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Polynomization of the Bessenrodt–Ono Type Inequalities for A-Partition Functions

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Abstract. For an arbitrary set or multiset A of positive integers, we associate the A-partition function $p_A(n)$ (that is the number of partitions of n whose parts belong to A). We also consider the analogue of the k-colored partition function, namely, $p_{A,-k}(n)$. Further, we define a family of polynomials $f_{A,n}(x)$ which satisfy the equality $f_{A,n}(k) = p_{A,-k}(n)$ for all $n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{N}$. This paper concerns a polynomialization of the Bessenrodt–Ono inequality, namely

$$f_{A,a}(x)f_{A,b}(x) > f_{A,a+b}(x),$$

where a, b are positive integers. We determine efficient criteria for the solutions of this inequality. Moreover, we also investigate a few basic properties related to both functions $f_{A,n}(x)$ and $f'_{A,n}(x)$.

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1. Introduction

Let A be a multiset of positive integers with each integer a having finite multiplicities $\mu(a)$ i.e., if $A = \{a_1, a_2, \ldots, a_k\}$ (where $k = \infty$ is allowed), then

$$\mu(j) := \#\{i \in \mathbb{N} : a_i = j\} < \infty.$$

We consider a sequence of polynomials $\{f_{A,n}(x)\}_n$:

$$\sum_{n=0}^{\infty} f_{A,n}(x)\, q^n := \prod_{a \in A} \left(\frac{1}{1-q^a}\right)^x. \tag{1.1}$$

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The nth polynomial evaluated at positive integers k provide the k-colored A-partition function $p_{A,-k}(n) = f_{A,n}(k)$. Some of the multisets we are interested in are given by $A_I := \{1\}$, $A_{II} := \{1_1, \ldots, 1_m\}$ ($\mu(1) = m$), $A_{III} := \mathbb{N}$, and $A_{IV} := \{1, 2_1, 2_2, 3_1, 3_2, 3_3, 4_1, \ldots\}$ with $\mu(a) = a$. It is noteworthy that we use indices to distinguish those members of A, whose values are equal. In particular, we have $p_{A_{III},-1}(n) = f_{A_{III},n}(1) = p(n)$ is the usual partition function, and more generally $p_{A_{III},-k}(n) = f_{A_{III},n}(k)$ is the k-colored partition function. Note that

$$f_{A_I,n}(x) = \frac{1}{n!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k,$$

where the coefficients are the unsigned Stirling numbers of the first kind. Furthermore, we have $f_{A_{II},n}(x) = f_{A_I,n}(m\,x)$ and $f_{A_{IV},n}(1) = pp(n)$, where $pp(n)$ denotes the plane partition function.

In this paper, we prove the following Bessenrodt–Ono type [9] inequalities. Let $a, b \in \mathbb{N}$ with $a + b > 2$. Let A be a multiset with $1 \in A$. We have

$$f_{A,a}(x)\, f_{A,b}(x) > f_{A,a+b}(x) \tag{1.2}$$

(i) for $x \geq 3$ and A an ordinary set. Note that $f_{A,1}(3)^2 - f_{A,2}(3) = 0$. We refer to Theorem 1.3.

(ii) for $x \geq 5$ and A a multiset with $\mu(a) \leq a$ for all $a \in A$. Note that $f_{A,1}(5)^2 - f_{A,2}(5) = 0$. We refer to Theorem 1.4.

Note that (ii) implies (i) for $x \geq 5$. Moreover, let A be a set, and B the multiset having the elements of A but with multiplicity m. Then

$$f_{B,n}(x) = f_{A,n}(m\,x). \tag{1.3}$$

Let us recall some basic definitions and known results. For a non-negative integer n, the partition function $p(n)$ enumerates all possible sequences

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$$

of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j \geq 1$ and

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_j.$$

Elements λ_i are called the parts of the partition λ. For example, there are 5 partitions of 4: (4), $(3,1)$, $(2,2)$, $(2,1,1)$ and $(1,1,1,1)$. Thus, we have $p(4) = 5$. It should be pointed out that $p(n) = 0$ if n is negative, and $p(0) = 1$—as there is only the empty partition in this case. In the 1740's Euler discovered the generating function for $p(n)$

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{i=1}^{\infty} \frac{1}{1 - q^i}.$$

There is a wealth of literature devoted to the partition theory. For a general introduction to the subject, we encourage the reader to see Andrews' books [4,5] as well as [1,24,31,34].

However, it also motivated to investigate other similar problems for combinatorial sequences. For instance, in 2016 Bessenrodt and Ono [9] proved the following result.

Theorem 1.1. (Bessenrodt, Ono) *Let a and b be natural numbers such that $a+b>9$. Then*

$$p(a)p(b)>p(a+b).$$

Their proof is based on asymptotic estimates due to Rademacher [32] and Lehmer [27]. It is worth noting that there are also other proofs. Alanazi, Gagola, and Munagi [2] show the above theorem in a combinatorial manner. On the other hand, Heim and Neuhauser [21] present the reasoning via induction on $a+b$ to obtain the aforementioned property.

It should be pointed out that Theorem 1.1 is not only art of art's sake, but it allows us to determine the value of

$$\max p(n)=\max\{p(\boldsymbol{\lambda}):\boldsymbol{\lambda}\text{ is a partition of }n\},$$

where $p(\boldsymbol{\lambda})=\prod_{i=1}^{j}p(\lambda_i)$ for $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\dots,\lambda_j)$. For more details, we encourage to see Bessenrodt and Ono's article [9].

Since 2016, a lot of people have examined analogous inequalities for various types of partition functions, among others, Chern, Fu, and Tang [10], Beckwith and Bessenrodt [7], Hou and Jagadeesan [25], Males [29], Dawsey and Masri [13], Heim, Neuhauser, and Tröger [23], Gajdzica [16].

However, it turns out that we can also pass from the discrete task to the continuous one by considering the appropriate family of polynomials—that is Gian-Carlo Rota's advice to study problems in combinatorics and number theory in terms of roots of polynomials. In 1955, Newman [30] began studying a family of polynomials $P_n(x)$ with exceptional properties. These polynomials are defined via the recursive formulae $P_0(x):=1$ and

$$P_n(x):=\frac{x}{n}\sum_{j=1}^{n}\sigma(j)P_{n-j}(x)$$

for $n\geq 1$, where $\sigma(m):=\sum_{\ell|m}\ell$ is the sum of divisors of $m\in\mathbb{N}$. The polynomial $P_n(x)$ is of degree n, and is integer-valued. Furthermore, these functions were investigated by Heim et al. [19,20,22]. In particular, they exhibited a practical formula for the derivatives $P_n'(x)$ for all $n\in\mathbb{N}$ [19], namely,

$$P_n'(x)=\sum_{j=1}^{n}\frac{\sigma(j)}{j}P_{n-j}(x).$$

Moreover, Heim and Neuhauser showed that the equality

$$P_n(k)=p_{-k}(n)$$

is true for all positive integers k and n, where $p_{-k}(n)$ denotes the number of k-colored partitions of n—in other words the generating function corresponding to $p_{-k}(n)$ is given by

$$\sum_{n=0}^{\infty}p_{-k}(n)q^n=\prod_{i=1}^{\infty}\frac{1}{(1-q^i)^k}.$$

Furthermore, Heim, Neuhauser, and Tröger [22] proved the Bessenrodt–Ono type inequality for polynomials $P_n(x)$. More specifically, they obtained the following result.

Theorem 1.2. (Heim, Neuhauser, Tröger) *Let a and b be positive integers such that $a+b>2$, and let $x>2$. Then, we have that*

$$P_a(x)P_b(x) > P_{a+b}(x).$$

If $x=2$, then the above inequality is valid for all $a+b>4$.

It should be pointed out that there are more results of that kind for other families of polynomials. For example, Heim, Neuhauser, and Tröger [23] considered a family polynomials associated to the so-called plane partition function $pp(n)$, that is, the number of partitions of n such that each part j might appear in j distinct colors. The generating function for $pp(n)$ takes the form

$$\sum_{n=0}^{\infty} pp(n)q^n = \prod_{i=1}^{\infty} \frac{1}{(1-q^i)^i}.$$

For more information about plane partitions and the plane partition function, we refer the reader to [4, Section 12], [35, Section 7.20], and [26]. Further, Li [28] has also obtained the Bessenrodt–Ono type inequalities for the overpartition function $\overline{p}(n)$ and its polynomization. Let us recall that the overpartition function $\overline{p}(n)$ counts the number of partitions of n such that the last occurrence of each distinct part may be overlined. In that case the corresponding generating function is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{i=1}^{\infty} \frac{1+q^i}{1-q^i}.$$

For more details related to overpartitions and the overpartition function, see [12].

In this paper, we deal with the Bessenrodt–Ono type inequalities for the so-called A-partition function and its polynomization. Let A be an arbitrary (finite or infinite) set or multiset of positive integers. By an A-partition of a non-negative integer n, we mean any partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_j)$ of n such that $\lambda_i \in A$ for each $1 \leq i \leq j$. For the sake of clarity, we also assume that two A-partitions of n are considered the same if there is only a difference in the order of their parts. Throughout the manuscript, we use subscripts to distinguish those members of A, whose values are the same. The A-partition function $p_A(n)$ enumerates all possible A-partitions of n. Additionally, we set $p_A(n)=0$ if n is negative, and $p_A(0)=1$. For example, if $A=\{1,2_1,2_2,3_1,3_2\}$, then $p_A(4)=8$ with: $(3_2,1)$, $(3_1,1)$, $(2_2,2_2)$, $(2_2,2_1)$, $(2_1,2_1)$, $(2_2,1,1)$, $(2_1,1,1)$, and $(1,1,1,1)$. The generating function for $p_A(n)$ satisfies the equality

$$\sum_{n=0}^{\infty} p_A(n)q^n = \prod_{a\in A} \frac{1}{1-q^a}. \tag{1.4}$$

There is an abundance of literature devoted to A-partitions, we refer the reader to, for instance, [3,6,8,11,14,15,17,33,36,37].

It is proven in the next section that one can assign unique polynomials $f_{A,n}(x)$ to the given A-partition function in such a way that $f_{A,n}(k) = p_{A,-k}(n)$ holds for every natural number k and non-negative integer n, where $p_{A,-k}(n)$ is, similarly to the case of the classical partition function, the k-colored A-partition function.

We proceed to the main part of the paper. Using the law of induction and methods analogous to those presented in Heim, Neuhauser, and Tröger's paper [22], we prove the following result.

Theorem 1.3. *For any set $A \subseteq \mathbb{N}$ with $1 \in A$, the inequality*

$$f_{A,a}(x)f_{A,b}(x) > f_{A,a+b}(x)$$

holds for all $x > 3$ and $a, b \in \mathbb{N}$. Moreover, we also have

$$f_{A,a}(3)f_{A,b}(3) \geq f_{A,a+b}(3)$$

for every $a, b \in \mathbb{N}$ with the equality if and only if $a = b = 1$ and $2 \in A$.

It is worth pointing out that the above property significantly extends Theorem 1.2—in a sense that instead of taking $\mathbb{N}$, we can consider any infinite or finite set A with $1 \in A$. For example, one may apply it to obtain the polynomization of the Bessenrodt–Ono type inequality for the k-regular partition function for every $k \geq 2$. That is an extension of Beckwith and Bessenrodt's theorem, see [7, Theorem 2.1].

Moreover, we also show a similar result for some family of multisets A. More precisely, we determine the following.

Theorem 1.4. *If A is such a multiset of natural numbers that 1 occurs exactly once in A, and each number j appears at most j times in A, then the inequality*

$$f_{A,a}(x)f_{A,b}(x) > f_{A,a+b}(x)$$

holds for all $x > 5$ and $a, b \in \mathbb{N}$. In addition, we have

$$f_{A,a}(5)f_{A,b}(5) \geq f_{A,a+b}(5)$$

for all $a, b \in \mathbb{N}$ with the equality if and only if $a = b = 1$ and 2 appears exactly two times in A.

The above theorem generalizes Heim, Neuhauser, and Tröger's result devoted to the plane partition function [23, Theorem 1.3].

This manuscript is organized as follows. In Sect. 2, we introduce necessary notations, define the appropriate family of polynomials $f_{A,n}(x)$ associated with the fixed A-partition function and show some of their basic properties. Section 3 deals with the Bessenrodt–Ono inequalities for both k-colored A-partition functions $p_{A,-k}(n)$ and the polynomials $f_{A,n}(x)$ for such sets A that $1 \in A$ ($\mu(a) = 1$ for each $a \in A$). Finally, Sect. 4 examines the analogous problems to those from Sect. 3 for such multisets A that $1 \in A$ and $\mu(j) \leq j$ for every $j \in \mathbb{N}$.

2. Preliminaries

At first, let us fix some notations and conventions. By $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}_{\geq 0}$ we denote the set of real numbers, the set of positive integers, and the set of non-negative integers, respectively. For an arbitrary multiset of positive integers A, $\mu(a)$ denotes the number of appearances of the number a in A. Furthermore, for a fixed finite or infinite set or multiset of positive integers A, the A-partition function $p_A(n)$ is the number of A-partitions of n (actually, if A is infinite, we have to require that $\mu(a) < \infty$ for every $a \in A$, otherwise there will be infinitely many numbers m such that $p_A(m) = \infty$). In such a setting, we also introduce a family of polynomials associated to the A-partition function via the power series expansion (1.1).

It is straightforward to see that $f_{A,0}(x) = 1$ and $f_{A,n}(1) = p_A(n)$. Furthermore, if we consider the k-colored A-partition function $p_{A,-k}(n)$, then we have that $p_{A,-k}(n) = f_{A,n}(k)$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}_{\geq 0}$. Moreover, there are recurrence formulae for the polynomials $f_{A,n}(x)$ and $f'_{A,n}(x)$ as well.

Proposition 2.1. *For a fixed set (or multiset) A of positive integers, we have*

$$f_{A,n}(x) = \frac{x}{n} \sum_{j=1}^{n} \sigma_A(j) f_{A,n-j}(x) \tag{2.1}$$

and

$$f'_{A,n}(x) = \sum_{j=1}^{n} \frac{\sigma_A(j)}{j} f_{A,n-j}(x), \tag{2.2}$$

for all $n \in \mathbb{N}$, where $\sigma_A(i)$ is a sum of those divisors of i that belong to A.

Proof. Let us denote the left-hand side of the equality (1.1) by $F(q,x)$. Since we know Maclaurin series of both the natural logarithm and the geometric series, and have

$$\frac{\partial F(q,x)}{\partial q} = \frac{\partial \log(F(q,x))}{\partial q} F(q,x) \quad \text{and} \quad \frac{\partial F(q,x)}{\partial x} = \frac{\partial \log(F(q,x))}{\partial x} F(q,x),$$

it follows that

$$\sum_{n=1}^{\infty} n f_{A,n}(x) q^n = \left(x \sum_{a \in A} \sum_{n=1}^{\infty} a q^{an} \right) \cdot \left(\sum_{n=0}^{\infty} f_{A,n}(x) q^n \right) \tag{2.3}$$

and

$$\sum_{n=0}^{\infty} f'_{A,n}(x) q^n = \left(\sum_{a \in A} \sum_{n=1}^{\infty} \frac{a q^{an}}{an} \right) \cdot \left(\sum_{n=0}^{\infty} f_{A,n}(x) q^n \right). \tag{2.4}$$

Hence, by comparing the appropriate coefficients standing next to q^n of both (2.3) and (2.4), we obtain the required properties. □

Now, it is not difficult to notice that if $1 \in A$, then $f_{A,n}(x)$ is a polynomial of degree n. Otherwise, we have that $f_{A,1}(x) = 0$. To omit such a confusion, we assume that henceforth $1 \in A$ and 1 occurs exactly one time in A. We also note that if $\#A < \infty$, then $\gcd(A) = 1$ is a necessary and sufficient condition for a Bessenrodt–Ono type inequality to exist for $p_A(n)$ (see [16]). Next, we introduce an additional notation, namely,

$$\Delta_{A,n}(x) := f_{A,n+1}(x) - f_{A,n}(x). \tag{2.5}$$

The following characterization follows from the above discussion.

Theorem 2.2. *Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$ with $x \geq 1$. Then, we have*

$$f_{A,n}(x) \leq f_{A,n+1}(x) \quad \text{and} \quad 1 \leq f'_{A,n}(x) < f'_{A,n+1}(x).$$

For $x > 1$, we additionally have that

$$f_{A,n}(x) < f_{A,n+1}(x).$$

Proof. First, we show that $f_{A,n}(x) \leq f_{A,n+1}(x)$. The proof is by induction on n under the assumption that $x \geq 1$. Let us notice that $f_{A,1}(x) = x$ and $f_{A,2}(x) = x(x + (1 + 2s_2))/2$, where $s_2 = \mu(2)$. Thus, we get that

$$\Delta_{A,1}(x) \geq \frac{x}{2}(x-1) \geq 0$$

for every $x \geq 1$. Now, we assume that the statement is valid for all $x \geq 1$ and $1 \leq m \leq n-1$, and check its correctness for $m = n$. The identity (2.2) and the induction hypothesis assert that

$$f'_{A,n+1}(x) > \sum_{j=1}^{n} \frac{\sigma_A(j)}{j} f_{A,n+1-j}(x) \geq \sum_{j=1}^{n} \frac{\sigma_A(j)}{j} f_{A,n-j}(x) = f'_{A,n}(x).$$

Thus, we have that the inequality $f'_{A,n+1}(x) > f'_{A,n}(x)$ is satisfied for every $x \geq 1$. One can also notice that

$$f_{A,n+1}(1) = p_A(n+1) = p_A(n) + p_{A\setminus\{1\}}(n+1) \geq f_{A,n}(1).$$

Hence, the function $\Delta'_{A,n}(x)$ is positive for every $x \geq 1$, and the inequality $f_{A,n+1}(x) \geq f_{A,n}(x)$ holds for all such x, as required. The second part of the statement follows directly from the above discussion. □

Now, we are ready to investigate the Bessenrodt–Ono inequality for the polynomials $f_{A,n}(x)$. We divide our consideration into a few sections depending on the structure of A.

3. The Bessenrodt–Ono Inequality for an Arbitrary Set A with $1 \in A$

In this section, we assume that $A \subseteq \mathbb{N}$—meaning that each number j might occur at most once in A. Moreover, we also require that $1 \in A$. At the beginning, let us consider a very useful example when A is a singleton.

Lemma 3.1. *For the singleton $A_I = \{1\}$, the equality*

$$f_{A_I,n}(x) = \frac{1}{n!}\prod_{i=0}^{n-1}(x+i) = \binom{x+n-1}{n}$$

holds for every $n \in \mathbb{Z}_{\geq 0}$.

Proof. To prove the required property, it is enough to systematically utilize the formula (2.1). □

It is worth pointing out that the above polynomials are naturally connected with a well-known combinatorial sequence. The unsigned Stirling number of the first kind $\left[{k \atop l}\right]$ is the number of permutations of k elements with exactly l cycles. Equivalently, it may be defined as the appropriate coefficient of the rising factorial (Pochhammer function), namely

$$x^{\overline{n}} := x(x+1)\cdots(x+n-1) = \sum_{i=0}^{n} \left[{n \atop i}\right] x^i.$$

For more information about Stirling numbers, we refer the reader to [18, Chapter 6]. Therefore, it is clear that the equality

$$f_{A_I,n}(x) = \frac{1}{n!}\sum_{i=0}^{n} \left[{n \atop i}\right] x^i$$

is valid for all $n \in \mathbb{Z}_{\geq 0}$.

Furthermore, let us notice that for any other set (or multiset) $B \subseteq \mathbb{N}$ with $1 \in B$, we have that

$$f_{A_I,n}(x) \leq f_{B,n}(x)$$

for all $n \geq 0$ and $x \geq 0$—that is a direct consequence of the equality (2.1). Moreover, we may easily extend Lemma 3.1 to any finite multiset of ones.

Corollary 3.2. *For an arbitrary positive integer m and the multiset $A_{II} = \{1_1, \ldots, 1_m\}$, the equality*

$$f_{A_{II},n}(x) = f_{A_I,n}(mx)$$

is satisfied for every $n \in \mathbb{Z}_{\geq 0}$.

Before we go to the main part of this section, let us take a look on a concrete family of polynomials $f_{A,n}(x)$. For instance, if we set $A = \{1, 2, 3, 4, 5\}$, then one can easily calculate the corresponding polynomials $f_{A,n}(x)$ for small values of n. We present all the cases for $1 \leq n \leq 7$ in Table 1.

To gain some intuition related to the behavior of the polynomials $f_{A,a}(x)f_{A,b}(x) - f_{A,a+b}(x)$, one can determine their roots. Figure 1 shows all their complex roots of $1 \leq a, b \leq 10$.

Now, our aim is to find a Bessenrodt–Ono type inequality for the 3-colored A-partition function—that is $p_{A,-3}(n) = f_{A,n}(3)$. That result will play a central role in the proof of the main theorem of this section. To deal with the task, we show the reasoning by induction on $a + b$. Actually, we combine methods recently utilized by Heim, Neuhauser, and Tröger in [21,22].

Theorem 3.3. *For every set $A \subseteq \mathbb{N}$ with $1 \in A$, the inequality*

$$f_{A,a}(3)f_{A,b}(3) > f_{A,a+b}(3)$$

holds for all pairs $(a, b) \in \mathbb{N}^2$ except for $(1, 1)$. In that case, we have that

$$f_{A,1}(3)f_{A,1}(3) \geq f_{A,2}(3)$$

with equality whenever $2 \in A$.

Table 1. The polynomials $f_{A,n}(x)$ for $A = \{1, 2, 3, 4, 5\}$ and $1 \le n \le 7$

n	$f_{A,n}(x)$
1	x
2	$\frac{1}{2}x(x+3)$
3	$\frac{1}{6}x(x+1)(x+8)$
4	$\frac{1}{24}x(x+1)(x+3)(x+14)$
5	$\frac{1}{120}x(x+3)(x+6)\left(x^2+21x+8\right)$
6	$\frac{1}{720}x(x+2)(x+5)\left(x^3+38x^2+289x+72\right)$
7	$\frac{1}{5040}x\left(x^6+63x^5+1225x^4+9345x^3+28294x^2+25872x+720\right)$

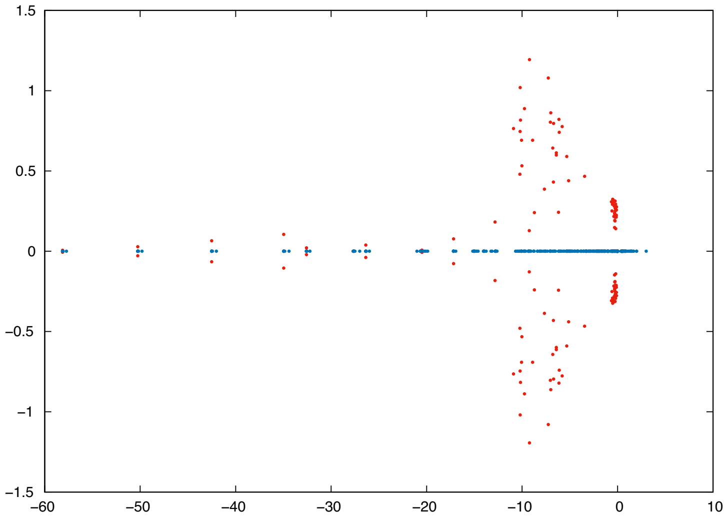

Figure 1. Complex roots of the polynomials $f_{A,a}(x)f_{A,b}(x) - f_{A,a+b}(x)$ for $A = \{1, 2, 3, 4, 5\}$ and $1 \le a, b \le 10$

Proof. At first, let us notice that for each $n \ge 0$, we have $f_{A,n}(3)f_{A,0}(3) = f_{A,n}(3)$. We prove the statement by induction on $a + b$. Let us assume that $a + b = 2$. Since, we have

$$f_{A,2}(3) = \begin{cases} 9, & \text{if } 2 \in A, \\ 6, & \text{if } 2 \notin A, \end{cases}$$

it is straightforward to see that

$$f_{A,1}(3)f_{A,1}(3) = 9 \ge f_{A,2}(3).$$

On the other hand, if $a+b=3$, then

$$f_{A,3}(3)=\begin{cases}22, & \text{if } 2,3\in A,\\ 19, & \text{if } 2\in A \text{ and } 3\notin A,\\ 13, & \text{if } 2\notin A \text{ and } 3\in A,\\ 10, & \text{if } 2,3\notin A,\end{cases}$$

and the inequality

$$f_{A,1}(3)f_{A,2}(3)>f_{A,3}(3)$$

follows. Next, we check the validity of the statement for all pairs (a,b) such that $a,b\leq 14$ and possible choices for the set A. The procedure which allows us to deal with this task is specifically described in Lemma 3.4. Now, we suppose that the statement is valid for all pairs $(a,b)\in\mathbb{N}^2$ with $3\leq a+b\leq n$, and check whether it is also true for $a+b=n+1$. Without loss of generality we may assume that $a\geq b\geq 1$ and $a\geq 15$. Employing the formula (2.1) to the difference $f_{A,a}(3)f_{A,b}(3)-f_{A,a+b}(3)$, we get

$$\begin{aligned}
&\frac{3}{a}\sum_{j=1}^{a}\sigma_A(j)f_{A,a-j}(3)f_{A,b}(3)-\frac{3}{a+b}\sum_{i=1}^{a+b}\sigma_A(i)f_{A,a+b-i}(3)\\
&\quad>\frac{3}{a}\sum_{j=1}^{a}\sigma_A(j)f_{A,a-j}(3)f_{A,b}(3)-\frac{3}{a+b}\sum_{i=1}^{a}\sigma_A(i)f_{A,a-i}(3)f_{A,b}(3)\\
&\qquad-\frac{3}{a+b}\sum_{i=1}^{b}\sigma_A(a+i)f_{A,b-i}(3)\\
&\quad=\frac{3}{a(a+b)}\left(b\sum_{j=1}^{a}\sigma_A(j)f_{A,a-j}(3)f_{A,b}(3)-a\sum_{i=1}^{b}\sigma_A(a+i)f_{A,b-i}(3)\right)
\end{aligned}$$

where the inequality follows directly from the induction hypothesis which is applied to $f_{A,a+b-i}(3)$ for each i between 1 and a. Thus, it is enough to prove that the expression in the bracket is non-negative. To make the text more transparent, let us set

$$\kappa_a:=\sum_{j=1}^{a}\sigma_A(j)f_{A,a-j}(3).$$

We obtain that

$$\begin{aligned}
&b\kappa_a f_{A,b}(3)-a\sum_{i=1}^{b}\sigma_A(a+i)f_{A,b-i}(3)\\
&\quad=3\kappa_a\sum_{i=1}^{b}\sigma_A(i)f_{A,b-i}(3)-a\sum_{i=1}^{b}\sigma_A(a+i)f_{A,b-i}(3)\\
&\quad=\sum_{i=1}^{b}\left(3\kappa_a\sigma_A(i)-a\sigma_A(a+i)\right)f_{A,b-i}(3).
\end{aligned}$$

Since $1 \le \sigma_A(n) \le \sigma(n) \le n(1+\ln n)$ for every $n \ge 1$ and $a \ge b$, we further have that

$$\begin{aligned} 3\kappa_a - 2a^2(1+\ln(2a)) &\ge 3\sum_{j=0}^{a-1} f_{A,j}(3) - 2a^2(1+\ln(2a)) \qquad (\star) \\ &\ge 3\binom{a+2}{3} - 2a^2(1+\ln(2a)), \end{aligned}$$

where the last inequality is a consequence of Lemma 3.1. Hence, it is enough to show that the function

$$\Psi(x) := (x+1)(x+2) - 4x(1+\ln(2x))$$

is non-negative for all $x \ge 15$; and it is not a difficult problem to solve. The proof is complete by the law of induction. □

For the sake of clarity, let us complete the proof by dealing with the remaining cases for $1 \le a, b \le 14$.

Lemma 3.4. *For any set $A \subseteq \mathbb{N}$ such that $1 \in A$, the inequality*

$$f_{A,a}(3) f_{A,b}(3) > f_{A,a+b}(3)$$

holds for all such pairs $a, b \in \mathbb{N}$ that $a, b \le 14$ and $a+b > 2$.

Proof. To prove the lemma, we carry out appropriate computations in Wolfram Mathematica [38]. At the beginning, we assume that $a, b \le 10$, $a+b \ge 3$, and consider all possible subsets A of the set $\{1, 2, \dots, a+b\}$ such that $1 \in A$. Observe that, in fact, it is enough to consider the aforementioned subsets. Otherwise, if $j \in A$ and $j > a+b$, then j can not appear as a part of the partitions of a, b and $a+b$. Hence, any such j does not influence the values of $f_{A,a}(3)$, $f_{A,b}(3)$, and $f_{A,a+b}(3)$. Considering all possible choices of A, and numbers a and b such that $a, b \le 10$, we check the correctness of the statement for every such a triple. Since in all of these cases the set A is finite, it follows from Bell's theorem (see, [8, 16]) that the function $f_{A,n}(3)$ is a quasi-polynomial, namely, the expression of the form

$$f_{A,n}(3) = c_l(n)n^l + c_{l-1}(n)n^{l-1} + \cdots + c_0(n),$$

where $l = 3|A| - 1$ and all of these coefficients $c_i(n)$ depend on the residue class of $n \pmod{*}\text{lcm}(A)$. Thus, we may explicitly determine its values and directly verify the claim.

Afterwards, we slightly modify our reasoning for all pairs a and b such that $a \ge b \ge 1$ and $11 \le a \le 14$. Mainly because the number of possible choices of the set A rapidly grows with the value of $a+b$, which, on the other hand, significantly influences the time of the calculations. Thus, the idea is to assume something more about the set A. For instance, let us suppose that $1, 2 \in A$, then $f_{A,n}(3) \ge f_{\{1,2\},n}(3)$ for each non-negative integer n. However, we can directly derive the formula for $f_{\{1,2\},n}(3)$ (due to Bell's theorem [8]) which takes the form

$$f_{\{1,2\},n}(3) = \begin{cases} \frac{n^5}{960} + \frac{3n^4}{128} + \frac{19n^3}{96} + \frac{25n^2}{32} + \frac{173n}{120} + 1, & \text{if } 2 \mid n, \\ \frac{n^5}{960} + \frac{3n^4}{128} + \frac{19n^3}{96} + \frac{49n^2}{64} + \frac{1249n}{960} + \frac{91}{128}, & \text{if } 2 \nmid n. \end{cases}$$

Now, taking minima of the corresponding coefficients of the above polynomials, it is straightforward to see that

$$f_{\{1,2\},n}(3) \geq \frac{n^5}{960} + \frac{3n^4}{128} + \frac{19n^3}{96} + \frac{49n^2}{64} + \frac{1249n}{960} + \frac{91}{128}$$

holds for every $n \geq 0$. Therefore, we might replace the lower bound in ($\star$) in the proof of Theorem 3.3 by the right hand side of the above inequality, and verify that

$$3\sum_{n=1}^{a-1}\left(\frac{n^5}{960} + \frac{3n^4}{128} + \frac{19n^3}{96} + \frac{49n^2}{64} + \frac{1249n}{960} + \frac{91}{128}\right) \geq 2a^2(1 + \ln(2a))$$

holds for every $a > 4$. Actually, we might present a similar reasoning also for $f_{\{1,r\},n}(3)$, where $r \in \{3, 4, 5, 6, 7\}$ and obtain that the required inequality for $f_{\{1,r\},n}(3)$ is satisfied for all $a \geq 11$ and $b \geq 1$. Hence, to deal with the remaining cases that are those corresponding to $11 \leq a \leq 14$ and $b \leq a$, it is enough to repeat the above method for all possible subsets A of $\{1, 8, 9, \dots, a+b\}$. $\square$

Before we pass to the central part of this section, let us point out some consequences of Theorem 3.3.

Corollary 3.5. *For any finite (or infinite) multiset B such that $1 \in B$ and $\mu(j) = 3$ for each number $j \in B$, we have that*

$$p_B(a)p_B(b) > p_B(a + b)$$

is true for all positive integers a and b (except possibly $a = b = 1$ whenever $2 \in B$).

Remark 3.6. It should be noted that Corollary 3.5 significantly improves Gajdzica's result [16, Theorem 5.4] related to the Bessenrodt–Ono inequality for all multisets B of the form $B = \{1_1, 1_2, 1_3, (a_2)_1, (a_2)_2, (a_2)_3, \dots, (a_k)_1, (a_k)_2, (a_k)_3\}$ with $1 < a_2 < \cdots < a_k$. More precisely, his theorem asserts that

$$p_B(a)p_B(b) > p_B(a + b)$$

holds for all $a, b \geq \frac{2}{3k}\prod_{i=1}^{3k}(1 + 3ik \cdot \operatorname{lcm}(a_2, a_3, \dots, a_k)) + 2$. On the other hand, Corollary 3.5 maintains that the above inequality is valid for all positive integers a and b (except possibly $a = b = 1$ whenever $2 \in B$).

Now, it is time to present the proof of Theorem 1.3.

3.1. Proof of Theorem 1.3

Proof. We show the claim by induction on $a+b$. For $a+b=2$, we have that

$$f_{A,1}(x)f_{A,1}(x)-f_{A,2}(x)\geq x^2-x(x+3)/2=x(x-3)/2,$$

which is a consequence of the equality

$$f_{A,2}(x)=\begin{cases}x(x+3)/2, & \text{if } 2\in A,\\ x(x+1)/2, & \text{if } 2\notin A.\end{cases}$$

For $a+b=3$, on the other hand, one can determine that

$$f_{A,3}(x)=\begin{cases}x(x+1)(x+8)/6, & \text{if } 2,3\in A,\\ x\left(x^2+9x+2\right)/6, & \text{if } 2\in A \text{ and } 3\notin A,\\ x\left(x^2+3x+8\right)/6, & \text{if } 2\notin A \text{ and } 3\in A,\\ x(x+1)(x+2)/6, & \text{if } 2,3\notin A.\end{cases}$$

Hence, we get that

$$f_{A,2}(x)f_{A,1}(x)-f_{A,3}(x)=\begin{cases}x(x-2)(x+2)/3, & \text{if } 3\in A,\\ x(x-1)(x+1)/3, & \text{if } 3\notin A,\end{cases}$$

and the claim follows for $a+b=3$. Thus, let us assume that the statement is valid for all $3\leq a+b\leq n$, and examine whether it is also correct for $a+b=n+1$ with a and b such that $1\leq b\leq a$ and $a\geq 2$. Theorem 3.3 points out that the inequality $f_{A,a}(3)f_{A,b}(3)>f_{A,a+b}(3)$ is true for all positive integers a and b such that $a+b>2$. Now, it suffices to show that the derivative

$$\frac{d}{dx}\left(f_{A,a}(x)f_{A,b}(x)-f_{A,a+b}(x)\right)$$

is positive for all $x\geq 3$, because this implies that $f_{A,a}(x)f_{A,b}(x)-f_{A,a+b}(x)>0$. Since $f_{A,n}(x)f_{A,0}(x)=f_{A,n}(x)$, we have

$$\begin{aligned}&f'_{A,a}(x)f_{A,b}(x)+f_{A,a}(x)f'_{A,b}(x)-f'_{A,a+b}(x)\\ &=\sum_{j=1}^{a}\frac{\sigma_A(j)}{j}f_{A,a-j}(x)f_{A,b}(x)+\sum_{j=1}^{b}\frac{\sigma_A(j)}{j}f_{A,a}(x)f_{A,b-j}(x)\\ &\quad-\sum_{j=1}^{a+b}\frac{\sigma_A(j)}{j}f_{A,a+b-j}(x)\\ &\quad>\sum_{j=1}^{a}\frac{\sigma_A(j)}{j}f_{A,a-j}(x)f_{A,b}(x)+\sum_{j=1}^{b}\frac{\sigma_A(j)}{j}f_{A,a}(x)f_{A,b-j}(x)\\ &\quad-\sum_{j=1}^{a}\frac{\sigma_A(j)}{j}f_{A,a-j}(x)f_{A,b}(x)-\sum_{j=1}^{b}\frac{\sigma_A(a+j)}{a+j}f_{A,b-j}(x)\\ &=\sum_{j=1}^{b}\left(\frac{\sigma_A(j)}{j}f_{A,a}(x)-\frac{\sigma_A(a+j)}{a+j}\right)f_{A,b-j}(x).\end{aligned}$$

Therefore, we can just show that

$$\frac{\sigma_A(j)}{j} f_{A,a}(x) - \frac{\sigma_A(a+j)}{a+j} \geq 0$$

for each j between 1 and b. Since $a \geq b$ and $1 \leq \sigma_A(n) \leq \sigma(n) \leq n(1+\ln n)$ for each $n \geq 1$, it is easy to see that the inequality

$$\frac{\sigma_A(j)}{j} f_{A,a}(x) - \frac{\sigma_A(a+j)}{a+j} \geq \frac{1}{a}\binom{x+a-1}{a} - (1+\ln(2a))$$

is valid for every such a j. However, the assumption that $x \geq 3$ asserts that

$$\frac{1}{a}\binom{x+a-1}{a} - (1+\ln(2a)) \geq (a+1)(a+2)/(2a) - 1 - \ln(2a).$$

Thus, it suffices to prove that the function

$$\psi(x) := (x+1)(x+2) - 2x(1+\ln(2x))$$

is positive for all real numbers $x \geq 3$, and this is not difficult to verify. □

Since Theorem 1.3 is a very general result, it delivers us a lot of meaningful consequences.

Corollary 3.7. *For the set $A_{III} = \mathbb{N}$, the inequality*

$$f_{A_{III},a}(x) f_{A_{III},b}(x) > f_{A_{III},a+b}(x)$$

is valid for all $x \geq 3$ and $a, b \in \mathbb{N}$ except $a = b = 1$. For $a = b = 1$, we have that

$$f_{A_{III},1}(3) f_{A_{III},1}(3) = f_{A_{III},2}(3) \quad \text{and} \quad f_{A_{III},1}(x) f_{A_{III},1}(x) > f_{A_{III},2}(x)$$

for all $x > 3$.

One can compare the obtained result with the theorem due to Heim, Neuhauser, and Tröger [22, Theorem 1.4]. More specifically, they proved that the inequality

$$f_{A_{III},a}(x) f_{A_{III},b}(x) > f_{A_{III},a+b}(x)$$

is satisfied for all $x > 2$, $a, b \in \mathbb{N}$ and $a + b > 2$. For $x = 2$, it is true whenever $a+b > 4$. Clearly, Theorem 1.3 has some additional requirements on the values of a, b and x, which is a consequence of its general character.

Remark 3.8. Let $k \geq 2$ be a natural number and let $A = \mathbb{N} \backslash \{jk : j \in \mathbb{N}\}$. Theorem 1.3 asserts the polynomization of the Bessenrodt–Ono inequality for the k-regular partition function $p_k(n)$ for any such a k. Recall that Beckwith and Bessenrodt [7, Theorem 2.1] proved the Bessenrodt–Ono inequality for $p_k(n)$ for $2 \leq k \leq 6$. Hence, to resolve the issue completely, it suffices to find the analogue of their result for any $k \geq 7$ and investigate the 2-colored k-regular partition function in that regard.

At the end of this section, let us show the general version of Corollary 3.5.

Table 2. The polynomials $f_{A,n}(x)$ for $A = \{1, 2_1, 2_2, 3, 5_1, 5_2, 5_3\}$ and $1 \leq n \leq 7$

n	$f_{A,n}(x)$
1	x
2	$\frac{1}{2}x(x+5)$
3	$\frac{1}{6}x\left(x^2+15x+8\right)$
4	$\frac{1}{24}x\left(x^3+30x^2+107x+30\right)$
5	$\frac{1}{120}x\left(x^4+50x^3+455x^2+550x+384\right)$
6	$\frac{1}{720}x\left(x^5+75x^4+1285x^3+4725x^2+5194x+960\right)$
7	$\frac{1}{5040}x\left(x^6+105x^5+2905x^4+22575x^3+49294x^2+55440x+720\right)$

Corollary 3.9. *Let $m > 3$ be a fixed natural number, and let B be an arbitrary multiset of positive integers such that $1 \in B$ and $\mu(j) = m$ for every $j \in B$. Then the inequality*

$$p_B(a)p_B(b) > p_B(a+b)$$

is true for all positive integers a and b.

4. The Bessenrodt–Ono Type Inequality for the A-Plane Partitions

In this section, we have two general assumptions on the multiset A. The first one is that 1 occurs exactly once in A. The second one, on the other hand, states that each number j might appear at most j times in A—in other words $\mu(j) \leq j$. In particular, observe that if every number j occurs exactly j times in A, then the corresponding A-partition function is just the plane partition function $p_A(n) = pp(n)$, and that is the reason for the title of this section. Nevertheless, there is a myriad possibilities to choose the multiset A. We can, for instance, consider any set A which satisfies the general assumptions from Sect. 3. However, we do not need to restrict ourselves only to sets.

Let us assume that $A = \{1, 2_1, 2_2, 3, 5_1, 5_2, 5_3\}$. The corresponding polynomials $f_{A,n}(x)$ might be easily calculated by employing the formula (2.1). Table 2 presents these polynomials for $1 \leq n \leq 7$.

Once again, one can ask about the localization of the complex roots of the polynomials $f_{A,a}(x)f_{A,b}(x) - f_{A,a+b}(x)$. Figure 2 shows all their complex roots for $1 \leq a, b \leq 10$.

Now, we wish to go to the main part of the section and find out the analogue of Theorem 3.3. However, in that case we consider the 5-colored A-partition function which agrees with the result due to Heim, Neuhauser, and Tröger [23] for the classical plane partition function.

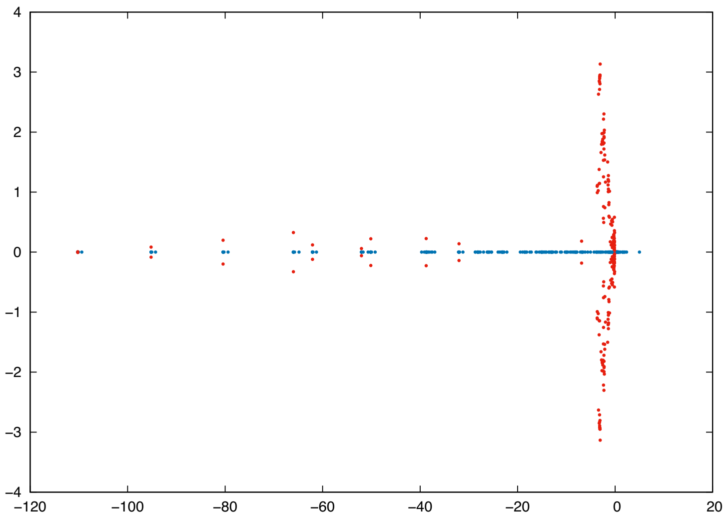

FIGURE 2. Complex roots of the polynomials $f_{A,a}(x)f_{A,b}(x) - f_{A,a+b}(x)$ for $A = \{1, 2_1, 2_2, 3, 5_1, 5_2, 5_3\}$ and $1 \le a, b \le 10$

Theorem 4.1. *If A is such a multiset of natural numbers that 1 occurs exactly once in A, and each number j appears at most j times in A. Then, the inequality*

$$f_{A,a}(5)f_{A,b}(5) > f_{A,a+b}(5)$$

holds for all $a, b \in \mathbb{N}$ with $a + b > 1$. For $a + b = 1$, we have that

$$f_{A,1}(5)f_{A,1}(5) \ge f_{A,2}(5)$$

with the equality whenever 2 occurs exactly two times in A.

Proof. As before, the proof is by the induction on $a + b$. If $a + b = 2$, then it is not difficult to check that

$$f_{A,2}(5) = \begin{cases} 25, & \text{if 2 appears exactly two times in } A, \\ 20, & \text{if 2 appears exactly one time in } A, \\ 15, & \text{if } 2 \notin A, \end{cases}$$

and, thus,

$$f_{A,1}(5)f_{A,1}(5) = 25 \ge f_{A,2}(5),$$

where the equality occurs if and only if 2 appears exactly two times in A. Let us also verify the statement for $a + b = 3$. In that case, it is not difficult to check that

$$f_{A,3}(5) \le \begin{cases} 100, & \text{if 2 appears exactly two times in } A, \\ 75, & \text{if 2 appears exactly one time in } A, \\ 50, & \text{if } 2 \notin A. \end{cases}$$

Therefore, we immediately obtain that

$$f_{A,2}(5)f_{A,1}(5) > f_{A,3}(5),$$

as required. Now, we use the same methods as in Lemma 3.4 to verify the claim for all $a, b \leq 8$ and possible choices for A. In fact, it turns out that the statement is valid for all the cases. Hence, let us assume that the claim is true for all pairs a and b such that $3 \leq a+b \leq n$, and check whether it is also true for $a+b = n+1$. Without loss of generality we may additionally require that $a \geq b \geq 1$ and $a \geq 9$. Once again, applying the formula (2.1) to the difference $f_{A,a}(5)f_{A,b}(5) - f_{A,a+b}(5)$, we get

$$\begin{aligned}
&\frac{5}{a}\sum_{j=1}^{a}\sigma_A(j)f_{A,a-j}(5)f_{A,b}(5) - \frac{5}{a+b}\sum_{i=1}^{a+b}\sigma_A(i)f_{A,a+b-i}(5) \\
&\quad > \frac{5}{a}\sum_{j=1}^{a}\sigma_A(j)f_{A,a-j}(5)f_{A,b}(5) - \frac{5}{a+b}\sum_{i=1}^{a}\sigma_A(i)f_{A,a-i}(5)f_{A,b}(5) \\
&\qquad - \frac{5}{a+b}\sum_{i=1}^{b}\sigma_A(a+i)f_{A,b-i}(5) \\
&\quad = \frac{5}{a(a+b)}\left(b\sum_{j=1}^{a}\sigma_A(j)f_{A,a-j}(5)f_{A,b}(5) - a\sum_{i=1}^{b}\sigma_A(a+i)f_{A,b-i}(5)\right),
\end{aligned}$$

where the second line follows from the induction hypothesis applied to $f_{A,a+b-i}(5)$ for each i between 1 and a. Thus, it suffices to show that the expression in the bracket is non-negative. Let us put

$$\kappa_a := \sum_{j=1}^{a}\sigma_A(j)f_{A,a-j}(5).$$

We see that

$$\begin{aligned}
&b\kappa_a f_{A,b}(5) - a\sum_{i=1}^{b}\sigma_A(a+i)f_{A,b-i}(5) \\
&\quad = 5\kappa_a\sum_{i=1}^{b}\sigma_A(i)f_{A,b-i}(5) - a\sum_{i=1}^{b}\sigma_A(a+i)f_{A,b-i}(5) \\
&\quad = \sum_{i=1}^{b}\left(5\kappa_a\sigma_A(i) - a\sigma_A(a+i)\right)f_{A,b-i}(5).
\end{aligned}$$

Since $1 \leq \sigma_A(n) \leq \sigma_2(n) \leq n(2n-1)$ for each $n \geq 1$ and $a \geq b$, we further simplify the above to

$$5\kappa_a - 2a^2(4a-1) \geq 5\sum_{j=0}^{a-1}f_{A,j}(5) - 2a^2(4a-1) \geq 5\binom{a+4}{5} - 2a^2(4a-1),$$

where the last inequality is the consequence of Lemma 3.1. Therefore, it is enough to prove that the function

$$\Psi(x) := (x+1)(x+2)(x+3)(x+4) - 48x(4x-1)$$

is non-negative for all $x \geq 9$ which is not a difficult task. This ends the proof by the law of induction. □

There are a few direct consequences of Theorem 4.1. Since they are very similar to Corollary 3.5 and Remark 3.6, we omit them and proceed to the proof of Theorem 1.4.

4.1. Proof of Theorem 1.4

Proof. We show the claim by induction on $a+b$. For $a+b=2$, we have that

$$f_{A,1}(x)f_{A,1}(x) - f_{A,2}(x) \geq x^2 - x/2(x+5) = x(x-5)/2,$$

which is a consequence of the equality

$$f_{A,2}(x) = \begin{cases} x(x+5)/2, & \text{if 2 appears exactly two times in } A, \\ x(x+3)/2, & \text{if 2 appears exactly one time in } A, \\ x(x+1)/2, & \text{if } 2 \notin A. \end{cases}$$

For $a+b=3$, on the other hand, one can verify that

$$f_{A,2}(x)f_{A,1}(x) - f_{A,3}(x) = \begin{cases} x\left(x^2-10\right)/3, & \text{if 3 occurs three times in } A, \\ x\left(x^2-7\right)/3, & \text{if 3 occurs two times in } A, \\ x(x-2)(x+2)/3, & \text{if 3 occurs one time in } A, \\ x(x-1)(x+1)/3, & \text{if } 3 \notin A, \end{cases}$$

and we see that the claim is true for $a+b=3$. Thus, let us suppose that the statement is valid for all $3 \leq a+b \leq n$, and investigate whether it is also correct for $a+b=n+1$ with a and b such that $1 \leq b \leq a$ and $a \geq 2$. Theorem 4.1 asserts that the inequality $f_{A,a}(5)f_{A,b}(5) > f_{A,a+b}(5)$ is true for all positive integers a and b such that $a+b>2$. Now, it suffices to show that the derivative

$$\frac{d}{dx}\left(f_{A,a}(x)f_{A,b}(x) - f_{A,a+b}(x)\right)$$

is positive for all $x \geq 5$, because it maintains that $f_{A,a}(x)f_{A,b}(x) - f_{A,a+b}(x) > 0$. Since $f_{A,n}(x)f_{A,0}(x) = f_{A,n}(x)$, we get

$$\begin{aligned} & f'_{A,a}(x)f_{A,b}(x) + f_{A,a}(x)f'_{A,b}(x) - f'_{A,a+b}(x) \\ &= \sum_{j=1}^{a} \frac{\sigma_A(j)}{j} f_{A,a-j}(x)f_{A,b}(x) + \sum_{j=1}^{b} \frac{\sigma_A(j)}{j} f_{A,a}(x)f_{A,b-j}(x) \\ &\quad - \sum_{j=1}^{a+b} \frac{\sigma_A(j)}{j} f_{A,a+b-j}(x) \\ &\quad > \sum_{j=1}^{a} \frac{\sigma_A(j)}{j} f_{A,a-j}(x)f_{A,b}(x) + \sum_{j=1}^{b} \frac{\sigma_A(j)}{j} f_{A,a}(x)f_{A,b-j}(x) \end{aligned}$$

$$- \sum_{j=1}^{a} \frac{\sigma_A(j)}{j} f_{A,a-j}(x) f_{A,b}(x) - \sum_{j=1}^{b} \frac{\sigma_A(a+j)}{a+j} f_{A,b-j}(x)$$
$$= \sum_{j=1}^{b} \left(\frac{\sigma_A(j)}{j} f_{A,a}(x) - \frac{\sigma_A(a+j)}{a+j} \right) f_{A,b-j}(x).$$

Therefore, it suffices to prove that

$$\frac{\sigma_A(j)}{j} f_{A,a}(x) - \frac{\sigma_A(a+j)}{a+j} \geq 0$$

for each j between 1 and b. Since $a \geq b$ and $1 \leq \sigma_A(n) \leq \sigma_2(n) \leq n(2n-1)$ for every $n \geq 1$, it is clear that

$$\frac{\sigma_A(j)}{j} f_{A,a}(x) - \frac{\sigma_A(a+j)}{a+j} \geq \frac{1}{a} \binom{x+a-1}{a} - 4a + 1$$

is valid for each such a j. However, the assumption that $x \geq 5$ points out that

$$\frac{1}{a} \binom{x+a-1}{a} - 4a + 1 \geq (a+1)(a+2)(a+3)(a+4)/(24a) - 4a + 1.$$

Now, we just examine when the function

$$\psi(x) := (x+1)(x+2)(x+3)(x+4) - 96x^2 + 24x$$

is positive. One can easily check that it is positive for every positive real number x. This completes the proof by the law of induction. □

It is worth pointing out that Theorem 1.4 agrees with the result of Heim, Neuhauser, and Tröger [23, Theorem 1.3], for the polynomization of the plane partition function $pp(n)$. Actually, Theorem 1.4 is a general property which might be successfully applied for various multisets A. In particular, it extends Theorem 1.3 for all $x \geq 5$.

Nevertheless, there is still a wide class of multisubsets for which we can utilize neither Theorem 1.3 nor Theorem 1.4, and that gap to our current knowledge will be investigated in the future.

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Data Availability Data will be made available on a reasonable request.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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