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The Maximum Number of Cliques in Graphs with Bounded Odd Circumference

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Abstract. In this work, we give the sharp upper bound for the number of cliques in graphs with bounded odd circumferences. This generalized Turán-type result is an extension of the celebrated Erdős and Gallai theorem and a strengthening of Luo's recent result. The same bound for graphs with bounded even circumferences is a trivial application of the theorem of Li and Ning.

1. Introduction

A central topic of extremal combinatorics is to investigate sufficient conditions for the appearance of given substructures. In particular, for a given graph H and a set of graphs $\mathcal{F}$, the *generalized Turán number* $\mathrm{ex}(n, H, \mathcal{F})$ denotes the maximum number of copies of H in a graph on n vertices containing no F as a subgraph, for every $F \in \mathcal{F}$. In 1959, Erdős and Gallai [2] determined the maximum number of edges in a graph with a small *circumference*, the length of a longest cycle. For integers n and k such that $n \geq k \geq 3$, they proved

$$\mathrm{ex}(n, K_2, \mathcal{C}_{\geq k}) \leq \frac{(k-1)(n-1)}{2},$$

where K_k denotes the complete graph with k vertices and $\mathcal{C}_{\geq k}$ denotes the family of cycles of length at least k. The bound is sharp for every n congruent to one modulo $k-2$. Equality is attained by graphs with $\frac{n-1}{k-2}$ maximal 2-connected blocks each isomorphic to K_{k-1}. Recently, Li and Ning [3] proved that to obtain the same upper bound, it is enough to forbid only long even cycles

$$\mathrm{ex}(n, K_2, \mathcal{C}^{even}_{\geq 2k}) \leq \frac{(2k-1)(n-1)}{2}, \tag{1}$$

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where $\mathcal{C}^{even}_{\geq 2k}$ denotes the family of even cycles of length at least $2k$, that is $\{C_{2k}, C_{2k+2}, \dots\}$. We also denote the family of odd cycles of length at least $2k+1$ by $\mathcal{C}^{odd}_{\geq 2k+1} := \{C_{2k+1}, C_{2k+3}, \dots\}$. For a graph G and a family of graphs $\mathcal{F}$, we say G is $\mathcal{F}$-free, if for all graphs $F \in \mathcal{F}$, G does not contain F as a subgraph, not necessarily induced. For graphs G and H, let us denote the number of copies of H in G by $H(G)$.

Note that graphs with bounded odd circumferences might have a quadratic number of edges (as a function of the number of vertices) since the n-vertex complete balanced bipartite graph is odd cycle-free with $\left\lfloor \frac{n^2}{2} \right\rfloor$ edges. On the other hand, Voss and Zuluga [6] proved that every 2-connected graph G with minimum degree at least $k \geq 3$, with at least $2k+1$ vertices, contains an even cycle of length at least $2k$. Even more, if G is not bipartite, then it contains an odd cycle of length at least $2k-1$.

There are numerous papers strengthening, generalizing, and extending the celebrated Erdős and Gallai theorem. Recently, Luo [4] proved

$$\mathrm{ex}(n, K_r, \mathcal{C}_{\geq k}) \leq \frac{(n-1)}{k-2}\binom{k-1}{r}, \tag{2}$$

for all $n \geq k \geq 4$. Chakraborti and Chen [1] strengthened Luo's result by obtaining a sharp upper bound for every n. This bound is a great tool for obtaining results in hypergraph theory. The bound was consequently reproved with different methods multiple times [5,7]. In this paper, we strengthen Luo's theorem. In particular, we obtain the same tight bounds for graphs with bounded odd circumferences. On the other hand, the result for graphs with bounded even circumference is trivial after applying (1) and a result of Luo [4, Cor.1.7],

$$\mathrm{ex}(n, K_r, P_{k+1}) \leq \frac{n}{k}\binom{k}{r},$$

where P_{k+1} denotes the path of length k. Since the graph G does not contain a cycle of length $2k$, the neighborhood of each vertex contains no path of length $2k-2$. In particular, for all $r \geq 3$ we have

$$\begin{aligned}\mathrm{ex}(n, K_r, \mathcal{C}^{even}_{\geq 2k}) &\leq \frac{1}{r}\sum_{v} \frac{d(v)}{2k-2}\binom{2k-2}{r-1} \\ &\leq \frac{1}{r}\frac{\mathrm{ex}(n, K_2, \mathcal{C}^{even}_{\geq 2k})}{k-1}\binom{2k-2}{r-1} \leq \frac{n-1}{2k-2}\binom{2k-1}{r}.\end{aligned}$$

For graphs with small odd circumferences, we have the following theorem.

Theorem 1. *For integers n, k, r satisfying $n \geq 2k \geq r \geq 3$ and $k \geq 14$,*

$$\mathrm{ex}(n, K_r, \mathcal{C}^{odd}_{\geq 2k+1}) \leq \frac{n-1}{2k-1}\binom{2k}{r}.$$

Equality holds only for connected n-vertex graphs which consisting of $\frac{n-1}{2k-1}$ maximal 2-connected blocks each isomorphic to K_{2k}.

It is interesting to ask for which integers and for which congruence classes the same phenomenon still holds. In this direction, we would like to raise a modest conjecture.

Conjecture 2. *For an integer $k \geq 2$ and a sufficiently large n. Let G be an n vertex $C_{3\ell+1}$-free graph for every integer $\ell \geq k$. Then for every r, $3k \geq r \geq 2$, the number of cliques of size r in G is at most*

$$\frac{n-1}{3k-1}\binom{3k}{r}.$$

Equality holds only for connected n-vertex graphs consisting of $\frac{n-1}{3k-1}$ maximal 2-connected blocks each isomorphic to K_{3k}.

Moreover, we would like to propose the following conjecture. Let p be a prime number and $\mathcal{C}^{prime}_{\geq p} := \{C_\ell : \ell \geq p \text{ and } \ell \text{ prime}\}$.

Conjecture 3. *For integers n, r and a prime p satisfying $r < p$, we have*

$$\mathrm{ex}(n, K_r, \mathcal{C}^{prime}_{\geq p}) \leq \frac{n-1}{p-2}\binom{p-1}{r}.$$

Equality holds only for connected n-vertex graphs consisting of $\frac{n-1}{p-2}$ maximal 2-connected blocks each isomorphic to K_{p-1}.

2. Proof of the Main Result

Proof. We prove Theorem 1 by induction on the number of vertices of the graph. The base cases for $n \leq 2k$ are trivial. Let G be a graph on n vertices where $n > 2k$. We assume that every $\mathcal{C}^{odd}_{\geq 2k+1}$-free graph on m vertices, for $m < n$, contains at most $\frac{m-1}{2k-1}\binom{2k}{r}$ copies of K_r and that equality is achieved for the class of graphs described in the statement of the theorem.

If $\delta(G)$, the minimum degree of G, is at most $k+2$, then we are done by the induction hypothesis

$$K_r(G) \leq K_r(G[V(G) \setminus \{v\}]) + \binom{k+2}{r-1} \leq \frac{n-2}{2k-1}\binom{2k}{r} + \binom{k+2}{r-1}$$
$$< \frac{n-1}{2k-1}\binom{2k}{r},$$

since $k \geq 14$ and $r \geq 3$. From here, we may assume G is a graph with $\delta(G) > k+2$, and each edge of G is in a copy of K_r.

Let $v_1v_2v_3\cdots v_m$ be a longest path of G such that v_1 is adjacent to v_t and t is the maximum possible among the longest paths. Consider the family $\mathcal{P}$ of all longest paths of G on the vertex set $\{v_1, v_2, v_3, \ldots, v_m\}$ such that $v_tv_{t+1}\cdots v_m$ is a sub-path with a terminal vertex v_m. Let T_1 be the set of terminal vertices, excluding v_m, of paths from $\mathcal{P}$.

Claim 1. *If $t \leq 2k$, then v_t is a cut vertex isolating $\{v_1, v_2, \ldots, v_{t-1}\}$ from the rest of the graph.*

Proof. If $T_1 = \{v_1, v_2, \dots, v_{t-1}\}$, then by the maximality of t, the vertex v_t is a cut vertex isolating $\{v_1, v_2, \dots, v_{t-1}\}$ from the rest of the graph. Hence we may assume there exists a v_i, $i \leq t-1$, which is not in T_1. Note that $v_{t-1} \in T_1$ thus we have $i < t-1$. Let the path $u_1 u_2 \cdots u_{t-1} u_t v_{t+1} \cdots v_m$ be a path from $\mathcal{P}$ such that $u_r \in T_1$ and $u_{r+1} \notin T_1$ minimizing r. Note that $u_t = v_t$ and $\{v_1, v_2, \dots, v_t\} = \{u_1, u_2, \dots, u_t\}$.

Here, we show that $r+1 \leq t/2$. At first, we assume $i \leq t/2$, then $v_1 v_2 \cdots v_m$ is a path in $\mathcal{P}$ thus $r+1 \leq i$, we are done since $i \leq t/2$. If $t/2 \leq i$, then consider the following path from $\mathcal{P}$ $v_{t-1} v_{t-2} \cdots v_i \cdots v_1 v_t \cdots v_m$ is a path in $\mathcal{P}$, thus $r+1 \leq t-1-i+1 = t-i$, we are done since $t-i \leq t/2$.

The vertex u_1 is not adjacent to two consecutive vertices from $\{u_{r+1}, u_{r+2}, \dots, u_t\}$ by the minimality of r. Indeed if u_1 is adjacent to u_i and u_{i+1} for $r+1 \leq i \leq t$, then

$$u_2 u_3 \cdots u_r u_{r+1} \cdots u_i u_1 u_{i+1} \cdots u_t v_{t+1} \cdots v_m$$

forms a path from $\mathcal{P}$ that contradicts the minimality of r.

The vertex u_1 is not adjacent to vertices $u_{r+2}, u_{r+3}, \dots, u_{2r+1}$ by the minimality of r. Indeed, suppose u_1 is adjacent to u_i for $r+2 \leq i \leq 2r+1$, then we get a path from $\mathcal{P}$

$$u_{i-1} u_{i-2} \dots u_{r+1} u_r \dots u_1 u_i u_{i+1} \dots u_t v_{t+1} \dots v_m,$$

which contradicts the minimality of r since $u_{r+1} \notin T_1$.

Finally, by the two observations from the preceding two paragraphs we have $d(u_1) \leq (r-1) + \frac{t-(2r+1)+1}{2} = \frac{t-2}{2} < k$, a contradiction. □

Claim 2. *We have $t \leq 2k$.*

Proof. Suppose otherwise let us assume $t > 2k$. Let C denote the cycle $v_1 v_2 \dots v_t v_1$. Since G is $\mathcal{C}^{odd}_{\geq 2k+1}$-free t is even and vertices v_{t-3} and v_{t-2} have no common neighbors in $G[V(G) \setminus V(C)]$. On the other hand, since every edge is in a K_r there is a vertex v_l adjacent to both v_{t-3} and v_{t-2}. Let us denote $c := t - 2k$.

In this paragraph, we show that $c-1 \leq l \leq 2k-4$. Consider the following two cycles of consecutive lengths $v_\ell v_{\ell+1} \dots v_{t-3} v_\ell$ and $v_\ell v_{\ell+1} \dots v_{t-3} v_{t-2} v_\ell$. One of them has an odd length, thus we have $t-2-\ell+1 \leq 2k$ since G is $\mathcal{C}^{odd}_{\geq 2k+1}$-free. Consider the following two cycles of consecutive lengths $v_\ell v_{\ell-1} \dots v_1 v_t v_{t-1} v_{t-2} v_{t-3} v_\ell$ and $v_\ell v_{\ell-1} \dots v_1 v_t v_{t-1} v_{t-2} v_\ell$. One of them has an odd length, thus we have $\ell + 4 \leq 2k$ since G is $\mathcal{C}^{odd}_{\geq 2k+1}$-free. Thus, we have $c-1 \leq l \leq 2k-4$.

There is no i such that v_i is adjacent with v_{t-1} and v_{i+1} is adjacent with v_1. Since otherwise, the following cycle has an odd length greater than $2k$,

$$v_1 v_2 \dots v_i v_{t-1} v_{t-2} \dots v_{i+1} v_1.$$

Note that, since $v_{t-1}, v_1 \in T_1$ and from maximality of t, we have $N(v_1), N(v_{t-1}) \subseteq V(C)$. Since G is $\mathcal{C}^{odd}_{\geq 2k+1}$-free and $t \geq 2k+2$ we have $N(v_1) \cap N^+(v_{t-1}) = \emptyset$, where $N^+(v_{t-1})$ denotes the following set $\{v_{i+1} : v_i \in N(v_{t-1})\}$.

There is no i such that both $l < i < l + c - 2$ and $v_1 v_i$ is an edge of G. Since otherwise, one of the following cycles is an odd cycle longer than $2k$,

$$v_1 v_2 \dots v_l v_{t-2} v_{t-3} \dots v_i v_1 \text{ or } v_1 v_2 \dots v_l v_{t-3} v_{t-4} \dots v_i v_1.$$

Thus, we have

$$N(v_1) \cap \{v_{l+2}, \dots, v_{l+c-3}\} = \emptyset.$$

Similarly, there is no i such that $l < i < l + c$ and $v_{t-1} v_i$ is an edge of G. Since otherwise, one of the following cycles is an odd cycle longer than $2k$,

$$v_{t-1} v_t v_1 \dots v_l v_{t-2} v_{t-3} \dots v_i v_{t-1} \text{ or } v_{t-1} v_t v_1 \dots v_l v_{t-3} v_{t-4} \dots v_i v_{t-1}.$$

Thus, we have

$$N^+(v_{t-1}) \cap \{v_{l+2}, \dots, v_{l+c-3}\} = \emptyset.$$

Recall we have $N(v_1) \cap N^+(v_{t-1}) = \emptyset$, hence, we get

$$\begin{aligned} |N(v_1)| &\leq \left| \left(\{v_1, v_2, \dots, v_{2k+c}\} \setminus N^+(v_{t-1}) \right) \setminus \{v_{l+2}, \dots, v_{l+c-3}\} \right| \\ &\leq 2k + c - (k+3) - (c-4) < \delta(G), \end{aligned}$$

a contradiction. □

From Claims 1 and 2, G contains a 2-connected block of size $x \leq 2k$. After contracting the block to a vertex we get a $\mathcal{C}^{odd}_{\geq 2k+1}$-free graph on $n - x + 1$ vertices. By the convexity of a binomial function $\binom{\cdot}{r}$ and induction hypothesis, we see that G contains at most $\frac{n-1}{2k-1}\binom{2k}{r}$ cliques of size r and equality is achieved if and only if G is a connected graph and every maximal 2-connected component of G is isomorphic to K_{2k}. □

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