i \div \ell, j))$ satisfy $\gamma(u_a) = \lambda(u_a)$ for all $a \in \mathbb{Z}_n$. Then, $\gamma^{-1} \circ \lambda \in C_n(i \div \ell, j)$ fixes all non-degree-2 vertices. Since each v_a^r is uniquely identified by its distances from u_a and u_{a+i}, all degree-2 vertices are also fixed by $\gamma^{-1} \circ \lambda$. Thus, $\gamma^{-1} \circ \lambda$ is the identity, and so, $\gamma = \lambda$.

To extend the action of $\alpha' = (s,t)$ to the degree-2 vertices, first assume $ti \equiv i$. Then, $(s,t) \cdot (u_{a+i}) = u_{s+t(a+i)} = u_{(s+ta)+i}$. In this case, we map the path of degree-2 vertices between u_a and u_{a+i} to the path of degree-2 vertices between u_{s+ta} and $u_{(s+ta)+i}$, in the same order. If $ti \equiv -i$, then $(s,t) \cdot (u_{a+i}) = u_{s+(a+i)t} = u_{(s+ta)-i}$. In this case, we map the path of degree-2 vertices between u_a and u_{a+i} to the path of degree-2 vertices between u_{s+ta} and $u_{(s+ta)-i}$, in 'reverse order.' It can be easily verified that α is respects adjacency and nonadjacency by checking that the action on the three types of edges: $u_a u_{a\pm j}$ for $a \in \mathbb{Z}_n$, $v_a^r v_a^{r\pm 1}$ for $a \in \mathbb{Z}_n$, and $u_a v_b^r$ with $a \in \mathbb{Z}_n$ and $(b,r) \in \{(a,1), (a-i, \ell)\}$. $\square$

Theorem 17. *Assume $C_n(i,j)$ is connected. Then, $\operatorname{Aut}(C_n(i \div \ell, j)) = \mathbb{Z}_n \rtimes H'$, with the action of elements of $\mathbb{Z}_n \rtimes H'$ as defined in Eq. 2.*

Proof. By Lemma 16, we have $\mathbb{Z}_n \rtimes H' \subseteq \operatorname{Aut}(C_n(i \div \ell, j))$. Thus, we show that $\operatorname{Aut}(C_n(i \div \ell, j)) \subseteq \mathbb{Z}_n \rtimes H'$. Let $\alpha \in \operatorname{Aut}(C_n(i \div \ell, j))$. Since automorphisms respect degree, α restricts to a bijection α' on the set of non-degree-2 vertices, $\{u_a \mid a \in \mathbb{Z}_n\}$. By definition, u_a and u_b are adjacent as vertices in $C_n(i,j)$ if and only if, as vertices in $C_n(i\div\ell, j)$, either they are adjacent or there is a unique path between them of length $\ell+1$, all of whose interior vertices have degree 2. These are properties respected by the automorphism α. Thus, u_a and u_b are adjacent as vertices in $C_n(i,j)$ if and only if $\alpha'(u_a)$ and $\alpha'(u_b)$ are adjacent as vertices in $C_n(i,j)$. Hence, α' is an automorphism of $C_n(i,j)$ that does not interchange edges of the form $\{a, a+i\}$ and $\{b, b+j\}$.

If $C_n(i,j)$ is twin-free and not $C_{10}(1,3)$, then by Theorem 12, $\alpha' \in \mathbb{Z}_n \rtimes H'$ and we are done. If $C_n(i,j)$ has twins, then by Theorem 14, α' is the composition of an element of $\mathbb{Z}_n \rtimes H'$ and an automorphism ρ that permutes twins. In each of the four special cases of Lemma 1, namely $C_4(1,2) = K_4$, $C_5(1,2) = K_5$, $C_6(1,3)$ and $C_8(1,3)$, as well as in the co-twin case $C_{10}(1,3)$, α preserves the Hamiltonian cycle in $C_n(i\div\ell, j)$ induced by edges having at least one endvertex of degree 2, so α' is an automorphism of the Hamiltonian cycle formed by arcs of voltage $i = 1$. Hence, ρ is trivial. Otherwise, $i + j = n/2$, but $n \neq 8$ and vertex a in $C_n(i,j)$ has unique twin vertex $a + i + j$. Using the fact that α respects the set of degree-4 neighbors of vertices in $C_n(i\div\ell, j)$, we can show that if ρ exchanges any twin pair in $C_n(i,j)$, then it exchanges all twin pairs, so either ρ is trivial or $\rho = (n/2, 1) \in \mathbb{Z}_n \rtimes H'$. □

Proposition 18. *If $H' = \{\pm 1\}$, then a minimum determining set of $C_n(i\div\ell, j)$ is $W = \{v_0^1\}$ if $\ell \geq 2$ and $W = \{u_0, v_0^1\}$ if $\ell = 1$.*

Proof. First, assume $\ell \geq 2$. Suppose $(s,t) \in \mathrm{Aut}(C_n(i\div\ell, j))$ fixes v_0^1. We must show $s \equiv 0$ and $t \equiv 1$ in $\mathbb{Z}_n$. If $t \not\equiv 1$, then $t \equiv -1$, so by Eq. 2 we have $(s,t)\cdot(v_0^1) = v_{s-i}^{\ell}$. Since v_0^1 is fixed, this implies $\ell = 1$, contradicting our assumption. Hence, $t \equiv 1$, and $(s,t)\cdot(v_0^1) = v_s^1$ yields that $s \equiv 0$.

Next, assume $\ell = 1$. Suppose (s,t) fixes both v_0^1 and u_0. Since (s,t) fixes u_0, $s \equiv 0$. If $t \not\equiv 1$, then $t \equiv 1$ and by Eq. 2, we have $(s,t)\cdot(v_0^1) = v_{-i}^1$. Since $0 < i < n/2$, $-i \not\equiv 0$, contradicting our assumption that (s,t) fixes v_0^1. Hence, $t \equiv 1$ and $\{u_0, v_0^1\}$ is a determining set.

For minimality, any determining set is nonempty as $C_n(i\div\ell, j)$ has nontrivial automorphisms. In the case $\ell = 1$, neither $\{u_a\}$ nor $\{v_a^1\}$ can be determining as they are fixed by the nontrivial automorphisms $(2a, -1)$ and $(i, -1)$, respectively. □

Now, assume $H' \neq \{\pm 1\}$. This means that $C_n(i,j)$ is not edge-transitive and $H' = \{\pm 1, \pm t\}$ for some $1 \neq t \in U(n)$, such that $ti \equiv i$ and $tj \equiv -j$. Also, neither i nor j is a unit. We require some additional algebraic results.

Lemma 19. *Let $0 < i < n/2$. If $i \notin U(n)$, then there exists a prime dividing n that does not divide i if and only if there exists $b \in \mathbb{N}$, such that $b \notin U(n)$ but $b - i \in U(n)$.*

Proof. First, assume there exists at least one prime dividing n that does not divide i. Let b be the product of all such primes. Then, $\gcd(n,b) = b > 1$, so $b \notin U(n)$. Since no prime dividing n divides both b and i, $\gcd(n, b-i) = 1$, meaning that $b - i \in U(n)$.

Conversely, assume that every prime dividing n also divides i. For any $b \notin U(n)$, some prime p divides both n and b. By assumption, p also divides i, and thus, p divides $b-i$. Thus $b-i \notin U(n)$. $\square$

Corollary 20. *Let n, i, j satisfy $\gcd(n, i, j) = 1$ and $0 < i < j \leq n/2$. If $H' \neq \{\pm 1\}$, then there exists $a \in U(n)$, such that $a + i \notin U(n)$.*

Proof. Assume $H' \neq \{\pm 1\}$. Since j is not a unit, there is a prime p dividing n that also divides j. However, p cannot divide i, because $\gcd(n, i, j) = 1$. Hence, by Lemma 19, there exists $b \notin U(n)$, such that $b - i \in U(n)$. Let $a \equiv b - i$. $\square$

Proposition 21. *Let n, i, j satisfy $\gcd(n, i, j) = 1$ and $0 < i < j \leq n/2$, and assume $H' \neq \{\pm 1\}$. Let $a \in U(n)$, such that $a+i \notin U(n)$. Then, $W = \{u_0, v_a^1\}$ is a minimum determining set of $C_n(i \div \ell, j)$.*

Proof. Suppose $(s, t) \in \mathrm{Aut}(C_n(i \div \ell, j))$ fixes $\{u_0, v_a^1\}$. Since (s, t) fixes u_0, $s \equiv 0$. By Eq. 2, $(0, t) \cdot (v_a^1)$ is v_{at}^1 or v_{at-i}^ℓ depending on whether $ti \equiv i$ or $ti \equiv -i$. In the former case, since v_a^1 is fixed, $at \equiv a$. Since $a \in U(n)$, this implies $t \equiv 1$. In the latter case, $\ell = 1$ and $at - i \equiv a$. By substitution, $at + it \equiv a$, so $(a + i)t = a$. By assumption, $a, t \in U(n)$, but $a + i \notin U(n)$, a contradiction. Thus, the only automorphism fixing $W = \{u_0, v_a^1\}$ is the identity automorphism $(0, 1)$.

We have already seen that for any $a \in \mathbb{Z}_n$, $(2a, -1)$ is a nontrivial automorphism fixing u_a, so $\{u_a\}$ is not a determining set. By Corollary 10, there exists $1 \neq t \in U(n)$, such that $ti \equiv i$ and $tj \equiv -j$. For any $a \in \mathbb{Z}_n$, $(a - at, t)$ is a nontrivial automorphism fixing v_a^r for all $r \in \{1, 2, \ldots, \ell\}$, and so, $\{v_a^r\}$ is not a determining set. Thus, $\mathrm{Det}(C_n(i \div \ell, j) \geq 2$. $\square$

Theorem 22. *If $\ell \geq 2$ and $H' = \{\pm 1\}$, then*

$$\mathrm{Det}(C_n(i \div \ell, j)) = 1,\ \mathrm{Dist}(C_n(i \div \ell, j)) = 2 \text{ and } \rho(C_n(i \div \ell, j)) = 1.$$

Otherwise, $\mathrm{Det}(C_n(i \div \ell, j)) = \mathrm{Dist}(C_n(i \div \ell, j)) = \rho(C_n(i \div \ell, j)) = 2.$

Proof. If $\ell \geq 2$ and $H' = \{\pm 1\}$, then by Proposition 18, $C_n(i \div \ell, j)$ has a one-element determining set. Any graph with determining number 1 has distinguishing number 2 and cost 1, as shown in [4].

If $\ell = 1$ and $H' = \{\pm 1\}$, then by Proposition 18, $W = \{u_0, v_0^1\}$ is a minimum determining set. If $H' \neq \{\pm 1\}$, then by Proposition 21, there exists $a \in \mathbb{Z}$, such that $\{u_a, v_0^1\}$ is a minimum determining set. In each case, the two vertices in the determining set have different degrees. Thus, coloring these vertices red and the other vertices blue produces a 2-distinguishing coloring with cost 2. Since $\mathrm{Det}(C_n(i \div \ell, j)) = 2$, no distinguishing 2-coloring can have a color class of size 1. $\square$

Our discussion of $C_n(i \div \ell, j)$ relies several times on the fact that $i \neq n/2$ under the overall assumption $0 < i < j \leq n/2$. For example, this allowed us to conclude that u_{a+i} and u_{a-i} are distinct vertices, and that the subscripts on the degree-2 vertices v_a^r are unambiguous. The other overall assumption of this paper is that $\gcd(n, i, j) = 1$, in which i and j play interchangeable roles. Thus, our results on $C_n(i \div \ell, j)$ carry over to the case $C_n(i, j \div \ell)$ when $j < n/2$.

Theorem 23. *Assume $C_n(i,j)$ is connected and $j < n/2$. For any $\alpha' = (s,t) \in \mathbb{Z}_n \rtimes H'$, there is a unique $\alpha \in \mathrm{Aut}(C_n(i, j \div \ell))$, such that $\alpha(u_a) = \alpha'(u_a)$ for all $a \in \mathbb{Z}_n$. The action of the unique extension α is defined as*

$$\alpha(x) = \begin{cases} u_{s+ta}, & x = u_a, \\ v^r_{s+at}, & x = v^r_a \text{ and } tj \equiv j, \\ v^{(\ell+1)-r}_{s+ta-j}, & x = v^r_a \text{ and } tj \equiv -j. \end{cases} \tag{3}$$

Then, $\mathrm{Aut}(C_n(i \div \ell, j)) = \mathbb{Z}_n \rtimes H'$, with the action of elements of $\mathbb{Z}_n \rtimes H'$ as defined in Eq. 3.

Proof. In general, we can simply modify the proofs of Lemma 16 and Theorem 17 by interchanging i and j. We must be more careful in the portion of the proof dealing with the case where $C_n(i,j)$ has twins or co-twins. The special cases $C_4(1,2)$ and $C_6(1,3)$ do not satisfy $j < n/2$. The other special cases, $C_5(1,2)$, $C_8(1,3)$ and $C10(1,3)$ have $\gcd(n,j) = 1$. Thus, the edges in $C_n(i, j \div \ell)$ having at least one endvertex of degree 2 again induce a Hamiltonian cycle. Since any $\alpha \in \mathrm{Aut}(C_n(i, j \div \ell))$ must respect this Hamiltonian cycle, the restriction α' must respect the n-cycle in $C_n(i,j)$ induced by the edges corresponding to the arcs of voltage j, and so, we can again conclude that α' is an element of the dihedral group D_{2n}. □

If $j < n/2$, then Proposition 18, Lemma 19, Corollary 20, and Proposition 21 all hold with the roles of i and j interchanged.

Theorem 24. *Assume $j < n/2$. If $\ell \geq 2$ and $H' = \{\pm 1\}$, then*

$$\mathrm{Det}(C_n(i, j \div \ell)) = 1, \ \mathrm{Dist}(C_n(i, j \div \ell)) = 2 \text{ and } \rho(C_n(i, j \div \ell)) = 1.$$

Otherwise, $\mathrm{Det}(C_n(i, j \div \ell)) = \mathrm{Dist}(C_n(i, j \div \ell)) = \rho(C_n(i, j \div \ell)) = 2.$

7.2. $C_n(i, j \div \ell)$, $j = n/2$

We recall some results for $C_n(i,j)$ with $j = n/2$. In this case, $n = 2j$ and $j \equiv -j$. From Lemma 1, $C_{2j}(i,j)$ is twin-free except in two cases: $C_4(1,2)$, in which any two vertices are twins, and $C_6(1,3)$, in which u_a and u_{a+2} are twins for all $a \in \mathbb{Z}_6$. These are also the only $C_{2j}(i,j)$ that are edge-transitive. From Lemma 6, if $C_{2j}(i,j)$ is twin-free, then for all $a \in \mathbb{Z}_{2j}$, we have $N(u_a) \cap N(u_{a+2i}) = \{u_{a+i}\}$ and $N(u_a) \cap N(u_{a+i+j}) = \{u_{a+i}, u_{a+j}\}$, and these are the only possibilities for two vertices to have common neighbors.

If we subdivide the loop with voltage j in the voltage graph with ℓ vertices of degree 2, then in the derived graph, the single (undirected) edge between u_a and u_{a+j} is replaced with two paths

$$(u_a, v^1_a, v^2_a, \ldots, v^\ell_a, u_{a+j}) \text{ and } (u_{a+j}, v^1_{a+j}, v^2_{a+j}, \ldots, v^\ell_{a+j}, u_{a+2j} = u_a).$$

Thus, $N(u_a) = \{v^1_a, v^\ell_{a+j}, u_{a-i}, u_{a+i}\}$. Examples $C_8(1, 4 \div 1)$ and $C_{10}(2, 5 \div 2)$ appear in Fig. 5.

Lemma 25. (1) *If $\ell \geq 2$, then $C_{2j}(i, j \div \ell)$ is twin-free.*

(2) *If $\ell = 1$, then for distinct a, b, $N(v^1_a) = N(v^1_b)$ if and only if $b = a + j$, and $N(u_a) = N(u_b)$ if and only if $(2j, i, j) = (4, 1, 2)$ and $b = a + 2$.*

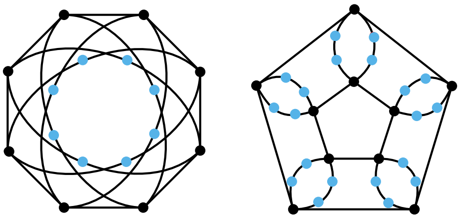

FIGURE 5. $C_8(1,4\div 1)$ and $C_{10}(2,5\div 2)$

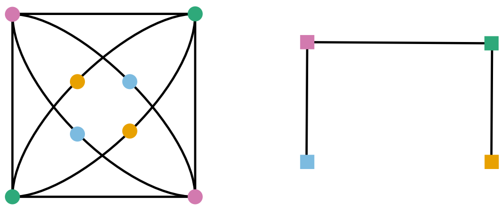

FIGURE 6. $C_4(1,2\div 1)$ and its twin quotient graph, P_4

Proof. First, assume $\ell \geq 2$. Then, each u_a has a distinct pair of degree-2 neighbors, and so, no two such vertices can be twins. Moreover, each vertex v_a^k has at least one neighbor of the form $v_a^{k\pm 1}$, meaning that for $a \not\equiv b$, vertices v_a^k and v_b^ℓ cannot be twins. Finally, vertices v_a^k and v_a^ℓ with $k < \ell$ cannot be twins, because either u_a (when $k = 1$) or v_a^{k-1} (when $k > 1$) is in $N(v_a^k)\backslash N(v_a^\ell)$.

Next, assume $\ell = 1$. Then, $N(v_a^1) = \{u_a, u_{a+j}\} = N(v_{a+j}^1)$. Next, assume $N(u_a) = N(u_b)$. For u_a and u_b to have the same degree-2 neighbors, $b = a + j$. For them to have the same degree-4 neighbors, $\{u_{a-i}, u_{a+i}\} = \{u_{a+j-i}, u_{a+j+i}\}$. Since $j \not\equiv 0$, this implies $a - i \equiv a + j + i$ and $a + i \equiv a + j - i$, so $2i \equiv j$. The general assumption that $\gcd(n, i, j) = 1$ implies $(2j, i, j) = (4, 1, 2)$. $\square$

Example 2. In the special case $C_4(1,2\div 1)$, each vertex is in a set of $k = 2$ mutual twins, and so, we can use the techniques of Sect. 3. See Fig. 6. The minimum twin cover $T = \{u_0, u_1, v_0^1, v_1^1\}$ is also a minimum determining set, so $\mathrm{Det}(C_4(1,2\div 1)) = 4$. The twin quotient graph is P_4, which has distinguishing number 2. By Theorem 4, $\mathrm{Dist}(C_4(1,2\div 1)) = 3$.

In general, if $\ell = 1$ and $n = 2j \geq 6$, then the set $T = \{v_0^1, v_1^1, \ldots, v_{j-1}^1\}$ is a minimum twin cover, but not a determining set. To see this, we must investigate the automorphism group of $C_{2j}(i, j\div\ell)$.

Applying Lemma 9 when $n = 2j$ yields $H' = \{\pm 1\}$, so $\mathbb{Z}_{2j} \rtimes H'$ is the dihedral group D_{4j}. If $C_{2j}(i,j)$ is twin-free, $\mathrm{Aut}(C_{2j}(i,j)) = \mathbb{Z}_{2j} \rtimes H' = D_{4j}$ by Theorem 12. For $C_6(1,3)$, by Theorem 14, every automorphism is the composition of an element of D_{12} and an automorphism that permutes twin pairs.

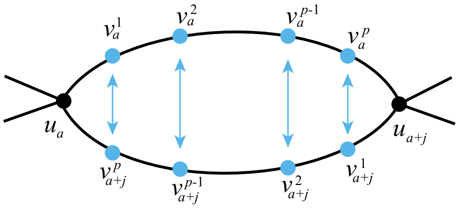

FIGURE 7. Action of the automorphism $\beta_a = \beta_{a+j}$

When $n = 2j$, $j \equiv -j$, so for any $a \in \mathbb{Z}_{2j}$ and $(s,t) \in \mathbb{Z}_{2j} \rtimes H'$, $(s,t) \cdot (u_{a+j}) = u_{s+ta+j}$. Thus, when extending $(s,t) \in \mathbb{Z}_{2j} \rtimes H'$ to the degree-2 vertices in $C_n(i, j{\div}\ell)$, we can simply set $(s,t) \cdot (v_a^r) = v_{s+ta}^r$.

Unlike the situation for $C_{2j}(i{\div}\ell, j)$, there are additional automorphisms of $C_{2j}(i, j{\div}\ell)$ beyond these extensions of the automorphisms in $\mathbb{Z}_{2j} \rtimes H'$. For each $a \in \mathbb{Z}_{2j}$, there is an automorphism β_a that 'flips' the degree-2 vertices on the two paths between u_a and u_{a+j} and leaves all other vertices fixed; see Fig. 7. More precisely, $\beta_a \cdot (u_b) = u_b$ for all $b \in \mathbb{Z}_n$ and

$$\beta_a \cdot (v_b^r) = \begin{cases} v_{b+j}^{(\ell+1)-r}, & \text{if } b \in \{a, a+j\}, \\ v_b^r, & \text{if } b \in \mathbb{Z}_{2j} \setminus \{a, a+j\}. \end{cases}$$

Since $\beta_a = \beta_{a+j}$, there are j such automorphisms. The subgroup generated by these automorphisms is $B = \langle \beta_a \mid 0 \le a < j \rangle \cong (\mathbb{Z}_2)^j$; we denote its identity by ι_B. An element $\beta \in B$ is of the form $\beta = \beta_0^{e_0} \cdots \beta_{j-1}^{e_{j-1}}$, where $e_0, \ldots, e_{j-1} \in \{0,1\}$.

Theorem 26. *The automorphism group of $C_{2j}(i, j{\div}\ell)$ is*

$$\mathrm{Aut}(C_{2j}(i, j{\div}\ell)) = (\mathbb{Z}_2)^j \rtimes (\mathbb{Z}_{2j} \rtimes \{\pm 1\}) = \mathbb{Z}_{2j} \rtimes D_{4j}.$$

An element is denoted $\beta \circ (s,t)$, where $\beta \in B$ and $(s,t) \in \mathbb{Z}_{2j} \rtimes \{\pm 1\}$.

Proof. It is routine to verify that the extension of each $(s,t) \in \mathbb{Z}_{2j} \rtimes \{\pm 1\}$ to the degree-2 vertices and each $\beta \in B$ are indeed automorphisms of $C_{2j}(i, j{\div}\ell)$. Thus, both $\mathbb{Z}_{2j} \rtimes \{\pm 1\}$ and B are subgroups of $\mathrm{Aut}(C_{2j}(i, j{\div}\ell))$. It is similarly routine to verify that for all $a \in \mathbb{Z}_{2j}$ and $(s,t) \in \mathbb{Z}_{2j} \rtimes \{\pm 1\}$, $(s,t) \circ \beta_a = \beta_{s+ta} \circ (s,t)$, so B is a normal subgroup of $\mathrm{Aut}(C_{2j}(i, j{\div}\ell))$. Notice, $B \cap \mathbb{Z}_{2j} \rtimes \{\pm 1\} = \emptyset$. It now suffices to show that every automorphism of $C_{2j}(i, j{\div}\ell)$ is the composition of an element of B and an element of $\mathbb{Z}_{2j} \rtimes \{\pm 1\}$.

Let $\alpha \in \mathrm{Aut}(C_{2j}(i, j{\div}\ell))$. Then, α restricts to a bijection α' on the set of degree-4 vertices, $\{u_a \mid a \in \mathbb{Z}_{2j}\}$. By the same reasoning as in the proof of Theorem 17, $\alpha' \in \mathrm{Aut}(C_{2j}(i,j))$. If $C_{2j}(i,j)$ is twin-free, then $\alpha' \in \mathrm{Aut}(C_{2j}(i,j)) = \mathbb{Z}_{2j} \rtimes \{\pm 1\}$ by Theorem 12. In the cases with twins, namely $C_4(1,2)$ and $C_6(1,3)$, the edges corresponding to arcs of voltage $i = 1$ induce a Hamiltonian $2j$-cycle in $C_{2j}(i,j)$. Thus, α' respects this $2j$-cycle. We can conclude that $\alpha' \in D_{4j} = \mathbb{Z}_n \rtimes \{\pm 1\}$.

Thus, in all cases, $\alpha' = (s,t)$ for some $s \in \mathbb{Z}_{2j}$ and $t \in \{\pm 1\}$. Since $\mathbb{Z}_{2j} \rtimes \{\pm 1\}$ is a subgroup of $\mathrm{Aut}(C_{2j}(i, j{\div}\ell))$, both (s,t) and $(-s,t)$ are in

$\mathrm{Aut}(C_{2j}(i, j_{\div}\ell))$. Then, $\gamma = \alpha \circ (-s, t)$ is an automorphism of $C_{2j}(i, j_{\div}\ell)$ that fixes every degree-4 vertex. We will show that $\gamma \in B$.

Let $a \in \mathbb{Z}_{2j}$. There are exactly two degree-2 vertices that are both adjacent to u_a and distance $\ell+1$ from u_{a+j}, namely v_a^1 and v_{a+j}^ℓ. Thus, to respect adjacency among degree-2 vertices, γ must either fix or interchange the two paths between u_a and u_{a+j} that have all interior vertices of degree 2. If it fixes them, let $e_a = 0$, and if it switches them, let $e_a = 1$. Note that $e_a = e_{a+j}$. Then, $\gamma = \beta_0^{e_0}\beta_1^{e_1}\cdots\beta_{j-1}^{e_{j-1}} \in B$. Since we defined $\gamma = \alpha \circ (-s, t)$, we have $\alpha = \gamma \circ (s, t)$. □

Proposition 27. *If $j \geq 3$, then a minimum determining set of $C_{2j}(i, j_{\div}\ell)$ is $W = \{v_0^1, v_1^1, \ldots, v_{j-1}^1\}$ if $\ell \geq 2$ and $W = \{u_0, v_0^1, v_1^1, \ldots, v_{j-1}^1\}$ if $\ell = 1$.*

Proof. First note that for each $a \in \mathbb{Z}_n$, a determining set must contain at least one degree-2 vertex on the two paths between u_a and u_{a+j}, for otherwise β_a is a nontrivial automorphism fixing the set.

Assume $\ell \geq 2$. The set $W = \{v_0^1, v_1^1, \ldots, v_{j-1}^1\}$ contains exactly one degree-2 vertex on each pair of paths between u_a and u_{a+j}. Thus, any automorphism fixing this set cannot include any β_a, so it is of the form $\iota_B \circ (s, t) = (s, t)$. Since u_0 is the only degree-4 vertex adjacent to v_0^1, any (s, t) fixing W also fixes u_0, and so, $s \equiv 0$. By definition, $(0, t) \cdot (v_1^1) = v_t^1$, so the assumption that $(0, t)$ fixes W implies $t \equiv 1$. The automorphism $\iota_B \circ (0, 1)$ is the identity of $\mathrm{Aut}(C_{2j}$ $(i, j_{\div}\ell)$, and so, W is determining.

Next, assume $\ell = 1$. By Lemma 25, v_a^1 and v_{a+j}^1 are twins for all $0 \leq a < j$. Note that β_a is an automorphism that interchanges these twin vertices and leaves all other vertices fixed. However, no minimum twin cover is determining, because it is fixed by the nontrivial automorphism $(\beta_0\beta_1\cdots\beta_{j-1}) \circ (j, 1)$. On the other hand, the set $W = \{u_0, v_0^1, v_1^1, \ldots, v_{j-1}^1\}$ is determining. Since $u_0 \in W$, any automorphism fixing W is of the form $\beta \circ (0, t)$, where $\beta = \beta_0^{e_0} \cdot \beta_1^{e_1} \cdot \cdots \cdot \beta_{j-1}^{e_{j-1}}$ for some $e_0, e_1, \ldots, e_{j-1} \in \{0, 1\}$, and $t \equiv \pm 1$. If $t \equiv 1$, then $(0, 1)$ fixes every element of W. If $e_a = 1$ for any $a \in \{0, 1, \ldots, j-1\}$, then $(\beta \circ (0, 1)) \cdot (v_a^1) = v_{j+a}^1 \neq v_a^1$, a contradiction. This would imply that $\beta = \iota_B$ and $\beta \circ (0, 1)$ is the identity. Therefore, for $\beta \circ (0, t)$ to be nontrivial, we require $t \equiv -1$. Then

$$[\beta \circ (0, -1)] \cdot (v_1^1) = \beta \cdot (v_{2j-1}^1) = \begin{cases} v_{2j-1}^1, & \text{if } e_1 = 0, \\ v_{j-1}^1, & \text{if } e_1 = 1. \end{cases}$$

Since $2j - 1 \not\equiv 1$ and $j - 1 \not\equiv 1$ (by the assumption that $j \geq 3$), there is no $\beta \in B$ for which $\beta \circ (0, -1)$ fixes W, and and so W is a determining set. □

Theorem 28. *Assume $C_{2j}(i, j_{\div}\ell)$ is connected.*

(1) *If $\ell \geq 2$, then* $\mathrm{Det}(C_{2j}(i, j_{\div}\ell)) = j$, $\mathrm{Dist}(C_{2j}(i, j_{\div}\ell)) = 2$, *and*

$$\rho(C_{2j}(i, j_{\div}\ell)) = \begin{cases} j+1, & \ell = 2 \text{ and } j \in \{2, 3, 4, 5\}, \text{ or } \ell = j = 3, \\ j, & \text{otherwise.} \end{cases}$$

(2) *If $\ell = 1$, then* $\mathrm{Det}(C_4(1, 2_{\div}1)) = 4$ *and* $\mathrm{Dist}(C_4(1, 2_{\div}1)) = 3$. *If $j \geq 3$,* $\mathrm{Det}(C_{2j}(i, j_{\div}1)) = j+1$, $\mathrm{Dist}(C_{2j}(i, j_{\div}1)) = 2$, *and* $\rho(C_{2j}(i, j_{\div}1)) = j+3$.

Proof. We first prove (2); assume $\ell = 1$. For each $a \in \mathbb{Z}_{2j}$, β_a interchanges twins v_a^1 and v_{a+j}^1, and fixes all other vertices. Thus, v_a^1 and v_{a+j}^1 must be in different color classes in any distinguishing coloring.

If $j = 2$, we have $C_4(1, 2 \div 1)$. The determining and distinguishing number are discussed in Example 2. The cost of 2-distinguishing is undefined in this case.

Now, suppose $j \geq 3$. By Proposition 27, $\mathrm{Det}(C_{2j}(i, j \div 1)) = j + 1$. Since i and j are nonadjacent in $C_{2j}(i, j)$, u_i and u_j are nonadjacent in $C_{2j}(i, j \div \ell)$. Color the vertices in $R = \{u_0, u_i, u_j, v_0^1, v_1^1 \cdots, v_{j-1}^1\}$ red and all other vertices blue, and assume $\beta \circ (s, t) \in B \rtimes (\mathbb{Z}_{2j} \rtimes \{\pm 1\})$ preserves these two color classes. Among the degree-4 vertices in R, u_0 and u_i are adjacent to each other but neither is adjacent to u_j. Thus, $[\beta \circ (s, t)] \cdot (u_j) = u_j$. This implies that $j \equiv s + tj \equiv s + j$, so $s \equiv 0$. Note that $[\beta \circ (0, t)] \cdot (u_0) = u_0$. Then, u_i must also be fixed by $\beta \circ (0, t)$ so $ti \equiv i$ and so $t \equiv 1$. In order for $\beta \circ (0, 1)$ to fix all of the degree-2 vertices in R, $\beta = \iota_B$. Thus, this is a 2-distinguishing coloring with $|R| = j + 3$ red vertices and (by the assumption that $j \geq 3$) at least $j + 3$ blue vertices. Hence, $\rho(C_{2j}(i, j \div \ell)) \leq j + 3$.

To find a lower bound on cost, first note that since v_a^1 and v_{a+j}^1 must be in different color classes for all $a \in \mathbb{Z}_{2j}$, each color class in a 2-distinguishing coloring contains exactly j degree-2 vertices. Suppose there is just one degree-4 vertex, u_a, in the minimum size color class. We know that $(s, t) = (2a, -1)$ fixes u_a. For each $b \in \mathbb{Z}_{2j}$, $(2a, -1)$ exchanges v_b^1 and v_{2a-b}^1. If v_b^1 and v_{2a-b}^1 have opposite colors, then let $e_b = 1$, and otherwise, let $e_b = 0$. By the first sentence of the paragraph, $e_{b+j} = e_b$. Then, let

$$\beta = \prod \{\beta_b \mid e_b = 1 \text{ and } 0 \leq b < j\}.$$

Then, $\beta \circ (2a, -1)$ preserves the color classes, so the coloring is not distinguishing. Suppose instead that there are exactly two degree-4 vertices, u_a and u_b, in the minimum color class. Using the same process as above, we can find $\beta \in B$, such that

$\beta \circ (a + b, -1)$ preserves the color classes, so the coloring is not distinguishing. Thus, $\rho(C_{2j}(i, j \div \ell)) > j + 2$.

We now prove (1); assume $\ell \geq 2$. In this case, $\mathrm{Det}(C_{2j}(i, j \div \ell)) = j$ by Proposition 27. To show that $C_{2j}(i, j \div \ell)$ is 2-distinguishable, color the vertices in $R = \{u_0, v_0^1, v_1^1, \ldots, v_{j-1}^1\}$ red and all other vertices blue and assume $\beta \circ (s, t)$ preserves the color classes. Since u_0 is the only degree-4 vertex in R, it is fixed by $\beta \circ (s, t)$, so $s \equiv 0$. If $t \equiv -1$

$$[\beta \circ (0, -1)] \cdot (v_1^1) = \beta \cdot (v_{2j-1}^1) = \begin{cases} v_{2j-1}^1, & \text{if } e_{j-1} = 0, \\ v_{j-1}^\ell, & \text{if } e_{j-1} = 1. \end{cases}$$

Since neither v_{2j-1}^1 nor v_{j-1}^ℓ is in R, no automorphism of the form $\beta \circ (0, -1)$ preserves R. Hence, $t \equiv 1$. Since $(s, t) = (0, 1)$ fixes every degree-2 vertex in R, $\beta = \iota_B$, and so, $\beta \circ (s, t)$ is the identity. This is therefore a 2-distinguishing coloring, so $\mathrm{Dist}(C_{2j}(i, j \div \ell)) = 2$ and $\rho(C_{2j}(i, j \div \ell)) \leq j + 1$.

For a lower bound on cost, note that in any 2-distinguishing coloring, v_a^r and $v_{a+j}^{\ell+1-r}$ must have different colors for at least one $1 \leq r \leq \ell$; otherwise, β_a preserves the color classes. Thus, each color class must have at least j vertices.

What remains is to establish when we can find a 2-distinguishing coloring with exactly j vertices in a color class. In any 2-distinguishing coloring, candidates for a color class of size j in a 2-distinguishing coloring are sets of the form

$$R = \{v_{a_0}^{r_0}, v_{a_1}^{r_1}, \ldots, v_{a_{j-1}}^{r_{j-1}}\}.$$

If every such R is preserved by some nontrivial automorphism, then the cost is $j+1$. Conversely, if we can find one such R preserved only by the identity, then the cost is j. In the case $\ell = 2$ and $j \in \{2, 3, 4, 5\}$, one can computationally verify the result. For $\ell = 2$, $j \geq 6$, the set $R = \{v_0^1, v_1^1, v_{j+2}^1, v_3^1, \ldots, v_{j-1}^1\}$ is only preserved by the identity; see [7] for details.

Now, assume $\ell \geq 3$. Let $R = \{v_0^2, v_1^1, \ldots, v_{j-2}^1, v_{j-1}^{\ell}\}$. Assume $\beta \circ (s,t)$ preserves R. The only element of R not adjacent to a degree-4 vertex is v_0^2, so it is fixed by $\beta \circ (s,t)$, implying that $s \equiv 0$ and $e_0 = 0$. Suppose that $t \equiv -1$. Then, $\beta = \beta_1 \beta_2 \cdots \beta_{j-1}$ and the image of R is

$$\{v_0^2, v_1^1, v_2^{\ell}, \ldots, v_{j-1}^{\ell}\}. \tag{4}$$

For $j = 2$ and $j > 3$, the set in (4) is not R. We conclude $t \equiv 1$, which in turn implies $\beta = \iota_B$, and so, the only automorphism preserving R is the identity. However, for $j = 3$, the set (4) is R. If $\ell \geq 4$, this problem is easily addressed by replacing R with $R' = \{v_0^1, v_1^2, v_2^2\}$. However, if $j = \ell = 3$, it can be computationally shown that the cost of 2-distinguishing cannot be j; see [7] for details. □

8. Future Work

A natural extension of our work is to find symmetry parameters for connected $C_n(A)$ where $|A| > 2$. Note that if $\overline{A}$ denotes the complement of A in $\{1, 2, \ldots, n\}$, then $C_n(\overline{A}) = \overline{C_n(A)}$. It is known that the determining number, distinguishing number and, if relevant, the cost of 2-distinguishing are equal for a graph and its complement. This means that it suffices to find these symmetry parameters for connected $C_n(A)$, where $2 < |A| \leq n/2$.

For two-generator circulant graphs, we have found it fruitful to divide into cases depending on the presence of twins or co-twins. The obvious generalization of Theorem 12 would be that if $C_n(A)$ is is twin-free and co-twin-free, then $\mathrm{Aut}(C_n(A)) = \mathbb{Z}_n \rtimes H$ where $H = \mathrm{Aut}(\mathbb{Z}_n, A)$. Equivalently, by Godsil's result, if $C_n(A)$ is twin-free and co-twin-free, then $\mathbb{Z}_n$ is a normal subgroup of $\mathrm{Aut}(C_n(A))$. For $C_n(A)$ with twins, we can use the results of Sect. 3 to compute the symmetry parameters in terms of those of the twin quotient graph $\widetilde{C_n(A)}$. By vertex transitivity, if one vertex in $C_n(A)$ has k twins, then so does every vertex. It follows that the degree of every vertex is a multiple of k. Together, these imply that the twin quotient graph is a twin-free circulant graph of order n/k with fewer generators. A similar approach is helpful in considering circulant graphs with co-twins.

Another direction for future research would be to investigate the symmetry parameters of other subdivisions of connected $C_n(i, j)$, such as $C_n(i_{\div}\ell, j_{\div}m)$ when $i \neq j$ and when $i = j$ but $\ell \neq m$.

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