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On a General Approach to Bessenrodt–Ono Type Inequalities and Log-Concavity Properties

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Abstract. In recent literature concerning integer partitions one can find many results related to both the Bessenrodt–Ono type inequalities and log-concavity properties. In this note, we offer some general approach to this type of problems. More precisely, we prove that under some mild conditions on an increasing function F of at most exponential growth satisfying the condition $F(\mathbb{N}) \subset \mathbb{R}_+$, we have $F(a)F(b) > F(a+b)$ for sufficiently large positive integers a, b. Moreover, we show that if the sequence $(F(n))_{n \geq n_0}$ is log-concave and $\limsup_{n\to+\infty} F(n+n_0)/F(n) < F(n_0)$, then F satisfies the Bessenrodt–Ono type inequality.

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1. Introduction

Let $\mathbb{N}$ be the set of non-negative integers, $\mathbb{N}_+$ the set of positive integers and $\mathbb{R}_+ = (0, +\infty)$. Moreover, for an integer $n_0 \geq 2$ we put $\mathbb{N}_{\geq n_0} = \{n \in \mathbb{N} : \ n \geq n_0\}$.

For given $A \subset \mathbb{N}_+$ and $n \in \mathbb{N}$ by $p_A(n)$ we denote the number of partitions of n with parts in the set A. As usual, we put $p_A(0) = 1$. It is well known that the ordinary generating function for the sequence $(p_A(n))_{n\in\mathbb{N}}$ takes the form

$$\sum_{n=0}^{\infty} p_A(n)x^n = \prod_{a\in A} \frac{1}{1-x^a}.$$

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If $A = \mathbb{N}_+$ then we simply write $p(n)$ instead of $p_{\mathbb{N}_+}(n)$. In this case the partition function is the famous Euler partition function thoroughly studied by Ramanujan and many others. The number of papers devoted to various properties of $p(n)$, or more generally, for $p_A(n)$ for various choice of the set A is enormous. The standard reference is the book of Andrews [2] (see also [3] for less advanced approach).

A few years ago there emerged a broad research devoted to log-behavior of partition statistics. Let us recall that a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers is said to be log-concave if the inequality

$$a_n^2 > a_{n-1}a_{n+1}$$

is valid for all sufficiently large values of n. On the other hand, it is called log-convex if the inverse inequality holds. The first who proved the log-concavity of classical partition function $p(n)$ was Nicolas in [27].

Recently, using analytic methods, DeSalvo and Pak [13] reproved the Nicolas' theorem.

Now, there is a wealth of literature devoted to the log-concavity property for other variations of the partition function. For instance, Bringmann, Kane, Rolen and Tripp [8] investigated the case of the k-colored partition function and partly proved the conjecture formulated by Heim and Nauhauser in [18]. This conjecture is a polynomial generalization of a one stated by Chern, Fu, and Tang in [10]. On the other hand, Dawsey and Masri [11] examined the Andrews *spt*-function in that direction. Further, Engel [14] proved that the overpartition function $\overline{p}(n)$ is log-concave for every $n \geq 2$. Gajdzica [15] discovered a similar phenomenon for the A-partition function when A is finite. O'Sullivan [28] investigated the number of partitions into powers and proved a conjecture of Ulas [30]. Ono, Pujahari and Rolen [29] showed that the plane partition function satisfies the log-concavity property as well.

However, the research devoted to log-behavior of partition functions is not only bounded by the log-concavity or log-convexity properties. Another interesting phenomenon is the so-called Bessenrodt–Ono inequality. More precisely, in 2016, Bessenrodt and Ono [7] showed that for $a, b \geq 2$ and $a + b > 9$ we have

$$p(a)p(b) > p(a + b). \tag{1.1}$$

Their proof is based on the asymptotic estimates due to Lehmer [23]. It is worth noting that there are two alternative approaches to derive the result. Alanazi, Gagola and Munagi [1] showed the Bessenrodt–Ono inequality in combinatorial manner by constructing appropriate injections between some sets of partitions. Heim and Neuhauser [19], on the other hand, presented a proof which is based on the induction on $a + b$.

Moreover, the inequality (1.1) can be effectively used in practice. For instance, one can apply it to determine the value of

$$\max p(n) = \max \{p(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \text{ is a partition of } n\},$$

where $p(\boldsymbol{\lambda})$ denotes the extended partition function defined as $p(\boldsymbol{\lambda}) := \prod_{i=1}^{j} p(\lambda_i)$ for $\lambda = (\lambda_{\mathbf{1}}, \lambda_{\mathbf{2}}, \ldots, \lambda_{\mathbf{j}})$. For additional information, we refer the reader to [7].

There are a lot of publications regarding Bessenrodt–Ono type inequalities. For example, Beckwith and Bessenrodt [5] found out similar properties for the k-regular partition function. Chern, Fu and Tang [10] considered the k-colored partition function. Dawsey and Masri [11] discovered the analogous phenomenon for the Andrews *spt*-function. Heim, Neuhauser and Tröger [21] examined the issue for the plane partition function. Moreover, Heim and Neuhauser widely generalized the property and investigated the so-called polynomization of the Bessenrodt–Ono inequality for a couple of partition functions [16,17,20,21]. Hou and Jagadeesan [22], and Males [26] discovered the analogues of the Bessenrodt–Ono inequality for the so-called partition rank function.

Actually, there are many more properties related to the log-behavior of partition statistics. We do not discuss them here, but focus on some general criteria for both Bessenrodt–Ono type inequalities and log-concavity problems. It turns out that a plenty of the aforementioned results are proved using some asymptotic estimates. Therefore, it would be convenient to possess several conditions which assert that a function with an appropriate growth is log-concave or fulfills the Bessenrodt–Ono inequality. Essentially, these goals were the main motivations for our investigation.

Let us describe the content of the paper in some details. In Sect. 2 we get a general criterion on the sequence $(F(n))_{n\in\mathbb{N}}$, which guarantees that it satisfies the Bessenrodt–Ono type inequality. In other words, for all sufficiently large $a, b \in \mathbb{N}, a \geq b$ we have $F(a)F(b) > F(a+b)$. In particular, as an application we reprove recent result of Heim and Neuhauser which says that the plane partition function satisfies the Bessenrodt–Ono type inequality. In Sect. 3 we present some conditions guaranteeing asymptotic log-concavity of the sequence $(F(n))_{n\in\mathbb{N}}$. As an application we reprove recent result of DeSalvo and Pak [13, Theorem 1.1]. Finally, in the last section, under some mild condition on the sequence $(F(n))_{n\in\mathbb{N}}$ we show that log-concavity implies the Bessenrodt–Ono type inequality. We also prove that the opposite implication is not true. More precisely, we show that for each $m \in \mathbb{N}_{\geq 2}$ the m-ary partition function $b_m(n) = p_A(n)$, where $A = \{m^i : \ i \in \mathbb{N}\}$, satisfies the Bessenrodt–Ono type inequality but is not log-concave.

2. Bessenrodt–Ono Type Inequality Holds for a Class of Sub-exponential Functions

In this section we offer a general approach to Bessenrodt–Ono type inequalities. More precisely, we prove the following general result.

Theorem 2.1. *Let $c_1, c_2, f, F : \mathbb{N}_+ \to \mathbb{R}_+$ be given and such that the inequalities*

$$c_1(n)e^{f(n)} < F(n) < c_2(n)e^{f(n)}$$

are valid for all positive integers $n \geq N_0$ *for some* $N_0 \in \mathbb{N}_+$. *Suppose further that the following conditions hold:*

(1) $\exists g : \mathbb{N}_+ \to \mathbb{R}_+, N_1 \in \mathbb{N}_+ \; \forall a \geq b \geq N_1 : f(a) + f(b) - f(a+b) \geq g(b)$;
(2) $\exists h : \mathbb{N}_+ \to \mathbb{R}_+, N_2 \in \mathbb{N}_+ \; \forall a \geq b \geq N_2 : c_2(a+b)/c_1(a) \leq h(b)$;
(3) $\exists N_3 \in \mathbb{N}_+ \; \forall n \geq N_3 : g(n) \geq \log h(n) - \log c_1(n)$.

Then, the inequality

$$F(a)F(b) > F(a+b)$$

is satisfied for all $a, b \geq \max\{N_0, N_1, N_2, N_3\}$.

Proof. Let us fix functions c_1, c_2, f, F and g as in the statement. Since both of the inequalities

$$F(a)F(b) > c_1(a)c_1(b)e^{f(a)+f(b)}$$

and

$$F(a+b) < c_2(a+b)e^{f(a+b)}$$

are true for any positive integers $a, b \geq N_0$, it is enough to prove that the following inequality

$$e^{f(a)+f(b)-f(a+b)} \geq \frac{c_2(a+b)}{c_1(a)c_1(b)}$$

is satisfied for all sufficiently large values of a and b. From (1), it follows that the inequality

$$f(a) + f(b) - f(a+b) \geq g(b)$$

holds for every $a \geq b \geq N_1$. On the other hand, (2) implies that the inequality

$$\frac{c_2(a+b)}{c_1(a)} \leq h(b)$$

is valid for any $a \geq b \geq N_2$. Hence, it is enough to show that the following

$$e^{g(b)} \geq h(b)/c_1(b)$$

holds for all large values of b—but that is a direct consequence of (3), as required. □

Although the above theorem is very easy we show that it is strong enough to be applied to classical examples of partition functions.

Example 2.2. Let us consider the partition function $p(n)$. It follows from the Bessenrodt and Ono's paper [7] that the inequalities

$$\frac{\sqrt{3}}{12n}\left(1 - \frac{1}{\sqrt{n}}\right)e^{\frac{\pi}{6}\sqrt{24n-1}} < p(n) < \frac{\sqrt{3}}{12n}\left(1 + \frac{1}{\sqrt{n}}\right)e^{\frac{\pi}{6}\sqrt{24n-1}}$$

hold for every positive integer n. Therefore, we will use Theorem 2.1 with $F(n) := p(n)$,

$$f(n) := \frac{\pi}{6}\sqrt{24n-1}, c_1(n) := \frac{\sqrt{3}}{12n}\left(1 - \frac{1}{\sqrt{n}}\right) \text{and} c_2(n) := \frac{\sqrt{3}}{12n}\left(1 + \frac{1}{\sqrt{n}}\right).$$

Now, if we assume that $a \geq b \geq 1$, then we have that

$$\begin{aligned} f(a)+f(b)-f(a+b) &= \frac{\pi}{6}\cdot\frac{2\sqrt{24a-1}\sqrt{24b-1}-1}{\sqrt{24a-1}+\sqrt{24b-1}+\sqrt{24(a+b)-1}} \\ &> \frac{\pi}{6}\cdot\frac{\sqrt{24a-1}\sqrt{24b-1}}{\sqrt{24a-1}+\sqrt{24b-1}}-\frac{\pi}{6}\cdot\frac{1}{2\sqrt{23}+\sqrt{47}} \\ &> \frac{\pi}{6}\cdot\frac{\sqrt{24a-1}\sqrt{24b-1}}{\sqrt{24a-1}+\sqrt{24b-1}}-\frac{1}{24} \\ &\geq \frac{\pi}{12}\sqrt{24b-1}-\frac{1}{24} =: g(b) \end{aligned}$$

On the other hand, if we assume that $a \geq 9$ and $1 \leq b \leq a$, then one can derive that

$$\begin{aligned} \frac{c_2(a+b)}{c_1(a)} &= \frac{12a}{12(a+b)}\left(1+\frac{1}{\sqrt{a+b}}\right)\left(1+\frac{1}{\sqrt{a}-1}\right) \\ &< \left(1+\frac{1}{\sqrt{10}}\right)\left(1+\frac{1}{\sqrt{9}-1}\right) < 2 =: h(b). \end{aligned}$$

Hence, it is enough to observe that

$$e^{g(n)} \geq 8\sqrt{3}n\left(1+\frac{1}{\sqrt{n}-1}\right)$$

is true for all sufficiently large values of n. In fact, one can show that the above is valid for every $n \geq 22$. Thus, Theorem 2.1 implies that the Bessenrodt–Ono inequality

$$p(a)p(b) > p(a+b)$$

holds for all $a \geq b \geq 22$.

Example 2.3. Let us recall that the number $pp(n)$ of plane partitions of n can be computed via the generating function (obtained by MacMahon [24])

$$\sum_{n=0}^{\infty} pp(n)x^n = \prod_{n=1}^{\infty}\frac{1}{(1-x^n)^n}.$$

From Wright's formula [32, Formula (2.21)] one can deduce the existence of (ineffective constants) $\alpha, \beta, \gamma, N > 0$ such that for all $n > N$ the following inequalities

$$\alpha n^{-\frac{25}{36}}\left(1-\frac{\beta}{\sqrt{n}}\right)e^{\gamma n^{2/3}} < pp(n) < \alpha n^{-\frac{25}{36}}\left(1+\frac{\beta}{\sqrt{n}}\right)e^{\gamma n^{2/3}}$$

hold. We set $F(n) := pp(n)$,

$$f(n) := \gamma n^{2/3}, \quad c_1(n) := \alpha n^{-\frac{25}{36}}\left(1-\frac{\beta}{\sqrt{n}}\right) \quad \text{and} \quad c_2(n) := \alpha n^{-\frac{25}{36}}\left(1+\frac{\beta}{\sqrt{n}}\right),$$

and apply Theorem 2.1. For every $a \geq b \geq N$, we have that

$$\begin{aligned} f(a)+f(b)-f(a+b) &= \gamma\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}-a^{\frac{2}{3}}\left(1+\frac{b}{a}\right)^{\frac{2}{3}}\right) \\ &\geq \gamma\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}-a^{\frac{2}{3}}\left(1+\frac{2b}{3a}\right)\right) \\ &= \gamma\left(b^{\frac{2}{3}}-\frac{2b}{3a^{1/3}}\right) \geq \frac{\gamma}{3}b^{\frac{2}{3}}, \end{aligned}$$

where the first inequality is a consequence of Bernoulli's inequality. On the other hand, it is not difficult to see that

$$\frac{c_2(a+b)}{c_1(a)} = \left(1+\frac{b}{a}\right)^{-\frac{25}{36}}\left(1+\frac{\beta}{\sqrt{a+b}}\right)\left(1+\frac{\beta}{\sqrt{a}-\beta}\right) < 1+\frac{2\beta}{\sqrt{a}-\beta} \leq 2\beta+1$$

whenever $a \geq b \geq \max\{N, (\beta+1)^2\}$. Now, it is straightforward to deduce that

$$e^{\frac{\gamma}{3}n^{\frac{2}{3}}} \geq \frac{2\beta+1}{\alpha}n^{\frac{25}{36}}\left(1+\frac{\beta}{\sqrt{n}-\beta}\right)$$

is satisfied for all but finitely many values of n. In conclusion, we get that $pp(n)$ fulfills the Bessenrodt–Ono type inequality for all large parameters a and b.

At this point, it is worth saying that Heim et al. [21, Theorem 1.1] completely solved the Bessenrodt–Ono type inequality for the plane partition function. More precisely, they showed that the inequality

$$pp(a)pp(b) > pp(a+b)$$

is satisfied for every $a, b \in \mathbb{N}_{\geq 2}$ with $a+b \geq 12$.

3. Log-Concavity Property Holds for a Class of Sub-exponential Functions

Let us recall that a sequence $(F(n))_{n\in\mathbb{N}}$ (or just a function $F: \mathbb{N} \to \mathbb{R}_+$) is log-concave if

$$F(n)^2 > F(n-1)F(n+1)$$

for all $n \geq n_0$, where n_0 is some positive integer.

In this section we get a general result which under mild conditions on the growth of the function $F: \mathbb{N} \to \mathbb{R}_+$ guarantees that F is log-concave.

Theorem 3.1. *Let $c_1, c_2, f, F: \mathbb{N} \to \mathbb{R}_+$ be given and such that the inequalities*

$$c_1(n)e^{f(n)} < F(n) < c_2(n)e^{f(n)}$$

are valid for all positive integers $n \geq N_0$ for some $N_0 \in \mathbb{N}$. Suppose further that the following conditions hold:

(1) $\exists N_1 \in \mathbb{N}_+ \; \exists h: \mathbb{N}_+ \to \mathbb{R}_+ \; \forall n \geq N_1: h(n) \leq 2f(n) - f(n-1) - f(n+1)$;
(2) $\exists N_2 \in \mathbb{N}_+ \; \forall n \geq N_2: c_2(n+1)c_2(n-1)/c_1^2(n) \leq e^{h(n)}$.

Then, for all the values of $n \geq \max\{N_0, N_1, N_2\}$ *the inequality*

$$F^2(n) > F(n-1)F(n+1)$$

is true.

Proof. Let c_1, c_2, f, F, h be as in the statement of the theorem. It follows that both of the inequalities

$$F^2(n) > c_1^2(n)e^{2f(n)}$$

and

$$F(n-1)F(n+1) < c_2(n-1)c_2(n+1)e^{f(n-1)+f(n+1)}$$

are valid for every $n \geq N_0$. Therefore, it is enough to show that the inequality

$$e^{h(n)} \geq \frac{c_2(n-1)c_2(n+1)}{c_1^2(n)}$$

is true for all sufficiently large values of n, but this is exactly (2). Hence, we conclude that the inequality

$$F^2(n) > F(n-1)F(n+1)$$

is satisfied for all $n \geq \max\{N_0, N_1, N_2\}$, as required. □

Example 3.2. For every positive integer $n \geq 37$, we have that

$$c_1(n)e^{\mu(n)} < p(n) < c_2(n)e^{\mu(n)}, \tag{3.1}$$

where

$$\begin{aligned}
\mu(n) &:= \frac{\pi}{6}\sqrt{24n-1}, \\
c_1(n) &:= \frac{\sqrt{12}}{24n-1}\left(1 - \frac{1}{\mu(n)} - \frac{1}{\mu^3(n)}\right), \\
c_2(n) &:= \frac{\sqrt{12}}{24n-1}\left(1 - \frac{1}{\mu(n)} + \frac{1}{\mu^3(n)}\right).
\end{aligned}$$

The inequalities (3.1) follow directly from Chen, Jia and Wang [9, Lemma 2.2] and some numerical computations carried out in Wolfram Mathematica [31].

Now, we want to apply Theorem 3.1 to deduce the log-concavity for $p(n)$. At first, let us observe that the generalized binomial theorem asserts that for every $|j| < |n|$ the following inequalities

$$t_{-j}(n) < \sqrt{n+j} < t_{+j}(n) \tag{3.2}$$

are true, where

$$t_{-j}(n) = n^{\frac{1}{2}} + \frac{1}{2}jn^{-\frac{1}{2}} - \frac{1}{8}j^2n^{-\frac{3}{2}} - 2 \cdot |j|^3 n^{-\frac{5}{2}}$$

and

$$t_{+j}(n) = n^{\frac{1}{2}} + \frac{1}{2}jn^{-\frac{1}{2}} - \frac{1}{8}j^2n^{-\frac{3}{2}} + 2 \cdot |j|^3 n^{-\frac{5}{2}}.$$

Therefore, we have that

$$\begin{aligned}2\mu(n)-\mu(n-1)-\mu(n+1)&=\frac{\sqrt{24}\pi}{6}\left(2\sqrt{n-\frac{1}{24}}-\sqrt{n-\frac{25}{24}}-\sqrt{n+\frac{23}{24}}\right)\\&\geq\frac{\sqrt{24}\pi}{6}\left(\frac{1}{4}n^{-\frac{3}{2}}-\frac{55588}{13824}n^{-\frac{5}{2}}\right).\end{aligned}$$

On the other hand, one can also calculate that

$$\begin{aligned}\frac{c_2(n-1)c_2(n+1)}{c_1^2(n)}&=\left(1+\frac{24^2}{(24n-1)^2-24^2}\right)^{\frac{5}{2}}\\&\quad\times\frac{\left(\mu^3(n-1)-\mu^2(n-1)+1\right)\left(\mu^3(n+1)-\mu^2(n+1)+1\right)}{\left(\mu^3(n)-\mu^2(n)-1\right)^2}\\&\leq\left(1+\frac{24^2}{(24n-1)^2-24^2}\right)^{\frac{5}{2}}\times\left(1+\frac{24\sqrt{6}}{7\pi^3}n^{-\frac{3}{2}}\right),\end{aligned}$$

where the last inequality is a consequence of (3.2) and some elementary but tiresome computations. Hence, it is enough to check under what conditions on n the inequality

$$\begin{aligned}1+\frac{\sqrt{24}\pi}{6}\left(\frac{1}{4}n^{-\frac{3}{2}}-\frac{55588}{13824}n^{-\frac{5}{2}}\right)&\geq\left(1+\frac{24^2}{(24n-1)^2-24^2}\right)^{\frac{5}{2}}\\\times\left(1+\frac{24\sqrt{6}}{7\pi^3}n^{-\frac{3}{2}}\right)&\end{aligned}$$

is true. One can verify that it holds for all $n\geq 94$. Therefore, if we examine the positivity of $p^2(n)-p(n-1)p(n+1)$ for every $1\leq n\leq 93$, then we obtain Nicolas' theorem (see, [13, Theorem 1.1] or [27]).

4. Log-Concavity (Usually) Implies Bessendrodt-Ono Type Inequality

In this section we show a strict connection between log-concavity property and the Bessenrodt–Ono type inequality.

Theorem 4.1. *Let $F:\mathbb{N}_+\to\mathbb{R}_+$ be fixed. Assume further that there exists $n_0\in\mathbb{N}_+$ such that $(F(n))_{n\geq n_0}$ is log-concave and*

$$\limsup_{n\to\infty}\frac{F(n+n_0)}{F(n)}<F(n_0).$$

Then, the inequality

$$F(a)F(b)>F(a+b)$$

holds for all sufficiently large numbers $a,b\in\mathbb{N}_+$.

Proof. At first, let us observe that the log-concavity property implies that the inequality

$$\frac{F(n)}{F(n-1)} > \frac{F(n+1)}{F(n)} \tag{4.1}$$

is valid for each $n \geq n_0$. Thus, it is enough to check the validity of the inequality in

$$\begin{aligned} F(a+b) &= \frac{F(a+b)}{F(a+b-1)} \times \cdots \times \frac{F(a+n_0+1)}{F(a+n_0)} F(a+n_0) \\ &< F(a)\frac{F(b)}{F(b-1)} \times \cdots \times \frac{F(n_0+1)}{F(n_0)} F(n_0) = F(a)F(b) \end{aligned}$$

for all sufficiently large a and b. However, (4.1) points out that

$$\frac{F(n_0+i+1)}{F(n_0+i)} > \frac{F(a+n_0+i+1)}{F(a+n_0+i)}$$

is true for any $i = 0, 1, \ldots, b-n_0-1$. Therefore, the task boils down to proving that the inequality

$$\frac{F(a+n_0)}{F(a)} < F(n_0)$$

is true for all large values of a, but it is a direct consequence of the leftover assumption from the statement. This ends the proof. □

One can easily notice that besides Theorem 4.1, the above proof implies the following property.

Proposition 4.2. *Let $F : \mathbb{N}_+ \to \mathbb{R}_+$ be given. Assume further that there exists $n_0 \in \mathbb{N}_+$ such that $(F(n))_{n \geq n_0}$ is log-concave and the inequality*

$$F(n)F(n_0) > F(n+n_0)$$

is valid for every $n \geq n_0$. Then, we have that

$$F(a)F(b) > F(a+b)$$

is satisfied for all $a, b \geq n_0$.

Example 4.3. Suppose that we know that the classical partition function satisfies

$$p(26)p(a) > p(a+26)$$

for every $a \geq 26$. Then, from [13, Theorem 1.1] or [27] and Proposition 4.2 we get that the inequality

$$p(a)p(b) > p(a+b)$$

holds for all $a, b \geq 26$.

Theorem 4.4. *Let $F : \mathbb{N} \to \mathbb{R}_+$ be fixed and such that $F(0) \geq 1$. Suppose further that the sequence $(F(n))_{n=0}^{\infty}$ is log-concave for every positive integer n. Then*

$$F(a)F(b) > F(a+b)$$

is true for all positive integers a and b.

Proof. The log-concavity property and the assumption that $F(0) \geq 1$ imply that

$$F(a+b) = \frac{F(a+b)}{F(a+b-1)} \times \frac{F(a+b-1)}{F(a+b-2)} \times \cdots \times \frac{F(a+1)}{F(a)} F(a)$$
$$< F(a)\frac{F(b)}{F(b-1)} \times \frac{F(b-1)}{F(b-2)} \times \cdots \times \frac{F(1)}{F(0)} F(0) = F(a)F(b)$$

holds for all positive integers a and b. This completes the proof. □

Before we proceed to some applications of Theorem 4.4, it is worth pointing out that its first proof was obtained by Asai et al. [4, Theorem 2.1 (a)]. More recently, Benfield and Roy [6] also investigated similar properties for sequences of positive real numbers.

Example 4.5. Since the sequence $(p(n))_{n\geq 26}$ is log-concave by [13, Theorem 1.1] or [27] and we have $p(26) = 2436 > 1$, Theorem 4.4 implies that the sequence $(p(n+26))_{n\in\mathbb{N}}$ satisfies the Bessenrodt–Ono type inequality.

Example 4.6. Let us set $q(n) := F_{2n}$, where F_j denotes the j-th Fibonacci number, where $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. From Cassini identity we deduce that

$$q^2(n) - q(n+1)q(n-1) = F_{2n}^2 - F_{2n-2}F_{2n+2} = 1 > 0.$$

Thus, the sequence $(q(n))_{n\in\mathbb{N}}$ is log-concave. However, it does not satisfy the Bessenrodt–Ono type inequality. Indeed, one can easily check that $\phi^{2n}/\sqrt{5} - 1 < F_{2n} < \phi^{2n}/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$ is the golden mean. Thus, for $a+b \geq 3$ we have

$$q(a+b) - q(a)q(b) = F_{2(a+b)} - F_{2a}F_{2b} > \frac{\sqrt{5}-1}{5}\phi^{2(a+b)} - 1 > 0.$$

As a consequence, despite the fact that the sequence $(q(n))_{n\in\mathbb{N}}$ is log-concave, it does not satisfy the Bessenrodt–Ono type inequality. In conclusion, we see that the assumption $F(0) \geq 1$ in Theorem 4.4 is crucial.

On the other hand, let us note that for any fixed $j \in \mathbb{N}_+$ the sequence $(q(n+j))_{n\in\mathbb{N}}$ satisfies the Bessenrodt–Ono type inequality.

Theorem 4.1 asserts that a positive log-concave sequence usually satisfies the Bessenrodt–Ono inequality. Therefore, there appears a natural question whether the inverse statement is true. In general, it is not the case as the following example shows.

Example 4.7. For an arbitrary positive integer $m \geq 2$, the m-ary partition function $b_m(n)$ is just $p_A(n)$ for $A = \{m^i : i \in \mathbb{N}\}$, i.e., $b_m(n)$ is the number of representations of n as sums of powers of m. Let us recall that $b_m(0) = 1, b_m(mn+i) = b_m(mn)$ for $i = 1, \ldots, m-1$ and $b_m(mn) = b_m(mn-1) + b_m(n)$.

The sequence $(b_m(n))_{n\in\mathbb{N}}$ is not log-concave. Indeed, if we have that $n \equiv -1 \pmod{*}m$, then $b_m(n-1) = b_m(n) < b_m(n+1)$.

On the other hand, the following is true.

Theorem 4.8. *For each $m \geq 2$, the inequality*

$$b_m(x)b_m(y) > b_m(x+y)$$

is valid for all $x, y \geq m^2 + m$.

Proof. Let us fix $m \geq 2$ and $x, y \geq m^2 + m$. In such a setting, there exist numbers $c \geq d \geq m+1$ and $i, j \in \{0, 1, \ldots, m-1\}$ such that $x = cm + i$ and $y = dm + j$. Hence, we get that

$$b_m(x)b_m(y) = b_m(cm+i)b_m(dm+j) = b_m(cm)b_m(dm).$$

The right hand side from the Bessenrodt–Ono type inequality might be estimated as follows

$$\begin{aligned} b_m(x+y) &= b_m((c+d)m+i+j) \leq b_m((c+d+1)m) \\ &= b_m(c+d+1) + b_m((c+d)m) \\ &= b_m(c+d+1) + b_m(c+d) + b_m((c+d-1)m) \\ &= \cdots = b_m(c+d+1) + \cdots + b_m(c+1) + b_m(cm) \\ &\leq (d+1)b_m(2c+1) + b_m(cm). \end{aligned}$$

Therefore, it suffices to show that

$$(b_m(dm)-1)b_m(cm) > (d+1)b_m(2c+1).$$

The above might be further simplified to

$$b_m(dm) > d+2 \tag{4.2}$$

$$b_m(cm) \geq b_m(2c+1). \tag{4.3}$$

At first, let us deal with (4.2). Since $d \geq m+1$, it follows that m can be taken l times as a part of dm, where $l \in \{0, 1, \ldots, d\}$. In such a setting, we may require that all of the remaining parts of dm are equal to 1. Furthermore, the assumption that $d \geq m+1$ also guarantees that m^2 might occur as a part of dm, and if that is the case, then we can either take m as a part or not, which deliver us at least two additional m-ary partitions of dm. Thus, we obtain that

$$b_m(dm) = d+1+2 > d+2.$$

In the case of (4.3), let us observe that if $m \geq 3$, then $b_m(cm) \geq b_m(3c) \geq b_m(2c+1)$. On the other hand, if $m = 2$, then we just have $b_2(2c) = b_2(2c+1)$.

In conclusion, both inequalities (4.2) and (4.3) are satisfied for all $c, d \geq m+1$, as required. □

At the end, it is worth noting that despite having Mahler's theorem (see [25]), which states that

$$\log b_m(n) \sim \frac{(\log n)^2}{2\log m}, \tag{4.4}$$

we can neither use it nor apply Theorem 2.1 to deduce the Bessenrodt–Ono type inequality for the m-ary partition function. In order to apply Theorem 2.1 we need to know functions $c_1, c_2, f : \mathbb{N}_+ \to \mathbb{R}_+$ such that

$$c_1(n)e^{f(n)} < b_m(n) < c_2(n)e^{f(n)}.$$

However, asymptotic equality (4.4) ensures only that

$$e^{d_1(n)f(n)} < b_m(n) < e^{d_2(n)f(n)},$$

where $\lim_{n\to\infty} d_1(n) = \lim_{n\to\infty} d_2(n) = 1$. This provides too little information in order to use Theorem 2.1. Even stronger estimation,

$$\begin{aligned}\log b_m(mn) =& \frac{1}{2\log m}\left(\log\frac{n}{\log n}\right)^2 + \left(\frac{1}{2} + \frac{1}{\log m} + \frac{\log\log m}{\log m}\right)\log n \\ & - \left(1 + \frac{\log\log m}{\log m}\right)\log\log n + \Psi\left(\frac{\log n - \log\log n}{\log m}\right) + o(1)\end{aligned}$$

given by [12] is not sufficient to conclude the Bessenrodt–Ono type inequality for the m-ary partition function. Here the main obstruction is the function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$, which is implicitly given. We only know that Ψ is periodic with period 1 and bounded.

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