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Some Infinite-Dimensional Representations of Certain Coxeter Groups

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Abstract. A Coxeter group admits infinite-dimensional irreducible complex representations if and only if it is not finite or affine. In this paper, we provide a construction of some of those representations for certain Coxeter groups using some topological information of the corresponding Coxeter graphs.

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1. Introduction

Let (W, S) be an irreducible Coxeter group of finite rank, i.e., its Coxeter graph is connected and $|S| < \infty$. If W is finite, then irreducible representations (over $\mathbb{C}$) of W are certainly finite dimensional. If W is an affine Weyl group, then it is also well known that its irreducible representations are of finite dimension (one may refer to [3], [4, proof of Prop. 5.13], [7, Prop. 1.2] for more details). In general, we have the following fact.

Theorem 1.1. *All irreducible complex representations of W are of finite dimension if and only if W is a finite group or an affine Weyl group.*

The author owes a proof of this theorem to an anonymous referee of a previous version of this paper (see the "Appendix"). Nevertheless, the proof only tells us the existence of infinite-dimensional irreducible representations of infinite non-affine Coxeter groups, but it fails to construct such representations.

The main aim of this paper is to construct some irreducible representations of infinite dimension of a Coxeter group (W, S) satisfying either of the following:

1. there are at least two circuits in the Coxeter graph;

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2. there is at least one circuit in the Coxeter graph, and $m_{st} \geq 4$ for some $s, t \in S$ (for $s, t \in S$, we denote by m_{st} the order of st).

The main idea is to glue together many copies of representations of different dihedral subgroups of W, so that they form a "big" representation of W. The way of gluing is encoded in some topological information of the Coxeter graph. This method is inspired by the author's previous work [1].

The paper is organized as follows. Section 2 records some basic facts about representations of dihedral groups, as well as coverings and fundamental groups of graphs. Section 3 deals with case 1, in which the fundamental group of the Coxeter graph is a non-abelian free group. We utilize an infinite-dimensional irreducible representation of this free group to do the "gluing". In Sect. 4, we use the universal covering of the Coxeter graph to achieve our goal for case 2. In Sect. 5, we give another example of infinite-dimensional irreducible representation of a specific Coxeter group whose Coxeter graph has no circuits. Finally, in the appendix, we present the proof of Theorem 1.1 which is given by an anonymous referee.

2. Preliminaries

In this section, we recollect some notations and terminology used in this paper. We use e to denote the identity in a group. Coxeter groups considered throughout this paper are all irreducible and of finite rank.

2.1. Representations of Dihedral Groups

For a finite dihedral group $D_m := \langle r, t \mid r^2 = t^2 = (rt)^m = e \rangle$, we denote by $\mathbb{1}$ and ε, respectively, the trivial and the sign representation, i.e., $\mathbb{1} : r, t \mapsto 1$, $\varepsilon : r, t \mapsto -1$. If m is even, there are two more representations of dimension 1, i.e.,

$$\varepsilon_r : r \mapsto -1, t \mapsto 1; \qquad \varepsilon_t : r \mapsto 1, t \mapsto -1.$$

Let $\mathbb{C}\beta_r \oplus \mathbb{C}\beta_t$ be a vector space with formal basis $\{\beta_r, \beta_t\}$. For any integer k satisfying $1 \leq k < m/2$, let ρ_k denote the irreducible representation of D_m on $\mathbb{C}\beta_r \oplus \mathbb{C}\beta_t$ defined by

$$\begin{aligned} r \cdot \beta_r &= -\beta_r, & r \cdot \beta_t &= \beta_t + 2\cos\frac{k\pi}{m}\beta_r, \\ t \cdot \beta_t &= -\beta_t, & t \cdot \beta_r &= \beta_r + 2\cos\frac{k\pi}{m}\beta_t. \end{aligned}$$

Intuitively, r and t act on the (real) plane by two reflections with respect to two lines with an angle of $\frac{k\pi}{m}$; see Fig. 1.

Remark 2.1. If m is even and $k = m/2$, we may define $\rho_{m/2}$ as well by the same formulas, but then $\rho_{m/2} \simeq \varepsilon_r \oplus \varepsilon_t$ is reducible.

Remark 2.2. We have described the full set of irreducible representations of D_m, namely,

$$\{\mathbb{1}, \varepsilon\} \cup \{\rho_1, \ldots, \rho_{\frac{m-1}{2}}\}, \quad \text{if } m \text{ is odd;}$$

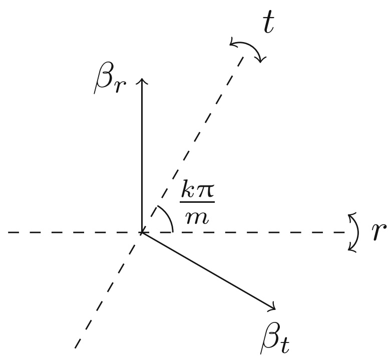


FIGURE 1. The representation $\rho_k : D_m \to \mathrm{GL}(\mathbb{C}\beta_r \oplus \mathbb{C}\beta_t)$

$$\{\mathbb{1}, \varepsilon, \varepsilon_r, \varepsilon_t\} \cup \{\rho_1, \ldots, \rho_{\frac{m}{2}-1}\}, \quad \text{if } m \text{ is even.}$$

The following lemma will be frequently used in Sects. 3 and 4 to prove our "gluing" is feasible.

Lemma 2.3. *The* $+1$*-eigenspaces of* r *and* t *in* ρ_k *are both one dimensional. However, there is no nonzero vector that can be fixed by* r *and* t *simultaneously.*

2.2. Graphs and the Universal Covering

The way we glue these representations will be encoded in some topological information of the Coxeter graph.

By definition, an *(undirected) graph* $G = (S, E)$ consists of a set S of vertices and a set E of edges, and elements in E are of the form $\{s, t\} \subseteq S$ (unordered). For our purpose, we only consider graphs without loops and multiple edges, i.e., there is no edge of the form $\{s, s\}$, and each edge $\{s, t\}$ occurs at most once in E. In a Coxeter graph, m_{st} is regarded as a label on the edge rather than a multiplicity. We say G is a *finite graph* if S is a finite set.

A sequence $(s_1, s_2, \ldots, s_n)$ of vertices is called a *path* in G if $\{s_i, s_{i+1}\} \in E, \forall i$. If $s_1 = s_n$, then we say such a path is a *closed path*. If further $s_1, \ldots, s_{n-1}$ are distinct in this closed path, then the path is called a *circuit*.

If every two vertices can be connected by a path, then we say G is a *connected graph*. A connected graph without circuits is called a *tree*. For a connected graph $G = (S, E)$, if $T = (S, E_0)$ is a tree with the same vertices set S and $E_0 \subseteq E$, then T is called a *spanning tree* of G. This condition is equivalent to say $|E_0| = |S| - 1$ when G is a connected finite graph. Any connected graph has a spanning tree, but not unique in general.

Let $G = (S, E)$ and $G' = (S', E')$ be two graphs. If $p : S \to S'$ is a map of sets such that for any edge $\{s, t\}$ in G, we have $p(s) \neq p(t)$ and $\{p(s), p(t)\}$ is also an edge in G', then we say that p is a *morphism* of graphs. We simply denote a morphism by $p : G \to G'$.

For any $s \in S$, we denote

$$E_s := \{t \in S \mid \{s, t\} \in E\}.$$

Suppose G' is connected and finite, and suppose $p : G \to G'$ is a morphism. If $p(S) = S'$, and if for any $s \in S$ the restriction of p to E_s gives rise to a bijection $E_s \xrightarrow{\sim} E'_{p(s)}$, then p is called a *covering*. It is natural to regard G' and G as locally finite simplicial complexes. Then p is also a covering of topological spaces.

Conversely, we view G as a topological space and suppose $p : X \to G$ is a covering of the topological space G. Then X has a graph structure such that p is a morphism of graphs (see [6, Theorem 83.4]). In particular, if p is the universal covering, then X is a tree. Thus for any connected finite graph, we can talk about its universal covering graph.

2.3. The Fundamental Group of a Graph

Let $G = (S, E)$ be a connected graph, and $T = (S, E_0)$ be a spanning tree. For any edge $\mathfrak{e} \in E \setminus E_0$, if we choose a vertex $s_\mathfrak{e}$ of $\mathfrak{e}$ to be its head and the other $t_\mathfrak{e}$ to be its tail, then $\mathfrak{e} = \{s_\mathfrak{e}, t_\mathfrak{e}\}$ and there is a unique circuit $c'_\mathfrak{e}$ in $(S, E_0 \cup \{\mathfrak{e}\})$ of the form $c'_\mathfrak{e} = (s_\mathfrak{e}, t_\mathfrak{e}, \ldots, s_\mathfrak{e})$.

Fix a vertex $s_0 \in S$. For any $c'_\mathfrak{e}$, there is a unique path in T without repetitive vertices from s_0 to $s_\mathfrak{e}$. We denote the path by $p_\mathfrak{e}$. Define $c_\mathfrak{e}$ to be the concatenation of $p_\mathfrak{e}, c'_\mathfrak{e}, p_\mathfrak{e}^{-1}$, where $p_\mathfrak{e}^{-1}$ is the inverse path in the obvious sense. Then each $c_\mathfrak{e}$ is a closed path from s_0 to itself. If we view G as a topological space, then we have the following result on its fundamental group $\pi_1(G)$.

Lemma 2.4. [6, Theorem 84.7] *$\pi_1(G)$ is a free group with a set of free generators $\{c_\mathfrak{e} \mid \mathfrak{e} \in E \setminus E_0\}$. In particular, if there is more than one circuit in G, then $\pi_1(G)$ is non-abelian.*

Remark 2.5. Note that $p_\mathfrak{e}$ is a path in T. Thus, all edges in $c_\mathfrak{e}$ except $\mathfrak{e}$ lie in E_0, and $\mathfrak{e}$ appears in $c_\mathfrak{e}$ only once. If $\mathfrak{e}' \in E \setminus E_0$ is another edge, then $\mathfrak{e}'$ does not appear in $c_\mathfrak{e}$.

3. Representations via Fundamental Groups of Coxeter Graphs

From now on, suppose (W, S) is an irreducible Coxeter group of finite rank with Coxeter graph $G = (S, E)$. Then, G is a connected finite graph.

In this section, we assume that

$$\text{there are at least two circuits in } G.$$

Thus, by Lemma 2.4, $\pi_1(G)$ is a non-abelian free group. In this section, we use an infinite-dimensional irreducible representation of $\pi_1(G)$ to construct another such representation for W.

For convenience, we may further assume

$$m_{st} < \infty, \quad \forall s, t \in S.$$

This is not essential. If some m_{st}'s are infinity, then we replace them by any integer larger than 2 (e.g., 3), so that we obtain another Coxeter group (W_1, S) and a surjective homomorphism $W \twoheadrightarrow W_1$, $s \mapsto s$. Ignoring labels on edges, the two Coxeter groups have the same Coxeter graph in a topological sense.

An irreducible representation of W_1 becomes an irreducible representation of W via pulling back by the homomorphism.

3.1. The Construction

We fix $s_0 \in S$. Let $T = (S, E_0)$ be a spanning tree of $G = (S, E)$. Then we obtain a set of free generators of $\pi_1(G)$ by the method in Sect. 2.3, say $c_1, \ldots, c_l$ ($l < \infty$ since G is a finite graph), with s_0 on each of them.

We have $|E \setminus E_0| \geq 2$ since we assume that there is more than one circuit in G. We then fix two distinct edges:

$$\{s_1, t_1\}, \{s_2, t_2\} \in E \setminus E_0.$$

We may assume that $\{s_1, t_1\}, \{s_2, t_2\}$ lie in c_1, c_2, respectively, and c_1 goes through t_1 first and then s_1, and c_2 goes through t_2 first and then s_2. In our choice of c_i, the edge $\{s_1, t_1\}$ appears in c_1 only once, while it does not appear in other c_i's (see Remark 2.5). Similar for $\{s_2, t_2\}$. The two edges might share a common vertex, like $s_1 = s_2$, but it does not matter.

We define a vector space V with formal basis $\{\alpha_{s,n} \mid n \in \mathbb{Z}, s \in S\}$,

$$V := \bigoplus_{n \in \mathbb{Z}, s \in S} \mathbb{C}\alpha_{s,n}.$$

For any $s, t \in S$ and $n \in \mathbb{Z}$, we define $s \cdot \alpha_{t,n}$ as follows:

(1) if $s = t$, then $s \cdot \alpha_{s,n} := -\alpha_{s,n}$;

(2) $s_1 \cdot \alpha_{t_1,n} := \alpha_{t_1,n} + 2\cos\frac{\pi}{m_{s_1t_1}}\alpha_{s_1,n+1}$,
$t_1 \cdot \alpha_{s_1,n+1} := \alpha_{s_1,n+1} + 2\cos\frac{\pi}{m_{s_1t_1}}\alpha_{t_1,n}$;
$s_2 \cdot \alpha_{t_2,n} := \alpha_{t_2,n} + 2^{n+1}\cos\frac{\pi}{m_{s_2t_2}}\alpha_{s_2,n+1}$,
$t_2 \cdot \alpha_{s_2,n+1} := \alpha_{s_2,n+1} + 2^{-n+1}\cos\frac{\pi}{m_{s_2t_2}}\alpha_{t_2,n}$;

(3) if the unordered pair $\{s, t\} \neq \{s_1, t_1\}$ or $\{s_2, t_2\}$, and if $s \neq t$, then $s \cdot \alpha_{t,n} := \alpha_{t,n} + 2\cos\frac{\pi}{m_{st}}\alpha_{s,n}$.

Looking at case (2), one can see that the vectors $\alpha_{t_2,n}$ and $2^n\alpha_{s_2,n+1}$ span an irreducible representation isomorphic to ρ_1 (see Sect. 2.1) of the dihedral subgroup $\langle s_2, t_2\rangle$. The vectors $\alpha_{t_2,n}$ and $2^n\alpha_{s_2,n+1}$ play the same roles as the β's in Sect. 2.1. Similarly, $\{\alpha_{t_1,n}, \alpha_{s_1,n+1}\}$ span an irreducible representation isomorphic to ρ_1 of $\langle s_1, t_1\rangle$.

In case (3), if $m_{st} = 2$, then $s \cdot \alpha_{t,n} = \alpha_{t,n}$; if $m_{st} \geq 3$, then $\{\alpha_{t,n}, \alpha_{s,n}\}$ span an irreducible representation isomorphic to ρ_1 of $\langle s, t\rangle$.

Lemma 3.1. *V is a representation of W with the action defined above.*

Proof. Obviously, s^2 acts as the identity for any $s \in S$.

For $s, t \in S$ and $m_{st} = 2$, we need to show $st \cdot \alpha_{r,n} = ts \cdot \alpha_{r,n}$, $\forall r \in S$. Note that we have

$$s \cdot \alpha_{t,n} = \alpha_{t,n} \text{ and } t \cdot \alpha_{s,n} = \alpha_{s,n} \text{ for any } n \in \mathbb{Z},$$

since $m_{st} = 2$. If $r = s$ or $r = t$, then clearly it holds $st \cdot \alpha_{r,n} = ts \cdot \alpha_{r,n}$. If $r \neq s$ and $r \neq t$, then

$$s \cdot \alpha_{r,n} = \alpha_{r,n} + c_1\alpha_{s,n_1} \text{ and } t \cdot \alpha_{r,n} = \alpha_{r,n} + c_2\alpha_{t,n_2}$$

for some $c_1, c_2 \in \mathbb{C}$ and $n_1, n_2 \in \{n, n \pm 1\}$, and then

$$st \cdot \alpha_{r,n} = s \cdot (\alpha_{r,n} + c_2 \alpha_{t,n_2}) = \alpha_{r,n} + c_1 \alpha_{s,n_1} + c_2 \alpha_{t,n_2},$$
$$ts \cdot \alpha_{r,n} = t \cdot (\alpha_{r,n} + c_1 \alpha_{s,n_1}) = \alpha_{r,n} + c_2 \alpha_{t,n_2} + c_1 \alpha_{s,n_1}.$$

Therefore, we have $st \cdot \alpha_{r,n} = ts \cdot \alpha_{r,n}$ as desired.

Now, assume $m_{st} \geq 3$. We need to verify that $(st)^{m_{st}} \cdot \alpha_{r,n} = \alpha_{r,n}$. This is also obvious if $r = s$ or $r = t$, since we are in the dihedral world. If s, t, r are distinct, then we have the following cases.

(1) If any two of s, t, r are not $\{s_1, t_1\}$ or $\{s_2, t_2\}$, then the three-dimensional subspace spanned by $\alpha_{r,n}$, $\alpha_{s,n}$, $\alpha_{t,n}$, denoted by U, stays invariant under the actions of s and t. We write $U_s := \{v \in U \mid s \cdot v = v\}$, $U_t := \{v \in U \mid t \cdot v = v\}$. Then $\dim U_s = \dim U_t = 2$, and thus there exists $0 \neq v_0 \in U$ such that $s \cdot v_0 = t \cdot v_0 = v_0$. Note that $3 \leq m_{st} < \infty$ and $\mathbb{C}\alpha_{s,n} \oplus \mathbb{C}\alpha_{t,n}$ forms a representation isomorphic to ρ_1 of $\langle s, t \rangle$. Hence, $v_0 \notin \mathbb{C}\alpha_{s,n} \oplus \mathbb{C}\alpha_{t,n}$ by Lemma 2.3, and then $\{v_0, \alpha_{s,n}, \alpha_{t,n}\}$ is a basis of U. Now, we can see that $(st)^{m_{st}} \cdot \alpha_{r,n} = \alpha_{r,n}$.

(2) If $m_{rt} = 2$, then there exists $k \in \{n-1, n, n+1\}$ and $q \in \{k-1, k, k+1\}$ such that $\alpha_{r,n}, \alpha_{s,k}, \alpha_{t,q}$ span an s, t-invariant subspace. By the same arguments in case (1), we have $(st)^{m_{st}} \cdot \alpha_{r,n} = \alpha_{r,n}$. The case $m_{rs} = 2$ is similar.

In the following cases, we assume s, t, r do not commute with each other.

(3) If $s = s_1$, $t = t_1$, while s_2, t_2 do not occur simultaneously in s, r, t, then $\alpha_{s_1,n+1}$, $\alpha_{t_1,n}$, $\alpha_{r,n}$, $\alpha_{s_1,n}$, $\alpha_{t_1,n-1}$ span a five-dimensional s, t-invariant subspace U. Define U_s, U_t as in case (1), then $\dim U_s = \dim U_t = 3$. The same argument shows that $(s_1 t_1)^{m_{s_1 t_1}} \cdot \alpha_{r,n} = \alpha_{r,n}$.

(4) If $s = s_1$, $r = t_1$, while s_2, t_2 do not occur simultaneously in s, r, t, then $\alpha_{s_1,n}$, $\alpha_{t,n}$, $\alpha_{r,n}$, $\alpha_{s_1,n+1}$, $\alpha_{t,n+1}$ span an s, t-invariant subspace. The same argument works.

(5) If $s = s_1$, $t = t_1 = t_2$, $r = s_2$, then $\alpha_{r,n}, \alpha_{s,n}, \alpha_{t,n-1}$ span an s, t-invariant subspace. The same argument works.

(6) If $s = s_1$, $t = t_1 = s_2$, $r = t_2$, then $\alpha_{t_1,n-1}$, $\alpha_{s_1,n}$, $\alpha_{r,n}$, $\alpha_{s_2,n+1}$, $\alpha_{s_1,n+2}$ span an s, t-invariant subspace. The same argument works.

(7) If $s = s_1$, $t = s_2$, $r = t_1 = t_2$, then $\alpha_{s_1,n+1}$, $\alpha_{r,n}$, $\alpha_{s_2,n+1}$ span an s, t-invariant subspace. The same argument works.

(8) If $s = s_1$, $t = t_2$, $r = t_1 = s_2$, then $\alpha_{t_2,n+1}$, $\alpha_{s_1,n+1}$, $\alpha_{r,n}$, $\alpha_{t_2,n-1}$, $\alpha_{s_1,n-1}$ span an s, t-invariant subspace. The same argument works.

In the above cases, if we exchange the letters s, t or the indices $1, 2$, then the arguments are totally the same. Thus, we always have $(st)^{m_{st}} \cdot \alpha_{r,n} = \alpha_{r,n}$.

□

3.2. The Infinite-Dimensional Irreducible Quotient

Theorem 3.2. *Recall that (W, S) is irreducible, and that there are at least two circuits in its Coxeter graph G. Let V be defined as in Sect. 3.1, and*

$$V_0 := \{v \in V \mid s \cdot v = v, \forall s \in S\}.$$

Then the representation V/V_0 of W is irreducible of infinite dimension.

Proof. We denote $V_-^s := \{v \in V \mid s \cdot v = -v\}$. Then $V_-^s = \bigoplus_n \mathbb{C}\alpha_{s,n}$. For any edge $\{s,t\}$ in G, $V_-^s \oplus V_-^t$ is a subrepresentation of $\langle s,t \rangle$ in V, isomorphic to an infinite direct sum of ρ_1. For any $v \in V_-^s$, let $f_{st}(v) := (t \cdot v - v)/2\cos\frac{\pi}{m_{st}}$. Then $f_{st}(v) \in V_-^t$, and the linear map $f_{st} : V_-^s \to V_-^t$ is a linear isomorphism of vector spaces. For example, when $\{s,t\} \neq \{s_1,t_1\}$ or $\{s_2,t_2\}$, we have $f_{st}(\alpha_{s,n}) = \alpha_{t,n}$. Moreover, $f_{st}(v)$ lies in the subrepresentation generated by v.

Let $0 \neq v \in V$, and let U be the subrepresentation generated by v. If $v \notin V_0$, say, $t \cdot v \neq v$, then $t \cdot v - v \in V_-^t \cap U$. Suppose $(r_0 = t, r_1, \dots, r_k = s_0)$ is a path connecting t and s_0. Here, s_0 is the vertex fixed in Sect. 3.1. Then,

$$v_0 := f_{r_{k-1}r_k} \cdots f_{r_1r_2} f_{r_0r_1}(t \cdot v - v) \in V_-^{s_0} \cap U \text{ and } v_0 \neq 0.$$

Apply the maps f_{**} along the closed paths $c_1, \dots, c_l$ chosen in Sect. 3.1. Then we obtain l linear isomorphisms of $V_-^{s_0}$, denoted by $X_1, \dots, X_l$, respectively. This makes $V_-^{s_0}$ form a representation of the free group $\pi_1(G)$. Except X_1 and X_2, other X_is are identity maps of $V_-^{s_0}$, and we have

$$X_1(\alpha_{s_0,n}) = \alpha_{s_0,n+1}, \quad X_2(\alpha_{s_0,n}) = 2^n \alpha_{s_0,n+1}.$$

It is easy to verify that $V_-^{s_0}$ is an irreducible representation of $\pi_1(G)$. Notice that $0 \neq v_0 \in V_-^{s_0} \cap U$, $X_i^{\pm 1}(v_0) \in U$. Thus, $V_-^{s_0} \subseteq U$. Since G is connected, $V_-^{s_0}$ generates the whole representation V. Hence, $V = U$. We have proved that V/V_0 is an irreducible representation of W.

Note that s_0 acts on $V_-^{s_0}$ by -1. So $V_-^{s_0} \cap V_0 = 0$, and thus $\dim V/V_0 = \infty$. □

4. Representations via Universal Coverings of Coxeter Graphs

In this section we assume that

$$\text{the Coxeter graph } G = (S,E) \text{ is not a tree,}$$
$$\text{and there exist } s_1, s_2 \in S \text{ such that } m_{s_1s_2} \geq 4.$$

In this section, we use the universal covering of G to construct an infinite-dimensional representation of W, then find an irreducible (sub)quotient in it.

For the same reason stated before Sect. 3.1, we may further assume

$$m_{st} < \infty, \quad \forall s,t \in S.$$

4.1. The Construction

We fix $s_1, s_2 \in S$ such that $m_{s_1s_2} \geq 4$. Let $p : G' \to G$ be the universal covering of G, where $G' = (S', E')$. Fix an edge $\{s_1', s_2'\}$ in G' such that $p(s_1') = s_1$, $p(s_2') = s_2$, as shown in Fig. 2.

We define a vector space V with formal basis $\{\alpha_a \mid a \in S'\}$,

$$V := \bigoplus_{a \in S'} \mathbb{C}\alpha_a.$$

For any $s \in S$ and $a \in S'$, we define $s \cdot \alpha_a$ as follows:

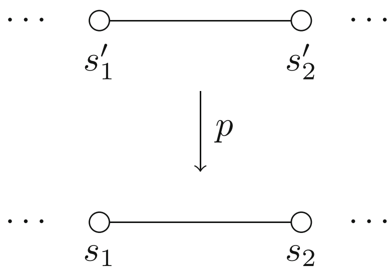


FIGURE 2. The edge $\{s_1', s_2'\}$

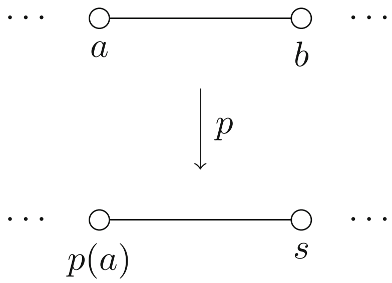


FIGURE 3. The vertex b in $p^{-1}(s)$ adjacent to a

(1) if $s = p(a)$, then $s \cdot \alpha_a := -\alpha_a$;
(2) if $s = s_1$, $a = s_2'$, then $s_1 \cdot \alpha_{s_2'} := \alpha_{s_2'} + 2\cos\frac{2\pi}{m_{s_1 s_2}}\alpha_{s_1'}$;
(3) if $s = s_2$, $a = s_1'$, then $s_2 \cdot \alpha_{s_1'} := \alpha_{s_1'} + 2\cos\frac{2\pi}{m_{s_1 s_2}}\alpha_{s_2'}$;
(4) if it is not in the cases above, and if s is not adjacent to $p(a)$ in G, then $s \cdot \alpha_a := \alpha_a$;
(5) if it is not in the cases above, and if s is adjacent to $p(a)$ in G, then we denote by b the vertex adjacent to a in $p^{-1}(s)$ (see Fig. 3), and $s \cdot \alpha_a := \alpha_a + 2\cos\frac{\pi}{m}\alpha_b$, where $m := m_{s,p(a)} \geq 3$.

In particular, $\mathbb{C}\alpha_{s_1'} \oplus \mathbb{C}\alpha_{s_2'}$ forms a representation of the dihedral subgroup $\langle s_1, s_2 \rangle$, isomorphic to ρ_2 (note that if $m_{s_1 s_2} = 4$, then this representation splits, see Remark 2.1). While for other pairs of adjacent vertices $\{a, b\}$ in G', $\mathbb{C}\alpha_a \oplus \mathbb{C}\alpha_b$ forms an irreducible representation of $\langle p(a), p(b) \rangle$ isomorphic to ρ_1.

Lemma 4.1. *V is a representation of W with the action defined above.*

Proof. From the construction, it is clear that s^2 acts by identity for any $s \in S$. Suppose $s, t \in S$ and $s \neq t$. We need to verify that $(st)^{m_{st}} \cdot \alpha_a = \alpha_a$ for any $a \in S'$. If $p(a) = s$ or t, then α_a lies in a subrepresentation of $\langle s, t \rangle$. Thus, we have $(st)^{m_{st}} \cdot \alpha_a = \alpha_a$. If $p(a) \neq s$ and $p(a) \neq t$, then the relationship of the three vertices $p(a), s, t$ in G is in one of the following cases (ignoring labels like

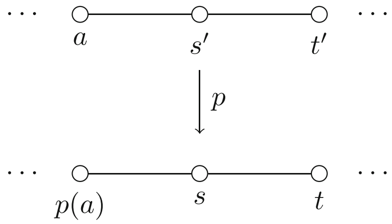


FIGURE 4. The vertices s' and t'

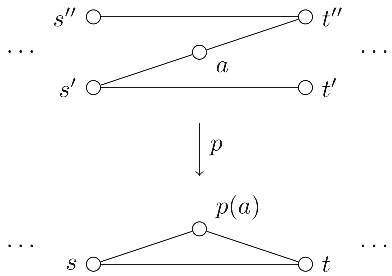
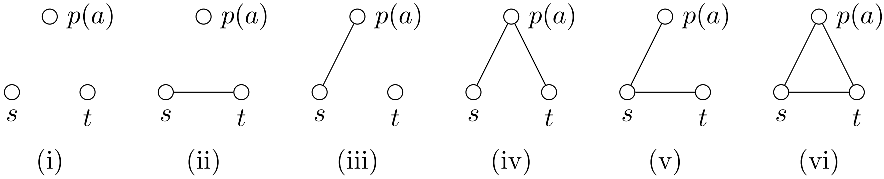


FIGURE 5. The vertices s', t', s'' and t''

m_{st} on edges),



(In cases (iii) and (v), exchanging letters s and t does not cause an essential difference. So we omit them.) In cases (i) (iii) (iv), we may verify directly by definition that $st \cdot \alpha_a = ts \cdot \alpha_a$. In case (ii), it is also clear that $(st)^{m_{st}} \cdot \alpha_a = \alpha_a$.

Suppose we are in case (v). We denote by s' the vertex adjacent to a in $p^{-1}(s)$, and by t' the vertex adjacent to s' in $p^{-1}(t)$. Then a is not adjacent to t', as shown in Fig. 4. The three-dimensional subspace spanned by $\alpha_a, \alpha_{s'}, \alpha_{t'}$ stays invariant under the action of s and t. By the same arguments as in the proof of Lemma 3.1, it holds that $(st)^{m_{st}} \cdot \alpha_a = \alpha_a$.

Suppose we are in case (vi). We denote by s' the vertex adjacent to a in $p^{-1}(s)$, by t' the vertex adjacent to s' in $p^{-1}(t)$, by t'' the vertex adjacent to a in $p^{-1}(t)$, and by s'' the vertex adjacent to t'' in $p^{-1}(s)$ (see Fig. 5). Then a, t', s'' are not adjacent to each other. Then $\alpha_a, \alpha_{s'}, \alpha_{s''}, \alpha_{t'}, \alpha_{t''}$ span an s,t-invariant subspace of dimension 5. The same arguments as in the proof of Lemma 3.1 yield $(st)^{m_{st}} \cdot \alpha_a = \alpha_a$. □

4.2. The Infinite-Dimensional Irreducible (Sub)quotient

Let a and b be two arbitrary vertices in G'. Since G' is a tree, there is a unique path $(a = t_0, t_1, \ldots, t_n = b)$ connecting a, b such that all t_is are distinct. We define $\mathrm{d}(a, b) := n$ to be the distance between a and b. We define also

$$S'_1 := \{a \in S' \mid \mathrm{d}(a, s'_1) < \mathrm{d}(a, s'_2)\}, \quad S'_2 := \{a \in S' \mid \mathrm{d}(a, s'_2) < \mathrm{d}(a, s'_1)\}.$$

Then one of them is an infinite set. Without loss of generality, we assume $|S'_1| = \infty$. Let

$$V_1 := \bigoplus_{a \in S'_1} \mathbb{C}\alpha_a, \quad V_0 := \{v \in V \mid s \cdot v = v, \forall s \in S\}.$$

Then $\dim V_1 = \infty$. If $m_{s_1 s_2} = 4$, then V_1 is a subrepresentation of W in V.

Lemma 4.2.

(1) *If $m_{s_1 s_2} > 4$ and $v \in V \setminus V_0$, then V is generated by v as a representation of W.*

(2) *If $m_{s_1 s_2} = 4$ and $v_1 \in V_1 \setminus V_0$, then V_1 is generated by v_1 as a representation of W.*

Proof. (1). Suppose $s \cdot v \neq v$, $s \in S$. Then by definition we know that $v - s \cdot v$ is a finite sum of the form $\sum_{a \in p^{-1}(s)} x_a \alpha_a$, where each x_a is a complex number. Let $u_0 := v - s \cdot v$, and U be the subrepresentation generated by v. Then $u_0 \in U$. We take $a_0 \in p^{-1}(s)$ such that $x_{a_0} \neq 0$. Suppose the shortest path in G' connecting a_0 and s'_1 is

$$(a_0, a_1, \ldots, a_n = s'_1).$$

Let $t_i := p(a_i) \in S$, $u_i := u_{i-1} - t_i \cdot u_{i-1}$. Inductively, we can see that u_i is of the form $\sum_{b \in p^{-1}(t_i)} x_{i,b} \alpha_b$, where $x_{i,b} \in \mathbb{C}$. Since p is a covering map, there is only one vertex (namely, a_{i-1}) in $p^{-1}(t_{i-1})$ adjacent to a_i. Thus in the expression of u_i, the coefficient x_{i,a_i} of α_{a_i} is nonzero. In particular, taking $i = n$, we know that $u_n \in U$ and the coefficient of $\alpha_{s'_1}$ is nonzero.

We view V as a representation of the finite dihedral group $D := \langle s_1, s_2 \rangle$. Since the group algebra $\mathbb{C}[D]$ is semisimple, V decomposes into a direct sum of some copies of irreducible representations of D. From the construction of V, the only irreducible representations of D which are possible to occur in V are $\mathbb{1}, \rho_1, \rho_2$. Moreover, ρ_2 appears only once, namely, $\mathbb{C}\alpha_{s'_1} \oplus \mathbb{C}\alpha_{s'_2}$. Therefore, there is an element $d \in \mathbb{C}[D]$ such that $d \cdot u_n = \alpha_{s'_1}$, and hence $\alpha_{s'_1} \in U$. Note that G' is a connected graph. From the definition of V, we know that $\alpha_{s'_1}$ generates the whole V. Thus, $U = V$.

(2). The proof is similar. As above, we take a vertex $a_0 \in p^{-1}(s) \cap S'_1$ such that α_{a_0} has nonzero coefficient in the linear expression of $v_1 - s \cdot v_1$ ($\neq 0$), and do the same discussion along the shortest path connecting a_0 and s'_1 (note that all of the vertices in this path belong to S'_1). Then we know that in the subrepresentation generated by v_1, there is a vector u_n with nonzero coefficient of $\alpha_{s'_1}$.

Decompose V_1 into a direct sum of irreducible representations of $D = \langle s_1, s_2 \rangle$. Then only $\mathbb{1}, \rho_1, \varepsilon_{s_1}$ occur. Moreover, the subrepresentation ε_{s_1}, spanned by $\alpha_{s'_1}$, is of multiplicity one. Thus, $\alpha_{s'_1}$ lies in the subrepresentation generated by v_1, while $\alpha_{s'_1}$ generates the whole representation V_1. □

Let

$$\overline{V} := V/V_0, \quad \overline{V_1} := V_1/(V_1 \cap V_0).$$

(V is defined in Sect. 4.1, while V_1 and V_0 are defined in Sect. 4.2.)

Theorem 4.3. *Recall that $m_{s_1 s_2} \geq 4$ and there is a circuit in the Coxeter graph. If $m_{s_1 s_2} > 4$, then $\overline{V}$ is an irreducible representation of W. If $m_{s_1 s_2} = 4$, then $\overline{V_1}$ is an irreducible representation of W. Moreover, $\overline{V}$ and $\overline{V_1}$ are both infinite dimensional.*

Proof. From Lemma 4.2, we already know that the representations mentioned in this theorem are irreducible. Thus, it suffices to show they are both infinite dimensional. Obviously, $\dim \overline{V} \geq \dim \overline{V_1}$. Thus, we only need to show $\dim \overline{V_1} = \infty$. Let $s \in S$ be arbitrary. Then, $p^{-1}(s) \cap S'_1$ is an infinite set. Let

$$U := \bigoplus_{a \in p^{-1}(s) \cap S'_1} \mathbb{C}\alpha_a.$$

Then $U \subseteq V_1$, and $\dim U = \infty$. For any $0 \neq v \in U$, we have $s \cdot v = -v$. Thus, $U \cap V_0 = 0$, and $\overline{V_1} = V_1/(V_1 \cap V_0)$ is infinite dimensional. □

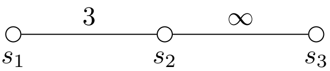
Remark 4.4. In the theorem, it is necessary to take the quotient by V_0 (V_0 might be nonzero). For example, consider the universal covering in Fig. 6, where $p(s'_i) = p(s''_i) = s_i$, $p(t'_i) = t_i$, $p(r'_i) = r_i$, $\forall i$, and all edges in the Coxeter graph are labelled by 3 (hence omitted) except the triangle $s_0 s_1 s_2$. In this situation, the vector

$$\alpha_{t'_1} + \alpha_{t'_2} + \alpha_{t'_3} + \alpha_{t'_4} + 2\alpha_{t'_0} - 2\alpha_{r'_0} - \alpha_{r'_1} - \alpha_{r'_2} - \alpha_{r'_3} - \alpha_{r'_4}$$

is fixed by every s_i, r_i and t_i. Besides, under the setting of Sect. 3 it is also necessary to take the quotient by V_0 in Theorem 3.2 (an example can be given in a similar way).

5. Another Example

In this section, we construct an infinite-dimensional irreducible representation of a specific Coxeter group whose Coxeter graph is a tree. Suppose the Coxeter graph G of (W, S) is



This Coxeter group is isomorphic to $\mathrm{PGL}(2, \mathbb{Z})$ (see [2, §5.1]).

Recall that the dihedral subgroup $\langle s_1, s_2 \rangle$ has an irreducible representation ρ_1 on $\mathbb{C}\beta_{s_1} \oplus \mathbb{C}\beta_{s_2}$ (see Sect. 2.1). There is a basis $\{u, v\}$ of this space such that u, v are eigenvectors of s_1 with eigenvalues $+1, -1$, respectively. The vectors u, v can be chosen so that $s_2 \cdot u = (3v - u)/2$, $s_2 \cdot v = (u + v)/2$, as illustrated in Fig. 7.

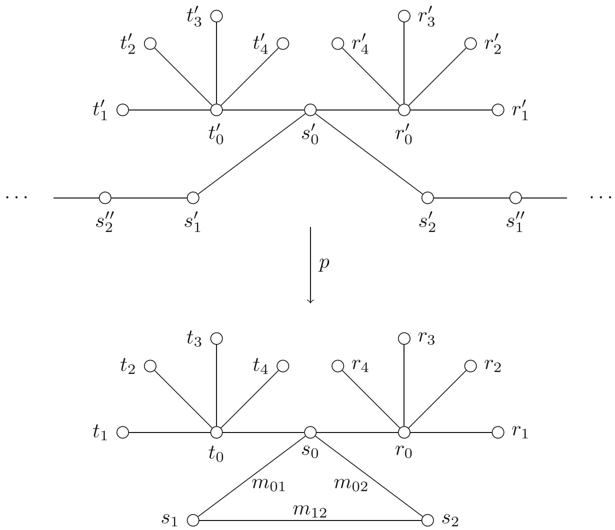


FIGURE 6. A universal covering

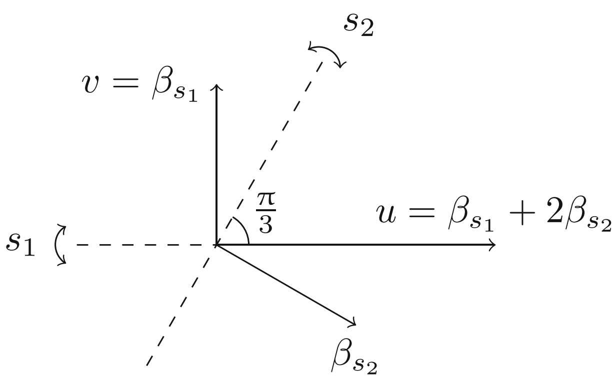


FIGURE 7. The basis vectors u, v of the representation ρ_1

Let

$$V := \mathbb{C}u_0 \oplus \bigoplus_{i \in \mathbb{N}_{>0}} (\mathbb{C}u_i \oplus \mathbb{C}v_i)$$

be the vector space with basis $\{u_i, v_j \mid i \in \mathbb{N}, j \in \mathbb{N}_{>0}\}$, and let s_1, s_2 act on V by

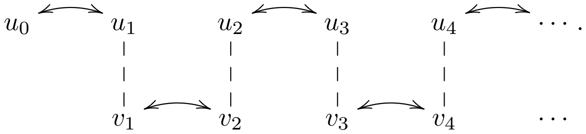
$$\begin{aligned} s_1 \cdot u_0 = s_2 \cdot u_0 &= u_0, \\ s_1 \cdot u_i =& u_i, \quad s_1 \cdot v_i = -v_i, \quad \forall i \in \mathbb{N}_{>0}, \\ s_2 \cdot u_i =& \frac{3v_i - u_i}{2}, \quad s_2 \cdot v_i = \frac{u_i + v_i}{2}, \quad \forall i \in \mathbb{N}_{>0}. \end{aligned}$$

Then as a representation of $\langle s_1, s_2 \rangle$, V is a direct sum of a trivial representation $\mathbb{1}$ and infinite many copies of ρ_1.

To make V be a representation of W, we only need to find an involution on V commuting with s_1, and let s_3 act by this involution. Let

$$s_3 \cdot u_{2k} = u_{2k+1}, \quad s_3 \cdot u_{2k+1} = u_{2k}, \quad \forall k \in \mathbb{N},$$
$$s_3 \cdot v_{2k} = v_{2k-1}, \quad s_3 \cdot v_{2k-1} = v_{2k}, \quad \forall k \in \mathbb{N}_{>0}.$$

Intuitively, s_3 permutes these basis vectors:



Obviously, the action of s_3 is an involution and commutes with the action of s_1. Thus, V forms a representation of W.

Similar to the proof of Lemma 4.2, u_0 lies in any nonzero subrepresentation of V. Note that u_0 generates the whole V. So V is an irreducible representation of W.

Remark 5.1. Similar constructions also give irreducible representations of infinite dimension when $m_{s_1 s_2} = 3$ is replaced by larger integers.

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Conflict of Interest The author has no relevant interests to declare.

Ethics Approval Not applicable.

6. Appendix: A Sketched Proof of Theorem 1.1

This proof is given by an anonymous referee of a previous version of this paper. The proof uses the following fact.

Lemma 6.1. [5, Corollary 2] *Suppose (W, S) is an infinite non-affine irreducible Coxeter group of finite rank. Then there exists a subgroup $Y' \subseteq W$ of finite index and a surjective homomorphism $\varphi' : Y' \twoheadrightarrow F'$, where F' is a non-abelian free group.*

Now we can prove Theorem 1.1, and we only need to prove the "only if" part.

Suppose (W, S) is infinite and non-affine. Let Y', φ', F' be as in Lemma 6.1. Let Y be the intersection of all conjugates of Y' in W. Then Y is a normal subgroup of W of finite index. Since Y is of finite index in Y', the image $\varphi'(Y)$ is a subgroup of F' of finite index. Therefore, $\varphi'(Y)$ is also a non-abelian free group (see, for example, [6, Theorem 85.1]). Let X_1, X_2 be two free generators of $\varphi'(Y)$, and let $F = \langle X_1, X_2 \rangle$. Then F is a free group of rank two, and we have a surjection $\varphi'(Y) \twoheadrightarrow F$. By composing this surjection with the map $\varphi'|_Y$, we obtain a surjective homomorphism $\varphi : Y \twoheadrightarrow F$.

Let V be an infinite-dimensional irreducible representation of F (for example, $V := \bigoplus_{n \in \mathbb{Z}} \mathbb{C}\alpha_n$, and let $X_1 \cdot \alpha_n = \alpha_{n+1}$, $X_2 \cdot \alpha_n = 2^n \alpha_{n+1}$, as in the proof of Theorem 3.2). By pulling back this F-representation along φ, V becomes an irreducible representation of Y.

We consider the induced representation $V_Y^W := \mathbb{C}[W] \otimes_{\mathbb{C}[Y]} V$ of W, and we choose $w_1, \ldots, w_n$ as coset representatives for Y in W. Then $V_Y^W = \bigoplus_{1 \leq i \leq n} w_i \otimes V$ as a vector space. Since Y is a normal subgroup of W, each summand $w_i \otimes V$ is an irreducible representation of Y. Thus, V_Y^W is a semisimple Noetherian and Artinian $\mathbb{C}[Y]$-module, and hence a Noetherian and Artinian $\mathbb{C}[W]$-module. Consequently, there exists an irreducible W-representation $M \subseteq V_Y^W$ as a subrepresentation.

If we view M as a $\mathbb{C}[Y]$-module, then, by semi-simplicity of the $\mathbb{C}[Y]$-module V_Y^W and Schur's lemma, M is isomorphic to a direct sum of some $\mathbb{C}[Y]$-modules $w_i \otimes V$. In particular, M is infinite dimensional. Theorem 1.1 is proved.

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