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Turán Problems for Oriented Graphs

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Abstract. A classical Turán problem asks for the maximum possible number of edges in a graph of a given order that does not contain a particular graph H as a subgraph. It is well-known that the chromatic number of H is the graph parameter which describes the asymptotic behavior of this maximum. Here, we consider an analogous problem for oriented graphs, where compressibility plays the role of the chromatic number. Since any oriented graph having a directed cycle is not contained in any transitive tournament, it makes sense to consider only acyclic oriented graphs as forbidden subgraphs. We provide basic properties of the compressibility, show that the compressibility of acyclic oriented graphs with out-degree at most 2 is polynomial with respect to the maximum length of a directed path, and that the same holds for a larger out-degree bound if the Erdős–Hajnal conjecture is true. Additionally, generalizing previous results on powers of paths and arbitrary orientations of cycles, we determine the compressibility of acyclic oriented graphs with restricted distances of vertices to sinks and sources.

1. Introduction

For a graph H, we denote by $\mathrm{ex}(n, H)$ the maximum possible number of edges in a graph on n vertices which does not contain H as a subgraph. The problem of determining the value of $\mathrm{ex}(n, H)$ for different graphs H is one of the most fundamental questions in Extremal Graph Theory. Erdős and Stone [12] proved that

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2),$$

which gives a tight asymptotic bound for $\mathrm{ex}(n, H)$ in terms of the chromatic number $\chi(H)$ of H whenever $\chi(H) > 2$. Whereas for $\chi(H) = 2$, i.e., when H

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is bipartite, no general asymptotic formula for $\mathrm{ex}(n, H)$ is known, and partial results include bounds for complete bipartite graphs [18], even cycles [7] or bipartite graphs with bounded degeneracy [1,16]. For more details, see the survey [15]. The notion of $\mathrm{ex}(n, H)$ naturally generalizes to the setting when a family $\mathcal{H}$ of graphs is forbidden. If we define $\chi(\mathcal{H})$ as the minimum of $\chi(H)$ for $H \in \mathcal{H}$, then the Erdős–Stone Theorem still holds and gives tight asymptotic bounds for $\chi(\mathcal{H}) > 2$.

The notion of $\mathrm{ex}(n, H)$ can be defined similarly for directed graphs and oriented graphs. Recall that by a *directed graph* we mean a pair $H = (V(H), E(H))$, where $V(H)$ is a set of vertices and $E(H)$ is a set of ordered pairs of different vertices called *arcs*. Whereas by an *oriented graph* we understand a directed graph in which any two vertices are joined by at most one arc, in other words, an orientation of a graph. To avoid ambiguity, we use notation ex_d in the setting of directed graphs and ex_o in the setting of oriented graphs.

Research on this problem in the directed setting can be traced back to the works of Brown, Erdős, Harary, Häggkvist, Simonovits, and Thomassen [4–6,17]. In particular, Brown, Erdős, and Simonovits [5] proved that for every family of directed graphs $\mathcal{H}$ there exists a sequence $(G_n)_{n\geq 1}$ of n-vertex graphs not containing any $H \in \mathcal{H}$ as a subgraph, such that each G_n is a blow-up of some fixed directed graph D and $\mathrm{ex}_\mathrm{d}(n, \mathcal{H}) = |E(G_n)| + o(n^2)$. Here, by a *blow-up* of an oriented graph D we mean a graph created by replacing each vertex v of D by some independent set I_v and each arc uv of D by all possible arcs from I_u to I_v. Even though the theorem does not give much information about the graph D itself, Valadkhan [23] observed that in the case of oriented graphs one may assume that D is a tournament. It is clear that D is the largest tournament whose blow-ups do not contain any $H \in \mathcal{H}$. In other words, D is the largest tournament to which there is no homomorphism from any $H \in \mathcal{H}$, where a homomorphism of oriented graphs is defined as a mapping between their sets of vertices that preserves arcs. This leads to the following crucial definition and theorem.

Definition 1. The *compressibility* of a family of oriented graphs $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the smallest $k \in \mathbb{N}$ such that for every tournament T on k vertices there exists a homomorphism from some $H \in \mathcal{H}$ to T. If no such k exists, we put $\tau(\mathcal{H}) = \infty$. For brevity, we define compressibility of an oriented graph H as $\tau(H) := \tau(\{H\})$.

Theorem 2. (Valadkhan [23]) *For any family $\mathcal{H}$ of oriented graphs,*

$$\mathrm{ex}_\mathrm{o}(n, \mathcal{H}) = \left(1 - \frac{1}{\tau(\mathcal{H}) - 1}\right)\binom{n}{2} + o(n^2).$$

Therefore, the compressibility plays the same role in the context of oriented graphs as the chromatic number in the context of graphs and the Erdős–Stone Theorem. In particular, determining the compressibility of a graph or a family of graphs is asymptotically solving the respective problem on the maximum number of arcs in oriented graphs of a given order not containing this graph or a family of graphs.

Here, we focus on properties of $\tau(\mathcal{H})$ when $\mathcal{H}$ has a single member. (In contrast to the chromatic number, in general $\tau(\mathcal{H})$ may differ from $\min\{\tau(H) : H \in \mathcal{H}\}$, see Example 5.) If an oriented graph contains a directed cycle, and therefore it cannot be mapped homomorphically to any transitive tournament, then its compressibility is infinite. Therefore, we shall consider only forbidding acyclic oriented graphs. It is also easy to notice that a transitive tournament on k vertices does not contain a homomorphic image of any acyclic oriented graph with a directed path of order (i.e., the number of vertices) greater than k, hence $\tau(H)$ is always at least the maximum order $p(H)$ of a directed path in H. In fact, $\tau(H)$ can grow exponentially in terms of $p(H)$ as witnessed by transitive tournaments (Example 4) or particular orientations of complete bipartite graphs (Proposition 7). Therefore, it is natural to ask, as in [23], for which families of acyclic oriented graphs the growth is polynomial, or for which the trivial lower bound is optimal, i.e., $\tau(H) = p(H)$.

In Sect. 3, we show that the compressibility of acyclic oriented graphs with out-degree at most 2 is polynomial with respect to the maximum order of a directed path (Theorem 12), and that the same holds for a larger out-degree bound under the additional assumption that the Erdős–Hajnal conjecture holds (Theorem 9). Additionally, generalizing results for the square of a path, we determine the compressibility of acyclic oriented graphs with out-degree at most 2 having restricted structure (Theorem 17). Finally, in Sect. 4, generalizing the result by Valadkhan [23] for acyclic orientations of cycles, we prove that the equality $\tau(H) = p(H)$ holds for oriented graphs H with restricted distances of all vertices to sinks and sources (Theorem 26).

2. Notation and Basic Properties of Compressibility

First, we shall introduce the notation used throughout the paper. Let $\overrightarrow{T_k}$ denote the transitive tournament on k vertices. Let $\overrightarrow{P_k}$ be the *directed path* on k vertices, i.e., an orientation of a path with all arcs directed toward the same end-point of the path. Similarly, let $\overrightarrow{C_k}$ be the *directed cycle* on k vertices, i.e., a cyclic orientation of a cycle of length k. Finally, let $\overrightarrow{K_{s,t}}$ denote the orientation of a complete bipartite graph $K_{s,t}$ with all arcs directed toward the part of size t. If G is an oriented graph and $v \in V(G)$, then we use the standard notation $d^+(v)$ and $d^-(v)$ for the out-degree and in-degree of a vertex v in G, respectively, and write $N^+(v)$ for the out-neighborhood of a vertex v in G.

Let G and H be oriented graphs. By $G \odot H$ we mean the *composition* of oriented graphs, i.e., an oriented graph created by replacing each vertex of G by a copy of H and each arc of G by $\overrightarrow{K_{|H|,|H|}}$ directed according to the direction of the arc of G. Also, define $G \Rightarrow H$ as the disjoint sum of G and H with all possible arcs from vertices of G to vertices of H. In particular, if G and H are independent sets of size s and t respectively, then $G \Rightarrow H$ is isomorphic to $\overrightarrow{K_{s,t}}$. We say that G is *H-free* if G does not contain a subgraph (not necessarily induced) isomorphic to H. If $\mathcal{H}$ is a family of graphs, we say that G is $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. If H is a subgraph of G isomorphic to H', we refer to H as a *copy* of H' in G. We write $H \to G$ if there exists a homomorphism from H to G, which is equivalent to saying that H is a subgraph of some blow-up of G.

The compressibility of some particular graphs can be easily derived, for instance for directed paths.

Example 3. For any $k \geq 1$, $\tau(\overrightarrow{P_k}) = k$, as by a classical theorem of Rédei [20] every tournament on k vertices contains a copy of $\overrightarrow{P_k}$, i.e., a Hamiltonian path, while there is no homomorphism $\overrightarrow{P_k} \to \overrightarrow{T_{k-1}}$.

If in the definition of compressibility we ask for the existence of an *injective* homomorphism from H to every tournament of a given order, then we obtain the definition of a *1-color oriented Ramsey number*. See [19] for more information on this concept. As some graphs have no homomorphism into smaller oriented graphs, bounds on their compressibility follow from known bounds on their 1-color oriented Ramsey number.

Example 4. Since the compressibility of a transitive tournament $\overrightarrow{T_k}$ is equal to the 1-color oriented Ramsey number of $\overrightarrow{T_k}$, standard arguments [11,22] imply that

$$c_1 2^{k/2} \leq \tau(\overrightarrow{T_k}) \leq c_2 2^k$$

for some constants $c_1, c_2 > 0$ and any $k \geq 1$. These are essentially the best known general bounds.

It is easy to notice that the compressibility of a family of graphs $\mathcal{H}$ is always not larger than $\min\{\tau(H) : H \in \mathcal{H}\}$. However, in general, it can differ from $\min\{\tau(H) : H \in \mathcal{H}\}$ significantly.

Example 5. If $\mathcal{H} = \{\overrightarrow{P_{2^k}}, \overrightarrow{T_k}\}$ for any $k \geq 1$, then $\tau(\overrightarrow{P_{2^k}}) = 2^k$ and $\tau(\overrightarrow{T_k}) \geq c2^{k/2}$ for some constant $c > 0$, but $\tau(\mathcal{H}) = k$, since each tournament T on k vertices either contains $\overrightarrow{C_3}$, and therefore there exists a homomorphism $\overrightarrow{P_{2^k}} \to T$, or is transitive.

Let $p(H)$ be the order of a longest directed path in an acyclic oriented graph H. It is easy to observe that $p(H)$ can be equivalently defined as the smallest k for which there exists a homomorphism $H \to \overrightarrow{T_k}$. In particular, Example 4 implies that the compressibility $\tau(H)$ is bounded exponentially in terms of $p(H)$. This motivates the following definition.

Definition 6. Let $\mathcal{G}$ be a family of acyclic oriented graphs. We say that $\mathcal{G}$ is *polynomially* τ*-bounded* if there exist constants c, d such that for every $H \in \mathcal{G}$, we have

$$\tau(H) \leq cp(H)^d.$$

Valadkhan [23] observed that containing a large transitive tournament is not a necessary condition to have $\tau(H)$ exponentially large in terms of $p(H)$. Even forbidding $\overrightarrow{T_3}$ is not enough to guarantee polynomial τ-boundedness.

Proposition 7. (Valadkhan [23]) *For $n \geq 1$, let H_n be the only acyclic orientation of $K_{n,n}$ such that $p(H_n) = 2n$. Then, $\tau(H_n) \geq 2^{n/2}$.*

Note that if $\tau(H) = 2$, i.e., H is a subgraph of $\overrightarrow{K_{s,t}}$ for some $s, t \in \mathbb{N}$, Theorem 2 implies only that $\mathrm{ex}_o(n, H) = o(n^2)$ and one may ask for the order of magnitude of $\mathrm{ex}_o(n, H)$. In some cases, $\mathrm{ex}_o(n, H)$ can be upper bounded by $c \cdot \mathrm{ex}(n, H')$ for some constant $c > 0$, where H' is the graph obtained from H by removing all orientations of arcs, hence the known bounds for $\mathrm{ex}(n, H')$ translate to the bounds for $\mathrm{ex}_o(n, H)$. In particular, the Kővári-Sós-Turán Theorem [18] gives an upper bound on $\mathrm{ex}_o(\overrightarrow{K_{s,t}})$ for any $s, t \geq 1$, while Bondy–Simonovits Theorem [7] gives a bound for even cycles with edges oriented in alternating directions.

3. Oriented Graphs with Bounded Out-Degree

For any integer $k \in \mathbb{N}$, let $\mathcal{D}_k$ be the family of all acyclic oriented graphs with out-degree bounded by k. In this section, we consider the question whether $\mathcal{D}_k$ is polynomially τ-bounded.

Following the technique used by Fox, He, and Wigderson in the proof of Theorem 1.4 in [14] one can show that there exists a constant c such that for every $H \in \mathcal{D}_k$ it holds

$$\tau(H) \leq (kp(H))^{ck \log p(H)}.$$

This means that for an acyclic oriented graph H with bounded out-degree the compressibility $\tau(H)$ is quasi-polynomially bounded in terms of $p(H)$. We prove that this can be improved to a polynomial bound if the following conjecture is true.

Conjecture 8. *For every tournament T there exists a constant $\varepsilon > 0$ such that every tournament on n vertices contains as a subgraph either T or a transitive tournament on at least n^{ε} vertices.*

Alon et al. [2] proved that Conjecture 8 is equivalent to the well-known Erdős–Hajnal Conjecture [10].

Theorem 9. *Conjecture 8 implies that $\mathcal{D}_k$ is polynomially τ-bounded for every $k \in \mathbb{N}$.*

Before we prove this theorem, let us introduce the following notion. For an oriented graph H, we say that a subset $X \subseteq V(H)$ is *dominated* in H if $X \subseteq N^+(v)$ for some $v \in V(H)$. We have the following easy observation.

Observation 10. *For any $k \geq 2$ and any tournament T, if all k-subsets of $V(T)$ are dominated in T, then for any $H \in \mathcal{D}_k$ there exists a homomorphism $H \to T$.*

Proof. Since H is acyclic, there is an order of the vertices of H in which all the arcs are directed backwards. We embed in T the vertices of H in this order using the fact that each vertex in H has out-degree at most k and each set of k vertices in T is dominated by some vertex of T. □

Proof of Theorem 9. It is enough to prove that for every $k \in \mathbb{N}$ there exists a tournament T such that for each $H \in \mathcal{D}_k$ there exists a homomorphism $H \to T$. If such T exists, then Conjecture 8 implies that for every $H \in \mathcal{D}_k$, each tournament on $p(H)^{1/\varepsilon}$ vertices contains a copy of either T or $\overrightarrow{T}_{p(H)}$. In both cases, it contains a homomorphic image of H. Thus, $\tau(H) \leq p(H)^{1/\varepsilon}$.

Existence of such a tournament T follows from a probabilistic argument due to Erdős [9] or explicit constructions using Paley graphs (see [3]), but we add it for completeness. Let $n \in \mathbb{N}$ be large enough and T be a random tournament on n vertices. For a k-vertex subset $X \subseteq V(T)$, let A_X be the event that X is not dominated in T. Then, $A = \bigcup_{X \in \binom{V(T)}{k}} A_X$ is the event that some k-vertex subset of $V(T)$ is not dominated by any vertex. The probability of A can be bounded as follows:

$$\mathbb{P}(A) \leq \sum_{X \in \binom{V(T)}{k}} \mathbb{P}(A_X) = \binom{n}{k} \left(1 - \left(\frac{1}{2}\right)^k\right)^{n-k} \xrightarrow{n \to \infty} 0.$$

Therefore, for large enough n, the probability of the complement of A is positive, i.e., there exists a tournament T in which every set of k vertices is dominated by some other vertex. By Observation 10, there exists a homomorphism $H \to T$ for any $H \in \mathcal{D}_k$. □

Remark 11. One may prove Theorem 9 in the same way for a slightly larger class of acyclic oriented graphs. We say that an oriented graph G is k-*degenerate* if its underlying graph G' is k-degenerate, i.e., every subgraph H of G' contains a vertex of degree at most k in H. Let $\mathcal{D}'_k$ be the family of all k-degenerate acyclic oriented graphs. An analog of observation 10 for $\mathcal{D}'_k$ holds if one assumes that the tournament T has the following property: for every sequence of signs $\varepsilon_i \in \{+, -\}$ and vertices $v_i \in V(T)$, $1 \leq i \leq k$, there exists a vertex $v \in V(T)$ such that $v \in N^{\varepsilon_i}(v_i)$ for every $1 \leq i \leq k$. In particular, Conjecture 8 implies that $\mathcal{D}'_k$ is polynomially τ-bounded for every $k \in \mathbb{N}$.

In the case $k = 2$, one can notice that the composition $\overrightarrow{C_3} \odot \overrightarrow{C_3}$, i.e., the graph consisting of three sets A_1, A_2, A_3 of vertices forming directed triangles with all edges from A_1 to A_2, from A_2 to A_3, and from A_3 to A_1, satisfies the assumption of Observation 10. It is so, because any two vertices in one set are dominated by any vertex in the preceding set, while vertices from different sets, say from A_1 and A_2, are dominated by the in-neighbor in A_1 of the first vertex. As [2, Theorem 2.1] implies that the tournament $\overrightarrow{C_3} \odot \overrightarrow{C_3}$ satisfies Conjecture 8 with the constant $\varepsilon = 1/148$, we have

$$\tau(H) \leq cp(H)^{148}$$

for any $H \in \mathcal{D}_2$. We prove a much better bound.

Theorem 12. *There exists a constant c such that for every $H \in \mathcal{D}_2$ we have*

$$\tau(H) \leq cp(H)^4.$$

Before we prove this result, recall the notion of a domination graph. The *domination graph* of a tournament T is defined as the spanning subgraph $\mathrm{dom}(T)$ of T consisting of those arcs from $E(T)$ that are not dominated in T. This may seem counter-intuitive that $\mathrm{dom}(T)$ consists of non-dominated arcs, but one can also view it from the other perspective, that an arc is in $\mathrm{dom}(T)$ if every non-incident vertex of the graph is dominated by at least one of the endpoints of the arc. One of the most basic properties of the domination graph is the following easy observation, the proof of which is included for completeness.

Observation 13. *If T is a tournament and $vw, v'w'$ are two vertex disjoint arcs from $E(\mathrm{dom}(T))$, then any arc between the sets $\{v, w\}$ and $\{v', w'\}$ completely determines the orientation of all the remaining arcs between those four vertices — either vv', $v'w$, ww', $w'v \in E(T)$, or vw', $w'w$, wv', $v'v \in E(T)$.*

Proof. If the arcs of the tournament T between the sets $\{v, w\}$ and $\{v', w'\}$ are not forming a directed cycle, then there exists a vertex such that either v and w or v' and w' are its out-neighbors, which contradicts the fact that arcs vw and $v'w'$ are not dominated. Thus, the arcs between the sets $\{v, w\}$ and $\{v', w'\}$ are forming a directed cycle. Depending on its direction, we obtain one of the two possibilities listed in the statement of the observation. □

We are ready now to prove Theorem 12.

Proof of Theorem 12. We use induction on $p(H)$. If $p(H) = 2$, then $\tau(H) = 2$. If $p(H) = 3$, then $H \to \overrightarrow{T_3}$, and since any tournament on 4 vertices contains $\overrightarrow{T_3}$, we have $\tau(H) \leq 4$. Thus, for $p(H) \leq 3$ the inequality $\tau(H) \leq cp(H)^4$ holds for any $c \geq 1$.

Let T be any tournament on $cp(H)^4$ vertices for some constant $c > 0$ and $p(H) \geq 4$. As there is a homomorphism from H to $\overrightarrow{T}_{p(H)}$, we may assume that T does not contain a transitive tournament on $p(H)$ vertices.

Assume first that $\mathrm{dom}(T)$ does not contain a matching on $2cp(H)^3$ vertices. By removing from T the vertices of any maximum matching in $\mathrm{dom}(T)$, we obtain a tournament T' on at least $cp(H)^4 - 2cp(H)^3$ vertices, which is greater than $c(p(H)-1)^4$ for $p(H) \geq 4$. Let H' be the subgraph of H obtained by removing all sources in H, i.e., vertices that do not have any in-neighbors. Then $p(H') = p(H) - 1$ as any maximal path in H' can be extended by one vertex to some maximal path in H, so we can apply the induction hypothesis to find a homomorphism $H' \to T'$. Since every pair of vertices from $V(T')$ is dominated in T and the maximum out-degree of H is at most two, we can extend this homomorphism to $H \to T$. Therefore, we may assume that there exists a subgraph M of $\mathrm{dom}(T)$ which is a matching on at least $2cp(H)^3$ vertices.

From Observation 13, it follows that for every arc $vw \in E(M)$ and every other vertex $u \in V(M)$, either $vu, uw \in E(T)$ or $wu, uv \in E(T)$. Let $\mathcal{T}_M$ denote the set of all subtournaments of T on $|E(M)|$ vertices which contain exactly one vertex from every arc of M. Then, $\mathcal{T}_M$ is closed under the operation of *flipping* a vertex, i.e., reversing the orientations of all arcs incident to this vertex. Indeed, this operation corresponds to replacing a vertex by its neighbor in M.

We want to prove that there exists a subgraph of some tournament in $\mathcal{T}_M$ isomorphic to $\overrightarrow{C_3} \odot \overrightarrow{C_3}$, because, as mentioned earlier, $\overrightarrow{C_3} \odot \overrightarrow{C_3}$ satisfies the assumption of Observation 10, so $H \to \overrightarrow{C_3} \odot \overrightarrow{C_3}$, which implies that $H \to T$. Note that $\overrightarrow{C_3} \odot \overrightarrow{C_3}$ consists of three clusters, each being a copy of $\overrightarrow{C_3}$. If we flip all vertices from one cluster, then this cluster will remain a copy of $\overrightarrow{C_3}$, but arcs between this cluster and remaining ones will reverse, resulting in a subgraph isomorphic to $\overrightarrow{T_3} \odot \overrightarrow{C_3}$. Therefore, it is enough to prove that every tournament on cn^3 vertices contains a copy of $\overrightarrow{T_3} \odot \overrightarrow{C_3}$ or $\overrightarrow{T_n}$. As $\overrightarrow{T_3} \odot \overrightarrow{C_3}$ is isomorphic to $(\overrightarrow{C_3} \Rightarrow \overrightarrow{C_3}) \Rightarrow \overrightarrow{C_3}$, we force its appearance in two steps using the following claim. □

Claim 14. *For any oriented graph D, real $c_0 > 0$, and integer $s \geq 1$, there exists $c > 0$ such that the following holds. If every $\overrightarrow{T_n}$-free tournament on $c_0 n^s$ vertices contains a copy of D, then every $\overrightarrow{T_n}$-free tournament on cn^{s+1} vertices contains a copy of $D \Rightarrow \overrightarrow{C_3}$.*

Proof. Let T' be any $\overrightarrow{T_n}$-free tournament on $m = an^{s+1}$ vertices for $a \geq \max(3\sqrt[4]{8c_0}, 6)$. Assume firstly that T' contains at most n^{3s+2} copies of $\overrightarrow{C_3}$. Since we need to find a copy of $D \Rightarrow \overrightarrow{C_3}$, we want to find a lower bound for the number t' of copies of $\overrightarrow{T_1} \Rightarrow \overrightarrow{C_3}$ in T'. As every tournament on m vertices contains at least $m/3$ vertices of out-degree at least $m/3$ (otherwise between the $2m/3$ vertices of out-degree less than $m/3$ we cannot have all the edges), we may choose the source of $\overrightarrow{T_1} \Rightarrow \overrightarrow{C_3}$ among those $m/3$ vertices. Now, since every $\overrightarrow{T_n}$-free tournament on $2n$ vertices contains at least n copies of $\overrightarrow{C_3}$ (otherwise after removing one vertex from each copy we are left with a transitive tournament on at least n vertices), we can count the number of subsets of size $2n$ in the out-neighborhood restricted to $\lceil m/3 \rceil$ vertices and obtain

$$t' \geq \frac{m}{3} \cdot \frac{n\binom{\lceil m/3 \rceil}{2n}}{\binom{\lceil m/3 \rceil - 3}{2n-3}} = \frac{mn}{3} \cdot \frac{\binom{\lceil m/3 \rceil}{3}}{\binom{2n}{3}} \geq \frac{mn}{3} \cdot \frac{m^3}{(2n)^3 \cdot 3^3} = \frac{m^4}{2^3 \cdot 3^4 n^2} = \frac{a^4 n^{4s+2}}{2^3 \cdot 3^4},$$

as every copy of $\overrightarrow{T_1} \Rightarrow \overrightarrow{C_3}$ will be counted this way at most $\binom{\lceil m/3 \rceil - 3}{2n-3}$ times. Since there are at most n^{3s+2} copies of $\overrightarrow{C_3}$ in T', there exists a copy of $\overrightarrow{C_3}$ which is dominated by at least

$$\frac{t'}{n^{3\,s+2}} \geq \frac{a^4}{2^3 \cdot 3^4} n^s \geq c_0 n^s$$

vertices of T'. Since any subtournament of T' of order at least $c_0 n^k$ contains a copy of D, we conclude that the tournament T' contains the desired copy of $D \Rightarrow \overrightarrow{C_3}$.

In order to prove the claim, consider any $\overrightarrow{T_n}$-free tournament T on cn^{s+1} vertices for some $c \geq \max(3^4 a^3 c_0, 3a)$. From the previous paragraph, we may assume that every subtournament on an^{s+1} vertices contains at least n^{3s+2} copies of $\overrightarrow{C_3}$. By the same counting argument, we get that the number t of copies of $\overrightarrow{T_1} \Rightarrow \overrightarrow{C_3}$ in T satisfies

$$t \geq \frac{cn^{s+1}}{3} \cdot \frac{n^{3s+2}\binom{\lceil cn^{s+1}/3\rceil}{an^{s+1}}}{\binom{\lceil cn^{s+1}/3\rceil-3}{an^{s+1}-3}} = \frac{cn^{4s+3}}{3} \cdot \frac{\binom{\lceil cn^{s+1}/3\rceil}{3}}{\binom{an^{s+1}}{3}}$$
$$\geq \frac{cn^{4s+3}}{3} \cdot \frac{(cn^{s+1})^3}{(an^{s+1})^3 \cdot 3^3} = \frac{c^4 n^{4s+3}}{a^3 \cdot 3^4}.$$

Since there are at most $|V(T)|^3 = c^3 n^{3s+3}$ copies of $\overrightarrow{C_3}$ in T, there exists a copy of $\overrightarrow{C_3}$ that is dominated by at least

$$\frac{t}{c^3 n^{3\,s+3}} \geq \frac{c}{a^3 \cdot 3^4} \cdot n^s \geq c_0 n^s$$

vertices of T. Thus, T contains the desired copy of $D \Rightarrow \overrightarrow{C_3}$. □

Applying the above claim for $n = p(H)$, $D = \overrightarrow{C_3}$, $s = 1$, and $c_0 > 1$, and afterward for $D = (\overrightarrow{C_3} \Rightarrow \overrightarrow{C_3})$ and $s = 2$ we conclude that any tournament in $\mathcal{T}_M$ on $cp(H)^3$ vertices contains a copy of $\overrightarrow{T_3} \odot \overrightarrow{C_3}$ or $\overrightarrow{T}_{p(H)}$, which ends the proof of Theorem 12.

For certain subclasses of $\mathcal{D}_k$, it is possible to find homomorphisms into tournaments of even linear order. For instance, for powers of paths. A *k-th power of a path* is an oriented graph obtained from a directed path by adding arcs between vertices at distance at most k.

Theorem 15. (Draganić et al. [8]) *For every $n, k \geq 2$, every tournament on n vertices contains a k-th power of a directed path of order $n/(2^{4k+6}k) + 1$. Moreover, for $k = 2$, every tournament on n vertices contains a square of a directed path of order $\lceil 2n/3 \rceil$ and this value is optimal.*

A generalization of a square of a directed path, considered in Theorem 15, is an oriented graph obtained from a directed path by adding arcs between vertices at some particular distance, but not necessarily 2.

Definition 16. For any $2 \leq \ell < k$, let $\overrightarrow{P_k}(\ell)$ be the oriented graph on k vertices $v_1, \ldots, v_k$ with arcs $v_i v_{i+1}$ for $1 \leq i \leq k-1$ and $v_i v_{i+\ell}$ for $1 \leq i \leq k-\ell$. In other words, $\overrightarrow{P_k}(\ell)$ is a directed path on k vertices with additional arcs between vertices at distance ℓ. Let also $\overrightarrow{C_k}(\ell)$ be the oriented graph on k vertices $w_0, w_1, \ldots, w_{k-1}$ with arcs $w_i w_{i+1 \pmod k}$ and $w_i w_{i+\ell \pmod k}$ for $0 \leq i < k$.

As $\overrightarrow{P_k}(\ell)$ is a subgraph of the ℓ-th power of $\overrightarrow{P_k}$, Theorem 15 implies that $\tau(\overrightarrow{P_k}(\ell))$ is linear in terms of $p(\overrightarrow{P_k}(\ell)) = k$. But the constant provided in Theorem 15 for large ℓ is very far from being optimal. The following theorem closes this gap and shows that for $\ell = 2$ and 3, the compressibility of $\overrightarrow{P_k}(\ell)$ differs from the compressibility of $\overrightarrow{P_k}$.

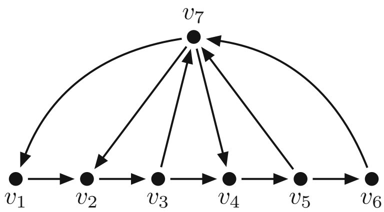

FIGURE 1. Tournament $\widetilde{T}$ from the proof of Theorem 17 for $\ell = 3$. The bottom vertices induce a transitive tournament

Theorem 17. *For $2 \leq \ell < k$, let $\overrightarrow{P_k}(\ell)$ be the oriented graph defined in Definition 16. Then,*

- $\tau(\overrightarrow{P_k}(2)) = \left\lfloor \frac{3k-1}{2} \right\rfloor$,
- $\left\lfloor \frac{7k-1}{6} \right\rfloor \leq \tau(\overrightarrow{P_k}(3)) \leq 3k$,
- $\tau(\overrightarrow{P_k}(\ell)) = k$ *if* $\ell \geq 4$.

Proof. For $\ell = 2$, the graph $\overrightarrow{P_k}(2)$ is just a square of a path, and Theorem 15 implies that every tournament on $\left\lfloor \frac{3k-1}{2} \right\rfloor$ vertices contains a copy of $\overrightarrow{P_k}(2)$. On the other hand, there are tournaments on $\left\lfloor \frac{3k-1}{2} \right\rfloor - 1$ vertices that do not have a homomorphism from $\overrightarrow{P_k}(2)$. For odd k, we consider the tournament $\overrightarrow{T}_{(k-1)/2} \odot \overrightarrow{C_3}$, while for even k consider the tournament $\overrightarrow{T_1} \Rightarrow (\overrightarrow{T}_{k/2-1} \odot \overrightarrow{C_3})$. The considered tournaments have exactly $\left\lfloor \frac{3k-1}{2} \right\rfloor - 1$ vertices and any homomorphism of $\overrightarrow{P_k}$ into them maps some three consecutive vertices into a copy of $\overrightarrow{C_3}$, which cannot happen for the homomorphism of $\overrightarrow{P_k}(2)$.

If $\ell \geq 4$, then $\tau(\overrightarrow{P_k}(\ell)) \geq k$ as there exists no homomorphism $\overrightarrow{P_k}(\ell) \to \overrightarrow{T}_{k-1}$. To prove the upper bound, consider any tournament T on k vertices. Then, T admits a decomposition $T_1 \Rightarrow \cdots \Rightarrow T_m$ into strongly connected components. If any of those components is of size at least $\ell - 1$, then it contains a copy of $\overrightarrow{C}_{\ell-1}$ (because every strongly connected tournament on t vertices contains a copy of $\overrightarrow{C_s}$ for any $3 \leq s \leq t$), and since there is a homomorphism $\overrightarrow{P_k}(\ell) \to \overrightarrow{C}_{\ell-1}$, we have $\overrightarrow{P_k}(\ell) \to T$. Otherwise, all strongly connected components are of size strictly smaller than $\ell - 1$. This means that any function that maps the Hamiltonian path of $\overrightarrow{P_k}(\ell)$ into any Hamiltonian path of T induces a homomorphism $\overrightarrow{P_k}(\ell) \to T$.

We are left with the hardest case $\ell = 3$. To prove the lower bound, consider a tournament $\widetilde{T}$ on 7 vertices $v_1, \ldots, v_7$, with arcs $v_i v_j$ for $1 \leq i < j \leq 6$ and $N^+(v_7) = \{v_1, v_2, v_4\}$, see Fig. 1. We want to prove that there exists no homomorphism $\overrightarrow{P_7}(3) \to \widetilde{T}$. This implies that there exists no homomorphism of $\overrightarrow{P}_{6a+1}(3) \to \overrightarrow{T_a} \odot \widetilde{T}$ for any integer $a \geq 1$, as otherwise 7 consecutive vertices of $\overrightarrow{P}_{6a+1}(3)$ would be mapped to a copy of $\widetilde{T}$, and the claimed lower bound follows. $\square$

Assume that $x_1, x_2, \ldots, x_7$ are the images of consecutive vertices of $\overrightarrow{P_7}(3)$ under some homomorphism $\overrightarrow{P_7}(3) \to \widetilde{T}$. As the vertices $v_1, \ldots, v_6$ induce a transitive tournament, there must exist the smallest i such that $x_i = v_7$. If $i = 1$, then since x_1x_4 is an arc and $x_1x_2x_3x_4$ is a path, we must have $x_4 = v_4$. But then it is not possible to find a path $x_4x_5x_6x_7$ with an arc x_4x_7. If $2 \leq i \leq 4$, then similarly $x_{i+3} = v_4$, hence $x_{i+2} \in \{v_1, v_2, v_3\}$. But since $x_{i-1}x_i$ is an arc, we have $x_{i-1} \in \{v_3, v_5, v_6\}$ and it is not possible for $x_{i-1}x_{i+2}$ to be an arc. If $5 \leq i \leq 6$, then by a symmetric argument we conclude that $x_{i-3} = v_3$, $x_{i-2} \in \{v_4, v_5, v_6\}$ and $x_{i+1} \in \{v_1, v_2, v_4\}$, hence $x_{i-2}x_{i+1}$ cannot be an arc. Finally, if $i = 7$, then we must have $x_j = v_j$ for every $1 \leq j \leq 7$, but in this case x_4x_7 is not an arc. This finishes the proof of the lower bound.

To prove the upper bound, we apply the following theorem that characterizes the general structure of the domination graphs of tournaments. Here, by a *directed caterpillar* we mean a directed path with possible additional outgoing pendant arcs.

Theorem 18. (Fisher et al. [13]) *The domination graph of a tournament is either an odd directed cycle with possible outgoing pendant arcs and isolated vertices, or a forest of directed caterpillars.*

We prove by induction on k that for every tournament T on $3k$ vertices there exists a homomorphism from $\overrightarrow{P_k}(3)$ to T. For $k \leq 3$, an oriented graph $\overrightarrow{P_k}(3)$ is just a directed path $\overrightarrow{P_k}$, which can be mapped homomorphically into any tournament on k vertices (Example 3).

For $k > 3$, let T be any tournament on $3k$ vertices. Note that $\overrightarrow{P_k}(3) \to \overrightarrow{C_5}(3)$, so we may assume that T does not contain $\overrightarrow{C_5}(3)$. Denote vertices of $\overrightarrow{P_k}(3)$ by $w_1, \ldots, w_k$ with arcs of the form w_iw_{i+1} and w_iw_{i+3}. Whenever we use the induction hypothesis to obtain a homomorphism $\overrightarrow{P_{k-1}}(3) \to T$, we think of this $\overrightarrow{P_{k-1}}(3)$ as of a subgraph of $\overrightarrow{P_k}(3)$ induced by vertices $w_2, w_3, \ldots, w_k$. In particular, to find a homomorphism $\overrightarrow{P_k}(3) \to T$, we only need to map w_1 to a vertex dominating the images of w_2 and w_4. This is possible exactly when the images of w_2 and w_4 induce an arc which does not belong to $E(\mathrm{dom}(T))$.

It turns out that if $\mathrm{dom}(T)$ contains a cycle of length at least five, two caterpillars, or a caterpillar with a directed path of length at least three, then T must contain $\overrightarrow{C_5}(3)$. This follows from the following observation.

Observation 19. *If* $\mathrm{dom}(T)$ *contains two vertex disjoint arcs, whose sources are not connected by an arc in* $\mathrm{dom}(T)$*, then T contains a copy of* $\overrightarrow{C_5}(3)$.

Proof. Let vw and $v'w'$ be the two arcs in $\mathrm{dom}(T)$, and without loss of generality let $v'v \in E(T) \setminus E(\mathrm{dom}(T))$. By Observation 13, all arcs between vw and $v'w'$ are then completely determined. Moreover, since $v'v \notin E(\mathrm{dom}(T))$, there exists a vertex u which dominates $v'v$, in particular it is neither w nor w'. Since vw and $v'w'$ are not dominated, we have that $wu, w'u \in E(T)$. Now, it is straightforward to check that vertices v, w, v', w' and u, in this order, induce a copy of $\overrightarrow{C_5}(3)$, as depicted in Fig. 2. □

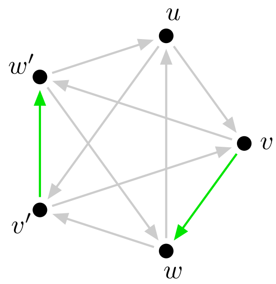

Figure 2. $\overrightarrow{C_5}(3)$ created in T using Observation 19. Green arcs belong to $E(\mathrm{dom}(T))$

By Theorem 18 and Observation 19, $\mathrm{dom}(T)$ must be either a directed triangle with some outgoing arcs or a directed caterpillar with a longest directed path of length at most 2. In particular, there exist at most three vertices with a positive out-degree in $\mathrm{dom}(T)$, hence it is possible to find a subset $D \subseteq V(T)$ of size at most 3 such that each arc from $E(\mathrm{dom}(T))$ is incident to at least one vertex from D. Let T' be the subtournament of T induced by $V(T) \setminus D$. Since $|V(T')| \geq 3(k-1)$, by the induction hypothesis there exists a homomorphism $\overrightarrow{P_{k-1}}(3) \to T'$. Moreover, the arc induced by the images of w_2 and w_4 cannot belong to $E(\mathrm{dom}(T))$, hence we can extend this homomorphism to $\overrightarrow{P_k}(3) \to T$. $\square$

We believe that the lower bound for $\tau(\overrightarrow{P_k}(3))$ proven in Theorem 17 gives the correct value of $\tau(\overrightarrow{P_k}(3))$, so we state the following conjecture.

Conjecture 20. *Let H be the oriented graph consisting of a path on k vertices and additional arcs between vertices at distance* 3*. Then* $\tau(H) = \left\lfloor \frac{7k-1}{6} \right\rfloor$.

4. Compressibility of ℓ-Layered Graphs

In this section, we study a class of acyclic oriented graphs H for which $\tau(H) = p(H)$. The considered class contains in particular graphs $\overrightarrow{P_k}(\ell)$ for $\ell \geq 4$, for which the equality holds by Theorem 17, as well as some graphs with out-degree not bounded by $p(H)$. It also generalizes the results of Valadkhan [23] for orientations of trees and cycles.

Definition 21. We say that an acyclic oriented graph H is *ℓ-layered* if for every vertex $v \in V(H)$ which is not a sink nor a source there exists a pair $(i, j) \in \mathbb{Z}_\ell^2$ such that the length of every directed path from any source of H to v is congruent to i modulo ℓ and the length of every directed path from v to any sink of H is congruent to j modulo ℓ. If a vertex v was assigned a pair (i, j), we will say that it is of type (i, j).

For $\ell \geq 2$, let $\mathcal{L}_\ell$ denote the family of all ℓ-layered acyclic oriented graphs.

Example 22. For any $3 \leq \ell < k$, the graph $\overrightarrow{P_k}(\ell)$ is $(\ell - 1)$-layered.

Example 23. Consider an acyclic oriented graph H and some integer $\ell \geq 2$, and replace each arc uv of H by a directed path of length ℓ from u to v. Then, the resulting graph, also called an $(\ell-1)$-subdivision of H, is ℓ-layered.

Example 24. For any integers $k \geq 3$ and $\ell \geq 2$, each acyclic orientation of a cycle on k vertices is ℓ-layered.

Example 25. An acyclic oriented graph obtained from a directed path $v_1 v_2 \dots v_k$ by adding a new vertex v and an arc $v_{k-2}v$ is not ℓ-layered for any $\ell \geq 2$. It follows from the fact that the distance from v_1 to v is $k-2$, while from v_1 to v_k it is $k-1$. On the other hand, it is easy to observe that any acyclic orientation of a tree can be mapped homomorphically to some directed path, which is ℓ-layered for every $\ell \geq 2$.

Because the oriented graph in Proposition 7 is 2-layered, the class $\mathcal{L}_2$ is not polynomially τ-bounded. However, for $\ell \geq 3$ the situation is completely different.

Theorem 26. *Let $\ell \geq 3$ and $H \in \mathcal{L}_\ell$ with $p(H) \geq 6$. Then, $\tau(H) = p(H)$.*

Proof. Firstly, observe that H can be mapped homomorphically into $\overrightarrow{C_\ell} \Rightarrow \overrightarrow{T_1}$. Indeed, if we denote the consecutive vertices of $\overrightarrow{C_\ell}$ by $w_0, w_1, \dots, w_{\ell-1}$ and the only vertex of $\overrightarrow{T_1}$ by w, then we can define a map $H \to (\overrightarrow{C_\ell} \Rightarrow \overrightarrow{T_1})$ in the following way: assign every source of H to w_0, every sink of H to w, and every vertex of type (i, j) to w_i. It is straightforward to check that this is indeed a homomorphism.

If T' is any tournament on 5 vertices containing a copy of $\overrightarrow{C_5}$, then it contains $\overrightarrow{C_4}$, which contains $\overrightarrow{C_3}$, so some vertex of T' is contained in a copy of $\overrightarrow{C_3}$ and a copy of $\overrightarrow{C_4}$. Thus, there is a homomorphism $\overrightarrow{C_\ell} \to T'$ for any $\ell \geq 3$. In particular, there always exists a homomorphism $H \to (T' \Rightarrow \overrightarrow{T_1})$. An analogous argument shows that there always also exists a homomorphism $H \to (\overrightarrow{T_1} \Rightarrow T')$.

Fix now a tournament T on $p(H)$ vertices. Assume first that T is not strongly connected. If at least one strongly connected component is of size at least $\min(5, \ell)$, then it contains a Hamiltonian cycle and every cycle of a smaller length. In particular, a copy of $\overrightarrow{C_\ell}$ or $\overrightarrow{C_5}$, and there exists a homomorphism $H \to T$ by the observation in the previous paragraph. Therefore, we may assume that all strongly connected components of T are of size smaller than $\min(5, \ell)$. For each $v \in V(H)$, let $\ell(v)$ denote the length of any longest directed path in H starting at v. Choose any Hamiltonian path P in T with vertices in order $v_{p(H)-1}, \dots, v_0$. Since every strongly connected component of T is of size smaller than ℓ, we have $v_i v_j \in E(T)$ for any $i - j > \ell$. Define a map $H \to T$ by assigning each $v \in V(H)$ to $v_{\ell(v)}$. Since for each arc $vw \in E(H)$ we have either $\ell(v) - \ell(w) = 1$ or $\ell(v) - \ell(w) > \ell$, it follows that this map is indeed a homomorphism.

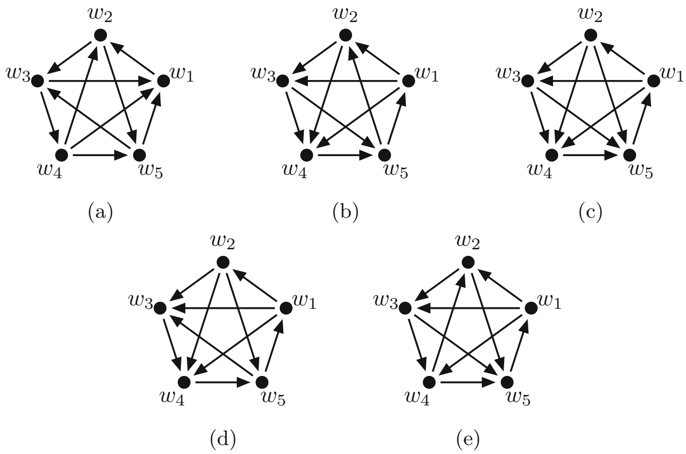

FIGURE 3. Tournaments used in the proof of Theorem 26

We are left with the case that T is strongly connected. Since any strongly connected tournament on $p(H)$ vertices contains a strongly connected subtournament on 6 vertices, it is enough to show that there exists a homomorphism from H to any strongly connected tournament on 6 vertices.

Let us introduce the following tournaments on 5 vertices:

- T_a, obtained from $\overrightarrow{C_5}(3)$ by reversing the arc w_1w_4;
- T_b, obtained from $\overrightarrow{C_5}(2)$ by reversing the arc w_1w_4;
- T_c, obtained from $\overrightarrow{T_5}$ by reversing the arc between the sink and the source;
- T_d, obtained from T_c by reversing the arc w_3w_5;
- T_e, obtained from T_c by reversing the arc w_2w_4.

All of them are depicted in Fig. 3. Let $\mathcal{T} = \{T_a, T_b, T_c, T_d, T_e\}$. By showing a series of claims we will prove that every strongly connected tournament on 6 vertices contains some tournament from $\mathcal{T}$, and that there exists a homomorphism from H to any tournament in $\mathcal{T}$.

Claim 27. *Every strongly connected tournament on* 5 *vertices is isomorphic to* $\overrightarrow{C_5}(2)$ *or some* $T \in \mathcal{T}$*.*

Proof. Let T be a strongly connected tournament on 5 vertices $w_1, \ldots, w_5$ with arcs w_5w_1 and w_iw_{i+1} for $1 \leq i \leq 4$. If there are no vertices in T with out-degree equal to 3, then $d^+(w_i) = 2$ for every $1 \leq i \leq 5$ and T is isomorphic to $\overrightarrow{C_5}(2)$ (since $\overrightarrow{C_5}(2)$ and $\overrightarrow{C_5}(3)$ are isomorphic).

Assume now that there is exactly one vertex v in T with out-degree 3. Then, there is also exactly one vertex w with out-degree 1. If $vw \in E(T)$, then by reversing an arc vw we obtain a tournament T' with all vertices having out-degree 2, hence T' is isomorphic to $\overrightarrow{C_5}(3)$ and T is isomorphic to either

T_a or T_b. If $wv \in E(T)$, then the three remaining vertices of T are in out-neighborhood of v and in-neighborhood of w. They must induce a copy of $\overrightarrow{C_3}$, since v is the only vertex with out-degree 1. But then, T is isomorphic to T_e.

We are left with the case when there are two vertices with out-degree 3. It is easy to see that they must be neighbors in a copy of $\overrightarrow{C_5}$ contained in T, which determines all but one arc in T. Depending on the orientation of this remaining arc, we conclude that T is isomorphic either to T_c or to T_d. ☐

Claim 28. *Every strongly connected tournament on* 6 *vertices contains a copy of some* $T \in \mathcal{T}$.

Proof. By Claim 27, it is enough to find a strongly connected subtournament with a vertex of in-degree or out-degree equal to 3. Let T be any strongly connected tournament on 6 vertices. It must contain a copy of $\overrightarrow{C_5}$ and vertices of this copy induce a strongly connected subtournament T'. If T' is isomorphic to some element of $\mathcal{T}$, then we are done. Otherwise, by Claim 27, it must be isomorphic to $\overrightarrow{C_5}(2)$; let $w_1, \ldots, w_5$ be consecutive vertices of the outer directed cycle of T', and let w denote the remaining vertex of T. Since T is strongly connected, w has in-neighbors and out-neighbors in T'; without loss of generality, we may assume that $w_1w, ww_2 \in E(T)$. If $ww_4 \in E(T)$, then the subtournament T_1 induced by vertices w, w_2, w_3, w_4, w_1 is strongly connected and in-degree of w_4 in T_1 is equal to 3. If $w_4w \in E(T)$, then the subtournament T_2 induced by vertices w, w_2, w_4, w_5, w_1 is strongly connected and out-degree of w_4 is equal to 3. In both cases, either T_1 or T_2 is isomorphic to some element of $\mathcal{T}$, which finishes the proof. ☐

To simplify the proof that H has a homomorphism to each $T \in \mathcal{T}$, we want to construct an oriented graph Q_ℓ such that H can be mapped homomorphically into Q_ℓ and then for each T provide a homomorphism from Q_ℓ to T. For every $0 \le i < \ell$, let D_i be a directed cycle on a vertex set $\{(j, i-j) \in \mathbb{Z}_\ell^2 : 0 \le j < \ell\}$ with arcs from $(j, i-j)$ to $(j+1, i-j-1)$ for every $0 \le j < \ell$, where addition is taken modulo ℓ. Define Q_ℓ as a disjoint union of D_i, over all $0 \le i < \ell$, and two additional vertices v_s, v_t, with arcs from v_s to v_t, from v_s to $(1, i)$, and from $(i, 1)$ to v_t for all $0 \le i < \ell$. Since the graph H is ℓ-layered, we have a natural homomorphism $H \to Q_\ell$ which maps all sources of H to v_s, all sinks of H to v_t, and all vertices of type (i, j) to the vertex (i, j) for every pair $(i, j) \in \mathbb{Z}_\ell^2$.

Claim 29. *Let T be a tournament on at least* 5 *vertices. Assume there exist vertices $u, v \in V(T)$ such that $uv \in E(T)$ and:*

- *$vw, wu, uz, zv \in E(T)$ for some $w, z \in V(T)$ and z is contained in a copy of $\overrightarrow{C_3}$,*
- *$ux, xy, yv \in E(T)$ for some $x, y \in V(T)$ and an arc xy is contained in a copy of $\overrightarrow{C_3}$ and $\overrightarrow{C_4}$.*

If $\ell = 3$ or $\ell = 4$, then there exists a homomorphism $Q_\ell \to T$.

Proof. Start defining the homomorphism $Q_\ell \to T$ by assigning v_s to u and v_t to v. It remains to define homomorphism $D_i \to T$ for every $0 \le i < \ell$ such

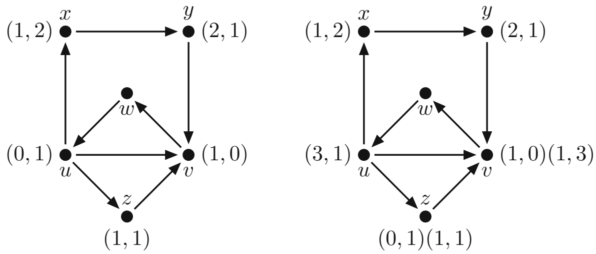

FIGURE 4. Partial homomorphisms $Q_3 \to T$ and $Q_4 \to T$ from the proof of Claim 29. To find a homomorphism $Q_3 \to T$, one needs to assume that vertex z and arc xy are contained in copies of $\overrightarrow{C_3}$. To find a homomorphism $Q_4 \to T$, one needs to assume that arc xy is contained in a copy of $\overrightarrow{C_4}$

that the image of $(1, i)$ is in out-neighborhood of u and the image of $(i, 1)$ is in the in-neighborhood of v. Assign $(1, 1)$ to z, $(1, 2)$ to x, and $(2, 1)$ to y. If $\ell = 3$, then assign $(0, 1)$ to u and $(1, 0)$ to v. If $\ell = 4$, then assign $(0, 1)$ and $(1, 1)$ to z, $(3, 1)$ to u, and $(1, 3)$ to v. All of these assignments are depicted in Fig. 4. Using the assumptions in the claim, it is straightforward to check that this can be extended to a homomorphism $Q_\ell \to T$. □

Claim 30. *For every* $3 \le \ell \le 5$ *and every* $T \in \mathcal{T}$, *there exists a homomorphism* $Q_\ell \to T$.

Proof. If $\ell = 3$ or $\ell = 4$, it is enough for every $T \in \mathcal{T}$ to find vertices $u, v \in V(T)$ satisfying the assumptions of Claim 29. It is straightforward to verify that one can choose:

- w_4 as u and w_5 as v for T_a,
- w_1 as u and w_4 as v for T_b,
- w_1 as u and w_4 as v for T_c if $\ell = 4$,
- w_5 as u and w_3 as v for T_d,
- w_1 as u and w_4 as v for T_e.

We only need to define separately a homomorphism $Q_3 \to T_c$, and we can do it in the following way. Assign v_s to w_2, v_t and $(1, 0)$ to w_4, $(1, 1)$ to w_3, and $(1, 2)$ to w_5. One can verify that this mapping can be extended to a homomorphism $Q_3 \to T_c$, which finishes the proof for $\ell = 3$ and $\ell = 4$.

Consider now $\ell = 5$. Note that for every $T \in \mathcal{T}$ there is a copy of $\overrightarrow{C_5}$ with consecutive vertices $w_1, w_2, \ldots, w_5$, and denote it by C_T. Each D_i for $0 \le i < 5$ can be mapped homomorphically into C_T in five different ways. We claim that for every $T \in \mathcal{T}$ there exists a homomorphism $Q_5 \to T$ that maps each D_i into C_T. Note that if the image of v_s is of out-degree k, then for every $0 \le i < 5$ there are k choices for a homomorphism $D_i \to C_T$ that agrees with v_s, and if the image of v_t is of in-degree k', then there are k' choices for a homomorphism $D_i \to C_T$ that agrees with v_t. Moreover, for every $T \in \{T_a, T_b, T_c, T_d\}$ there exist vertices $u, v \in V(T)$ such that $d^+(u) = 3$, $d^-(v) = 3$, and $uv \in E(T)$.

Therefore, if we choose u as the image of v_s and v as the image of v_t, then for each $0 \le i < 5$ there exists a homomorphism $D_i \to C_T$ agreeing with v_s and v_t simply by the pigeonhole principle. Finally, for T_e it is straightforward to verify that one can map v_s to w_1, v_t to w_4, $(1,0)$ to w_4, $(1,1)$ to w_3, $(1,2)$ to w_2, $(1,3)$ to w_4, and $(1,4)$ to w_3. □

Claims 28 and 30 together imply for every $3 \le \ell \le 5$ that Q_ℓ can be mapped into any strongly connected tournament on 6 vertices. Hence, to finish the proof of the theorem, it is enough to show that for $\ell \ge 6$ the graph Q_ℓ also can be mapped homomorphically into every $T \in \mathcal{T}$. Note that for every $T \in \mathcal{T}$, each vertex of T is contained in a copy of $\overrightarrow{C_3}$. Therefore, if $v_1v_2v_3v_4$ is a directed path in D_i for some $0 \le i < \ell$ and neither v_2 nor v_3 are neighbors of v_s or v_t in Q_ℓ, we can aim to find a homomorphism $D_i \to T$ which maps v_1 and v_4 to the same vertex of T, thus essentially reducing the length of D_i by 3. Since we can always perform this operation as long as the length of the cycle is at least 6, we can reduce the problem to the case when $3 \le \ell \le 5$, which was proved in Claim 30. □

Note that the assumed bound $p(H) \ge 6$ in Theorem 26 cannot be improved. Indeed, $\overrightarrow{C_5}(2)$ does not contain two vertices u and v with paths of length 1, 2 and 3 from u to v, so the oriented graph H consisting of paths of lengths 1, 2, 3 and 4 with common endpoints is ℓ-layered with $p(H) = 5$ and $\tau(H) \ge 6$. Analogous constructions can be provided for $p(H) = 4$ and $p(H) = 3$. In the cases $p(H) \le 5$ one can easily show that the best bounds are $\tau(H) \le 2$ when $p(H) = 2$, $\tau(H) \le 4$ when $p(H) = 3$, and $\tau(H) \le 6$ when $p(H) \in \{4, 5\}$ and H is ℓ-layered.

5. Concluding Remarks

For given $k \ge 1$ we construct a sequence of acyclic oriented graphs $(H_n)_{n\ge1}$ by taking H_n for $n \le k$ to be a transitive tournament on n vertices, and for each $n > k$ creating H_n by adding a vertex v_S for each set S of k vertices in H_{n-1} and connecting it by arcs to the vertices in S. Then $H_n \in \mathcal{D}_k$, $p(H_n) = n$, and for every $H \in \mathcal{D}_k$ there exists a homomorphism $H \to H_{p(H)}$. Therefore, to understand the asymptotic behavior of the compressibility of acyclic oriented graphs with out-degree at most k, it suffices to examine the sequence $(H_n)_{n\ge1}$. However, even for $k = 2$ we were able to compute $\tau(H_n)$ only for a few initial values of n, and we were unable to find a superlinear lower bound for $\tau(H_n)$. It might be useful to prove some better lower bounds for the compressibility of this family.

Let T be a tournament on 11 vertices $v_0, \dots, v_{10}$ with arcs v_iv_{i+j} for $j \in \{1, 3, 4, 5, 9\}$ and indices taken modulo 11. One can verify that every copy of $\overrightarrow{C_3}$ in T is dominated by some vertex, hence every $H \in \mathcal{D}_3$ can be mapped homomorphically into $T \odot \overrightarrow{C_3}$. Therefore, to prove that $\mathcal{D}_3$ is polynomially τ-bounded, it suffices to show that T satisfies Conjecture 8. It would be interesting to prove Conjecture 8 for this graph, especially with some low exponent.

Since bounding compressibility of a graph H is bounding the Turán function $\mathrm{ex_o}(n,H)$ and polynomial bounds on compressibility are related with the well-known Erdős–Hajnal Conjecture [10], it would be interesting to determine which families of acyclic oriented graphs are polynomially τ-bounded. For instance, families of graphs defined by forbidding a particular graph or having a restricted structure.

Problem 31. *For which acyclic oriented graphs F is the family of F-free acyclic oriented graphs polynomially τ-bounded?*

Theorem 12 shows that it holds for $F = \overrightarrow{K_{1,3}}$. Also, by Proposition 7, if the family of F-free acyclic oriented graphs is polynomially τ-bounded, then F must be bipartite.

The following definitions and notation are taken from [21]. We say that an oriented graph H is an *o-clique* if every two vertices of H are joined by a directed path of length at most 2. Define the *absolute oriented clique number* of H, denoted by $\omega_{ao}(H)$, as the maximum size of an o-clique contained in H, and the *relative oriented clique number* of H, denoted by $\omega_{ro}(H)$, as the maximum size of a subset $S \subseteq V(H)$ such that every two vertices of S are joined in H by a directed path of length at most 2. It is clear that if H is an o-clique and T is a tournament, then any homomorphism $H \to T$ must be injective, and for a general oriented graph H we have $\omega_{ao}(H) \leq \omega_{ro}(H) \leq |V(T)|$. For $k \geq 3$, let $\mathcal{A}_k$ denote the family of all acyclic oriented graphs with absolute clique number at most k, and let $\mathcal{R}_k$ denote the family of all acyclic oriented graphs with relative clique number at most k. We have $\mathcal{R}_k \subseteq \mathcal{A}_k$ and one may observe that $\mathcal{D}_k \subseteq \mathcal{R}_{k^2+1}$.

Conjecture 32. *For $k \geq 3$, the families $\mathcal{A}_k$ and $\mathcal{R}_k$ are polynomially τ-bounded.*

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Declarations

Conflict of interest The authors state that there is no conflict of interest. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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