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Some Results for Bipartition Difference Functions

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Abstract. Inspired by a recent work of Kim, Kim and Lovejoy on two overpartition difference functions, we study some bipartition difference functions, four of which are related to Ramanujan's identities recorded in his lost notebook. We show that they are always positive by elementary q-series transformations.

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1. Introduction

Throughout the paper, we adopt the standard q-Pochhammer symbol

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$$

for $|q|<1$, and

$$(a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

for any integer n.

We will use the following fundamental tools in the subject of q-series

$$\sum_{n\geq 0}\frac{(a;q^m)_n(b;q)_{mn}}{(q^m;q^m)_n(c;q)_{mn}}t^n = \frac{(b;q)_\infty(at;q^m)_\infty}{(c;q)_\infty(t;q^m)_\infty}\sum_{n\geq 0}\frac{(c/b;q)_n(t;q^m)_n}{(q;q)_n(at;q^m)_n}b^n, \tag{1.1}$$

$$\sum_{n\geq 0}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}t^n = \frac{(c/b;q)_\infty(bt;q)_\infty}{(c;q)_\infty(t;q)_\infty}\sum_{n\geq 0}\frac{(abt/c;q)_n(b;q)_n}{(bt;q)_n(q;q)_n}(c/b)^n, \tag{1.2}$$

$$\sum_{n\geq 0}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}t^n = \frac{(abt/c;q)_\infty}{(t;q)_\infty}\sum_{n\geq 0}\frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(q;q)_n}(abt/c)^n. \tag{1.3}$$

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whose proof can be found in [4, Theorem 1.2.1, Corollaries 1.2.4 and 1.2.5].

A partition λ of a positive integer n is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $|\lambda| = \sum_{i=1}^{r} \lambda_i = n$. The terms λ_i are called the parts of λ. A bipartition of n is a pair of partitions (λ_r, λ_b) with $|\lambda_r| + |\lambda_b| = n$, where the parts of λ_r are colored red and the parts of λ_b are colored blue. An overpartition of n is a partition of n in which the first occurrence of a number may be overlined [9].

Recently, Kim, Kim and Lovejoy [10] focused on the following identity due to Ramanujan [4, Entry 1.4.9]

$$\sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q;q)_n^2} = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} (-1)^n \frac{q^{n(n+1)/2}}{(q^2;q^2)_n}. \tag{1.4}$$

It is not hard to see that the series on the left-hand side of (1.4) is the generating function for overpartitions where there are no nonoverlined parts larger than the number of overlined parts. It is natural to consider $\overline{p}(m,n)$, the number of overpartitions of n in which there are exactly m nonoverlined parts larger than the number of overlined parts. Then

$$\sum_{n\geq 0} \overline{p}(m,n) z^m q^n = \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q;q)_n^2 (zq^{n+1};q)_\infty}.$$

Apply (1.2) with $a = zq, b = q/\tau, c = q, t = -\tau$, and then let $\tau \to 0$. After dividing both sides of the resulting identity by $(zq;q)_\infty$, we can conclude that

$$\sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q;q)_n^2 (zq^{n+1};q)_\infty} = \frac{(-q;q)_\infty}{(q;q)_\infty (zq;q)_\infty} \sum_{n\geq 0} (-1)^n \frac{(-zq;q)_n q^{n(n+1)/2}}{(q^2;q^2)_n}. \tag{1.5}$$

If we set $z = 0$, then (1.5) reduces to (1.4). If we set $z = -1$ in (1.5), the left-hand side can be interpreted as

$$\sum_{n\geq 0} (\overline{p}_e(n) - \overline{p}_o(n)) q^n = \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q;q)_n^2 (-q^{n+1};q)_\infty}, \tag{1.6}$$

where $\overline{p}_e(n)$ and $\overline{p}_o(n)$ counts the number of overpartitions of n which have an even and odd number of nonoverlined parts larger than the number of overlined parts, respectively. Despite the sign in the generating function, Kim, Kim and Lovejoy [10] showed that the coefficient of q^n for $n > 1$ in (1.6) is always positive.

Theorem 1.1. (Kim et al.) *For $n > 1$, we have*

$$\overline{p}_e(n) > \overline{p}_o(n).$$

Kim, Kim and Lovejoy [10] also discussed similar inequalities for other types of partitions. For example, let $q_e(n)$ and $q_o(n)$ denote the number of bipartitions of n which have an even and odd number of blue parts larger than the number of red parts, respectively. Clearly,

$$\sum_{n\geq 0} (q_e(n) - q_o(n)) q^n = \sum_{n\geq 0} \frac{q^n}{(q;q)_n^2 (-q^{n+1};q)_\infty}. \tag{1.7}$$

They proved that (1.7) has nonnegative coefficients.

Theorem 1.2. (Kim et al.) *For $n > 1$, we have*

$$q_e(n) > q_o(n).$$

In fact, (1.7) is closely related to

$$\sum_{n\geq 0} \frac{q^n}{(q;q)_n^2} = \frac{1}{(q;q)_\infty^2} \sum_{n\geq 0} (-1)^n q^{n(n+1)/2}, \tag{1.8}$$

which is recorded in Ramanujan's lost notebook [4, Entry 1.4.10]. It is readily seen that the series on the left-hand side of (1.8) is the generating function for bipartitions where there are no blue parts larger than the number of red parts.

In this work, we aim to explore several partition inequalities of this nature, four of which have close connections with Ramanujan's identities [4, Entries 1.4.3, 1.4.4, 1.4.5 and 1.4.11]

$$\sum_{n\geq 0} \frac{a^n q^n}{(q;q)_n (bq;q^2)_n} = \frac{1}{(aq;q)_\infty (bq;q^2)_\infty} \sum_{n\geq 0} \frac{(aq;q)_{2n} (-b)^n q^{n^2}}{(q^2;q^2)_n}, \tag{1.9}$$

$$\sum_{n\geq 0} \frac{a^n q^{2n}}{(q^2;q^2)_n (bq;q)_{2n}} = \frac{1}{(aq^2;q^2)_\infty (bq;q)_\infty} \sum_{n\geq 0} \frac{(aq^2;q^2)_n (-b)^n q^{n(n+1)/2}}{(q;q)_n}, \tag{1.10}$$

$$\sum_{n\geq 0} (-aq;q)_n (-q;q)_n q^n = (-q;q)_\infty (-aq;q)_\infty \sum_{n\geq 0} \frac{(q;q^2)_n q^{2n}}{(-aq;q)_{2n+1}}, \tag{1.11}$$

$$\sum_{n\geq 0} \frac{q^{2n}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty^2} \left(1 + 2\sum_{n\geq 0} (-1)^n q^{n(n+1)/2}\right). \tag{1.12}$$

2. Bipartitions Related to Ramanujan's Identities

Setting $a = 1$ and $b = q$ in (1.9), we have

$$\sum_{n\geq 0} \frac{q^n}{(q;q)_n (q^2;q^2)_n} = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty} \sum_{n\geq 0} (-1)^n \frac{(q;q)_{2n} q^{n^2+n}}{(q^2;q^2)_n}. \tag{2.1}$$

It is easy to see that the series on the left-hand side of (2.1) is the generating function for bipartitions where each blue part is even and no blue part is larger than twice the number of red parts. Let $a_e(n)$ and $a_o(n)$ denote the number of bipartitions of n with each blue part being even and having an even and odd number of blue parts larger than twice the number of red parts, respectively. Then

$$\sum_{n\geq 0} (a_e(n) - a_o(n)) q^n = \sum_{n\geq 0} \frac{q^n}{(q;q)_n (q^2;q^2)_n (-q^{2n+2};q^2)_\infty}.$$

Theorem 2.1. *For $n \geq 1$, we have*

$$a_e(n) > a_o(n).$$

Proof. It is clear that

$$\sum_{k\geq 0}\frac{q^k}{(q;q)_k(q^2;q^2)_k(-q^{2k+2};q^2)_\infty}=\frac{1}{(-q^2;q^2)_\infty}\sum_{k\geq 0}\frac{(-q^2;q^2)_kq^k}{(q;q)_k(q^2;q^2)_k}.$$

Setting $a=-1, b=q, c=0, m=2, t=-q^2$ in (1.1) and then dividing both sides by $(q;q)_\infty(q^2;q^2)_\infty$, we find that

$$\begin{aligned}
&\frac{1}{(-q^2;q^2)_\infty}\sum_{k\geq 0}\frac{q^k(-q^2;q^2)_k}{(q;q)_k(q^2;q^2)_k}\\
&\quad=\frac{1}{(q;q)_\infty(q^2;q^2)_\infty}\sum_{k\geq 0}\frac{(-1)^kq^{2k}(-1;q^2)_k(q;q)_{2k}}{(q^2;q^2)_k}\\
&\quad=\frac{1}{(q^2;q^2)_\infty}\sum_{k\geq 0}\frac{(-1)^kq^{2k}(-1;q^2)_k}{(q^2;q^2)_k(q^{2k+1};q)_\infty}\\
&\quad=\frac{1}{(q^2;q^2)_\infty}\sum_{k\geq 0}\left(\frac{q^{4k}(-1;q^2)_{2k}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}-\frac{q^{4k+2}(-1;q^2)_{2k+1}}{(q^2;q^2)_{2k+1}(q^{4k+3};q)_\infty}\right)\\
&\quad=\frac{1}{(q^2;q^2)_\infty}\sum_{k\geq 0}\left(\frac{q^{4k}(-1;q^2)_{2k}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}-\frac{q^{4k+2}(-1;q^2)_{2k}(1+q^{4k})}{(q^2;q^2)_{2k}(q^{4k+2};q)_\infty}\right)\\
&\quad=\frac{1}{(q^2;q^2)_\infty}\sum_{k\geq 0}\frac{q^{4k}(-1;q^2)_{2k}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}\left(1-q^2(1+q^{4k})(1-q^{4k+1})\right)\\
&\quad=\frac{1}{(q^2;q^2)_\infty}\sum_{k\geq 0}\frac{q^{4k}(-1;q^2)_{2k}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}(1-q^2-q^{4k+2}+q^{4k+3}+q^{8k+3})\\
&\quad=\frac{1}{(q^2;q^2)_\infty}\sum_{k\geq 0}\frac{q^{4k}(-1;q^2)_{2k}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}(1-q^2-q^{4k+2}+q^{4k+3})\\
&\qquad+\frac{1}{(q^2;q^2)_\infty}\sum_{k\geq 0}\frac{q^{12k+3}(-1;q^2)_{2k}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}.
\end{aligned}\tag{2.2}$$

We claim that for $k\geq 0$,

$$F_k(q):=\frac{1-q^2-q^{4k+2}+q^{4k+3}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}$$

is a series with nonnegative coefficients.

First, we have

$$\begin{aligned}
F_0(q)&=\frac{1-q^2-q^2+q^3}{(q;q)_\infty}\\
&=\frac{1-2q^2+q^4+q^3-q^4}{(q;q)_\infty}\\
&=\frac{(1-q^2)^2}{(q;q)_\infty}+\frac{q^3(1-q)}{(q;q)_\infty}
\end{aligned}$$

$$= \frac{1+q}{(q^3;q)_\infty} + \frac{q^3}{(q^2;q)_\infty},$$

in which the coefficient of q^n is positive for $n \neq 2$. Hence, $F_0(q)/(q^2;q^2)_\infty$ is a series with positive coefficients.

For $k \geq 1$, we have

$$F_k(q) = \frac{(1-q^2)(1-q^{4k+2}) + q^{4k+3} - q^{4k+4}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}$$
$$= \frac{1}{(q^4;q^2)_{2k-1}(1-q^{4k+1})(q^{4k+3};q)_\infty} + \frac{q^{4k+3}(1-q)}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}.$$

Suppose that

$$\sum_{n\geq 0} f(n)q^n = \frac{1}{(q^{4k+1};q)_\infty},$$

where $f(n)$ is the number of partitions of n into parts at least $4k+1$. Given such a partition of $n-1$, adding 1 to the largest part, we obtain a partition of n. This map is injective, which implies that $f(n) \geq f(n-1)$. Thus,

$$\frac{1-q}{(q^{4k+1};q)_\infty} = \sum_{n\geq 0}(f(n) - f(n-1))q^n$$

has nonnegative coefficients, so does for

$$\frac{q^{4k+3}(1-q)}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}.$$

Therefore, each $F_k(q)$ has nonnegative coefficients.

The desired statement follows from the conclusion that the first sum on the right-hand side of (2.2) is a series with positive coefficients, and the second sum is a series with nonnegative coefficients. □

Setting $a = b = 1$ in (1.10), we obtain

$$\sum_{n\geq 0} \frac{q^{2n}}{(q;q)_{2n}(q^2;q^2)_n} = \frac{1}{(q^2;q^2)_\infty(q;q)_\infty} \sum_{n\geq 0} \frac{(q^2;q^2)_n(-1)^n q^{n(n+1)/2}}{(q;q)_n}. \quad (2.3)$$

The series on the left-hand side of (2.3) is the generating function for bipartitions where there are an even number of red parts, and each blue part is even and no blue part is larger than the number of red parts. Let $b_e(n)$ and $b_o(n)$ be the number of bipartitions of n where the number of red parts is even, each blue part is even and there are an even and odd number of blue parts larger than the number of red parts, respectively. Then

$$\sum_{n\geq 0}(b_e(n) - b_o(n))q^n = \sum_{n\geq 0} \frac{q^{2n}}{(q;q)_{2n}(q^2;q^2)_n(-q^{2n+2};q^2)_\infty}.$$

Theorem 2.2. *For $n > 2$, we have*

$$b_e(n) > b_o(n).$$

In fact, we have the following more general result, which includes Theorems 1.2 and 2.2 as special cases.

Theorem 2.3. *For $m \geq 1$, the coefficient of q^n in the series*

$$G_m(q) := \sum_{n\geq 0} \frac{q^{mn}}{(q;q)_{mn}(q^m;q^m)_n(-q^{mn+m};q^m)_\infty}$$

is always positive for $n > m$.

Proof. Note that

$$\begin{aligned} G_m(q) &= \frac{1}{(q^{2m};q^{2m})_\infty} \sum_{n\geq 0} \frac{q^{mn}(-q^m;q^m)_n}{(q;q)_{mn}} (q^{mn+m};q^m)_\infty \\ &= \frac{1}{(q^{2m};q^{2m})_\infty} \left((q^m;q^m)_\infty + \frac{q^m(1+q^m)}{(q;q)_m} (q^{2m};q^m)_\infty + \cdots \right). \end{aligned}$$

It is easy to see that the coefficient of q^n in $G_m(q)$ is positive for $m < n < 2m$. It is immediate that $1/(q^{2m};q^{2m})_\infty$ has nonnegative coefficients. Thus, it suffices to show that

$$\sum_{n\geq 0} \frac{q^{mn}(-q^m;q^m)_n}{(q;q)_{mn}} (q^{mn+m};q^m)_\infty.$$

has nonnegative coefficients, where the coefficient of q^n is positive if $n \geq 2m$. To prove this, we construct an injection

$$\Phi : \bigcup_{n\geq 0} O_{m,n} \times D^o_{m,n} \to \bigcup_{n\geq 0} O_{m,n} \times D^e_{m,n},$$

where $O_{m,n}$ is the set of overpartitions into parts $\leq mn$ such that there is a nonoverlined part mn and the parts may be overlined when they are multiplies of m, and $D^o_{m,n}$ (respectively, $D^e_{m,n}$) is the set of partitions into distinct parts $\equiv 0 \pmod m$ with an odd (respectively, even) number of parts and each part at least $mn+m$. For $(\mu,\nu) \in O_{m,n} \times D^o_{m,n}$, we move the smallest part of ν, say ν_1, to μ. Then it is clear that the resulting partition is in $O_{m,\nu_1/m} \times D^e_{m,\nu_1/m}$ as $\nu_1 > mn$. Since the resulting partition (μ',ν') has an even number of parts in ν', we see that their contribution toward the summation cancel out each other. Moreover, ν_1 is the unique largest part of μ', and is nonoverlined. We choose $(\alpha,\beta) \in O_{m,n} \times D^e_{m,n}$ such that α has at least two nonoverlined parts of size mn. Then (α,β) has no preimage under Φ. Hence, the coefficient of q^n in

$$\sum_{n\geq 0} \frac{q^{mn}(-q^m;q^m)_n}{(q;q)_{mn}} (q^{mn+m};q^m)_\infty.$$

is larger than or equal to the coefficient of q^n in $H_m(q)$, which is defined by

$$H_m(q) := \sum_{n\geq 0} \frac{q^{2mn}(-q^m;q^m)_n}{(q;q)_{mn}} \frac{(q^{mn+m};q^m)_\infty + (-q^{mn+m};q^m)_\infty}{2}.$$

It is routine to see that the coefficient of q^n in $H_m(q)$ is positive for $n \geq 2m$. □

Remark 2.4. In the original version of this paper, Theorems 2.2 and 2.3 were two conjectured results. The present proof of Theorem 2.3 was provided by one of anonymous referees.

Setting $a = 1$ in (1.11) and then multiplying both sides by q, we have

$$\sum_{n\geq 0} q^{n+1}(-q;q)_n(-q;q)_n = (-q;q)_\infty^2 \sum_{n\geq 0} \frac{(q;q^2)_n q^{2n+1}}{(-q;q)_{2n+1}}. \tag{2.4}$$

It is easy to see that the series on the left-hand side of (2.4) generates the bipartitions where red parts must appear and are distinct, blue parts are distinguishable and every blue part is smaller than the largest red part. Let $c_e(n)$ and $c_o(n)$ denote the number of bipartitions where red parts must appear and are distinct, blue parts are distinguishable and there are an even and odd number of blue parts larger than or equal to the largest red part, respectively. Then

$$\sum_{n\geq 0}(c_e(n) - c_o(n))q^n = \sum_{n\geq 0} q^{n+1}(-q;q)_n(-q;q)_n(q^{n+1};q)_\infty. \tag{2.5}$$

Theorem 2.5. *For $n > 2$,*

$$c_e(n) > c_o(n).$$

Proof. It is clear that

$$\sum_{n\geq 0} q^{n+1}(-q;q)_n(-q;q)_n(q^{n+1};q)_\infty = (q;q)_\infty \sum_{n\geq 0} \frac{(-q;q)_n^2}{(q;q)_n} q^{n+1}.$$

Setting $a = b = -q, t = q$ and letting $c \to 0$ in (1.2), and then multiplying both sides by $q(q;q)_\infty$ gives

$$\begin{aligned}(q;q)_\infty \sum_{n\geq 0} \frac{(-q;q)_n^2}{(q;q)_n} q^{n+1} =& (-q^2;q)_\infty \sum_{n\geq 0} \frac{(-q;q)_n}{(q;q)_n(-q^2;q)_n} q^{(n+1)(n+2)/2} \\ =& \sum_{n\geq 0} \frac{(-q;q)_n(-q^{n+2};q)_\infty}{(q;q)_n} q^{(n+1)(n+2)/2}.\end{aligned}$$

Obviously, each summand on the right-hand side of the above equation has nonnegative coefficients. Moreover, the first summand is $q(-q^2;q)_\infty$, which implies that $c_e(n) - c_o(n) > 0$ for $n > 2$.

Corollary 2.6. *The difference $c_e(n) - c_o(n)$ is almost always even, and is odd if and only if $n = j(3j \pm 1)$ or $n = j(3j \pm 1)/2$ for some j.*

Proof. With the fact that $(-q;q)_n \equiv (q;q)_n \pmod 2$, we can rewrite (2.5) as

$$\sum_{n\geq 0}(c_e(n) - c_o(n))q^n \equiv (q;q)_\infty \sum_{n\geq 0}(-q;q)_n q^{n+1} \pmod 2.$$

It is routine to see that

$$\sum_{n\geq 0}(-q;q)_n q^{n+1} = (-q;q)_\infty - 1,$$

from which we get

$$\sum_{n\geq 0}(c_e(n)-c_o(n))q^n \equiv (q;q)_\infty((-q;q)_\infty-1)$$
$$\equiv (q^2;q^2)_\infty+(q;q)_\infty \pmod 2.$$

It follows from Euler's pentagonal number theorem [1, Corollary 1.7]

$$(q;q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}$$

that

$$(q;q)_\infty\equiv\sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} \pmod 2.$$

We now arrive at

$$\sum_{n\geq 0}(c_e(n)-c_o(n))q^n \equiv \sum_{n=-\infty}^{\infty} q^{n(3n+1)}+\sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} \pmod 2,$$

which yields the desired result. □

We now turn to (1.12), the series on the left-hand side of which denotes the generating function for bipartitions where each red part is at least two and no blue part is larger than the number of red parts. We now allow blue parts to be larger than the number of red part. Let $d_e(n)$ and $d_o(n)$ be the number of such bipartitions which have an even and odd number of blue parts larger than the number of red parts, respectively. Then

$$\sum_{n\geq 0}(d_e(n)-d_o(n))q^n=\sum_{n\geq 0}\frac{q^{2n}}{(q;q)_n^2(-q^{n+1};q)_\infty}.$$

Theorem 2.7. *For $n>1$,*

$$d_e(n)>d_o(n).$$

Proof. Clearly, we have

$$\sum_{n\geq 0}\frac{q^{2n}}{(q;q)_n^2(-q^{n+1};q)_\infty}=\frac{1}{(-q;q)_\infty}\sum_{n\geq 0}\frac{(-q;q)_n}{(q;q)_n^2}q^{2n}.$$

Setting $b=-q, c=q, t=q^2$ and letting $a\to 0$ in (1.3), we obtain

$$\sum_{n\geq 0}\frac{(-q;q)_n}{(q;q)_n^2}q^{2n}=\frac{1}{(q^2;q)_\infty}\sum_{n\geq 0}\frac{(-1;q)_n}{(q;q)_n^2}q^{n(n+5)/2}.$$

Dividing both sides of the above equation by $(-q;q)_\infty$, we find that

$$\frac{1}{(-q;q)_\infty}\sum_{n\geq 0}\frac{q^{2n}(-q;q)_n}{(q;q)_n^2}$$
$$=\frac{1}{(-q;q)_\infty(q^2;q)_\infty}\left(1+\frac{2q^3}{(1-q)^2}+\sum_{n\geq 2}\frac{q^{n(n+5)/2}(-1;q)_n}{(q;q)_n^2}\right)$$

$$= \frac{1}{(1+q)(q^4;q^2)_\infty}\left(1+\frac{2q^3}{(1-q)^2}+\sum_{n\geq 2}\frac{q^{n(n+5)/2}(-1;q)_n}{(q;q)_n^2}\right)$$

$$= \frac{1}{(q^4;q^2)_\infty}\left(\frac{1}{1+q}+\frac{2q^3}{(1-q)(1-q^2)}\right)$$

$$+\frac{1}{(q^4;q^2)_\infty}\sum_{n\geq 2}\frac{2q^{n(n+5)/2}(-q^2;q)_{n-2}}{(q;q)_n^2}. \tag{2.6}$$

It is not hard to see that

$$\frac{1}{1+q}+\frac{2q^3}{(1-q)(1-q^2)}$$

is a series with the coefficient of q being -1 and the coefficient of q^n being positive for $n \geq 2$.

Therefore, the right-hand side of (2.6) is a series with positive coefficients except for the term q. □

It appears that Lebesgue [11] obtained

$$\sum_{n\geq 0}\frac{(-zq;q)_n}{(q;q)_n}q^{n(n+1)/2} = \frac{(-zq^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.7}$$

which is a bivariate form of Ramanujan's theta function

$$\sum_{n\geq 0} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

Setting $z = 1$ in (2.7), we get

$$\sum_{n\geq 0}\frac{q^{n(n+1)/2}}{(q;q)_n}(-q;q)_n = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

The series on the left-hand side generates bipartitions where red parts are distinct, blue parts are also distinct and no blue part is larger than the number of red parts. Let $f_e(n)$ and $f_o(n)$ denote the number of bipartitions of n where both red and blue parts are distinct and there are an even and odd number of blue parts larger than the number of red parts, respectively. Then we have

$$\sum_{n\geq 0}(f_e(n)-f_o(n))q^n = \sum_{n\geq 0}\frac{q^{n(n+1)/2}}{(q;q)_n}(-q;q)_n(q^{n+1};q)_\infty.$$

In fact, the difference $f_e(n)-f_o(n)$ is the number of certain partitions. We first give the necessary notions and notation. Recently, Andrews and Newman [5] introduced the minimal excludant to study partitions, which is defined to be the smallest positive integer not appearing in a partition. Define $mexd_o(n)$ to be the number of partitions of n into distinct parts with the minimal excludant being odd. Obviously, $mexd_o(n) > 0$ for each $n \geq 2$ since the partition formed by a unique part of size n is a partition of n into distinct parts with 1 being its minimal excludant.

Theorem 2.8. *For $n > 0$,*

$$f_e(n) - f_o(n) = mexd_o(n),$$

which implies $f_e(n) > f_o(n)$ except that $n = 1$. Moreover,

$$\sum_{n\geq 0}(f_e(n) - f_o(n))q^n = (-q;q)_\infty \sum_{n\geq 0} q^{n(3n+1)/2}(1-q^{2n+1}).$$

Proof. Clearly, we have

$$\sum_{n\geq 0} \frac{q^{n(n+1)/2}(-q;q)_n(q^{n+1};q)_\infty}{(q;q)_n} = (q;q)_\infty \sum_{n\geq 0} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q)_n^2}.$$

Setting $a = -q, c = q, t = -q/b$ and letting $b \to \infty$ in (1.2), we find that

$$\sum_{n\geq 0} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q)_n^2} = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n}.$$

Therefore, we have

$$\begin{aligned}
(q;q)_\infty \sum_{n\geq 0} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q)_n^2} &= (-q;q)_\infty \sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n} \\
&= \sum_{n\geq 0} (-1)^n q^{n(n+1)/2}(-q^{n+1};q)_\infty \\
&= \sum_{n\geq 0} q^{n(2n+1)}(-q^{2n+1};q)_\infty \\
&\quad - \sum_{n\geq 0} q^{(2n+1)(n+1)}(-q^{2n+2};q)_\infty \\
&= \sum_{n\geq 0} q^{n(2n+1)}(-q^{2n+2};q)_\infty(1+q^{2n+1}-q^{2n+1}) \\
&= \sum_{n\geq 0} q^{1+2+\cdots+2n}(-q^{2n+2};q)_\infty \\
&= \sum_{n\geq 0} mexd_o(n)q^n.
\end{aligned}$$

Thus,

$$\sum_{n\geq 0}(f_e(n) - f_o(n))q^n = \sum_{n\geq 0} mexd_o(n)q^n.$$

Equating the coefficient of q^n, we find that $f_e(n) - f_o(n) = mexd_o(n)$. Invoking the false theta identity [3, Entry 9.4.2]

$$\sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n} = \sum_{n\geq 0} q^{n(3n+1)/2}(1-q^{2n+1}),$$

we conclude the second statement in the theorem immediately. □

Corollary 2.9. *The difference $f_e(n) - f_o(n)$ is almost always even, and is odd exactly when $n = j(3j \pm 1)$ for some j.*

Proof. By Euler's pentagonal number theorem [1, Corollary 1.7], we see that

$$(q;q)_\infty \equiv \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}$$
$$\equiv \sum_{n\geq 0} q^{n(3n+1)/2}(1+q^{2n+1}) \pmod 2.$$

From Theorem 2.8, we get

$$\begin{aligned}\sum_{n\geq 0}(f_e(n)-f_o(n))q^n &\equiv (-q;q)_\infty \sum_{n\geq 0} q^{n(3n+1)/2}(1+q^{2n+1})\\ &\equiv (-q;q)_\infty (q;q)_\infty\\ &\equiv (q^2;q^2)_\infty\\ &\equiv \sum_{n=-\infty}^{\infty} q^{n(3n+1)} \pmod 2,\end{aligned}$$

which finishes the proof. □

3. Another Two Bipartition Difference Functions

In this section, we discuss another two types of bipartitions and establish the positivity of the relevant difference functions.

Define $bo(n)$ to be the number of bipartitions into odd parts, where each blue part (if exists) is larger than twice the number of red parts. Let $bo_e(n)$ and $bo_o(n)$ be the number of bipartitions counted by $bo(n)$ with an even and odd number of blue parts, respectively.

Theorem 3.1. *We have*

$$\sum_{n\geq 0}(bo_e(n)-bo_o(n))q^n = \frac{(-q^2;q^2)_\infty}{(q^2;q^4)_\infty}. \tag{3.1}$$

Proof. Using standard combinatorial arguments, we have

$$\begin{aligned}\sum_{n\geq 0}(bo_e(n)-bo_o(n))q^n &= \sum_{n\geq 0}\frac{q^n}{(q^2;q^2)_n(-q^{2n+1};q^2)_\infty}\\ &= \frac{1}{(-q;q^2)_\infty}\sum_{n\geq 0}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^n\\ &= \frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty(q;q^2)_\infty}\\ &= \frac{(-q^2;q^2)_\infty}{(q^2;q^4)_\infty}.\end{aligned}$$

where we used the famous q-binomial theorem [1, Theorem 2.1] in the penultimate step. □

As an immediate consequence of Theorem 3.1, we have the following result.

Corollary 3.2. *For $n > 0$, we have $bo_e(n) = bo_o(n)$ if n is odd, and $bo_e(n) > bo_o(n)$ if n is even.*

It is unexpected to obtain the following Ramanujan-type congruence for $bo_e(n)$.

Theorem 3.3. *For $n \geq 0$,*

$$bo_e(10n+9) \equiv 0 \pmod 5.$$

Proof. It is straightforward to see that

$$\begin{aligned}\sum_{n\geq 0}(bo_e(n)+bo_o(n))q^n &= \sum_{n\geq 0} bo(n)q^n \\ &= \sum_{n\geq 0}\frac{q^n}{(q^2;q^2)_n(q^{2n+1};q^2)_\infty} \\ &= \frac{1}{(q;q^2)_\infty}\sum_{n\geq 0}\frac{(q;q^2)_n}{(q^2;q^2)_n}q^n \\ &= \frac{1}{(q;q^2)_\infty}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \\ &= \frac{(q^2;q^2)^3_\infty}{(q;q)^2_\infty}, \qquad (3.2)\end{aligned}$$

where we used the q-binomial theorem [1, Theorem 2.1] in the penultimate step.

It follows from [2, Theorem 3.3] that the coefficients of q^{5n+4} in $(q^2;q^2)^3_\infty/(q;q)^2_\infty$ are divisible by 5. Thus,

$$bo_e(5n+4) + bo_o(5n+4) \equiv 0 \pmod 5.$$

Furthermore, combining with Corollary 3.2, we see that $bo_e(10n+9) \equiv 0 \pmod 5$. □

Remark 3.4. In [7], Chern obtained an infinite family of congruences modulo powers of 5 for the coefficients of q^n in $(q^2;q^2)^3_\infty/(q;q)^2_\infty$. Thus, there exists an infinite family of congruences modulo powers of 5 for $bo_e(n)$. In addition, some congruences modulo 4 for the coefficients of q^n in $(q^4;q^4)^3_\infty/(q^2;q^2)^2_\infty$ are presented in [6]. Hence, there will exist similar congruences modulo 2 for $bo_e(n)$.

Remark 3.5. In [2], Andrews studied a kind of restricted partitions where each even part is smaller than each odd part and only the largest even part appears an odd number of times. The set of such partitions of n is denoted $\overline{\mathcal{EO}}(n)$. Chern [8] undertook an extensive analysis of $\overline{\mathcal{EO}}(n)$. Let $\overline{eo}_0(n)$ and $\overline{eo}_2(n)$ be the number of partitions in $\overline{\mathcal{EO}}(n)$ with the largest even part congruent to 0 and 2 modulo 4, respectively. Chern [8] proved that

$$\sum_{n\geq 0}(\overline{eo}_0(n) - \overline{eo}_2(n))q^n = \frac{(-q^4;q^4)_\infty}{(q^4;q^8)_\infty}.$$

Combining (3.2) and the above identity, we get

$$bo_e(n) - bo_o(n) = \overline{eo}_0(2n) - \overline{eo}_2(2n).$$

In addition, combining [2, Corollary 3.2] and (3.2) yields that

$$bo_e(n) + bo_o(n) = \overline{eo}_0(2n) + \overline{eo}_2(2n).$$

Therefore, we have $bo_e(n) = \overline{eo}_0(2n)$ and $bo_o(n) = \overline{eo}_2(2n)$.

Define $\mathcal{PDO}(n)$ to be the set of bipartitions of n where all blue parts are distinct and odd. Let $pdo_e(n)$ and $pdo_o(n)$ be the number of bipartitions in $\mathcal{PDO}(n)$ which have an even and odd number of blue parts larger than twice the number of red parts, respectively.

Theorem 3.6. *For $n \geq 2$,*

$$pdo_e(n) > pdo_o(n).$$

Proof. The standard combinatorial arguments reveal directly that

$$\begin{aligned}\sum_{n\geq 0}(pdo_e(n) - pdo_o(n))q^n &= \sum_{n\geq 0}\frac{q^n}{(q;q)_n}(-q;q^2)_n(q^{2n+1};q^2)_\infty \\ &= (q;q^2)_\infty \sum_{n\geq 0}\frac{q^n(-q;q^2)_n}{(q;q)_n(q;q^2)_n}.\end{aligned}$$

Setting $a=-1, b=q, c=0, m=2, t=-q$ in (1.1) first and then multiplying both sides by $(-q;q^2)_\infty/(q;q)_\infty$, we derive that

$$\begin{aligned}&(q;q^2)_\infty \sum_{k\geq 0}\frac{q^k(-q;q^2)_k}{(q;q)_k(q;q^2)_k} \\ &= \frac{(-q;q^2)_\infty}{(q;q)_\infty}\sum_{k\geq 0}\frac{(-1)^k q^k(-1;q^2)_k(q;q)_{2k}}{(q^2;q^2)_k} \\ &= (-q;q^2)_\infty \sum_{k\geq 0}\frac{(-1)^k q^k(-1;q^2)_k}{(q^2;q^2)_k(q^{2k+1};q)_\infty} \\ &= (-q;q^2)_\infty \sum_{k\geq 0}\left(\frac{q^{2k}(-1;q^2)_{2k}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty} - \frac{q^{2k+1}(-1;q^2)_{2k+1}}{(q^2;q^2)_{2k+1}(q^{4k+3};q)_\infty}\right) \\ &= (-q;q^2)_\infty \sum_{k\geq 0}\frac{q^{2k}(-1;q^2)_{2k}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}\left(1 - q(1+q^{4k})(1-q^{4k+1})\right) \\ &= (-q;q^2)_\infty \sum_{k\geq 0}\frac{q^{2k}(-1;q^2)_{2k}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}\left((1-q)(1-q^{4k+1}) + q^{8k+2}\right) \\ &= (-q;q^2)_\infty \sum_{k\geq 0}\left(\frac{q^{2k}(-1;q^2)_{2k}(1-q)}{(q^2;q^2)_{2k}(q^{4k+2};q)_\infty} + \frac{q^{10k+2}(-1;q^2)_{2k}}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}\right) \\ &= (-q^3;q^2)_\infty \sum_{k\geq 0}\left(\frac{q^{2k}(-1;q^2)_{2k}(1-q^2)}{(q^2;q^2)_{2k}(q^{4k+2};q)_\infty} + \frac{q^{10k+2}(-1;q^2)_{2k}(1+q)}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty}\right)\end{aligned}$$

$$
\begin{aligned}
&= (-q^3;q^2)_\infty \Bigg(\frac{1-q^2}{(q^2;q)_\infty} + \sum_{k\geq 1} \frac{q^{2k}(-1;q^2)_{2k}}{(q^4;q^2)_{2k-1}(q^{4k+2};q)_\infty} \\
&\quad + \sum_{k\geq 0} \frac{q^{10k+2}(-1;q^2)_{2k}(1+q)}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty} \Bigg) \\
&= (-q^3;q^2)_\infty \Bigg(\frac{1}{(q^3;q)_\infty} + \sum_{k\geq 1} \frac{q^{2k}(-1;q^2)_{2k}}{(q^4;q^2)_{2k-1}(q^{4k+2};q)_\infty} \\
&\quad + \sum_{k\geq 0} \frac{q^{10k+2}(-1;q^2)_{2k}(1+q)}{(q^2;q^2)_{2k}(q^{4k+1};q)_\infty} \Bigg). \qquad (3.3)
\end{aligned}
$$

It is easy to see that each summand on the right-hand side of (3.3) has nonnegative coefficients, which implies that $pdo_e(n) \geq pdo_o(n)$ for all n. Clearly, the coefficient of q^2 is 3. Note that the series $1/(q^3;q)_\infty$ ensures that the coefficient of q^n for $n \geq 3$ is positive. □

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Declarations

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