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Chainlink Polytopes and Ehrhart Equivalence

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Abstract. We introduce a class of polytopes that we call chainlink polytopes and show that they allow us to construct infinite families of pairs of non-isomorphic rational polytopes with the same Ehrhart quasipolynomial. Our construction is based on circular fence posets, a recently introduced class of posets, which admit a non-obvious and nontrivial symmetry in their rank sequences. We show that this symmetry can be lifted to the level of polyhedral models (which we call chainlink polytopes) for these posets. Along the way, we introduce the related class of chainlink posets and show that they exhibit analogous nontrivial symmetry properties. We further prove an outstanding conjecture on the unimodality of rank polynomials of circular fence posets.

Keywords. Ehrhart theory, Posets, Rank polynomials, Unimodality.

1. Introduction

This paper is about a class of polytopes that naturally arise from poset theory, specifically, in the study of fence posets and related objects. They are easy to describe, amenable to analysis, and possess certain unexpected and interesting properties. These polytopes will be indexed by compositions; let $\bar{a} = (a_1, \ldots, a_s)$ be a composition of n and let l be a non-negative integer. The *chainlink polytope* $\mathrm{CL}(\bar{a}, l)$ with *chain composition* $\bar{a}$ and *link number* l is defined to be the polytope

$$\mathrm{CL}(\bar{a}, l) = \{x \in \mathbb{R}^s \mid 0 \leq x_i \leq a_i,\, x_i - x_{i \,(\mathrm{mod}\, s)+1} \leq a_i - l,\, i \in [s]\}.$$

This is a polytope that naturally lies in $\mathbb{R}^s$ and has a maximum of $3s$ facets. When the link number l is equal to zero, the second set of constraints becomes

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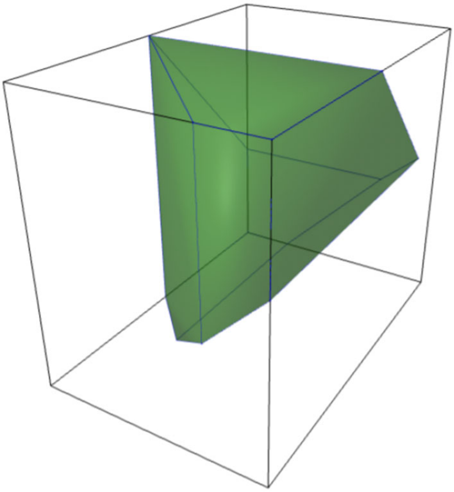


FIGURE 1. The polytope $\mathrm{CL}(\bar{a} = (6, 4, 5),\, l = 2)$

redundant and the polytope becomes a cuboid

$$\mathrm{CL}(\bar{a}, 0) = [0, a_1] \times [0, a_2] \times \cdots \times [0, a_s].$$

When the link number is larger, new facets emerge. For example, see Fig. 1.

We will also work with certain special sections of these chainlink polytopes. For a positive real number t, we set

$$\mathrm{CL}^t(\bar{a}, l) := \mathrm{CL}(\bar{a}, l) \cap \{x_1 + \cdots + x_s = t\}.$$

The polytopes $\mathrm{CL}^t(\bar{a}, l)$ are empty for $t \notin [0, n]$, where $n = a_1 + \cdots + a_s$. One of the main results in this paper is the following (unexpected) symmetry property of these sections.

Theorem 1. *Let $\bar{a} = (a_1, \ldots, a_s)$ be a composition of n, and let l be a positive integer, such that $2l \le \min\{a_i\}_{i\in[s]}$. Then, the complementary sections of the chainlink polytope $CL(\bar{a}, l)$ have the same volume. In other words, for any integer t, we have that*

$$\mid CL^t(\bar{a}, l) \mid = \mid CL^{n-t}(\bar{a}, l) \mid,$$

where $|\cdot|$ denotes the relative volume of the polytope.[1]

This theorem is a special case of the following more general theorem. The terms used will be formally defined in the next section.

Theorem 2. *Let $\bar{a} = (a_1, \ldots, a_s)$ be a composition of n, and let l be a non-negative integer, such that $2l \le \min\{a_i\}_{i\in[s]}$. Then, complementary sections*

[1] There will be no ambiguity in the definition of the relative volume for us. All our polytopes will lie on hyperplanes of the form $\{x_1 + \cdots + x_s = t\}$ and we will work with the volume form that assigns volume 1 to the polytope $\mathcal{P} = \mathrm{conv}\{0, e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_s\}$.

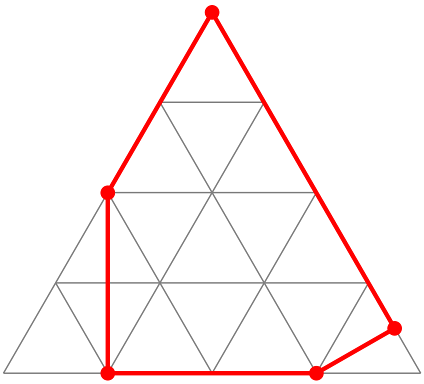


Figure 2. The sections $\mathrm{CL}^4(\bar{a}, l)$ and $\mathrm{CL}^{11}(\bar{a}, l)$ where $\bar{a} = (6, 4, 5)$ and $l = 2$

of the chainlink polytope have the same Ehrhart quasipolynomial, meaning, for any integer t, we have

$$\mathrm{Ehr}\ CL^t(\bar{a}, l) = \mathrm{Ehr}\ CL^{n-t}(\bar{a}, l).$$

To exemplify why this is unexpected, consider the chainlink polytope $\mathrm{CL}((6, 4, 5), 2)$ depicted in Fig. 2. The sections at $t = 4$ and $t = 11$ have the same volume, but are non-isomorphic.

Remark 1.1. Theorems 1 and 2 may fail when $2\,l > \min\{a_i\}_{i\in[s]}$. For instance, take our running example $\bar{a} = (6, 4, 5)$, but with $l = 3$ rather than 2. Let $t = 7$ with complementary section corresponding to $n - t = 15 - 7 = 8$. In this case, we have that even the number of lattice points (the Ehrhart quasipolynomials evaluated at 1) are different

$$\#\mathrm{CL}^t(\bar{a}, l) = 9, \qquad \#\mathrm{CL}^{n-t}(\bar{a}, l) = 10,$$

as are the volumes. We have that

$$\mid \mathrm{CL}^t(\bar{a}, l) \mid = \frac{71}{12}\sqrt{3}, \qquad \mid \mathrm{CL}^{n-t}(\bar{a}, l) \mid = 6\sqrt{3}.$$

We do not have a conceptual explanation for why we may lose symmetry when $2l > \min\{a_i\}_{i\in[s]}$. However, in Sect. 3, we will relate these polytopes with a natural class of posets, the so-called *circular fence posets*, and this perspective will provide some insight into this phenomenon.

2. Background

Where do these chainlink polytopes come from? At first sight, they (might perhaps) seem unmotivated, if (again perhaps) natural. We were led to these following the paper by the first and third authors [1] on fence posets, and in particular, a tricky problem that they had been unable to solve. We first recall the definition of fence posets.

Definition 2.1. Given a composition $\bar{c} = (c_1, \ldots, c_k)$, the *fence poset* is the poset on $n+1$ nodes, where $n = c_1 + \cdots + c_k + 1$ defined by the cover relations

$$x_1 \prec x_2 \prec \cdots \prec x_{c_1+1} \succ x_{c_1+2} \succ \cdots \succ x_{c_1+c_2+1} \prec x_{c_1+c_2+2} \prec \cdots .$$

These posets arise in a number of contexts including cluster algebras, quiver representation theory, and combinatorics. They also appear in recent work of Morier-Genoud and Ovisenko [2], where the authors introduce and study a q-deformation of the rational numbers. In this same paper, the authors conjectured the following (see also the paper by McConville et al. [3]), which was proved by the first and third authors; see [1].

Theorem 2.2. *The rank polynomials of fence posets are unimodal.*

The main step in the proof of this theorem involved the introduction of an ancillary class of posets, the so-called *circular fence posets*, and an unexpected property of these posets.

Definition 2.3. Given a composition with an even number of parts $\bar{c} = (c_1, \ldots, c_{2\,s})$, the circular fence poset $\bar{F}(\bar{c})$ is the poset on n nodes where $n = c_1 + \cdots + c_{2s}$, defined by the cover relations

$$x_1 \prec \cdots \prec x_{c_1+1} \succ x_{c_1+2} \succ \cdots \succ x_{c_1+c_2+1} \prec \cdots \prec x_{1+\sum_1^{2s-1} c_i} \succ \cdots \succ x_n \succ x_1.$$

In other words, this is what we get by identifying the two end points of a regular fence poset.

In [1, Theorem 1.2], the authors showed that circular fence posets satisfy an apriori unexpected property.

Theorem 2.4. (Kantarcı Oğuz, Ravichandran) *Rank polynomials of circular fence posets are symmetric.*

We make here a comment on why this result is unexpected. Given a composition $\bar{c} = (c_1, \ldots, c_{2s})$, let $\operatorname{shft} \bar{c}$ be the composition that is the cyclical shift of $\bar{c}$, that is $\operatorname{shft} \bar{c} = (c_{2\,s}, c_1, c_2, \ldots, c_{2\,s-1})$. A calculation shows that the symmetry of the rank polynomials of $\bar{F}(\bar{c})$ is equivalent to the statement that the posets $\bar{F}(\bar{c})$ and $\bar{F}(\operatorname{shft} \bar{c})$ have the same rank polynomial. It is also possible to see that this same rank symmetry may also be expressed as saying that the poset of lower ideals (our $\bar{F}(\bar{a})$) and the poset of upper ideals of the same fence poset have the same rank polynomial. However, except under very special cases, the two posets *are not isomorphic*. For instance, take $\bar{c} = (2, 1, 1, 2)$ as in Fig. 3, where we also plot the relevant Hasse diagrams (Table 1).

A second, this time bijective proof of Theorem 2.4 was given by Elizalde and Sagan in [4]. Interestingly, both proofs of this result are intricate and we felt it natural to seek a transparent proof of this basic result. We present such a proof in this paper; see Corollary 4.7.

As mentioned above, in [1], the symmetry of the rank polynomials of circular fence posets was used to prove Theorem 2.2, i.e., that rank polynomials of (regular) fence posets are unimodal. Generically, rank polynomials of

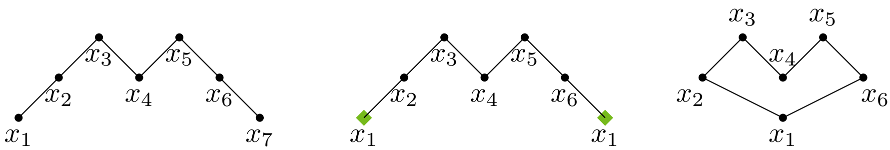
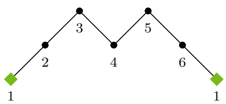
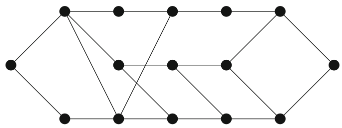
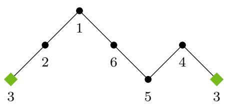
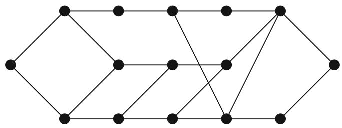


FIGURE 3. The fence poset $F(2,1,1,2)$ (left) and two depictions of the circular fence poset $\bar{F}(2,1,1,2)$(center and right). In the middle one, the two nodes marked x_1 are identified

TABLE 1. Example showing that lattices of upper and lower ideals can be non-isomorphic (the inclusion in the Hasse diagrams given is in the direction left to right)

Composition	Fence poset	Hasse diagram of lattice of lower ideals
$\bar{c}$ $(2,1,1,2)$		
$\bar{c}_{\text{shift}}$ $(2,2,1,1)$		

circular fence posets seemed to be unimodal as well, though there are certain exceptions; a calculation shows that

$$\overline{R}((1,1,1,1);q) = 1 + 2q + q^2 + 2q^3 + q^4.$$

Extensive computer calculations, however, suggested the following conjecture.

Conjecture 2.5. [1] *The rank polynomial $\overline{R}(\bar{a};q)$ of a circular fence poset $\overline{F}(\bar{a})$ is unimodal except when $\bar{a} = (a,1,a,1)$ or $(1,a,1,a)$ for some positive integer a.*

In this same paper, the authors were able to use the close connection between circular fences posets and regular fence posets to show that if $\bar{R}(\bar{a})$ is not unimodal, then the composition $\bar{a}$ must be of the form

$$\bar{a} = \left(a_1, \overbrace{1}^{b_1}, a_2, \overbrace{1}^{b_2}, \ldots, a_s, \overbrace{1}^{b_s}\right), \tag{1}$$

where any two entries larger than 1 are separated by at least one 1. In other words, the b_i are at least 1.

While the authors were able to prove certain additional necessary conditions for nonunimodality, they were unable to settle the conjecture. In this paper, we will focus on the case where each $b_i = 1$, so that we get a composition of the form

$$\bar{a} = (a_1, 1, a_2, 1, \ldots, a_s, 1). \tag{2}$$

It is interesting that such compositions also play an important role in the bijective proof of symmetry of Elizalde and Sagan in [4], where the authors refer to such circular fence posets as *gate posets* (or *gates* for short).

It turns out that ideals of gate posets corresponding to compositions $\bar{a} = (a_1, 1, a_2, 1, \ldots, a_s, 1)$ are precisely the lattice points in the chainlink polytope $\mathrm{CL}(\bar{a}, 1)$, where $\bar{a} = (a_1, a_2, \ldots, a_s)$. Further, the number of rank k ideals is precisely the number of lattice points in $\mathrm{CL}^k(\bar{a}, 1)$. The symmetry of the rank polynomials of gates may then be written as

$$\#\mathrm{CL}^k(\bar{a}, 1) = \#\mathrm{CL}^{n-k}(\bar{a}, 1),$$

where $n = a_1 + a_2 + \cdots + a_s + s$ and $\#P$ is the number of lattice points contained in a polytope P. We will see in Proposition 3.2 that the polytopes $\mathrm{CL}^k(\bar{a}, 1)$ while not necessarily integral, are always rational, in particular, half-integral.

A classical theorem of Ehrhart, see [5, Theorem 3.23], says that for any rational polytope P, there is a quasipolynomial which we denote Ehr P, such that for every positive integer n,

$$[\mathrm{Ehr}\ P](n) = \#\, nP.$$

In other words, the number of lattice points in positive integral dilates of P agrees with the evaluation of the quasipolynomial Ehr P at these points.

We were naturally led to investigate whether the syntactic generalization

$$\mathrm{Ehr}\ \mathrm{CL}^k(\bar{a}, 1) = \mathrm{Ehr}\ \mathrm{CL}^{n-k}(\bar{a}, 1) \tag{3}$$

is true as well. Well, it is! And this is the content of the main theorem in this paper. The equality of Ehrhart quasipolynomials yields as a corollary the equality of volumes of these polytopes; see Theorem 2.

Proving Theorem 2 needed several new ideas. Denote the Ehrhart polynomial Ehr $\mathrm{CL}^k(\bar{a}, 1)$ by f_k. This polynomial evaluated at 1 counts the number of ideals of size k in a certain circular fence poset. When evaluated at other integers, say $f_k(m)$, we will show that the value can again be interpreted as the number of lower ideals of size mk in a certain poset, which we call a *chainlink poset*. These posets share a familial resemblance to circular fence posets. They are introduced in Sect. 4, where we also discuss the connections to circular fence posets. An example of a chainlink poset is given below (Fig. 4).

In this figure, cover relations are represented as usual by upward sloping lines: We have for instance that $5 \prec 3$ and $6 \prec 7$. When two vertices have the same label, this means that they are identified. The rank polynomial of the above chainlink poset $C((5,4),2)$ is

$$\bar{R}(C((5,4),2)) = 1 + 2q + 3q^2 + 3q^3 + 3q^4 + 3q^5 + 3q^6 + 3q^7 + 2q^8 + q^9.$$

Note that this polynomial is symmetric.

We will show that all chainlink posets have symmetric rank polynomials. The strategies for showing symmetry for circular fence posets in [1,4] do not

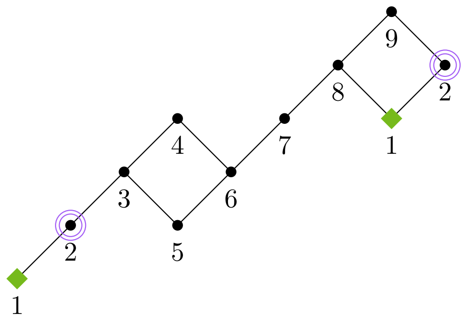


FIGURE 4. A chainlink poset: $C((4,5),2)$

carry over and we needed to approach the problem differently. The new ingredient is a linear algebraic approach coming from the theory of oriented posets (see [6]) that arguably yields a transparent proof. We note that this yields a new (third) proof of rank symmetry for circular fences as well.

This approach has as its starting point the following basic feature of fence posets: They can be built up by gluing chains in an iterative manner. We review in Sect. 4.1 how the rank polynomials of fence (and chainlink) posets can be computed by multiplying certain 2×2 matrices: The entries of these matrices are certain polynomials that encode refined order relations.

Using this approach has led to another felicitous consequence: We discovered new recurrences, that we could then use to prove Conjecture 2.5. We include a proof of this in Sect. 5.

3. Chainlink Polytopes

We repeat here the description of the chainlink polytopes.

Definition 3.1. Let $\bar{a} = (a_1, \ldots, a_s)$ be a composition of n and let $l \geq 0$. The *chainlink polytope* $\mathrm{CL}(\bar{a}, l)$ is defined as

$$\mathrm{CL}(\bar{a}, l) = \{x \in \mathbb{R}^s \mid 0 \leq x_i \leq a_i,\, i \in [s],\, x_i - x_{i \,(\mathrm{mod}\, s)+1} \leq a_i - l,\, i \in [s]\}.$$

Given $t \in \mathbb{R}_{\geq 0}$, the t-section of the chainlink polytope is the polytope defined as

$$\mathrm{CL}^t(\bar{a}, l) = \mathrm{CL}(\bar{a}, l) \cap \{x_1 + \cdots + x_s = t\}.$$

We note here a basic integrality property of these polytopes.

Proposition 3.2. *Let $\bar{a} = (a_1, \ldots, a_s)$ be a composition of n, let $l, t \in \mathbb{Z}_{\geq 0}$. Then*

- *The vertices of $CL(\bar{a}, l)$ are integral.*
- *Assume that $2l \leq \min \bar{a}$. Then, for every integer t, the vertices of $CL^t(\bar{a}, l)$ are either integral or half-integral.*

We prove this following Lemma 6.3. Some remarks are in order.

1. Half-integral vertices can indeed occur. For instance, the polytope $\mathrm{CL}^2((2,2),1)$ is given by
$$0 \le x_1, x_2 \le 2, \quad x_1 - x_2 \le 1, \quad x_2 - x_1 \le 1, \quad x_1 + x_2 = 2,$$
has dimension 1, and has as its two vertices $(3/2, 1/2)$ and $(1/2, 3/2)$.
2. The condition on l is necessary. Take, for instance, the polytope $\mathrm{CL}^5((3,3,3),2)$. One may check that the point $(8/3, 5/3, 2/3)$ is a vertex of the polytope.
3. It is possible to show that if the vertices of $\mathrm{CL}^t(\bar{a}, l)$ are all integral or half-integral for every integer t, then we must have that $2l \le \min\{a_i\}_{i\in[s]}$.

These polytopes are closely related to a class of posets we call *chainlink posets* which we describe in Sect. 4.

As noted in the introduction, one of our motivations for introducing these polytopes was to better understand circular fence posets; there are many examples of polyhedral models for combinatorial objects leading to interesting structural insights; see for instance [7–10]. Now, well-studied polyhedral models for posets already exist. Indeed, there are two common polytopes associated with posets—the order and chain polytopes; wouldn't they have served our purpose as well? We now address this question.

Given a poset $\mathcal{P}$ on $[n]$, the order polytope is defined as
$$\mathcal{O}(\mathcal{P}) := \{x \in [0,1]^n \mid \text{ for all } i \prec_{\mathcal{P}} j,\ x_i \le x_j\}.$$
A chain in a poset is a totally ordered subset; let us denote the set of all chains in the poset $\mathcal{P}$ by $\mathrm{Ch}(\mathcal{P})$. The chain polytope is defined as
$$\mathcal{C}(\mathcal{P}) := \{x \in [0,1]^n \mid \text{ for all } C \in \mathrm{Ch}(\mathcal{P}),\ \sum_{i\in C} x_j \le 1\}.$$
These polytopes (both of which are lattice polytopes) became popular following Stanley's proof that these polytopes (unexpectedly) have the same volume (and more generally, the same Ehrhart polynomial); see [7]. Stanley's proof exhibits a unimodular triangulation of the chain polytope as well, indexed by the linear extensions of $\mathcal{P}$.

The first attempt at getting a polyhedral perspective on fence posets would be to consider the order polytope of the fence poset (call the fence poset F and the order polytope $\mathcal{O}(F)$), and note that the number of rank k ideals is the number of lattice points in
$$\mathcal{O}^k(F) = \mathcal{O}(F) \cap \{x_1 + \cdots + x_n = k\}.$$
However, though we have that $\mathcal{O}^k(F)$ and $\mathcal{O}^{n-k}(F)$ have the same number of lattice points, it is *not true* that they have the same Ehrhart polynomial.

Remark 3.3. Consider the gate poset for the composition $\bar{c} = (3,1,2,1,1,1)$, let $P = \mathcal{O}(\bar{F}(\bar{c}))$ be the order polytope and let $P^k = P \cap \{x_1 + \cdots + x_n = k\}$. Then, the lattice points in the second dilates of P^4 and the complementary section P^5 number, respectively, 84 and 83, i.e., we do not have equality of Ehrhart polynomials. This indicates the subtlety involved in this problem.

So far, we have proposed a polytopal model for *gates*, which are circular fences coming from compositions of the form $(c_1, 1, \dots, c_s, 1)$, i.e., where all the down steps have size 1. As mentioned above in Remark 3.3, the order polytope for the gate poset *does not* obey symmetry, which is why we have worked with chainlink polytopes instead.

What about general circular fences, i.e., those coming from compositions of the form $(c_1, d_1, \dots, c_s, d_s)$ with differing lengths of down steps $d_1, \dots, d_s$? A natural proposal is as follows. The polytope will consist of all real tuples $(x_1, y_1, \dots, x_s, y_s)$, such that

$$0 \le x_i \le c_i + l, \quad 0 \le y_i \le d_i - 1, \quad (d_i - 1)(x_i - l) \le y_i, \quad y_i \le (d_i - 1)x_{i+1},$$

where the indices are taken cyclically.

When all the down steps are 1, all save the first set of inequalities become trivial. However, if we agree to combine the last two set of inequalities, we get $x_i - l \le x_{i+1}$, which are the defining equations in the chainlink polytope with link number l.

Unfortunately, these polytopes do not generally have the symmetry that chainlink polytopes have. We do not know if there is a way of defining polyhedral models for general circular fence posets, so that this symmetry does hold.

4. Chainlink Posets and Rank Symmetry

Let $\mathrm{CL}(\bar{a}, l)$ be a chainlink polytope with $2\,l \le \min_i(a_i)$. Consider the integer points that lie inside the polytope. When $l = 1$, these points correspond to ideals of the circular fence poset $F(a_1 - 1, 1, a_2 - 1, 1, \dots, a_s - 1, 1)$, where the rank of the ideal corresponds to the sum of the coordinates of the point. For general l, the integer points can be interpreted as ideals of a poset $F_l(\bar{a})$ formed by adding extra edges to the Hasse diagram of $F(a_1 - 1, 1, a_2 - 1, 1, \dots, a_s - 1, 1)$, as shown in Fig. 5. More precisely, we can define chainlink posets as follows:

Definition 4.1. Let $\bar{a} = (a_1, \dots, a_s)$ be a composition, and l be a positive integer satisfying $2l \le \min_i a_i$ as in the case for chainlink polytopes. The *chainlink poset* $\mathrm{P}_{\mathrm{CL}}(\bar{a}, l)$ is given by points $x_{i,j}$ for $1 \le i \le s$ and $0 \le j \le a_i$ with the generating relations $x_{i,0} \succeq x_{i,1} \succeq \cdots \succeq x_{i,a_i}$ and $x_{i,a_i - l} \ge x_{i+1,l}$ for each i where $i + 1$ is calculated cyclically.

We will use $\mathrm{CL}^t(\bar{a}, l)$ to denote the slice of the polytope with respect to the hyperplane $x_1 + \cdots + x_s = t$. Note that this slice can be non-empty only when $t \in [0, n]$. Furthermore, the number of integer points in $\mathrm{CL}^t(\bar{a}, l)$ is given by the coefficient of q^t in the rank polynomial of $\mathrm{P}_{\mathrm{CL}}(\bar{a}, l)$.

The fact that

$$\#\mathrm{CL}^t(\bar{a}, 1) = \#\mathrm{CL}^{n-t}(\bar{a}, 1),$$

where $\#P$ denotes the number of lattice points in a polytope P, is as a result equivalent to the symmetry of the rank polynomial of $\mathrm{P}_{\mathrm{CL}}(\bar{a}, 1)$ (Since $l = 1$,

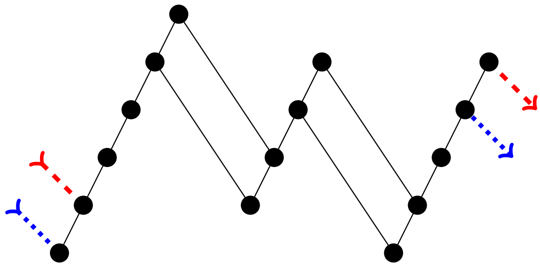


FIGURE 5. The chainlink poset with $\bar{a} = (6, 4, 5)$ and $l = 2$; the two top right nodes are connected to the two bottom left nodes

this poset is also a circular fence poset). The symmetry of this rank polynomial was recently proved using an inductive argument by Kantarcı Oğuz and Ravichandran in [1] and then bijectively by Elizalde and Sagan in [4].

The connection between integer points of the polytope and rank polynomial of the corresponding poset still holds if we multiply all by a positive integer k. This allows us to describe the coefficients of the Ehrhart quasipolynomial of slices of the chainlink polytope in terms of coefficients of rank polynomials of some chainlink posets. That means a general statement about the symmetry of rank polynomials of all chainlink posets can be used to prove the main theorem in this paper (Theorem 2) and this is precisely what we do in the next few sections.

4.1. Matrix Formulation

An oriented poset $P\nearrow = (P, x_L, x_R)$ consists of a poset P with two specialized vertices x_L and x_R which can be thought as the target (left) vertex $\odot$ and the source (right) vertex $\rightarrow$. One can think of an oriented poset as a poset with an upwards arrow coming out of the source vertex x_R. One can combine oriented posets by linking the arrow of one poset with target of another via $x_R \preceq y_L$ ($x_R \nearrow y_L$) to get $(P \nearrow Q)\nearrow$.[2]

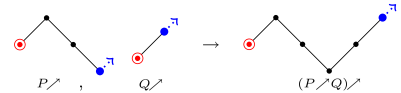
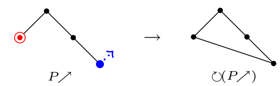
The effect of this operation on the rank polynomial can be calculated easily by 2×2 matrices. A *rank matrix* of an oriented poset $P\nearrow$ is defined as follows:

$$\mathfrak{M}_q(P\nearrow) := \begin{bmatrix} \mathfrak{R}(P;w)\mid_{x_R \in I} & \mathfrak{R}(P;w)\mid_{x_R \notin I} \\ \mathfrak{R}(P;w)\mid_{\substack{x_R \in I \\ x_L \notin I}} & \mathfrak{R}(P;w)\mid_{\substack{x_R \notin I \\ x_L \notin I}} \end{bmatrix}.$$

The entries are partial rank polynomials, where we are restricting to the ideals of the poset P satisfying the given constraints. We also use the notation $\circlearrowright (P\searrow)$ (resp. $\circlearrowright (P\nearrow)$) to denote the structure obtained by adding the relation $x_R \succeq x_L$ (resp. $x_R \preceq x_L$). On the rank matrix level, this corresponds to taking the trace. See Table 2 for precise formulas and examples of these operations.

[2]Linking via $x_R \succeq y_L$ is also an option; see [6].

Table 2. The linking operations through examples

	Formula	Example
$P \nearrow Q$	$\mathfrak{M}_q((P \nearrow Q)\nearrow) = \mathfrak{M}_q(P\nearrow) \cdot \mathfrak{M}_q(Q\nearrow)$	
$\circlearrowright(P\nearrow)$	$\mathfrak{R}(\circlearrowright(P\nearrow); w) = \mathrm{tr}(\mathfrak{M}_q(P\nearrow))$	

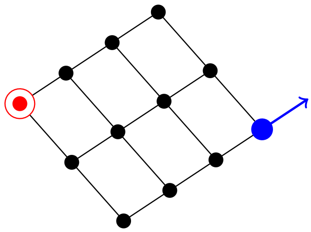


FIGURE 6. The box poset $B_{3\times 4}\nearrow$

In particular, consider the case where P is formed of a single node equal to both x_R and x_L. We call this oriented poset an *up step* and denote the corresponding matrix by U

$$U := \mathfrak{M}_q(\bullet\nearrow) := \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix}.$$

Note that combining $k+1$ such posets gives us a chain of length k with x_L corresponding to the minimal element, and x_L to the maximal. In the matrix level, we have

$$\mathfrak{M}_q(C_k\nearrow) = U^{k-1}.$$

Let $B_{a\times b}\nearrow$ denote the ab-element oriented box poset given by the direct product of two chains C_{a-1} and C_{b-1} with the left vertex given by $(a-1,0)$ and the right vertex is given by $(0,b-1)$ ($B_{3\times 4}\nearrow$ is shown in Fig. 6).

The rank matrix of a box poset is given as follows:

$$\begin{aligned}\mathfrak{M}_q(B_{a\times b}\nearrow) &= \begin{bmatrix} q^b\begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q & \begin{bmatrix} a+b-1 \\ b-1 \end{bmatrix}_q \\ q^b\begin{bmatrix} a+b-2 \\ b \end{bmatrix}_q & \begin{bmatrix} a+b-2 \\ b-1 \end{bmatrix}_q \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} a+b \\ b \end{bmatrix}_q - \begin{bmatrix} a+b-1 \\ b-1 \end{bmatrix}_q & \begin{bmatrix} a+b \\ b \end{bmatrix}_q - q^b\begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q \\ q^b\begin{bmatrix} a+b-2 \\ b \end{bmatrix}_q & \begin{bmatrix} a+b-2 \\ b-1 \end{bmatrix}_q \end{bmatrix}.\end{aligned}$$

Next, we will see that any given chainlink poset can be realized by combining copies of the up step, and the box poset $B_{2\times l}\nearrow$.

Proposition 4.2. *Consider the chainlink poset $P_{\mathrm{CL}}(\bar{a},l)$ with $2l \leq \min_i(a_i)$. Let $\bar{d}$ be the weak composition formed by taking $d_i = a_i - 2l$. The rank polynomial of $P_{\mathrm{CL}}(\bar{a},l)$ is given by*

$$\mathfrak{R}(P_{\mathrm{CL}}(\bar{a},l);q) = \mathrm{tr}(U^{d_1}\cdot B\cdot U^{d_2}\cdot B\cdot \cdots \cdot U^{d_1}\cdot B), \tag{4}$$

where B denotes the rank matrix $\mathfrak{M}_q(B_{2\times l}\nearrow)$.

Proof. The description of $\mathrm{P}_{\mathrm{CL}}(\bar{a},l);q$ is given by taking chains and connecting the l maxima of a chain of the l minima of the next. The condition on l assures that these are always different vertices. Each pair of $2l$ vertices has the shape of the poset $(B_{2\times l}\nearrow$ where the right endpoint of ith one is connected to the left endpoint of the $i+1$st one from below, with d_i points in between. This is the poset given by $\circlearrowright(B_{2\times l}\nearrow C_{d_1-1}\nearrow B_{2\times l}\nearrow UC_1 - d_2 - 1\nearrow \cdots B_{2\times l}\nearrow$

$UC_{d_s-1}\nearrow)$. As the rank polynomial can be calculating by taking the trace of the corresponding matrices, and $\mathfrak{M}_q(C_{a-1}\nearrow) = U^a$ the result follows. ☐

Example 4.3. The chainlink poset given in Fig. 5 with $\bar{a} = (6, 4, 5)$ and $l = 2$ can be formed by combining 2×2 boxes with up steps and then taking the closure: $\circlearrowleft(B_{2\times 2}\nearrow \cdot\bullet\nearrow \cdot\bullet\nearrow \cdot B_{2\times 2}\nearrow \cdot B_{2\times l}\nearrow \cdot\bullet\nearrow)$. The corresponding rank polynomial is given by

$$\begin{aligned}
&\mathrm{tr}\left(\begin{bmatrix} q^2[3]_q & [3]_q \\ q^2 & [2]_q \end{bmatrix} \cdot \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} q^2[3]_q & [3]_q \\ q^2 & [2]_q \end{bmatrix} \cdot \begin{bmatrix} q^2[3]_q & [3]_q \\ q^2 & [2]_q \end{bmatrix} \cdot \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix}\right) \\
&\quad = 1 + 3q + 6q^2 + 9q^3 + 12q^4 + 14q^5 + 16q^6 + 17q^7 \\
&\qquad + 17q^8 + 16q^9 + 14q^{10} + 12q^{11} + 9q^{12} + 6q^{13} + 3q^{14} + q^{15}.
\end{aligned}$$

Note that the rank polynomial given in this instance is symmetric. Next, we will show that this is always the case.

4.2. Recurrence Relations and Rank Symmetry

One advantage of building posets via matrices is that the characteristic equations of matrices give us recurrence relations in the rank polynomial level. For example, consider the characteristic polynomial of U. Plugging U in the place of x gives us the following identity:

$$U^2 = (q+1)U + q,$$

Note that the coefficient of U is symmetric around $q^{1/2}$ and q is trivially symmetric around q.

Lemma 4.4. *Let $B = \mathfrak{M}_q(B_{a\times b}\nearrow)$ for some fixed a, b. The characteristic polynomials of B as well as well as BU have coefficients that are symmetric polynomials in q. In particular, the trace and determinant of B and BU are symmetric about $ab/2$, ab, $(ab+1)/2$, and $ab+1$, respectively.*

Proof. As the matrix B satisfies

$$\mathfrak{M}_q(B_{a\times b}\nearrow) = \begin{bmatrix} q^b\begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q & \begin{bmatrix} a+b-1 \\ b-1 \end{bmatrix}_q \\ q^b\begin{bmatrix} a+b-2 \\ b \end{bmatrix}_q & \begin{bmatrix} a+b-2 \\ b-1 \end{bmatrix}_q \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a+b \\ b \end{bmatrix}_q - \begin{bmatrix} a+b-1 \\ b-1 \end{bmatrix}_q & \begin{bmatrix} a+b-1 \\ b-1 \end{bmatrix}_q \\ \begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q - \begin{bmatrix} a+b-2 \\ b-1 \end{bmatrix}_q & \begin{bmatrix} a+b-2 \\ b-1 \end{bmatrix}_q \end{bmatrix},$$

we will define another matrix B' to simplify some of our calculations

$$B' := B\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a+b \\ b \end{bmatrix}_q & -q^b\begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q \\ \begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q & -q^b\begin{bmatrix} a+b-2 \\ b \end{bmatrix}_q \end{bmatrix}.$$

Consider the trace of B

$$\begin{aligned}
tr(B) &= q^b\begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q + \begin{bmatrix} a+b-2 \\ b-1 \end{bmatrix}_q \\
&= \begin{bmatrix} a+b \\ b \end{bmatrix}_q - \begin{bmatrix} a+b-1 \\ b-1 \end{bmatrix}_q + \begin{bmatrix} a+b-1 \\ b-1 \end{bmatrix}_q - q^a\begin{bmatrix} a+b-2 \\ a \end{bmatrix}_q \\
&= \begin{bmatrix} a+b \\ b \end{bmatrix} - q^a\begin{bmatrix} a+b-2 \\ a \end{bmatrix}_q.
\end{aligned}$$

As both polynomials on the right are symmetric about $ab/2$ the result follows. Similarly, we can show that the determinant gives us a symmetric polynomial around ab

$$\det(B) = -\det(B') = q^b \begin{bmatrix} a+b \\ b \end{bmatrix}_q \begin{bmatrix} a+b-2 \\ b \end{bmatrix}_q - q^b \left(\begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q \right)^2 .$$

Now, let us do the same verification for the trace and determinant of the matrix BU

$$BU = \begin{bmatrix} q^{b+1} \begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q & q^b \begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q + \begin{bmatrix} a+b-1 \\ b-1 \end{bmatrix}_q \\ q^{b+1} \begin{bmatrix} a+b-2 \\ b \end{bmatrix}_q & q^b \begin{bmatrix} a+b-2 \\ b \end{bmatrix}_q + \begin{bmatrix} a+b-2 \\ b-1 \end{bmatrix}_q \end{bmatrix} = \begin{bmatrix} q^{b+1} \begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q & \begin{bmatrix} a+b \\ b \end{bmatrix}_q \\ q^{b+1} \begin{bmatrix} a+b-2 \\ b \end{bmatrix}_q & \begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q \end{bmatrix}.$$

One can verify that the trace of this matrix is symmetric around $(ba+1)/2$, whereas the determinant is symmetric around $(ba+1)$

$$\operatorname{tr}(BU) = (q^{b+1}+1) \begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q ,$$

$$\det(BU) = q^{b+1} \left(\begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q \right)^2 - q^{b+1} \begin{bmatrix} a+b-2 \\ b \end{bmatrix}_q \begin{bmatrix} a+b \\ b \end{bmatrix}_q .$$

□

Lemma 4.5. *Let X be a matrix whose trace is given by a symmetric polynomial on q with center C. Then, we have the following:*

1. *If $\operatorname{tr}(BX)$ is symmetric around $C + (ab)/2$, then for all k, $\operatorname{tr}(B^k X)$ is symmetric with center $C + k(ab)/2$.*
2. *If $\operatorname{tr}(UX)$ is symmetric around $C + 1/2$, then for all k, $\operatorname{tr}(U^k X)$ is symmetric with center $C + k/2$.*
3. *If $\operatorname{tr}(BUX)$ is symmetric around $C+(ab+1)/2$, then for all k, $\operatorname{tr}((BU)^k X)$ is symmetric with center $C + k(ab+1)/2$.*

Proof. For the first claim, consider the characteristic polynomial of B.

$$B^2 = \operatorname{tr}(B)B - \det(B).$$

Substitution into the trace equation gives the following identity for any $k \geq 2$:

$$\operatorname{tr}(B^k X) = \operatorname{tr}(B)\operatorname{tr}(B^{k-1}X) - \det(B)\operatorname{tr}(B^{k-2}X).$$

We have shown in Lemma 4.4 $\operatorname{tr}(B)$ is a symmetric polynomial with center $ab/2$ and $\det(B)$ is symmetric with center ab, the result follows by induction. The other claims follow similarly as $\operatorname{tr}(U)$, $\det(U)$ and $\operatorname{tr}(BU)$ and $\det(BU)$ are symmetric polynomials with centers of symmetry given by $1/2$, 1, $(ab+1)/2$, and $ab+1$, respectively. □

Theorem 4.6. *Let $(d_1, d_2, \ldots, d_s)$ be a weak composition and $B_{a\times b} \nearrow$ be an oriented box poset where B denotes the rank matrix $\mathfrak{M}_q(B_{a\times b} \nearrow)$. Then, the following polynomial is symmetric:*

$$\operatorname{tr}(U^{d_1} \cdot B \cdot U^{d_2} \cdot B \cdots U^{d_1} \cdot B).$$

Proof. We can use Lemma 4.5 to simplify the statement of our theorem:

- By Lemma 4.5 (2), it is enough to consider the cases where each $d_i = 0$ or 1.
- If, however, some d_i=0, we get consecutive copies of B. By Lemma 4.5 (1), these cases can be simplified, so that we can assume all d_i are equal to 1, leaving us with $\mathrm{tr}((U \cdot B)^k)$ for some k.
- By Lemma 4.5 (3), this can be further reduced to the symmetry of $\mathrm{tr}(U \cdot B)$ and $\mathrm{tr}(I)$.

As we have already shown that the trace of $B \cdot U$ is symmetric in Lemma 4.4 and the trace of the identity matrix is just a constant, we are done. □

Note that when $l = 1$, we recover the rank symmetry of gate posets.

Corollary 4.7. *The rank polynomial of any chainlink poset is symmetric.*

We can now use this machinery to prove our main theorem.

Theorem 2. *Let $\bar{a}$ be a composition of n, let l be a positive integer such that $2l \leq \min\{a_i\}_{i\in[s]}$ and let t be a positive real number. Then, complementary sections of the chainlink polytope have the same Ehrhart quasipolynomial*

$$\mathrm{Ehr}\ CL^t(\bar{a}, l) = \mathrm{Ehr}\ CL^{n-t}(\bar{a}, l).$$

As an illustration, we revisit the example we described in Fig. 2. The two complementary sections in this example are different. Nevertheless, they have the same Ehrhart quasipolynomial, given by

$$\left[\mathrm{Ehr}\ \mathrm{CL}^t((6,4,5),2)\right](t) = \begin{cases} \frac{27}{4}t^2 + \frac{9}{2}t + 1, & n \in \{0,2,4,\ldots\}, \\ \frac{27}{4}t^2 + \frac{9}{2}t + \frac{3}{4}, & n \in \{1,3,5,\ldots\}. \end{cases}$$

Proof. Consider the polytope $k\mathrm{CL}(\bar{a}, l) = \mathrm{CL}(k\bar{a}, kl)$. The number of integer points in $\mathrm{CL}^t(k\bar{a}, kl)$ is given by the coefficient of q^t in the rank polynomial of the corresponding chainlink poset $\mathrm{P}_{\mathrm{CL}}(k\bar{a}, kl)$. As by Corollary 4.7, the rank polynomial of any chainlink poset is symmetric, and the number of integer points in $\mathrm{CL}^t(k\bar{a}, kl)$ is the same as the number of integer points in $\mathrm{CL}^{n-t}(k\bar{a}, kl)$ for any k. As a consequence, they have the same Ehrhart quasipolynomial. □

The equality of volumes follows as a corollary.

Theorem 1. *Let $\bar{a} = (a_1, \ldots, a_s)$ be a composition of n, and let l be a positive integer, such that $2l \leq \min\{a_i\}_{i\in[s]}$. Then, the complementary sections of the chainlink polytope $CL(\bar{a}, l)$ have the same volume. In other words, for any integer t, we have*

$$\mid CL^t(\bar{a}, l) \mid = \mid CL^{n-t}(\bar{a}, l) \mid,$$

where $\mid \cdot \mid$ denotes the relative volume of the polytope.

Proof. It is a well-known fact (or see [5, Lemma 3.19] for a proof) that for a d-dimensional rational polytope P, we have that

$$\mid P \mid = \lim_{k\to\infty} \frac{\# kP}{k^d}.$$

Theorem 2 now yields the result.24 □

5. Unimodality and Multimodality

5.1. Unimodality

The recurrence relations from characteristic matrices have other applications, as well. In this subsection, we prove the following result.

Theorem 5.1. *Rank polynomials of circular fence posets $\bar{F}(\bar{a})$ are unimodal except when $\bar{a} = (a, 1, a, 1)$ or $(1, a, 1, a)$ for some positive integer a.*

We define the matrix for a *down step* denoted by D as follows:

$$D := \begin{bmatrix} 1+q & -q \\ 1 & 0 \end{bmatrix}.$$

The following lemma is an easy consequence of the work in [6]. The interested reader is referred there to learn about how down steps fit into the framework of oriented posets.

Lemma 5.2. *Let DC_n denote a decreasing chain, i.e., an n-element chain poset oriented by taking the maximal vertex as the target and the minimal vertex as the source. Then, we have*

$$\mathfrak{M}_q(DC_n \nearrow) = D^{n-1} \cdot U.$$

That means the above theorem may be restated as follows:

Theorem 5.1. *For any composition $\bar{a}$ with an even number of parts, the following polynomial is unimodal except when $\bar{a} = (a, 1, a, 1)$ or $(1, a, 1, a)$ for some positive integer a:*

$$\overline{R}(\bar{a}, q) = \mathrm{tr}(D^{a_1}U^{a_2}D^{a_3}U^{a_4}\cdots D^{a_{s-1}}U^{a_s}) = \mathrm{tr}(U^{a_1}D^{a_2}U^{a_3}D^{a_4}\cdots U^{a_{s-1}}D^{a_s}).$$

In [1] where the above statement was first stated as a conjecture, several cases were settled. It was shown that if $\bar{R}(\bar{a}, q)$ is *not* unimodal, then it must be of the following form.

- $\bar{a}$ should have even length.
- No two consecutive parts of $\bar{a}$ may both be greater than 1.
- If $\bar{a} = (X, a, 1, b, Y)$ for sequences X and Y, then $|a - b| \le 1$.

It was further shown that if $\bar{a}$ has length $2n$, then unimodality may only fail at the middle. In other words, if we let $\bar{R}(\bar{a}, q) = \sum_{i=0}^{2n} a_i q^i$, then

$$a_0 = a_{2n} \le a_1 = a_{2n-1} \le \cdots \le a_{n-1} = a_{n+1}.$$

Despite much effort, the last step in proving Theorem 5.1 resisted resolution. In this section, we will settle the outstanding cases using a new recurrence identity. The linear algebraic perspective introduced in this paper was critical in discovering this—the identity is *non-linear* and difficult to discover apriori. However, the linear algebraic perspective made discovering this identity easy.

We first note that one can easily show unimodality by direct calculation for the cases of $\bar{a} = (a, b)$ and $\bar{a} = (a, 1, a+b, 1)$ with $a, b \ge 1$. Our first identity will be the following:

$$DUD = DU + UD - U + D^3 - D^2. \tag{Id 1}$$

In terms of rank polynomials, (Id 1) translates to the following:

$$\begin{aligned}\overline{R}((a,1,b,X);q) = {} & \overline{R}((a-1,1,b,X);q) + \overline{R}((a,1,b-1,X);q) \\ & - \overline{R}((a-1,1,b-1,X);q) \\ & + \overline{R}((a+b+1,X);q) - \overline{R}((a+b,X);q).\end{aligned} \tag{Id 1$'$}$$

Here, X can be any odd-length composition. Note that we allow $a = 1$ or $b = 1$, in which case, with the assumption that a zero part means the parts to the left and right combine.

Proposition 5.3. *For an odd-length sequence $X = (x_1, x_2, \ldots, x_k)$ of positive integers, suppose that $a, b \geq 1$ and $\bar{R}(a-1,1,b-1,X)$ is unimodal. If $a > 1$ or $\ell(X) > 1$ with $x_1 > 1$ or $b \geq x_2$, then $\bar{R}(a,1,b,X)$ is also unimodal.*

Proof. As unimodality holds for compositions of odd size, we can focus on the case where $2n = \mid (a,1,b,X) \mid$ for some n. We can further assume that $\mid a-b \mid$ is at most 1, as otherwise again, we know we have unimodality. We will show that under these hypotheses

$$[q^n]\overline{R}((a,1,b,X);q) \geq [q^{n-1}]\overline{R}((a,1,b,X);q). \tag{Id 2}$$

Let $\overline{R}(a-1,1,b-1,X)$ be unimodal. Since $(a-1,1,b-1,X)$ is a composition of $2n-2$, it has a peak at $n-1$. Thus,

$$[q^n]\left(-\overline{R}(a-1,1,b-1,X)\right) \geq [q^{n-1}]\left(-\overline{R}(a-1,1,b-1,X)\right).$$

We also have the following by the symmetry of rank polynomials:

$$\begin{aligned}&[q^n]\left(\overline{R}(a-1,1,b,X) + \overline{R}(a,1,b-1,X)\right) \\ &\quad = [q^{n-1}]\left(\overline{R}(a-1,1,b,X) + \overline{R}(a,1,b-1,X)\right).\end{aligned}$$

By (Id 1$'$), to prove (Id 2), all that is left to show is that

$$[q^n]\left(\overline{R}(a+b+1,X) - \overline{R}(a+b,X)\right) \geq [q^{n-1}]\left(\overline{R}(a+b+1,X) - \overline{R}(a+b,X)\right).$$

By the symmetry, negative terms are equal and this is equivalent to showing that $\overline{R}(a+b+1,X)$ is unimodal. We consider the following cases:

- If X has one part only, $a > 1$ and $b = 1$, then we end up with a 2 part composition that is unimodal.
- Now, suppose X has at least three parts, $X = (x_1, x_2, \ldots)$. We inspect the composition $(a+b+1, x_1, x_2, \ldots)$ for unimodality. If $x_1 > 1$, then we have two consecutive parts greater than 1 as $a+b+1 \geq 1$. This gives us unimodality.
- If $x_1 = 1$, we may assume that $|b - x_2| \leq 1$, since, otherwise, we have unimodality. If $b \geq x_2$, we get a trio $a+b+1, 1, x_2$ with difference between $a+b+1$ and x_2 at least 2, which gives us unimodality.
- Finally, if $a > 1$ even if $b = x_2 - 1$, the difference between $a+b+1$ and x_2 is at least 2, so we have unimodality.

□

Proof of Theorem 5.1. As the other cases are already resolved, we will focus our attention to the case of $\bar{a}$ having at least 6 parts. By Proposition 5.3 and the preceding work, it is sufficient to show unimodality when all parts of $\bar{a}$ are 2 or 1. We can further suppose we have no consecutive $2, 2$ or $2, 1, 2$ as the former is unimodal, and the latter can be simplified. Then, $\bar{a}$ either contains consecutive parts $2, 1, 1, 2$ or $1, 1, 1, 2$, or it consists entirely of 1s.

If $\bar{a}$ contains consecutive parts $2, 1, 1, 2$, then $\bar{a} = (2, 1, 1, 2, 1, 1, X)$ for some X by our assumptions. As $\overline{R}((1, 3, 1, 1, X); q)$ is unimodal, so is $\overline{R}(\bar{a}; q)$.

If $\bar{a}$ contains consecutive parts $1, 1, 1, 2$, then either $\bar{a} = (1, 1, 1, 2, 1, 1)$ or $\bar{a} = (X, b, 1, 1, 1, 2, 1, 1)$ for some X and for some $b \in \{1, 2\}$ by our assumptions. The former case can be directly calculated. For the latter case, we can use Proposition 5.3 with the three 1s in the middle. As $\overline{R}((X, b + 3, 1, 1); q)$ is unimodal, so is $\overline{R}(\bar{a}; q)$.

That only leaves the case where $\bar{a}$ contains 1's only. Again, it is easy to show $\overline{R}((1, 1, 1, 1, 1, 1); q)$ is unimodal by direct calculation. Otherwise, $\bar{a} = (1, 1, 1, 1, 1, 1, 1, Y)$ for some Y. We may now apply Proposition 5.3 (under the condition $l(X) > 1$ and $b \geq x_2$, where $X = (1, 1, 1, 1, Y)$) to conclude that $\bar{a}$ will be unimodal provided that the rank polynomial of the circular fence poset corresponding to $(1, 0, 1, 0, 1, 1, 1, Y)$ is unimodal. This is the same as $\overline{R}((3, 1, 1, X); q)$, which we have shown is unimodal. We conclude that $\overline{R}(\bar{a}; q)$ is unimodal as well.

5.2. Multimodality

The counterexample $(k, 1, k, 1)$ to unimodality for rank polynomials of circular fence posets can be extended in the case of chainlink posets to obtain any number of peaks. We will use the generic term *multimodality* to describe situations where unimodality of sequences fails to hold. Consider $\mathrm{P_{CL}}((2k, 2k), k)$ for example. We get the following rank sequences:

- $k = 1$: $[1, 2, 1, 2, 1]$.
- $k = 2$: $[1, 2, 3, 2, 3, 2, 3, 2, 1]$.
- $k = 3$: $[1, 2, 3, 4, 3, 4, 3, 4, 3, 4, 3, 2, 1]$.
- $k = 4$: $[1, 2, 3, 4, 5, 4, 5, 4, 5, 4, 5, 4, 5, 4, 3, 2, 1]$.

Proposition 5.4. *The chainlink poset $P_{\mathrm{CL}}((2k, 2k), k)$ has the following rank sequence with $k + 2$ peaks:*

$$\left[1, 2, 3, \ldots, k+2, \overbrace{k+1, k+2}^{k+1}, k+1, k, \ldots, 2, 1\right].$$

Proof. Consider the corresponding chainlink polytope:

$$\mathrm{P_{CL}}((k+1, k+1), k) = \{(x_1, x_2) \mid 0 \leq x_1, x_2 \leq 2k,\ x_1 - x_2 \in [-k, k]\}.$$

This is a hexagon with vertices $(0, 0)$, $(k, 0)$,$(0, k)$, $(2k, k)$, $(k, 2k)$, and $(2k, 2k)$. The hexagon for $k = 3$ can be seen in Fig. 7. The rank sequence of the poset is in bijection with the numbers of integer points in the sections of the polytope. The rectangular region $(k, 0)$,$(0, k)$, $(2k, k)$, $(k, 2k)$ has alternating sections with k and $k - 1$ integer points which gives the multimodular behavior to the rank lattice. $\square$

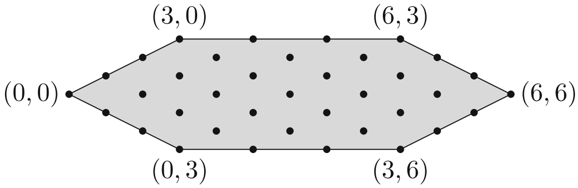


FIGURE 7. The chainlink polytope $\mathrm{P}_{\mathrm{CL}}((6,6),3)$ with integer points marked

6. Properties of Chainlink Polytopes

In this section, we examine some properties of chainlink polytopes.

Lemma 6.1. *Let $\bar{a} \in \mathbb{N}^s$ be a composition of n and let $l \in \mathbb{R}$. The chainlink polytope $CL(\bar{a}, l)$ is full dimensional when $l < \min(\bar{a})$.*

Proof. Let $\{\epsilon_i\}_{i\in[s]}$ be small positive real numbers all less than $\min(\bar{a}) - l$. Consider the point

$$x = (a_1 - \epsilon_1, \ldots, a_s - \epsilon_s).$$

We have that

$$\begin{aligned}(a_i - \epsilon) - (a_{i(\mathrm{mod}(s))+1} - \epsilon) &= a_i - a_{i(\mathrm{mod}(s))+1} + \epsilon_i - \epsilon_{i(\mathrm{mod}(s))+1} \\ &< a_i - \min \bar{a} + \epsilon_i \\ &< a_i - l,\end{aligned}$$

by the condition we have imposed on the ϵ_i. Consequently, all points of the form x above are in the polytope and constitute a full-dimensional subset. $\square$

Determining exactly when these polytopes are non-empty is a tricky problem and does not seem to have a nice solution. We note though that a routine application of LP duality shows that the condition $l \leq (a_1 + \cdots + a_s)/s$ is necessary.

Lemma 6.2. *Let $\bar{a} \in \mathbb{R}^s_{>0}$ and $l \in \mathbb{R}_{\geq 0}$. Suppose that $0 < l < \min_{i\in[s]} a_i$. Then, the polytope $CL(\bar{a}, l)$ has exactly $3s$ facets, defined by the equalities $x_i = 0$, $x_i = a_i$ and $x_i - x_{i+1} = a_i - l$.*

Proof. Let $i \in [s]$. Let $\epsilon = (\epsilon_1, \ldots, \epsilon_s) \in \mathbb{R}^s_{\geq 0}$ be any point, such that $\epsilon_i = 0$ and so that $\epsilon_j \leq \min_{i\in[s]} a_i - l$ for $j \neq i$. Then, it is readily verified that both ϵ and $(a_1, \ldots, a_s) - \epsilon$ are in $\mathrm{CL}(\bar{a}, l)$. Thus, the faces of $\mathrm{CL}(\bar{a}, l)$ defined by $x_i = 0$ and $x_i = a_i$ are $(s-1)$-dimensional.

Take now a small positive δ, so that $\min\{l, a_{i+1} - l\} > \delta$. Consider the point

$$p = (\epsilon_1, \ldots, \epsilon_{i-1}, a_i - l + \delta, \delta, \epsilon_{i+1}, \ldots, \epsilon_s).$$

Then, $p \in \mathrm{CL}(\bar{a}, l)$, as well. This is because we have that $p \in \mathbb{R}^s_{\geq 0}$ and

$$\epsilon_j \leq a_j,\ j \in \{1, \ldots, i-1, i+2, \ldots\}, \quad a_i - l + \delta \leq a_i, \quad \delta \leq a_{i+1},$$

showing that the point p satisfies the first two defining inequalities of $\mathrm{CL}(\bar{a}, l)$. Next, we see that

$$\epsilon_j - \epsilon_{j+1} \leq \epsilon_j \leq a_j - l,\ j \in \{1, \ldots, i-2, i+2, \ldots\},$$

as well as

$$\epsilon_{i-1} - (a_i - l + \delta) \leq \epsilon_{i-1} \leq a_{i-1} - l, \quad \delta - \epsilon_{i+1} \leq \delta \leq a_i - l,$$

showing that the third set of inequalities are also satisfied. Thus the face defined by $x_i - x_{i+1} = a_i - l$ is also $s-1$-dimensional. □

Lemma 6.3. *Let $\bar{a} \in \mathbb{R}^s_{>0}$ and $l \in \mathbb{R}_{\geq 0}$. Suppose that $2\,l \leq \min_{i\in[s]} a_i$. The vertices v of $CL(\bar{a}, l)$ have the form $v_i \in \{0, l, a_i - l, a_i\}$. Moreover, each edge must be parallel either to the standard base vectors e_i or to $e_i + e_{i+1}$ for some $i \in [s]$.*

Proof. Take $i \in [s]$. Consider the facet F_i defined by $x_i - x_{i+1} = a_i - l$ and let $p \in F_i$ be an arbitrary point. If $a_i > p_i > a_i - l$, take

$$p^+ := p + (a_i - p_i)(e_i - e_{i+1}), \quad p^- := p + (a_i - l - p_i)(e_i - e_{i+1}).$$

One sees easily that both p^+ and p^- are in $\mathrm{CL}(\bar{a}, l)$, and since p is a convex sum of p^+ and p^-, p cannot be a vertex. Thus, a vertex $v \in F_i$ must satisfy $v_i \in \{a_i - l, a_i\}$, and so, $v_{i+1} \in \{0, l\}$. Since all the other facets are given by $x_i = 0$ or $x_i = a_i$, we get that the coordinates of a vertex v must be of the form $v_i \in \{0, l, a_i - l, a_i\}$.

We will show that the edges must be parallel to the e_i or $e_i + e_{i+1}$ assuming that $2\,l < \min_{i\in[s]} a_i$. However, since any $\bar{a}$ with $2l \leq \min_{i\in[s]} a_i$ can be approximated by $\bar{a}'$ satisfying the strict inequality, the statement holds in this case as well.

The kernels of the functionals that define the chainlink polytope have the following form:

$$\begin{aligned}
\mathcal{A}_i &= \ker x_i = \mathrm{span}\{e_j : j \in [s] - \{i\}\}. \\
\mathcal{B}_i &= \ker x_i - x_{i+1} = \mathrm{span}\{e_i + e_{i+1}, e_j : j \in [s] - \{i, i+1\}\}.
\end{aligned}$$

Call $A_i = \{e_j : j \in [s] - \{i\}\}$ and $B_i = \{e_i + e_{i+1}, e_j : j \in [s] - \{i, i+1\}\}$. Let $I \subset [s]$, we have

$$\bigcap_{i\in I} \mathcal{A}_i = \mathrm{span}\left(\bigcap_{i\in I} A_i\right).$$

Let $J \subset [s]$ be a subset which contains no (cyclically) adjacent elements, then

$$\bigcap_{j\in J} \mathcal{B}_j = \mathrm{span}\left(\bigcap_{j\in J} B_j\right).$$

Let v be a vertex and e be an incident edge. Since $2\,l < \min_{i\in[s]} a_i$, the functionals $x_{i-1} - x_i$ and $x_i - x_{i+1}$ cannot both be maximized at v. Hence, the set $K = \{j \in [s] : v_j - v_{j+1} = a_j - l\}$ does not contain any (cyclically) adjacent elements. The edge e must be parallel to a one-dimensional subspace that is the intersection of the kernels of some of the functionals that are maximized at v. Thus, e is parallel to a one-dimensional space of the form

$$L = \operatorname{span}\left(\bigcap_{i\in I} A_i\right) \cap \operatorname{span}\left(\bigcap_{j\in J} B_j\right).$$

From this, it is easy to see that $L = \mathbb{R}e_i$ or $\mathbb{R}(e_i + e_{i+1})$ for some $i \in [s]$. $\square$

Proof of Proposition 3.2. By Lemma 6.3, the vertices are integral as any vertex v must be of the form $v_i \in \{0, l, a_i - l, a_i\}$.

For the second part, notice that the edges are transverse to the hyperplanes $H^t = \{x_1 + \cdots + x_s = t\}$. Hence, the vertices of $\mathrm{CL}^t(\bar{a}, l\}$ are the intersection of the edges with H^t. These intersections must be of the form $v + ae_i$ or $v + b(e_i + e_{i+1})$ for some vertex v and some $a, b \in \mathbb{R}$. Since the vertices are integral, we must have $a \in \mathbb{Z}$ or $b \in \frac{1}{2}\mathbb{Z}$.

Proposition 6.4. *If we have the strict inequality $2l < \min_{i\in[s]} a_i$, then the polytope $CL(\bar{a}, l)$ is simple and the combinatorial structure does not depend on $\bar{a}$ or l.*

Proof. Let v be a vertex. Denote the set of defining functionals which are maximized in $\mathrm{CL}(\bar{a}, l)$ on v by $F(v)$. We will show that v is simple by constructing a bijection

$$f : [s] \to F(v).$$

If $v_i = 0$, map $f : i \mapsto -x_i$ and if $v_i = a_i$ map $f : i \mapsto x_i$. If we have $v_i \notin \{0, a_i\}$, then either $v_i = a_i - l$ or $v_i = l$. In the former case, we must have $v_{i+1} = 0$ and in the latter $v_{i-1} = a_{i-1}$. In the former case map $f : i \mapsto x_i - x_{i+1}$ and in the latter $f : i \mapsto x_{i-1} - x_i$. Suppose that a linear functional $\phi \in F(v)$ is not in the image of the function f. Clearly, it cannot be either of the functionals x_i or $-x_i$ for any i. Therefore, $\phi = x_i - x_{i+1}$ for some i. Hence, either $v_i = a_i$ and $v_{i+1} = l$ or $v_i = a_i - l$ and $v_{i+1} = 0$. But then, in the former case, $f(i+1) = x_i - x_{i+1}$, and in the latter case, $f(i) = x_{i-1} - x_i$. Therefore, v is simple.

Let F be a subset of the defining linear functionals of size s. Suppose that F satisfies the following for each $i \in [s]$:

- at most one of x_i and $-x_i$ is in F,
- at most one of $x_i - x_{i+1}$ and $x_{i-1} - x_i$ is in F,
- if $x_i - x_{i+1} \in F$, then either $x_i \in F$ or $-x_{i+1} \in F$,
- if $x_i \in F$, then $-x_{i+1} \notin F$.

Then, there is a unique vertex v whose set of maximized functionals is F, that is, $F = F(v)$. Since the possible sets do not depend on $\bar{a}$ or l, we get a combinatorial equivalence between any two s-dimensional chainlink polytopes satisfying $2\,l < \min_{i\in[s]} a_i$. $\square$

Proposition 6.5. *Let $\bar{a} \in \mathbb{R}^s_{>0}$ and $l \in \mathbb{R}_{\geq 0}$. Suppose that $2\,l \leq \min_{i\in[s]} a_i$. The number of vertices of the chainlink polytope $CL(\bar{a}, l)$ is given by*

$$\boldsymbol{Vert}(CL(\bar{a}, l)) = \operatorname{tr}(A_1 \cdots A_s),$$

where $A_i = A$ *if* $a_i > 2\,l$ *and* $A_i = B$ *if* $a_i = 2l$, *with*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Proof. Let v be a vertex of $\mathrm{CL}(\bar{a}, l)$. From Proposition 6.3, we know that $v_i \in \{0, l, a_i - l, a_i\}$. Moreover, if $v_i \notin \{0, a_i\}$, then there are two possibilities.

1. The vertex v is contained in the facet defined by $x_i - x_{i+1} = a_i - l$. In this case, we see that $v_i = a_i - l$ and $v_{i+1} = 0$.
2. The vertex v is contained in the facet defined by $x_{i-1} - x_i = a_{i-1} - l$. And in this case, $v_{i-1} = a_{i-1}$ and $v_i = l$.

In light of this, we encode the vertices as follows:

- If $v_i = 0$ or if $v_i = l$, the ith index is called **small**. Note that in the latter case, we will have that $v_{i-1} = a_{i-1}$ by (2) above.
- If $2\,l < a_i$ and $v_i = a_i - l$, then the ith index is called **medium**. Notice that a medium index must be followed by a small index by (1) above.
- If $v_i = a_i$, then the ith index is called **big**.

From a vertex v, we construct a word $w_v : [s] \to \{t, m, b\}$, where

$$w_v(i) = \begin{cases} t & \text{if the index } i \text{ is small.} \\ m & \text{if the index } i \text{ is medium.}\ . \\ b & \text{if the index } i \text{ is big.} \end{cases}$$

The correspondence $v \mapsto w_v$ is 1–1, given $\bar{a}$ and w_v, we can reconstruct v. The words w_v obey two simple rules:

1. An m is followed by a t.
2. If $a_i = 2l$ and if $w_v(i-1) = b$, then the ith index cannot be medium, that is, $w_v(i) \neq m$.

Consider words of length $s+1$ that satisfy the above two rules. Construct matrices $M(\bar{a}) = (M_{ij}(\bar{a}))_{i,j \in \{t,m,b\}}$, so that $M_{ij}(\bar{a})$ are the number of length $(s+1)$-words that begin with i and end with j.

Let $\bar{a}'$ denote the first $s-1$ terms of $\bar{a}$. If the last entry $a_s = 2l$, we have

$$M(\bar{a}) = M(\bar{a}') \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

And if $a_i > 2l$, we have

$$M(\bar{a}) = M(\bar{a}') \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Since the vertices of $\mathrm{CL}(\bar{a}, l)$ correspond to the words that begin and end with the same letter, we get the trace formula in the proposition. $\square$

Corollary 6.6. *In particular if* $2l < \min_{i \in [s]} a_i$, *then*

$$\textit{\textbf{Vert}}(CL(\bar{a}, l)) = \mathrm{tr}(A^s),$$

which satisfies the linear recurrence

$$\operatorname{tr}(A^s) = 2\operatorname{tr}(A^{s-1}) + \operatorname{tr}(A^{s-2}),$$

for $s \geq 3$. It can be seen easily that $\operatorname{tr}(A) = 2$ and $\operatorname{tr}(A^2) = 6$. These are dubbed the "Companion Pell Numbers" in A002203 [11].

Corollary 6.7. *If $a_i = 2l$ for each $i \in [s]$, then*

$$\textbf{\textit{Vert}}(CL(\bar{a}, l)) = \operatorname{tr}(B^s),$$

which satisfies the linear recurrence

$$\operatorname{tr}(B^s) = 2\operatorname{tr}(B^{s-1}) + \operatorname{tr}(B^{s-2}) - \operatorname{tr}(B^{s-3})$$

for $s \geq 3$. It is easy to see that $\operatorname{tr}(B^0) = 3$, $\operatorname{tr}(B) = 2$ and $\operatorname{tr}(B^2) = 6$. This is the sequence A033304 in OEIS [12]. *Note that the matrix B shows up, but it is conjugated by a symmetric matrix.*

The calculation of the volume of a chainlink polytope has quite a straightforward formula in the case $2l \leq \min_{i\in[s]} a_i$.

Proposition 6.8. *Let $\bar{a} \in \mathbb{R}^s_{>0}$ and $l \in \mathbb{R}_{\geq 0}$. Suppose that $2l \leq \min_{i\in[s]} a_i$. The volume of the chainlink polytope $CL(\bar{a}, l)$ is given by the following trace formula:*

$$\mathbf{Vol}(CL(\bar{a}, l)) = \operatorname{tr}\left(\begin{bmatrix} a_1 & \frac{-l^2}{2} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_2 & \frac{-l^2}{2} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_3 & \frac{-l^2}{2} \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_s & \frac{-l^2}{2} \\ 1 & 0 \end{bmatrix}\right).$$

Proof. We shall give a description of the volume of the polytope in terms of the matchings of the cyclic graph on $[s]$. We define a *matching* to be any subset of edges that are pairwise disjoint, and denote by $\mathcal{M}_k([s])$ matchings of the cyclic graph on $[s]$ with exactly k edges. We will additionally use the shorthand $i \in M$ to denote when $i \in [s]$ is covered by an edge of a matching M. The chainlink polytope is the rectangular prism $P = \prod_{i=1}^s [0, a_i]$ with the sets $S_{(i,i+1)} : i \in [s]$ removed, where

$$S_{(i,i+1)} = \{x \in P : x_i - x_{i+1} > a_i - l\}.$$

We will think of the indexing of the sets as edges of the cyclic graph on $1, \ldots, s$. By inclusion–exclusion, one gets

$$\mathbf{Vol}(\mathrm{CL}(\bar{a}, l)) = a_1 \cdots a_s + \sum_{k=1}^{s} \sum_{1\leq i_1<i_2<\cdots<i_k\leq s} \mathbf{Vol}\left(\bigcap_{j=1}^{k} S_{(i_j, i_{j+1})}\right)(-1)^k.$$

If two edges e and f of the cyclic graph C_s intersect, then the intersection $S_e \cap S_f = \emptyset$. Thus, we only need to be concerned with the terms that come from matchings $m \in \mathcal{M}_k(s)$ in the above formula. For a matching $M \in \mathcal{M}_k(s)$, we have after permuting the coordinates

$$\bigcap_{e\in M} S_e = \prod_{i\notin M} [0, a_i] \times \prod_{\substack{(i,i+1) \\ \in M}} \{(x, y) \in \mathbb{R}^2 : a_i \geq x > a_i - l,$$
$$l > y \geq 0, x - y > a_i - l\}.$$

Here, we consider $(i, i+1)$ modulo s as usual. The volume of the above is clearly $\prod_{i \notin m} a_i \frac{l^{2k}}{2^k}$, which matches the trace formula. □

7. Remarks and Further Work

There are several questions about these chainlink polytopes that naturally arise.

- **Ehrhart equivalence:** Two rational polytopes $P, Q \in \mathbb{R}^d$ are said to be Ehrhart equivalent if they have the same Ehrhart quasipolynomial. They are said to be GL equidecomposable if we may partition $P = U_1 \cup \cdots \cup U_n$ and $Q = V_1 \cup \cdots \cup V_n$ into relatively open simplices, such that for each i, U_i and V_i are $GL_d(\mathbb{Z})$ equivalent. In [13], it was conjectured that Ehrhart equivalent polytopes are GL equidecomposable. This is known to be true for dimensions 2 [14] and 3 [15]. Sections of chainlink polytopes provide us with a large class of examples to test this conjecture.
- **Multimodality:** Theorem 5.1 can be expressed in the following way: Let $\bar{a}$ be a composition of n. Then, the function from $\{0, \ldots, n\}$ to $\mathbb{N}$ given by

  $$k \to \#\mathrm{CL}^k(\bar{a}, 1),$$

  is unimodal save when $\bar{a} = (a, 1, a, 1)$ or $(1, a, 1, a)$ and is bimodal in these cases. If we instead fix a positive integer l, such that $2l \leq \min\{a_i\}$ and look at

  $$k \to \#\mathrm{CL}^k(\bar{a}, l),$$

  the function may be multimodal. Indeed, we have that when $\bar{a} = (2k, 2k)$ and $l = k$, we have $k+1$ peaks. Can one describe the maximal number of modes that may arise for fixed l and when these are attained?
- **The General Chainlink Polytope:** In the case $2l > \min_{i \in [s]} a_i$, several interesting properties of the polytope $\mathrm{CL}(\bar{a}, l)$ no longer hold. Namely, the vertices of the sections are no longer half-integral, we lose the equality of volumes of complementary sections and our formulae for the number of the vertices and the volume of $\mathrm{CL}(\bar{a}, l)$ no longer hold. Given that the chainlink polytope $\mathrm{CL}(\bar{a}, l)$ is full-dimensional when $l < \min_{i \in [s]} a_i$, this leaves a lot to be investigated, both combinatorially and geometrically.

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References

[1] Kantarcı Oğuz, E., Ravichandran, M.: Rank polynomials of fence posets are unimodal. Discrete Math. **346** (2023). https://doi.org/10.1016/j.disc.2022.113218

[2] Morier-Genoud, S., Ovsienko, V.: q-deformed rationals and q-continued fractions. In: Forum of Mathematics, Sigma, vol. 8 (2020). Cambridge University Press

[3] McConville, T., Sagan, B.E., Smyth, C.: On a rank-unimodality conjecture of Morier-Genoud and Ovsienko. Discrete Mathematics 344(8), 112483 (2021)

[4] Elizalde, S., Sagan, B.E.: Partial rank symmetry of distributive lattices for fences. Ann. Comb. https://doi.org/10.1007/s00026-022-00600-8

[5] Beck, M., Robins, S.: Computing the continuous discretely. Undergraduate Texts in Mathematics 61, 1–364 (2007)

[6] Kantarcı Oğuz, E.: Oriented posets and rank matrices (2022) arXiv:2206.05517 [math.CO]

[7] Stanley, R.P.: Two poset polytopes. Discrete & Computational Geometry 1(1), 9–23 (1986)

[8] Konvalinka, M., Pak, I.: Geometry and complexity of O'Hara's algorithm. Advances in Applied Mathematics 42(2), 157–175 (2009)

[9] Stanley, R.P.: Two combinatorial applications of the Aleksandrov-Fenchel inequalities. Journal of Combinatorial Theory, Series A 31(1), 56–65 (1981)

[10] Edmonds, J.: Maximum matching and a polyhedron with 0, 1-vertices. Journal of research of the National Bureau of Standards B 69(125-130), 55–56 (1965)

[11] OEIS Foundation Inc.: The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A002203 (2022)

[12] OEIS Foundation Inc.: The On-Line Encyclopedia of Integer Sequences. Published electronically at https://oeis.org/A033304 (2022)

[13] Haase, C., McAllister, T.B.: Quasi-period collapse and $gl_n(\mathbb{Z})$-scissors congruence in rational polytopes. Contemporary Mathematics 452, 115–122 (2008)

[14] Greenberg, P.: Piecewise $sl_2(\mathbb{Z})$ geometry. Transactions of the American Mathematical Society 335(2), 705–720 (1993)

[15] Erbe, J., Haase, C., Santos, F.: Ehrhart-equivalent 3-polytopes are equidecomposable. Proceedings of the American Mathematical Society 147(12), 5373–5383 (2019)

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