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Two Enriched Poset Polytopes

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Abstract. Stanley introduced and studied two lattice polytopes, the order polytope and chain polytope, associated with a finite poset. Recently, Ohsugi and Tsuchiya introduce an enriched version of them, called the enriched order polytope and enriched chain polytope. In this paper, we give a piecewise-linear bijection between these enriched poset polytopes, which is an enriched analogue of Stanley's transfer map and bijectively proves that they have the same Ehrhart polynomials. Also, we construct explicitly unimodular triangulations of two enriched poset polytopes, which are the order complexes of graded posets.

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1. Introduction

We assume that readers are familiar with the definition of a poset presented in [9, Chapter 3]. Let P be a finite poset with d elements. We denote by $\mathbb{R}^P$ the vector space of all real-valued functions on P, and identify $\mathbb{R}^P$ with the Euclidean space $\mathbb{R}^d$. The *order polytope* $\mathcal{O}(P)$ of P is the subset of $\mathbb{R}^P$ consisting of all functions $f : P \to \mathbb{R}$ satisfying the following two conditions:

(i) $0 \le f(v) \le 1$ for all $v \in P$;
(ii) If $x < y$ in P, then we have $f(x) \le f(y)$.

And the *chain polytope* $\mathcal{C}(P)$ of P is the subset of $\mathbb{R}^P$ consisting of all functions $g : P \to \mathbb{R}$ satisfying the following two conditions:

(i) $g(v) \ge 0$ for all $v \in P$;
(ii) If $v_1 > \cdots > v_r$ is a chain in P, then we have $g(v_1) + \cdots + g(v_r) \le 1$.

Then it is known (see [10, Corollary 1.3 and Theorem 2.2]) that $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are convex polytopes whose vertex sets are given by

$$\mathcal{F}(P) = \{\chi_F : F \text{ is an order filter of } P\},$$

$$\mathcal{A}(P) = \{\chi_A : A \text{ is an antichain of} P\},$$

respectively, where χ_S is the characteristic function of a subset $S \subset P$ defined by $\chi_S(v) = 1$ if $v \in S$ and 0 otherwise. Here, an *order filter* of P is a subset $F \subset P$, such that if $v \in F$ and $v < w$, then $w \in F$. In particular, we have

$$\mathcal{O}(P) = \operatorname{conv} \mathcal{F}(P), \quad \mathcal{C}(P) = \operatorname{conv} \mathcal{A}(P),$$

where $\operatorname{conv} S$ denotes the convex hull of S. These poset polytopes are related via the transfer map.

Theorem 1.1. (Stanley [10, Theorem 3.2]) *We define a piecewise-linear map* $\Phi : \mathbb{R}^P \to \mathbb{R}^P$*, called the* transfer map*, by*

$$(\Phi f)(v) = \begin{cases} f(v) & \text{if } v \text{ is minimal in} P, \\ f(v) - \max\{f(w) : v \text{ covers} w \text{ in} P\} & \text{if } v \text{ is not minimal in} P \end{cases} \tag{1}$$

for $f \in \mathbb{R}^P$ *and* $v \in P$*. Then* Φ *induces a continuous bijection from* $\mathcal{O}(P)$ *to* $\mathcal{C}(P)$*. In particular,* Φ *provides a bijection between* $m\mathcal{O}(P) \cap \mathbb{Z}^P$ *and* $m\mathcal{C}(P) \cap \mathbb{Z}^P$ *for any nonnegative integer* m*, where* $m\mathcal{P} = \{mf : f \in \mathcal{P}\}$ *is the* m*th dilation of a polytope* $\mathcal{P}$ *and* $\mathbb{Z}^P$ *is the set of all integer-valued functions on* P*.*

The transfer map enables us to compare certain properties of $\mathcal{O}(P)$ and $\mathcal{C}(P)$. For example, the two polytopes $\mathcal{O}(P)$ and $\mathcal{C}(P)$ have the same Ehrhart polynomials, that is

$$\#\left(m\mathcal{O}(P) \cap \mathbb{Z}^P\right) = \#\left(m\mathcal{C}(P) \cap \mathbb{Z}^P\right). \tag{2}$$

We note that the polynomial $\#\left(m\mathcal{O}(P) \cap \mathbb{Z}^P\right)$ in m is the order polynomial (with a shifted argument) of the poset P, which counts the number of P-partitions. A map $h : P \to \mathbb{Z}_{\geq 0}$, where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers, is called a *P-partition* if $v \leq w$ implies $h(v) \leq h(w)$. Then $m\mathcal{O}(P) \cap \mathbb{Z}^P$ is the set of all P-partitions $h : P \to \mathbb{Z}_{\geq 0}$, such that $h(v) \leq m$ for all $v \in P$. Hence, the transfer map Φ gives a bijection between such P-partitions and lattice points in the mth dilation of the chain polytope $\mathcal{C}(P)$. In a very recent work, Higashitani [4] proves that $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are combinatorially mutation-equivalent using the transfer map Φ. The notion of combinatorial mutation was introduced from viewpoints of mirror symmetry for Fano manifolds. Also, we can transfer a canonical triangulation of $\mathcal{O}(P)$, which is the order complex of a graded poset as simplicial complexes, to $\mathcal{C}(P)$ via the transfer map Φ.

Theorem 1.2. (Stanley [10, Section 5]) *For a chain* $C = \{F_1 \supsetneq F_2 \supsetneq \cdots \supsetneq F_k\}$ *of order filters of* P*, we put*

$$S_C = \operatorname{conv}\{\chi_{F_1}, \dots, \chi_{F_k}\}, \quad T_C = \operatorname{conv}\{\Phi(\chi_{F_1}), \dots, \Phi(\chi_{F_k})\}. \tag{3}$$

Then we have

(a) *The collection* $\mathcal{S}_P = \{S_C : C \text{ is a chain of order filters}\}$ *is a unimodular triangulation of the order polytope* $\mathcal{O}(P)$*.*

(b) *The collection* $\mathcal{T}_P = \{T_C : C \text{ is a chain of order filters}\}$ *is a unimodular triangulation of the chain polytope* $\mathcal{C}(P)$*.*

In the last decade, many authors have generalized order polytopes and chain polytopes together with their connecting transfer maps. These generalizations include marked poset polytopes [1] and double poset polytopes [2,3].

Recently from a viewpoint of (left) enriched P-partitions, Ohsugi–Tsuchiya [7,8] introduced an enriched version of order polytopes and chain polytopes as follows. We define $\mathcal{F}^{(e)}(P)$ and $\mathcal{A}^{(e)}(P)$ by putting

$$\mathcal{F}^{(e)}(P) = \left\{ f \in \mathbb{Z}^P : \begin{array}{l} \text{(i) } f(v) \in \{1, 0, -1\} \ \text{ for any } v \in P, \\ \text{(ii) } \operatorname{supp}(f) \ \text{ is an order filter of } P, \ \text{ and} \\ \text{(iii) if } f(v) = -1, \ \text{ then } v \ \text{ is minimal in } \operatorname{supp}(f) \end{array} \right\}, \tag{4}$$

$$\mathcal{A}^{(e)}(P) = \left\{ f \in \mathbb{Z}^P : \begin{array}{l} \text{(i) } f(v) \in \{1, 0, -1\} \ \text{ for any } v \in P, \ \text{ and} \\ \text{(ii) } \operatorname{supp}(f) \ \text{ is an antichain of } P \end{array} \right\}, \tag{5}$$

where $\operatorname{supp}(f) = \{v \in P : f(v) \neq 0\}$. Then the *enriched order polytope* $\mathcal{O}^{(e)}(P)$ and the *enriched chain polytope* $\mathcal{C}^{(e)}(P)$ are defined as the convex hulls of $\mathcal{F}^{(e)}(P)$ and $\mathcal{A}^{(e)}(P)$, respectively:

$$\mathcal{O}^{(e)}(P) = \operatorname{conv} \mathcal{F}^{(e)}(P), \quad \mathcal{C}^{(e)}(P) = \operatorname{conv} \mathcal{A}^{(e)}(P).$$

(Note that enriched chain polytopes were independently introduced by Kohl, Olsen, and Sanyal [5, Section 7] under the name of *unconditional chain polytopes*.) Since $\mathcal{F}(P) = \mathcal{F}^{(e)}(P) \cap \{0,1\}^P$ and $\mathcal{A}(P) = \mathcal{A}^{(e)}(P) \cap \{0,1\}^P$, we have $\mathcal{O}(P) \subset \mathcal{O}^{(e)}(P)$ and $\mathcal{C}(P) \subset \mathcal{C}^{(e)}(P)$. Then, as an enriched version of (2), Ohsugi–Tsuchiya [8] used a commutative algebra technique to prove

$$\# \left(m\mathcal{O}^{(e)}(P) \cap \mathbb{Z}^P \right) = \# \left(m\mathcal{C}^{(e)}(P) \cap \mathbb{Z}^P \right) \tag{6}$$

for any nonnegative integer m. It is a natural problem to find a bijective proof of this equality (6).

For a nonnegative integer m, we denote by $\mathcal{E}_m(P)$ the set of all left enriched P-partitions $h : P \to \mathbb{Z}$, such that $|h(v)| \leq m$ for any $v \in P$ (see Sect. 2.4 for the definition of left enriched P-partitions). Then we have $\mathcal{F}^{(e)}(P) = \mathcal{O}^{(e)}(P) \cap \mathbb{Z}^P = \mathcal{E}_1(P)$, and if $m \geq 2$, then a map $h \in m\mathcal{O}^{(e)}(P) \cap \mathbb{Z}^P$ is not always a left enriched P-partition (see [8, Example 4.2]). However it is known ([7, Theorem 0.2 and its proof]) that there is an explicit bijection between $m\mathcal{C}^{(e)}(P) \cap \mathbb{Z}^P$ and $\mathcal{E}_m(P)$. This and (6) are the reasons why we call $\mathcal{O}^{(e)}(P)$ and $\mathcal{C}^{(e)}(P)$ "enriched" poset polytopes.

One of the main results of this paper is the following theorem, which gives a bijective proof of (6).

Theorem 1.3. *We define a piecewise-linear map* $\Phi^{(e)} : \mathbb{R}^P \to \mathbb{R}^P$*, which we call the* enriched transfer map*, inductively on the ordering of* P*, such that the value of* $\Phi^{(e)}(f)$ *at* v *is equal to*

$$\begin{cases} f(v) & \text{if } v \text{ is minimal in } P, \\ f(v) - \max \left\{ \displaystyle\sum_{i=1}^{r} \left| \left(\Phi^{(e)} f \right) (v_i) \right| : v > v_1 > \cdots > v_r \ \text{ is a chain in } P \right\} & \\ \qquad \text{if } v \text{ is not minimal in } P. \end{cases} \tag{7}$$

Then $\Phi^{(e)}$ induces a continuous bijection from $\mathcal{O}^{(e)}(P)$ to $\mathcal{C}^{(e)}(P)$. In particular, $\Phi^{(e)}$ provides a bijection between $m\mathcal{O}^{(e)}(P) \cap \mathbb{Z}^P$ and $m\mathcal{C}^{(e)}(P) \cap \mathbb{Z}^P$ for any nonnegative integer m.

Moreover, by composing with $\Phi^{(e)}$, we also obtain an explicit bijection between $m\mathcal{O}^{(e)}(P) \cap \mathbb{Z}^P$ and $\mathcal{E}_m(P)$ for any nonnegative integer m (Proposition 2.9).

It can be shown (see Proposition 2.7) that the restriction of $\Phi^{(e)}$ to $\mathcal{O}(P)$ gives a continuous piecewise-linear bijection between $\mathcal{O}(P)$ and $\mathcal{C}(P)$, which coincides with the restriction of Stanley's transfer map Φ in Theorem 1.1. Also, by the same technique of [4], we can show that $\mathcal{O}^{(e)}(P)$ and $\mathcal{C}^{(e)}(P)$ are combinatorially mutation-equivalent using the enriched transfer map $\Phi^{(e)}$ (see [4, Section 5]).

Ohsugi–Tsuchiya [7,8] constructed triangulations of enriched order and chain polytopes using the algebraic technique of Gröbner bases. Also, Kohl–Olsen–Sanyal [5] constructed triangulations of enriched chain polytopes from a viewpoint of convex geometry. Another main result of this paper is an explicit combinatorial description of triangulations of two enriched poset polytopes, which are the order complexes of graded posets as simplicial complexes and are transferred by the enriched transfer map $\Phi^{(e)}$. Our result is analogous to Stanley's canonical triangulations of two poset polytopes (see Theorem 1.2).

Theorem 1.4. *We equip $\mathcal{F}^{(e)}(P)$ with a poset structure by the partial ordering given in Definition 3.1. For a chain K in $\mathcal{F}^{(e)}(P)$, we define*

$$S_K^{(e)} = \operatorname{conv} K, \quad T_K^{(e)} = \operatorname{conv} \Phi^{(e)}(K). \tag{8}$$

Then we have the following:

(a) *The set $\mathcal{S}_P^{(e)} = \{S_K^{(e)} : K \text{ is a chain in } \mathcal{F}^{(e)}(P)\}$ is a unimodular triangulation of $\mathcal{O}^{(e)}(P)$.*
(b) *The set $\mathcal{T}_P^{(e)} = \{T_K^{(e)} : K \text{ is a chain in } \mathcal{F}^{(e)}(P)\}$ is a unimodular triangulation of $\mathcal{C}^{(e)}(P)$.*

Remark that the partial ordering on $\mathcal{F}^{(e)}(P)$ given in Definition 3.1 is an extension of the inclusion ordering on the set of order filters of P, so the poset $\mathcal{F}(P)$ is the induced subposet of $\mathcal{F}^{(e)}(P)$. Stanley gave the defining inequalities of facets of the canonical triangulations $\mathcal{S}_P$ and $\mathcal{T}_P$ of $\mathcal{O}(P)$ and $\mathcal{C}(P)$ ([10, Section 5]). We also give sets of defining inequalities of facets of the triangulation $\mathcal{S}_P^{(e)}$ and $\mathcal{T}_P^{(e)}$ of $\mathcal{O}^{(e)}(P)$ and $\mathcal{C}^{(e)}(P)$ (Corollary 3.16 and Proposition 3.18). On the other hand, we identify these triangulations with Ohsugi–Tsuchiya's triangulations algebraically obtained in [7,8] (Propositions 4.2 and 4.4).

The rest of this paper is organized as follows. In Sect. 2, we prove Theorem 1.3, and give an explicit bijection between left enriched P-partitions and lattice points of the dilated enriched order polytope. Sect. 3 is devoted to the proof of Theorem 1.4. We also give sets of defining inequalities for the maximal faces. In Sect. 4, we prove that the triangulations described in Theorem 1.4 coincide with the Ohsugi–Tsuchiya's triangulations.

2. Enriched Transfer Map

In this section, we give a proof of Theorem 1.3, and we use the enriched transfer map to describe a bijection between left enriched P-partitions and lattice points of the dilated enriched order polytope.

2.1. Notations

In what follows, we use the following notations and terminologies. Let P be a finite poset. For v, $w \in P$, we say that v covers w, written $v \gtrdot w$, if $v > w$ and there is no element u such that $v > u > w$. Given an antichain A, we denote by $\langle A \rangle$ the smallest order filter containing A. Given an element $v \in P$, we put

$$P_{\leq v} = \{w \in P : w \leq v\}, \quad P_{<v} = \{w \in P : w < v\}.$$

For a subposet Q of P, we denote by $\max Q$ and $\min Q$ the set of maximal and minimal elements of Q, respectively. For a chain $C = \{v_1 > v_2 > \cdots > v_r\}$ of Q, we say that

- C is *saturated* if $v_i \gtrdot v_{i+1}$ for $i = 1, \ldots, r-1$;
- C is *maximal* if it is saturated and $v_1 \in \max Q$ and $v_r \in \min Q$.

Let $\mathrm{C}(Q)$, $\mathrm{SC}(Q)$, and $\mathrm{MC}(Q)$ be the sets of all chains, all saturated chains and all maximal chains, respectively. We denote by $\operatorname{top} C$ the maximum element of a chain C. For $f \in \mathbb{R}^P$ and a chain $C = \{v_1 > \cdots > v_r\}$, we define

$$\begin{aligned} S(f;C) &= |f(v_1)| + \cdots + |f(v_r)|, \\ T^+(f;C) &= -f(v_1) - 2f(v_2) - \cdots - 2^{r-2}f(v_{r-1}) + 2^{r-1}f(v_r), \\ T^-(f;C) &= -f(v_1) - 2f(v_2) - \cdots - 2^{r-2}f(v_{r-1}) - 2^{r-1}f(v_r). \end{aligned}$$

Note that, if C is a one-element chain $\{v\}$, then $T^+(f;\{v\}) = f(v)$ and $T^-(f;\{v\}) = -f(v)$.

2.2. Defining Inequalities for Enriched Poset Polytopes

Our proof of Theorem 1.3 is based on the defining inequalities of $\mathcal{O}^{(e)}(P)$ and $\mathcal{C}^{(e)}(P)$ given by [8].

Proposition 2.1. ([7, Lemma 1.1], [8, Proposition 6.1 and Theorem 6.2]) *We have*

$$\mathcal{O}^{(e)}(P) = \left\{ f \in \mathbb{R}^P : \begin{array}{l} T^+(f;C) \leq 1 \textit{ for all } C \in \mathrm{SC}(P) \textit{ with } \operatorname{top} C \in \max(P) \\ T^-(f;C) \leq 1 \textit{ for all } C \in \mathrm{MC}(P) \end{array} \right\}, \tag{9}$$

and

$$\mathcal{C}^{(e)}(P) = \left\{ g \in \mathbb{R}^P : S(g;C) \leq 1 \textit{ for all } C \in \mathrm{MC}(P) \right\}. \tag{10}$$

Example 2.2. Let Λ be the three-element poset on $\{u, v, w\}$ with covering relations $u \lessdot w$ and $v \lessdot w$. If we identify $\mathbb{R}^\Lambda$ with $\mathbb{R}^3$ by the correspondence $f \leftrightarrow (f(u), f(v), f(w))$, we have

$$\mathcal{F}^{(e)}(\Lambda) = \left\{ \begin{array}{l} (0,0,0), (0,0,1), (0,0,-1), (1,0,1), (-1,0,1), (0,1,1), (0,-1,1) \\ (1,1,1), (1,-1,1), (-1,1,1), (-1,-1,1) \end{array} \right\},$$

$$\mathcal{A}^{(e)}(\Lambda) = \left\{ \begin{array}{l} (0,0,0),(1,0,0),(-1,0,0),(0,1,0),(0,-1,0),(0,0,1),(0,0,-1) \\ (1,1,0),(1,-1,0),(-1,1,0),(-1,-1,0) \end{array} \right\},$$

and

$$\mathcal{O}^{(e)}(\Lambda) = \left\{ f \in \mathbb{R}^{\Lambda} : \begin{array}{c} f(w) \le 1 \\ -f(u)+2f(w) \le 1,\ -f(v)+2f(w) \le 1 \\ -f(u)-2f(w) \le 1,\ -f(u)-2f(w) \le 1 \end{array} \right\},$$

$$\mathcal{C}^{(e)}(\Lambda) = \left\{ g \in \mathbb{R}^{\Lambda} : |g(u)|+|g(w)| \le 1,\ |g(v)|+|g(w)| \le 1 \right\}.$$

2.3. Proof of Theorem 1.3

In this subsection, we prove Theorem 1.3. The inductive definition (7) of $\Phi^{(e)}$ can be written as

$$\left(\Phi^{(e)}(f)\right)(v) = \begin{cases} f(v) & \text{if } v \text{ is minimal in } P, \\ f(v) - \max\{S(\Phi^{(e)}(f); C) : C \in \mathrm{C}(P_{<v})\} & \text{if } v \text{ is not minimal in } P. \end{cases} \tag{11}$$

It is easy to see that the map $\Phi^{(e)} : \mathbb{R}^P \to \mathbb{R}^P$ is a bijection.

Lemma 2.3. *The map $\Phi^{(e)} : \mathbb{R}^P \to \mathbb{R}^P$ is a bijection with inverse map $\Psi^{(e)}$ given by*

$$\left(\Psi^{(e)}(g)\right)(v) = \begin{cases} g(v) & \text{if } v \text{ is minimal in } P, \\ g(v) + \max\{S(g; C) : C \in \mathrm{C}(P_{<v})\} & \text{if } v \text{ is not minimal in } P. \end{cases} \tag{12}$$

Here, we note that

$$\max\{S(g;C) : C \in \mathrm{C}(P_{<v})\} = \max\{S(g;C) : C \in \mathrm{MC}(P_{<v})\};$$

hence, we may replace $\mathrm{C}(P_{<v})$ with $\mathrm{MC}(P_{<v})$ in (11) and (12). The following proposition follows from the definitions of $\Phi^{(e)}$ and $\Psi^{(e)}$.

Proposition 2.4. (a) *For $f \in \mathcal{F}^{(e)}(P)$, we have*

$$(\Phi^{(e)}(f))(v) = \begin{cases} f(v) & \text{if } v \text{ is minimal in } \operatorname{supp}(f), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\Phi^{(e)}(f) \in \mathcal{A}^{(e)}(P)$ and $\operatorname{supp} \Phi^{(e)}(f) = \min(\operatorname{supp}(f))$.

(b) *For $g \in \mathcal{A}^{(e)}(P)$, we have*

$$(\Psi^{(e)}(g))(v) = \begin{cases} 1 & \text{if } v \in \langle \operatorname{supp}(g) \rangle \setminus \min\langle \operatorname{supp}(g) \rangle, \\ g(v) & \text{if } v \in \min\langle \operatorname{supp}(g) \rangle, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\Psi^{(e)}(g) \in \mathcal{F}^{(e)}(P)$ and $\operatorname{supp} \Psi^{(e)}(g) = \langle \operatorname{supp}(g) \rangle$.

(c) *The map $\Phi^{(e)}$ induces a bijection between $\mathcal{F}^{(e)}(P)$ and $\mathcal{A}^{(e)}(P)$.*

To prove Theorem 1.3, we need to prepare two lemmas. We put

$$M(g; P_{\le v}) = \max\{S(g; C) : C \in \mathrm{MC}(P_{\le v})\},$$
$$M(g; P_{< v}) = \max\{S(g; C) : C \in \mathrm{MC}(P_{< v})\}.$$

Lemma 2.5. *Let $f \in \mathbb{R}^P$ and $v \in P$. We put*

$$\mathcal{T}(f; v) = \{T^+(f; C) : C \in \mathrm{SC}(P_{\le v}) \text{ with } \operatorname{top} C = v\} \cup \{T^-(f; C) : C \in \mathrm{MC}(P_{\le v})\}.$$

Then, for any $C \in \mathrm{MC}(P_{\le v})$, there exists an element $T \in \mathcal{T}(f; v)$ such that $S(\Phi^{(e)}(f); C) \le T$.

Proof. We write $g = \Phi^{(e)}(f)$. We proceed by induction on the ordering of P. If v is a minimal element, then C is a one-element chain $\{v\}$ and

$$S(g; C) = |g(v)| = |f(v)| = \begin{cases} f(v) = T^+(f; C) & \text{if } f(v) \ge 0, \\ -f(v) = T^-(f; C) & \text{if } f(v) \le 0. \end{cases}$$

If v is not a minimal element, then by definition

$$g(v) = f(v) - M(g; P_{<v}).$$

Let $C = \{v = v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r\}$. Since $C\backslash\{v\} = \{v_2 \gtrdot \cdots \gtrdot v_r\} \in \mathrm{MC}(P_{<v})$, we have

$$S(g; C \setminus \{v\}) \le M(g; P_{<v}).$$

If $g(v) = f(v) - M(v; P_{<v}) \ge 0$, then we have

$$S(g; C) = f(v) - M(g; P_{<v}) + S(g; C \setminus \{v\}) \le f(v) = T^+(f; \{v\}).$$

If $g(v) \le 0$, then we have

$$S(g; C) = -f(v) + M(g; P_{<v}) + S(g; C \setminus \{v\}) \le -f(v) + 2M(g; P_{<v}).$$

Let $C' \in \mathrm{MC}(P_{<v})$ be a chain which attains the maximum $M(g; P_{<v})$. Then, by applying the induction hypothesis to C' and $w = \operatorname{top} C'$, there exists a chain C'' satisfying one of the following conditions:

(i) $C'' \in \mathrm{SC}(P_{\le w})$ with $\operatorname{top} C'' = w$ and $S(g; C') \le T^+(g; C'')$;
(ii) $C'' \in \mathrm{MC}(P_{\le w})$ and $S(g; C') \le T^-(g; C'')$.

In the case (i), we have

$$S(g; C) \le -f(v) + 2S(g; C') \le -f(v) + 2T^+(g; C'') = T^+(g; \{v\} \cup C''),$$

and in the case (ii), we have

$$S(g; C) \le -f(v) + 2S(g; C') \le -f(v) + 2T^-(g; C'') = T^-(g; \{v\} \cup C'').$$

Since $v \gtrdot w$, we can complete the proof. □

Lemma 2.6. *Let $g \in \mathbb{R}^P$ and $v \in P$. For a chain $C = \{v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r\} \in \mathrm{SC}(P_{\leq v_1})$, we have*

$$2^{r-1}\left(|g(v_r)| + M(g; P_{<v_r})\right) + \sum_{i=1}^{r-1} 2^{r-i-1}\left(|g(v_{r-i})| - M(g; P_{<v_{r-i}})\right) \leq M(g; P_{\leq v_1}).$$

Proof. We proceed by induction on r. If $r = 1$, then

$$\begin{aligned} |g(v_1)| + M(g; P_{<v_1}) &= |g(v_1)| + \max\{S(g; C') : C' \in \mathrm{MC}(P_{<v_1})\} \\ &= \max\{|g(v_1)| + S(g; C') : C' \in \mathrm{MC}(P_{<v_1})\} \\ &= \max\{S(g; C) : C \in \mathrm{MC}(P_{\leq v_1})\} \\ &= M(g; P_{\leq v_1}). \end{aligned}$$

Let $r \geq 2$. Since $\{v_r\} \cup C' \in \mathrm{MC}(P_{<v_{r-1}})$ for any $C' \in \mathrm{MC}(P_{<v_r})$, we have

$$\begin{aligned} |g(v_r)| + M(g; P_{<v_r}) &= |g(v_r)| + \max\{S(g; C') : C' \in MC(P_{<v_r})\} \\ &\leq \max\{S(g; C'') : C'' \in \mathrm{MC}(P_{<v_{r-1}})\} = M(g; P_{<v_{r-1}}). \end{aligned}$$

Hence, we have

$$\begin{aligned} &2^{r-1}\left(|g(v_r)| + M(g; P_{<v_r})\right) + 2^{r-2}\left(|g(v_{r-1})| - M(g; P_{<v_{r-1}})\right) \\ &\quad \leq 2^{r-1} M(g; P_{<v_{r-1}}) + 2^{r-2}\left(|g(v_{r-1})| - M(g; P_{<v_{r-1}})\right) \\ &\quad = 2^{r-2}\left(|g(v_{r-1})| + M(g; P_{<v_{r-1}})\right). \end{aligned}$$

Therefore, using the induction hypothesis, we see that

$$\begin{aligned} &2^{r-1}\left(|g(v_r)| + M(g; P_{<v_r})\right) + \sum_{i=1}^{r-1} 2^{r-i-1}\left(|g(v_{r-i})| - M(g; P_{<v_{r-i}})\right) \\ &\quad \leq 2^{r-2}\left(|g(v_{r-1})| + M(g; P_{<v_{r-1}})\right) + \sum_{i=2}^{r-1} 2^{r-i-1}\left(|g(v_{r-i})| - M(g; P_{<v_{r-i}})\right) \\ &\quad \leq M(g; P_{\leq v_1}). \end{aligned}$$

This completes the proof. □

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. First, we shall prove that $f \in \mathcal{O}^{(e)}(P)$ implies $\Phi^{(e)}(f) \in \mathcal{C}^{(e)}(P)$. Let $f \in \mathcal{O}^{(e)}(P)$ and put $g = \Phi^{(e)}(f)$. We show that $S(g; C) \leq 1$ for all maximal chains $C = \{v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r\} \in \mathrm{MC}(P)$. By Lemma 2.5, there exists a chain C' satisfying one of the following conditions:

(i) $C' \in \mathrm{SC}(P_{\leq v_1})$ with $\mathrm{top}\, C' = v_1$ and $S(g; C) \leq T^+(f; C')$;
(ii) $C' \in \mathrm{MC}(P_{\leq v_1})$ and $S(g; C) \leq T^-(f; C')$.

Then it follows from (9) in Proposition 2.1 that $S(g; C) \leq 1$. Hence, using (10), we conclude that $g \in \mathcal{C}^{(e)}(P)$.

Conversely, we show that $g \in \mathcal{C}^{(e)}(P)$ implies $\Psi^{(e)}(g) \in \mathcal{O}^{(e)}(P)$. Let $g \in \mathcal{C}^{(e)}(P)$ and put $f = \Psi^{(e)}(g)$. We need to prove that $T^+(f; C) \leq 1$ for all $C \in \mathrm{SC}(P)$ with $\mathrm{top}\, C \in \max(P)$ and that $T^-(f; C) \leq 1$ for all $C \in \mathrm{MC}(P)$.

Suppose $C = \{v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r\} \in \mathrm{SC}(P)$ with $v_1 \in \max(P)$. Then by definition

$$T^+(f;C) = 2^{r-1} f(v_r) - \sum_{i=1}^{r-1} 2^{r-i-1} f(v_{r-i})$$
$$= 2^{r-1} \left(g(v_r) + M(g;P_{<v_r})\right) - \sum_{i=1}^{r-1} 2^{r-i-1} \left(g(v_{r-i}) + M(g;P_{<v_{r-i}})\right).$$

Using $x \le |x|$ and $-x \le |x|$, we see that

$$T^+(f;C) \le 2^{r-1} \left(|g(v_r)| + M(g;P_{<v_r})\right) + \sum_{i=1}^{r-1} 2^{r-i-1} \left(|g(v_{r-i})| - M(g;P_{<v_{r-i}})\right).$$

Then, using Lemma 2.6, we obtain

$$T^+(f;C) \le M(g;P_{\le v_1}) = \max\{S(g;C') : C' \in \mathrm{MC}(P_{\le v_1})\}.$$

Since $S(g;C') \le 1$ for all $C' \in \mathrm{MC}(P_{\le v_1})$ by (10), we have $T^+(f;C) \le 1$.

Suppose $C = \{v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r\} \in \mathrm{MC}(P)$. Then $v_1 \in \max(P)$ and $v_r \in \min(P)$. It follows from the definition that

$$T^-(f;C) = -2^{r-1} f(v_r) - \sum_{i=1}^{r-1} 2^{r-i-1} f(v_{r-i})$$
$$= -2^{r-1} g(v_r) - \sum_{i=1}^{r-1} 2^{r-i-1} \left(g(v_{r-i}) + M(g;P_{<v_{r-i}})\right).$$

Using $x \le |x|$ and $-x \le |x|$, we see that

$$T^-(f;C) \le 2^{r-1} |g(v_r)| + \sum_{i=1}^{r-1} 2^{r-i-1} \left(|g(v_{r-i})| - M(g;P_{<v_{r-i}})\right).$$

Since $\{v_r\} \in \mathrm{MC}(P_{<v_{r-1}})$, we have $|g(v_r)| \le M(g;P_{<v_{r-1}})$. Hence, we have

$$T^-(f;C) \le 2^{r-1} M(g;P_{<v_{r-1}}) + 2^{r-2} \left(|g(v_{r-1})| - M(g;P_{<v_{r-1}})\right)$$
$$+ \sum_{i=2}^{r-1} 2^{r-i-1} \left(|g(v_{r-i})| - M(g;P_{<v_{r-i}})\right)$$
$$= 2^{r-2} \left(|g(v_{r-1})| + M(g;P_{<v_{r-1}})\right)$$
$$+ \sum_{i=2}^{r-1} 2^{r-i-1} \left(|g(v_{r-i})| - M(g;P_{<v_{r-i}})\right).$$

Now, we can use Lemma 2.6 and (9) to obtain $T^-(f;C) \le M(g;P_{v_1}) \le 1$.

Therefore, we conclude that $f \in \mathcal{O}^{(e)}(P)$. □

Here, we show that the bijection $\Phi^{(e)} : \mathcal{O}^{(e)}(P) \to \mathcal{C}^{(e)}(P)$ restricts to the bijection $\Phi : \mathcal{O}(P) \to \mathcal{C}(P)$.

Proposition 2.7. *The restriction of the enriched transfer map $\Phi^{(e)}$ to $\mathcal{O}(P)$ coincides with the restriction of the transfer map Φ to $\mathcal{O}(P)$.*

Proof. Let $f \in \mathcal{O}(P)$ and put $g = \Phi(f)$, $\widetilde{g} = \Phi^{(e)}(f)$. By using the induction on the ordering of P, we prove

$$\max\{f(w) : w \lessdot v\} = \max\{g(v_1) + \cdots + g(v_r) : \{v_1 \gtrdot \cdots \gtrdot v_r\} \in \mathrm{MC}(P_{<v})\}, \tag{13}$$

$$\widetilde{g}(v) = g(v) \geq 0. \tag{14}$$

If v is minimal in P, then $f(v) = g(v) = \widetilde{g}(v)$. If v is not minimal in P and $\{w \in P : w \lessdot v\} = \{w_1, \ldots, w_k\}$, then it follows from the induction hypothesis for (14) that

$$\begin{aligned}&\max\{|\widetilde{g}(v_1)| + \cdots + |\widetilde{g}(v_r)| : \{v_1 \gtrdot \cdots \gtrdot v_r\} \in \mathrm{MC}(P_{<v})\}\\ &\quad= \max_{1\leq i\leq k}\left\{g(w_i) + \max\{g(v_2) + \cdots + v(v_r) : \{v_2 \gtrdot \cdots \gtrdot v_r\} \in \mathrm{MC}(P_{<w_i})\}\right\}.\end{aligned}$$

Using the induction hypothesis for (13) and (1), we obtain

$$\begin{aligned}&\max\{|\widetilde{g}(v_1)| + \cdots + |\widetilde{g}(v_r)| : \{v_1 \gtrdot \cdots \gtrdot v_r\} \in \mathrm{MC}(P_{<v})\}\\ &\quad= \max_{1\leq i\leq k}\left\{g(w_i) + \max\{f(u_i) : u_i \lessdot w_i\}\right\}\\ &\quad= \max_{1\leq i\leq k} f(w_i) = \max\{f(w) : w \lessdot v\}.\end{aligned}$$

Hence, comparing (11) with (1), we obtain (13) and (14). □

2.4. Left Enriched *P*-Partitions

In this subsection, we use the enriched transfer map to find a bijection from left enriched P-partitions to lattice points of the dilated enriched order polytope.

Recall the definition of left enriched P-partition introduced by Petersen [6]. A map $h : P \to \mathbb{Z}$ is called a *left enriched P-partition* if it satisfies the following two conditions:

(i) If $v \leq w$, then $|h(v)| \leq |h(w)|$;
(ii) If $v \leq w$ and $|h(v)| = |h(w)|$, then $h(w) \geq 0$.

We denote by $\mathcal{E}_m(P)$ the set of left enriched P-partitions $h : P \to \mathbb{Z}$, such that $|h(v)| \leq m$ for all $v \in P$. Note that $\mathcal{F}^{(e)}(P) = \mathcal{E}_1(P)$. Ohsugi–Tsuchiya [7] gave an explicit bijection between $\mathcal{E}_m(P)$ and $m\mathcal{C}^{(e)}(P) \cap \mathbb{Z}^P$.

Proposition 2.8. ([7, Theorem 0.2 and its proof]) *Let $\Pi : \mathcal{E}_m(P) \to \mathbb{R}^P$ be the map defined by*

$$(\Pi(h))(v) = \begin{cases} h(v) & \text{if } v \text{ is minimal in } P,\\ h(v) - \max\{|h(w)| : w \lessdot v\} & \text{if } v \text{ is not minimal in } P \text{ and } h(v) \geq 0,\\ h(v) + \max\{|h(w)| : w \lessdot v\} & \text{if } v \text{ is not minimal in } P \text{ and } h(v) < 0.\end{cases} \tag{15}$$

Then Π gives a bijection from $\mathcal{E}_m(P)$ to $m\mathcal{C}^{(e)}(P) \cap \mathbb{Z}^P$.

By composing this bijection Π with the inverse enriched transfer map $\Psi^{(e)} : m\mathcal{C}^{(e)}(P) \cap \mathbb{Z}^P \to m\mathcal{O}^{(e)}(P) \cap \mathbb{Z}^P$, we obtain an explicit bijection from $\mathcal{E}_m(P)$ to $m\mathcal{O}^{(e)}(P) \cap \mathbb{Z}^P$.

Proposition 2.9. *Let* $\Theta : \mathcal{E}_m(P) \to \mathbb{R}^P$ *be the map defined by*

$$(\Theta(h))\,(v) = \begin{cases} h(v) & \text{if } v \text{ is minimal in } P \text{ or } h(v) \geq 0, \\ h(v) + 2\max\{|h(w)| : w \lessdot v\} & \text{if } v \text{ is not minimal in } P \text{ and } h(v) < 0. \end{cases} \tag{16}$$

Then Θ *gives a bijection from* $\mathcal{E}_m(P)$ *to* $m\mathcal{O}^{(e)}(P) \cap \mathbb{Z}^P$.

Proof. We show that $\Theta = \Psi^{(e)} \circ \Pi$. Let $h \in \mathcal{E}_m(P)$ and put $g = \Pi(h)$. By comparing (11) with (15) and (16), it is enough to show

$$\max\{S(g;C) : C \in \mathrm{MC}(P_{<v})\} = \max\{|h(w)| : w \lessdot v\}. \tag{17}$$

We proceed by induction on the ordering of P. If v is minimal in P, there is nothing to prove. Suppose that v is not minimal in P. Since $h \in \mathcal{E}_m(P)$, we have $|h(v)| \geq \max\{|h(w)| : w \lessdot v\}$. Then it follows from (15) that

$$|g(v)| = |h(v)| - \max\{|h(w)| : w \lessdot v\}. \tag{18}$$

If $\{w \in P : w \lessdot v\} = \{w_1, \dots, w_k\}$, then we have

$$\begin{aligned} &\max\{S(g;C) : C \in \mathrm{MC}(P_{<v})\} \\ &\quad = \max_{1 \leq i \leq k} \{|g(w_i)| + \max\{S(g;C') : C' \in \mathrm{MC}(P_{<w_i})\}\}. \end{aligned}$$

Using (18), we have

$$\begin{aligned} \max\{S(g;C) : C \in \mathrm{MC}(P_{<v})\} &= \max_{1 \leq i \leq k} \Big\{|g(w_i)| + \max\{|h(u_i)| : u_i \lessdot w_i\}\Big\} \\ &= \max_{1 \leq i \leq k} \{|h(w_i)|\}, \end{aligned}$$

from which (17) follows. □

2.5. Vertices of Enriched Poset Polytopes

In this subsection, we determine the vertex sets of the enriched order polytope $\mathcal{O}^{(e)}(P)$ and the enriched chain polytope $\mathcal{C}^{(e)}(P)$.

To state the result, we need a partial ordering $\preceq$ on $\mathcal{F}^{(e)}(P)$ or $\mathcal{A}^{(e)}(P)$. For f, $f' \in \mathcal{F}^{(e)}(P)$ (or $\mathcal{A}^{(e)}(P)$), we write $f \preceq f'$ if $\mathrm{supp}(f) \subset \mathrm{supp}(f')$ and $f|_{\mathrm{supp}(f)} = f'|_{\mathrm{supp}(f)}$.

Example 2.10. If $\Lambda = \{u, v, w\}$ is the three-element chain with covering relations $u \lessdot w$ and $v \lessdot w$, then the Hasse diagrams of $\mathcal{F}^{(e)}(P)$ and $\mathcal{A}^{(e)}(P)$ with respect to $\preceq$ are shown in Figs. 1 and 2 respectively.

The enriched order polytope $\mathcal{O}^{(e)}(\Lambda)$ is the pyramid with five vertices $(1,1,1), (1,-1,1), (-1,-1,1), (-1,1,1)$, and $(0,0,-1)$, while the enriched chain polytope $\mathcal{C}^{(e)}(\Lambda)$ is the bipyramid with six vertices $(1,1,1), (1,-1,1)$, $(-1,-1,1), (-1,1,1), (0,0,1)$, and $(0,0,-1)$.

Proposition 2.11. (a) *A point* $f \in \mathcal{F}^{(e)}(P)$ *is a vertex of* $\mathcal{O}^{(e)}(P)$ *if and only if* f *is maximal with respect to the ordering* $\preceq$.

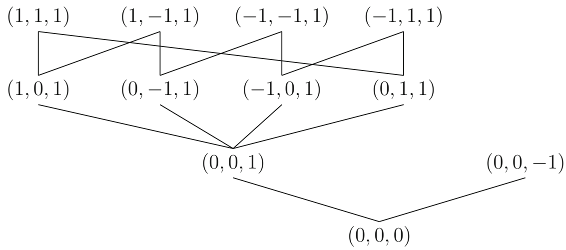

FIGURE 1. Hasse diagram of $(\mathcal{F}^{(e)}(\Lambda), \preceq)$

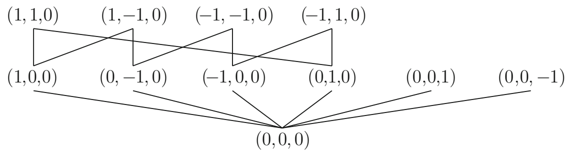

FIGURE 2. Hasse diagram of $(\mathcal{A}^{(e)}(\Lambda), \preceq)$

(b) *A point* $f \in \mathcal{A}^{(e)}(P)$ *is a vertex of* $\mathcal{C}^{(e)}(P)$ *if and only if* f *is maximal with respect to the ordering* $\preceq$.

Note that $f \in \mathcal{A}^{(e)}(P)$ is maximal with respect to $\preceq$ if and only if $\mathrm{supp}(f)$ is a maximal antichain.

Proof. (a) Let f be a maximal element of $\mathcal{F}_P^{(e)}$ with respect to $\preceq$. Assume to the contrary that f is not a vertex of $\mathcal{O}^{(e)}(P)$. Then there exist elements $g_1, \dots, g_r \in \mathcal{F}^{(e)}(P)$ and positive real numbers $\lambda_1, \dots, \lambda_r$, such that $g_i \neq f$ and

$$f = \sum_{i=1}^{r} \lambda_i g_i, \quad \sum_{i=1}^{r} \lambda_i = 1.$$

Considering the value at $v \in P$, we have

$$\sum_{i=1}^{r} \lambda_i g_i(v) = f(v) = \sum_{i=1}^{r} \lambda_i f(v).$$

If $f(v) = 1$, then we see that $\sum_{i=1}^{r} \lambda_i (1 - g_i(v)) = 0$. Since $\lambda_i > 0$ and $1 - g_i(v) \geq 0$, we obtain $g_i(v) = 1$ for all i. By a similar reasoning, we see that, if $f(v) = -1$, then we have $g_i(v) = -1$ for all i. Hence, we have $\mathrm{supp}(f) \subset \mathrm{supp}(g_i)$ and $f|_{\mathrm{supp}(f)} = g_i|_{\mathrm{supp}(f)}$. Since f is maximal with respect to $\preceq$, we have $f = g_i$, which contradicts to the assumption $g_i \neq f$. Therefore, f is a vertex of $\mathcal{O}^{(e)}(P)$.

Conversely, suppose that f is not maximal with respect to $\preceq$. Then there exists $g \in \mathcal{F}^{(e)}(P)$ such that $\operatorname{supp}(f) \subsetneq \operatorname{supp}(g)$ and $f|_{\operatorname{supp}(f)} = g|_{\operatorname{supp}(f)}$. We take a maximal element u of $\operatorname{supp}(g) \setminus \operatorname{supp}(f)$ and define f', $f'' : P \to \mathbb{R}$ by

$$f'(v) = \begin{cases} f(v) = g(v) & \text{if } v \in \operatorname{supp}(f), \\ g(u) & \text{if } v = u, \\ 0 & \text{otherwise}, \end{cases} \qquad f''(v) = \begin{cases} f(v) = g(v) & \text{if } v \in \operatorname{supp}(f), \\ -g(u) & \text{if } v = u, \\ 0 & \text{otherwise}. \end{cases}$$

Then $\operatorname{supp}(f') = \operatorname{supp}(f'') = \operatorname{supp}(f) \sqcup \{u\}$ is an order filter of P and u is an minimal element of $\operatorname{supp}(f') = \operatorname{supp}(f'')$. Hence, $f' \in \mathcal{F}^{(e)}(P)$. Since $f = (f' + f'')/2$, we see that f is not a vertex of $\mathcal{O}^{(e)}(P)$.

(b) Similar to (a). □

Remark 2.12. (1) A characterization of the vertex set of the enriched chain polytope $\mathcal{C}^{(e)}(P)$ is also given in [5, Section 7].

(2) In general, the image $\Phi^{(e)}(f)$ of a vertex f of $\mathcal{O}^{(e)}(P)$ under the enriched transfer map $\Phi^{(e)}$ is not a vertex of $\mathcal{C}^{(e)}(P)$, and the number of vertices of $\mathcal{O}^{(e)}(P)$ is different from that of $\mathcal{C}^{(e)}(P)$ (see Example 2.10).

3. Triangulations

In this section, we prove Theorem 1.4, which describes triangulations of enriched order and chain polytopes.

3.1. Poset Structure on $\mathcal{F}^{(e)}(P)$

We introduce a partial ordering $\geq$ on $\mathcal{F}^{(e)}(P)$, which is an extension of the inclusion ordering on the set of order filters of P. Note that this ordering $\geq$ is different from the ordering $\succeq$ used in Sect. 2.5.

Definition 3.1. For f, $g \in \mathcal{F}^{(e)}(P)$, we write $f > g$ if the following three conditions hold:

(i) $\operatorname{supp}(f) \supsetneq \operatorname{supp}(g)$;
(ii) $f(v) \geq g(v)$ for any $v \in \operatorname{supp}(g)$;
(iii) If $v \in \operatorname{supp}(g)$ and v is minimal in $\operatorname{supp}(f)$, then $f(v) = g(v)$.

Also, we write $f \geq g$ if $f = g$ or $f > g$.

The following lemma is obvious, but will be used in several places.

Lemma 3.2. *If $F \supset G$ are order filters of P and $v \in G$ is minimal in F, then v is minimal in G.*

Using this lemma, we can prove that $\mathcal{F}^{(e)}(P)$ is equipped with a poset structure with respect to the binary relation $\geq$.

Lemma 3.3. *The binary relation $\geq$ given in Definition 3.1 is a partial ordering on $\mathcal{F}^{(e)}(P)$.*

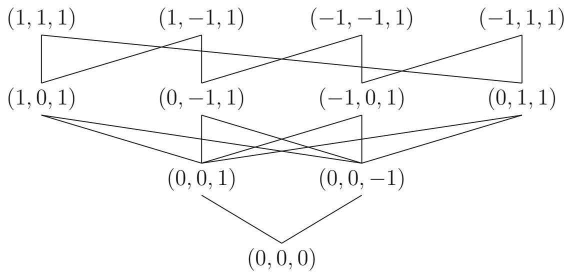

FIGURE 3. Hasse diagram of $(\mathcal{F}^{(e)}(\Lambda), \geq)$

Proof. It is enough to show the transitivity. Let f, g, $h \in \mathcal{F}^{(e)}(P)$ satisfy $f > g$ and $g > h$. Then it is clear that $\operatorname{supp}(f) \supsetneq \operatorname{supp}(h)$ and $f(v) \geq h(v)$ for any $v \in \operatorname{supp}(h)$. Since $\operatorname{supp}(f) \supset \operatorname{supp}(g) \supset \operatorname{supp}(h)$, it follows from Lemma 3.2 that, if $v \in \operatorname{supp}(h)$ is minimal in $\operatorname{supp}(f)$, then we have $f(v) = g(v) = h(v)$. □

Example 3.4. Let Λ be the three-element poset on $\{u, v, w\}$ with covering relations $u \lessdot w$ and $v \lessdot w$. Figure 3 shows the Hasse diagram of $(\mathcal{F}^{(e)}(\Lambda), \geq)$.

We collect several properties of this partial ordering on $\mathcal{F}^{(e)}(P)$.

Proposition 3.5. *The resulting poset $\mathcal{F}^{(e)}(P)$ has the following properties:*

(a) *For order filters F and G, we have $F \supset G$ if and only if $\chi_F \geq \chi_G$ in $\mathcal{F}^{(e)}(P)$, where χ_S is the characteristic function of S.*
(b) *The zero map 0 is the unique minimal element of $\mathcal{F}^{(e)}(P)$.*
(c) *If f covers g in $\mathcal{F}^{(e)}(P)$, then $\# \operatorname{supp}(f) = \# \operatorname{supp}(g) + 1$.*
(d) *If f is a maximal element in $\mathcal{F}^{(e)}(P)$, then $\operatorname{supp}(f) = P$.*
(e) *All maximal chains of $\mathcal{F}^{(e)}(P)$ have the same length $d = \#P$.*

Proof. (a) and (b) are obvious.

(c) It is enough to show that, if $f > g$, then there exists $h \in \mathcal{F}^{(e)}(P)$, such that $f \geq h > g$ and $\# \operatorname{supp}(h) = \# \operatorname{supp}(g) + 1$.

Since $\operatorname{supp}(f) \supsetneq \operatorname{supp}(g)$ and they are order filters of P, there exists $u \in \operatorname{supp}(f)$ such that $\operatorname{supp}(g) \cup \{u\}$ is an order filter of P. Then we define $h : P \to \{1, 0, -1\}$ by putting

$$h(v) = \begin{cases} f(v) & \text{if } v \in \operatorname{supp}(g) \cup \{u\}, \\ 0 & \text{otherwise.} \end{cases}$$

We see that $h \in \mathcal{F}^{(e)}(P)$, $\operatorname{supp}(h) = \operatorname{supp}(g) \cup \{u\}$, and $f \geq h > g$.

(d) Suppose that $\operatorname{supp}(g) \neq P$. Since $\operatorname{supp}(g)$ is a proper order filter of P, there exists $u \notin \operatorname{supp}(g)$ such that $\operatorname{supp}(g) \cup \{u\}$ is an order filter. Define $f : P \to \{1, 0, -1\}$ by putting

$$f(v) = \begin{cases} 1 & \text{if } v = u \text{ or } v \text{ covers } u, \\ g(v) & \text{otherwise.} \end{cases}$$

Then we have $f \in \mathcal{F}^{(e)}(P)$, $\mathrm{supp}(f) = \mathrm{supp}(g) \cup \{u\}$ and $f > g$.

(e) follows from (b), (c) and (d). □

Next, we consider chains in the poset $\mathcal{F}^{(e)}(P)$.

Definition 3.6. Given a chain $K = \{f_1 > f_2 > \cdots > f_k\}$ of $\mathcal{F}^{(e)}(P)$, we define its *support* $\mathrm{supp}(K)$ and *signature* $\mathrm{sgn}(K)$ as follows. The support $\mathrm{supp}(K)$ is the chain $\{\mathrm{supp}(f_1) \supsetneqq \mathrm{supp}(f_2) \supsetneqq \cdots \supsetneqq \mathrm{supp}(f_k)\}$ of order filters. The signature $\mathrm{sgn}(K)$ is the map $\varphi : P \to \{1, 0, -1\}$ given by the following:

(i) If v is not minimal in $\mathrm{supp}(f_i)$ for any i, then $\varphi(v) = 0$;

(ii) If v is minimal in $\mathrm{supp}(f_i)$ for some i, then $\varphi(v) = f_i(v)$.

The following lemma guarantees that the definition of $\varphi(v)$ in the case (ii) is independent of the choice of i.

Lemma 3.7. *Let* $K = \{f_1 > f_2 > \cdots > f_k\}$ *be a chain of* $\mathcal{F}^{(e)}(P)$*. If* v *is minimal in both* $\mathrm{supp}(f_i)$ *and* $\mathrm{supp}(f_j)$*, then we have* $f_i(v) = f_j(v)$*.*

Proof. We may assume $i < j$. Then $f_i > f_j$ and $\mathrm{supp}(f_i) \supset \mathrm{supp}(f_j)$. Since $v \in \mathrm{supp}(f_j)$ and minimal in $\mathrm{supp}(f_i)$, we have $f_i(v) = f_j(v)$ by the condition (iii) in Definition 3.1.

A key property of support and signature is the following.

Proposition 3.8. *Let* $X(P)$ *be the set of all chains of* $\mathcal{F}^{(e)}(P)$ *(including the empty chain), and* $Y(P)$ *the set of all pairs* (C, φ) *of chains* $C = \{F_1 \supsetneqq F_2 \supsetneqq \cdots \supsetneqq F_k\}$ *of order filters of* P *and maps* $\varphi : P \to \{1, 0, -1\}$ *satisfying*

$$\mathrm{supp}(\varphi) = \bigcup_{i=1}^{k} \min F_i, \tag{19}$$

where $\min F_i$ *is the set of minimal elements of* F_i*. Then the map* $X(P) \ni K \mapsto (\mathrm{supp}(K), \mathrm{sgn}(K)) \in Y(P)$ *is a bijection. In particular, maximal chains in* $\mathcal{F}^{(e)}(P)$ *are in bijection with pairs* (C, φ) *of maximal chains* C *of order filters and maps* $\varphi : P \to \{1, -1\}$*.*

It follows that the number of maximal chains in $\mathcal{F}^{(e)}(P)$ is equal to $2^d e(P)$, where $d = \#P$ and $e(P)$ is the number of linear extensions of P.

Proof. It follows from Definition 3.6 that $(\mathrm{supp}(K), \mathrm{sgn}(K)) \in Y(P)$ for $K \in X(P)$.

Given a chain $C = \{F_1 \supsetneqq \cdots \supsetneqq F_k\}$ of order filters and a map $\varphi : P \to \{1, 0, -1\}$ satisfying (19), we define $f_1, \cdots, f_k \in \mathbb{R}^P$ by

$$f_i(v) = \begin{cases} 1 & \text{if } v \in F_i \text{ and } v \text{ is not minimal in } F_i, \\ \varphi(v) & \text{if } v \in F_i \text{ and } v \text{ is minimal in } F_i, \\ 0 & \text{if } v \notin F_i. \end{cases}$$

Then we see that $f_i \in \mathcal{F}^{(e)}(P)$ and $\mathrm{supp}(f_i) = F_i$.

We show that $f_i > f_{i+1}$ for $1 \le i \le k-1$. First, one has $\mathrm{supp}(f_i) = F_i \supsetneqq F_{i+1} = \mathrm{supp}(f_{i+1})$. Second, we check that $f_i(v) \ge f_{i+1}(v)$ for $v \in \mathrm{supp}(f_{i+1})$.

Since $v \in \mathrm{supp}(f_{i+1}) \subset \mathrm{supp}(f_i)$, we have $f_i(v),\ f_{i+1}(v) \in \{1,-1\}$, and there is nothing to prove in the case $f_i(v) = 1$. If $f_i(v) = -1$, then v is minimal in $\mathrm{supp}(f_i)$, so v is minimal in $\mathrm{supp}(f_{i+1})$ by Lemma 3.2. Then we have $\varphi(v) = -1$ and $f_{i+1}(v) = -1 = f_i(v)$. Finally, if $v \in \mathrm{supp}(f_{i+1})$ and v is minimal in $\mathrm{supp}(f_i)$, then v is minimal in $\mathrm{supp}(f_{i+1})$ by Lemma 3.2 and $f_i(v) = \varphi(v) = f_{i+1}(v)$.

Therefore, $K = \{f_1 > f_2 > \cdots > f_k\}$ is a chain in $\mathcal{F}^{(e)}(P)$, and $\mathrm{supp}(K) = \{F_1 \supsetneq F_2 \supsetneq \cdots \supsetneq F_k\}$, $\mathrm{sgn}(K) = \varphi$. □

3.2. Triangulation of $\mathcal{C}^{(e)}(P)$

In this subsection, we use the triangulation of $\mathcal{C}(P)$ given in Theorem 1.2 to construct a unimodular triangulation of $\mathcal{C}^{(e)}(P)$. We transfer this triangulation of $\mathcal{C}^{(e)}(P)$ to $\mathcal{O}^{(e)}(P)$ via the inverse enriched transfer map $\Psi^{(e)}$ in the next subsection.

A (lattice) *triangulation* of a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d is a finite collection Δ of (lattice) simplices, such that

(i) every face of a member of Δ is in Δ,
(ii) the union of the simplices in Δ is $\mathcal{P}$, and
(iii) any two elements of Δ intersect in a common (possibly empty) face.

We say that a triangulation Δ is *unimodular* if all maximal faces of Δ are unimodular, i.e., have the Euclidean volume $1/d!$.

Recall that the simplices $S_C = \mathrm{conv}\{\chi_F : F \in C\}$ and $T_C = \mathrm{conv}\{\Phi(\chi_F) : F \in C\}$ of the triangulation given in Theorem 1.2 are described as follows.

Proposition 3.9. (Stanley [10, Section 5]) *If $C = \{F_1 \supsetneq F_2 \supsetneq \cdots \supsetneq F_k\}$ is a chain of order filters of P, then we have*

$$S_C = \left\{ \begin{array}{l} f \in \mathbb{R}^P : \\ \quad \textit{(i)}\, f \textit{ is constant on the subsets } P \setminus F_1, F_1 \setminus F_2, \ldots, F_{k-1} \setminus F_k, F_k, \\ \quad \textit{(ii)}\, 0 = f(P \setminus F_1) \le f(F_1 \setminus F_2) \le \cdots \le f(F_{k-1} \setminus F_k) \le f(F_k) = 1. \end{array} \right\}, \tag{20}$$

and

$$T_C = \Phi(S_C). \tag{21}$$

If $C = \{F_1 \supsetneq F_2 \supsetneq \cdots \supsetneq F_k\}$ is a chain of order filters of P, then $\chi_C = \{\chi_{F_1} > \chi_{F_2} > \cdots > \chi_{F_k}\}$ is a chain in $\mathcal{F}^{(e)}(P)$ by Proposition 3.5 (a), and

$$S^{(e)}_{\chi_C} = S_C, \quad T^{(e)}_{\chi_C} = T_C.$$

First, we show that any $T^{(e)}_K = \mathrm{conv}\, \Phi^{(e)}(K)$ is obtained from T_C by a composition of reflections. For $\varphi : P \to \{1, 0, -1\}$, we define a linear map $R_\varphi : \mathbb{R}^P \to \mathbb{R}^P$ by

$$(R_\varphi g)(v) = \begin{cases} g(v) & \text{if } \varphi(v) = 1 \text{ or } 0, \\ -g(v) & \text{if } \varphi(v) = -1. \end{cases}$$

The linear map R_φ is a composition of reflections along coordinate hyperplanes.

Proposition 3.10. *For a chain K in $\mathcal{F}_P^{(e)}$, we obtain*

$$T_K^{(e)} = R_{\mathrm{sgn}(K)}(T_{\mathrm{supp}(K)}). \tag{22}$$

Proof. Let $K = \{f_1 > \cdots > f_k\}$ and put $C = \mathrm{supp}(K) = \{F_1 \supsetneq \cdots \supsetneq F_k\}$ ($F_i = \mathrm{supp}(f_i)$) and $\varphi = \mathrm{sgn}(K)$. Since $T_C = \mathrm{conv}\,\Phi(\chi_C)$, we have

$$R_\varphi T_C = R_\varphi(\mathrm{conv}\,\Phi^{(e)}(\chi_C)) = \mathrm{conv}(R_\varphi(\Phi^{(e)}(\chi_C))).$$

Hence, it is enough to show that $R_\varphi(\Phi^{(e)}(\chi_{F_i})) = \Phi^{(e)}(f_i)$ for each i.

By the definition of the enriched transfer map, we have

$$\Phi^{(e)}(\chi_{F_i})(v) = \begin{cases} 1 & \text{if}\, v \ \text{ is minimal in}\, F_i, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi^{(e)}(f_i)(v) = \begin{cases} f_i(v) & \text{if}\, v \ \text{ is minimal in}\, F_i, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, it follows from the definition of $\varphi = \mathrm{sgn}(K)$ that

$$\varphi(v) = \begin{cases} f_i(v) & \text{if}\, v \ \text{ is minimal in some}\ \mathrm{supp}(f_i), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we obtain $R_\varphi(\Phi^{(e)}(\chi_{F_i})) = \Phi^{(e)}(f_i)$. □

To prove Theorem 1.4 (b), we prepare several lemmas. Given $\varphi \in \{1, 0, -1\}^P$, we put

$$V_\varphi = \left\{ g \in \mathbb{R}^P : \begin{array}{l} \text{(i) if}\, \varphi(v) = 1, \ \text{then}\, g(v) \geq 0, \\ \text{(ii) if}\, \varphi(v) = 0, \ \text{then}\, g(v) = 0, \\ \text{(iii) if}\, \varphi(v) = -1, \ \text{then}\, g(v) \leq 0 \end{array} \right\}.$$

For $\varepsilon \in \{1, -1\}^P$, we put

$$\mathcal{C}_\varepsilon^{(e)}(P) = \mathcal{C}^{(e)}(P) \cap V_\varepsilon, \quad \mathcal{A}_\varepsilon^{(e)}(P) = \mathcal{A}^{(e)}(P) \cap V_\varepsilon.$$

Since $\mathbb{R}^P = \bigcup_{\varepsilon \in \{1,-1\}^P} V_\varepsilon$, we have

$$\mathcal{C}^{(e)}(P) = \bigcup_{\varepsilon \in \{1,-1\}^P} \mathcal{C}_\varepsilon^{(e)}(P), \quad \mathcal{A}^{(e)}(P) = \bigcup_{\varepsilon \in \{1,-1\}^P} \mathcal{A}_\varepsilon^{(e)}(P).$$

Lemma 3.11. ([7, Lemma 1.1]) *For $\varepsilon \in \{1, -1\}^P$, we have*

$$\mathcal{C}_\varepsilon^{(e)}(P) = \mathrm{conv}(\mathcal{A}_\varepsilon^{(e)}(P)) = R_\varepsilon(\mathcal{C}(P)).$$

Proof. The first equality is proved in [7, Lemma 1.1]. We prove the second equality. Let ε_0 be the map given by $\varepsilon_0(v) = 1$ for all $v \in P$. Then $\mathcal{A}_{\varepsilon_0}^{(e)}(P) = \mathcal{A}(P)$ and $\mathcal{C}_{\varepsilon_0}^{(e)}(P) = \mathrm{conv}(\mathcal{A}(P)) = \mathcal{C}(P)$. Since $\mathcal{A}_\varepsilon^{(e)}(P) = R_\varepsilon(\mathcal{A}_{\varepsilon_0}^{(e)}(P)) = R_\varepsilon(\mathcal{A}(P))$, we have

$$\mathcal{C}_\varepsilon^{(e)}(P) = \mathrm{conv}(R_\varepsilon(\mathcal{A}(P))) = R_\varepsilon(\mathrm{conv}(\mathcal{A}(P))) = R_\varepsilon(\mathcal{C}(P)).$$

□

Lemma 3.12. *Suppose that* $\varphi \in \{1, 0, -1\}^P$ *and* $\varepsilon \in \{1, -1\}^P$ *satisfy* $\varphi|_{\mathrm{supp}(\varphi)} = \varepsilon|_{\mathrm{supp}(\varphi)}$. *Then we have*

(a) $V_\varphi \subset V_\varepsilon$.
(b) $R_\varphi|_{V_{|\varphi|}} = R_\varepsilon|_{V_{|\varphi|}}$, *where* $|\varphi|$ *is defined by* $|\varphi|(v) = |\varphi(v)|$.

Proof. (a) Let $g \in V_\varphi$. If $\varepsilon(v) = 1$, then $\varphi(v) = 1$ or 0 and $g(v) \geq 0$. If $\varepsilon(v) = -1$, then $\varphi(v) = -1$ or 0 and $g(v) \leq 0$. Hence, $g \in V_\varepsilon$.
(b) Let $g \in V_{|\varphi|}$. If $v \in \mathrm{supp}(\varphi)$, then we have $\varphi(v) = \varepsilon(v)$ and $(R_\varphi g)(v) = \varphi(v)g(v) = \varepsilon(v)g(v) = (R_\varepsilon g)(v)$. If $v \notin \mathrm{supp}(\varphi)$, then we have $\varphi(v) = g(v) = 0$, thus $(R_\varphi g)(v) = g(v) = 0$ and $(R_\varepsilon g)(v) = \varepsilon(v)g(v) = 0$. □

Lemma 3.13. (a) *Let* $C = \{F_1 \supsetneq \cdots \supsetneq F_k\}$ *be a chain of order filters of* P. *If* $\varphi \in \{1, 0, -1\}^P$ *satisfies* $\mathrm{supp}(\varphi) = \bigcup_{i=1}^k \min F_i$, *then* $T_C \subset V_{|\varphi|}$.
(b) *If* K *is a chain in* $\mathcal{F}^{(e)}(P)$, *then we have* $T_K^{(e)} \subset V_{\mathrm{sgn}(K)}$.
(c) *If* K *is a chain in* $\mathcal{F}^{(e)}(P)$ *and* $\varepsilon \in \{1, -1\}^P$ *satisfies* $\mathrm{sgn}(K)|_{\mathrm{supp}(\mathrm{sgn}(K))} = \varepsilon|_{\mathrm{supp}(\mathrm{sgn}(K))}$, *then we have* $T_K^{(e)} = R_\varepsilon T_{\mathrm{supp}(K)}$.

Proof. (a) Let $g \in T_C$. It is enough to show that $\varphi(v) = 0$ implies $g(v) = 0$. By Theorem 1.2 (a), there exists $f \in S_C$ such that $g = \Phi(f)$. Let i be the largest index such that $v \in F_i$, where we use the convention $F_0 = P$. If $i = 0$, then $f(v) = 0$ and $g(v) = 0$. Suppose that $i \geq 1$ and $\varphi(v) = 0$. Then, $v \in F_i \backslash F_{i+1}$ by the maximality of i. Since v is not minimal in F_i, there exists $w \in F_i$ such that $w \lessdot v$. If $w \in F_{i+1}$, then $v \in F_{i+1}$ (since F_{i+1} is an order filter), which contradicts to the maximality of i. Hence, we have $w \in F_i \backslash F_{i+1}$. Then, by (20), we have $f(v) = f(w)$. Therefore, $g(v) = (\Phi(f))(v) = f(v) - \max\{f(u) : u \lessdot v\} = f(v) - f(w) = 0$.
(b) Let $C = \mathrm{supp}(K)$ and $\varphi = \mathrm{sgn}(K)$. By (a), we have $T_C \subset V_{|\varphi|}$. Since $R_\varphi V_{|\varphi|} = V_\varphi$, we obtain $T_K = R_\varphi(T_C) \subset V_\varphi$.
(c) follows from Proposition 3.10, (a) and Lemma 3.12 (b). □

Note that $\Phi^{(e)}$ gives a bijection between $\mathcal{F}^{(e)}(P)$ and $\mathcal{A}^{(e)}(P)$ (Proposition 2.4 (c)), and that R_φ preserves $\mathcal{A}^{(e)}(P)$ for any $\varphi \in \{1, 0, -1\}^P$.

Lemma 3.14. *Given* $f_1, f_2 \in \mathcal{F}^{(e)}(P)$ *and* $\varphi \in \{1, 0, -1\}^P$, *we define* $f_1', f_2' \in \mathcal{F}^{(e)}(P)$ *by the condition*

$$R_\varphi \Phi^{(e)}(f_1) = \Phi^{(e)}(f_1'), \quad R_\varphi \Phi^{(e)}(f_2) = \Phi^{(e)}(f_2').$$

Then $f_1 > f_2$ *implies* $f_1' > f_2'$.

Proof. We may assume that there exists a unique $u \in P$ such that $\varphi(u) = -1$, that is

$$(R_\varphi g)(v) = \begin{cases} g(v) & \text{if } v \neq u, \\ -g(v) & \text{if } v = u. \end{cases}$$

Then it follows from Proposition 2.4 that, if $R_\varphi(\Phi^{(e)}(f)) = \Phi^{(e)}(f')$, then

$$f'(v) = \begin{cases} -f(v) & \text{if } u \in \min(\mathrm{supp}(f)) \text{ and } v = u, \\ f(v) & \text{otherwise,} \end{cases}$$

and $\mathrm{supp}(f') = \mathrm{supp}(f)$.

Now, we assume that $f_1 > f_2$. Then it is enough to prove the following two claims:

(1) If $u \in \mathrm{supp}(f_2')$, then $f_1'(u) \geq f_2'(u)$.
(2) If $u \in \mathrm{supp}(f_2')$ and u is minimal in $\mathrm{supp}(f_1')$, then $f_1'(u) = f_2'(u)$.

First, we prove (1) by dividing into four cases. If $u \in \min(\mathrm{supp}(f_1))$ and $u \in \min(\mathrm{supp}(f_2))$, then we have $f_1(u) = f_2(u)$; thus, $f_1'(u) = -f_1(u) = -f_2(u) = f_2'(u)$. If $u \in \min(\mathrm{supp}(f_1))$ and $u \notin \min(\mathrm{supp}(f_2))$, then it follows from Lemma 3.2 that $u \notin \mathrm{supp}(f_2)$, which contradicts to the assumption $u \in \mathrm{supp}(f_2') = \mathrm{supp}(f_2)$. If $u \notin \min(\mathrm{supp}(f_1))$ and $u \in \min(\mathrm{supp}(f_2))$, then we have $f_1'(u) = f_1(u) = 1$, thus $f_1'(u) \geq f_2'(u)$. If $u \notin \min(\mathrm{supp}(f_1))$ and $u \notin \min(\mathrm{supp}(f_2))$, then we have $f_1' = f_1$ and $f_2' = f_2$; thus, $f_1'(u) \geq f_2'(u)$.

Next, we prove (2). If $u \in \mathrm{supp}(f_2')$ and u is minimal in $\mathrm{supp}(f_1')$, then it follows from Lemma 3.2 that u is minimal in $\mathrm{supp}(f_2')$, and hence, we see that $f_1'(u) = -f_1(u) = -f_2(u) = f_2'(u)$. This completes the proof. □

Now, we are in position to prove Theorem 1.4 (b).

Proof of Theorem 1.4 (b). We need to show the following four claims:

(1) If K is a chain in $\mathcal{F}^{(e)}(P)$, then $T_K^{(e)}$ is a unimodular simplex.
(2) If K is a chain in $\mathcal{F}^{(e)}(P)$, then $T_K^{(e)} \subset \mathcal{C}^{(e)}(P)$.
(3) $\bigcup_K T_K^{(e)} = \mathcal{C}^{(e)}(P)$, where K runs over all chains in $\mathcal{F}^{(e)}(P)$.
(4) If K and L are chains in $\mathcal{F}^{(e)}(P)$, then $T_K^{(e)} \cap T_L^{(e)} = T_{K\cap L}^{(e)}$.

Recall that $T_K^{(e)} = \mathrm{conv}\, \Phi^{(e)}(K)$ and $R_\varphi : \mathbb{R}^P \to \mathbb{R}^P$ is a linear map given by

$$(R_\varphi g)(v) = \begin{cases} g(v) & \text{if } \varphi(v) = 1 \text{ or } 0, \\ -g(v) & \text{if } \varphi(v) = -1, \end{cases}$$

for $\varphi : P \to \{1, 0, -1\}$.

(1) If we put $C = \mathrm{supp}(K)$ and $\varphi = \mathrm{sgn}(K)$, then $T_K^{(e)} = R_\varphi(T_C)$ by Proposition 3.10. Since T_C is a unimodular simplex (Theorem 1.2 (b)) and R_φ is a composition of reflections, we see that $T_K^{(e)}$ is a unimodular simplex.
(2) We put $C = \mathrm{supp}(K)$ and $\varphi = \mathrm{sgn}(K)$, and take $\varepsilon \in \{1, -1\}^P$ such that $\varphi|_{\mathrm{supp}(\varphi)} = \varepsilon|_{\mathrm{supp}(\varphi)}$. Then, using Lemma 3.13 (c) and Lemma 3.11, we have
$$T_K^{(e)} = R_\varepsilon(T_C) \subset R_\varepsilon(\mathcal{C}(P)) = \mathcal{C}_\varepsilon^{(e)}(P) \subset \mathcal{C}^{(e)}(P).$$
(3) Using Lemma 3.11 and Theorem 1.2 (b), we have
$$\mathcal{C}^{(e)}(P) = \bigcup_{\varepsilon\in\{1,-1\}^P} \mathcal{C}_\varepsilon^{(e)}(P) = \bigcup_{\varepsilon\in\{1,-1\}^P} R_\varepsilon(\mathcal{C}(P)) = \bigcup_{\varepsilon\in\{1,-1\}^P} \bigcup_C R_\varepsilon(T_C),$$
where C runs over all chain of order filters of P. Given a chain C of order filters of P and $\varepsilon \in \{1, -1\}^P$, we define $\varphi : P \to \{1, 0, -1\}$ by putting
$$\varphi(v) = \begin{cases} \varepsilon(v) & \text{if } v \text{ is minimal in some } F_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows from Lemma 3.13 (c) that $R_\varepsilon T_C = T_K^{(e)}$, where K is the chain in $\mathcal{F}^{(e)}(P)$ corresponding to (C, φ) under the bijection of Proposition 3.8.

(4) We put $C = \operatorname{supp}(K)$, $\varphi = \operatorname{sgn}(K)$, $D = \operatorname{supp}(L)$ and $\psi = \operatorname{sgn}(L)$. Then we have $T_K^{(e)} \subset V_\varphi$ and $T_L^{(e)} \subset V_\psi$ by Lemma 3.13 (b). If we define $\eta : P \to \{1, 0, -1\}$ by putting

$$\eta(v) = \begin{cases} 1 & \text{if } \varphi(v) = \psi(v) = 1, \\ -1 & \text{if } \varphi(v) = \psi(v) = -1, \\ 0 & \text{otherwise}, \end{cases}$$

then we have $V_\varphi \cap V_\psi = V_\eta$. Hence, we have

$$T_K^{(e)} \cap T_L^{(e)} = T_K^{(e)} \cap T_L^{(e)} \cap V_\eta.$$

Since $T_K^{(e)} = \operatorname{conv}(\Phi^{(e)}(K))$ by definition, and V_η is a "boundary" of V_φ, we see that

$$T_K^{(e)} \cap V_\eta = \operatorname{conv}(\Phi^{(e)}(K)) \cap V_\eta = \operatorname{conv}(\Phi^{(e)}(K) \cap V_\eta).$$

We take $\varepsilon \in \{1, -1\}^P$ satisfying $\eta|_{\operatorname{supp}(\eta)} = \varepsilon|_{\operatorname{supp}(\eta)}$. Then we have $\Phi^{(e)}(K) \cap V_\eta$ and $\Phi^{(e)}(L) \cap V_\eta \subset \mathcal{A}_\varepsilon^{(e)}(P)$. Since R_ε gives a bijection between $\mathcal{A}_\varepsilon^{(e)}(P)$ and $\mathcal{A}(P)$, it follows from Lemma 3.14 that there exists a chain C' of order filters of P, such that $R_\varepsilon(\Phi^{(e)}(\chi_{C'})) = \Phi^{(e)}(K) \cap V_\eta$. Hence, we have

$$T_K^{(e)} \cap V_\eta = \operatorname{conv}(R_\varepsilon(\Phi^{(e)}(\chi_{C'}))) = R_\varepsilon \operatorname{conv}(\Phi^{(e)}(\chi_{C'})).$$

Similarly, there exists a chain D' of order filters of P, such that

$$T_L^{(e)} \cap V_\eta = \operatorname{conv}(R_\varepsilon(\Phi^{(e)}(\chi_{D'}))) = R_\varepsilon \operatorname{conv}(\Phi^{(e)}(\chi_{D'})).$$

Therefore, we have

$$\begin{aligned} T_K^{(e)} \cap T_L^{(e)} &= (T_K^{(e)} \cap V_\eta) \cap (T_L^{(e)} \cap V_\eta) \\ &= R_\varepsilon \operatorname{conv}(\Phi^{(e)}(\chi_{C'})) \cap R_\varepsilon \operatorname{conv}(\Phi^{(e)}(\chi_{D'})) \\ &= R_\varepsilon \left(\operatorname{conv}(\Phi^{(e)}(\chi_{C'})) \cap \operatorname{conv}(\Phi^{(e)}(\chi_{D'})) \right) \\ &= R_\varepsilon(T_{C'} \cap T_{D'}). \end{aligned}$$

By Theorem 1.2 (b), we see that $T_{C'} \cap T_{D'} = T_{C' \cap D'} = \operatorname{conv}(\Phi^{(e)}(\chi_{C' \cap D'}))$. Hence, we have

$$\begin{aligned} T_K^{(e)} \cap T_L^{(e)} &= R_\varepsilon \left(\operatorname{conv}(\Phi^{(e)}(\chi_{C' \cap D'})) \right) \\ &= R_\varepsilon \left(\operatorname{conv}(\Phi^{(e)}(\chi_{C'}) \cap \Phi^{(e)}(\chi_{D'})) \right) \\ &= \operatorname{conv} \left(R_\varepsilon(\Phi^{(e)}(\chi_{C'})) \cap R_\varepsilon(\Phi^{(e)}(\chi_{D'}) \right) \\ &= \operatorname{conv} \left((\Phi^{(e)}(K) \cap V_\eta) \cap (\Phi^{(e)}(L) \cap V_\eta) \right) \\ &= \operatorname{conv} \left(\Phi^{(e)}(K) \cap \Phi^{(e)}(L) \cap V_\eta \right) \end{aligned}$$

$$= \operatorname{conv}\left(\Phi^{(e)}(K) \cap \Phi^{(e)}(L)\right)$$
$$= \operatorname{conv}\left(\Phi^{(e)}(K \cap L)\right)$$
$$= T^{(e)}_{K \cap L}.$$

This completes the proof of Theorem 1.4 (b). □

We conclude this subsection with giving a set of defining inequalities of a facet $T^{(e)}_K$, where K is a maximal chain in $\mathcal{F}^{(e)}(P)$. Recall the result of Stanley [10] on the defining inequalities of facets of the triangulations of $\mathcal{O}(P)$ and $\mathcal{C}(P)$. To a maximal chain $C = \{F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_d\}$ of order filters of P, we associate a linear extension $(v_1, \dots, v_d)$ and chains $C_1, \dots, C_d$ of P as follows. The linear extension $(v_1, \dots, v_d)$ is defined by

$$F_i = F_{i-1} \cup \{v_i\} \quad (i = 1, \dots, d).$$

The chain C_i is given inductively by

(i) If v_i is minimal, then we put $C_i = \{v_i\}$;
(ii) If v_i is not minimal and j is the largest index satisfying $v_j \lessdot v_i$, then we put $C_i = \{v_i\} \cup C_j$.

Proposition 3.15. (Stanley [10, Section 5]) *Let C be a maximal chain of order filters of P. Let $(v_1, \dots, v_d)$ be the associated linear extension of P and $C_1, \dots, C_d$ the associated chains of P. Then we have*

(a) *The facet S_C of the triangulation $\mathcal{S}_P$ of $\mathcal{O}(P)$ is given by*
$$S_C = \{f \in \mathbb{R}^P : 0 \le f(v_1) \le f(v_2) \le \cdots \le f(v_d) \le 1\}.$$
(b) *If $f \in S_C$, then we have*
$$(\Phi(f))(v_i) = f(v_i) - f(v_j),$$
where j is the largest index satisfying $v_j \lessdot v_i$.
(c) *If we define*
$$L^C_i(g) = \sum_{v \in C_i} g(v),$$
then the facet T_C of the triangulation $\mathcal{T}_P$ of $\mathcal{C}(P)$ is given by
$$T_C = \{g \in \mathbb{R}^P : 0 \le L^C_1(g) \le L^C_2(g) \le \cdots \le L^C_d(g) \le 1\}.$$
(d) *If $g \in T_C$, then we have*
$$(\Psi(g))(v_i) = \sum_{v \in C_i} g(v),$$
where $\Psi : \mathcal{C}(P) \to \mathcal{O}(P)$ is the inverse transfer map.

Corollary 3.16. *Let K be a maximal chain in $\mathcal{F}^{(e)}(P)$ and put $C = \operatorname{supp}(K)$, $\varepsilon = \operatorname{sgn}(K)$. Let $C_1, \dots, C_d$ be the chains of P associated with C, and define*

$$\widetilde{L}^K_i(g) = \sum_{v \in C_i} \varepsilon(v) g(v) \quad (g \in \mathbb{R}^P).$$

Then the face $T_K^{(e)}$ *of the triangulation* $\mathcal{T}_P^{(e)}$ *of* $\mathcal{C}^{(e)}(P)$ *is given by*

$$T_K^{(e)} = \{g \in \mathbb{R}^P : 0 \le L_1^K(g) \le L_2^K(g) \le \cdots \le L_d^K(g) \le 1\}. \tag{23}$$

Proof. It follows from Proposition 3.10 and Proposition 3.15 (c). □

3.3. Triangulation of $\mathcal{O}^{(e)}(P)$

In this subsection, we transfer the triangulation of $\mathcal{C}^{(e)}(P)$ to $\mathcal{O}^{(e)}(P)$ via the inverse map $\Psi^{(e)}$ of the enriched transfer map $\Phi^{(e)}$. To prove Theorem 1.4 (a), it is enough to show that $S_K^{(e)} = \operatorname{conv} K = \Psi^{(e)}(T_K^{(e)})$ and it is a unimodular simplex.

Lemma 3.17. *Let* K *be a maximal chain in* $\mathcal{F}^{(e)}(P)$ *and put* $C = \operatorname{supp}(K)$, $\varepsilon = \operatorname{sgn}(K)$. *Let* $(v_1, \dots, v_d)$ *be the linear extension and* $C_1, \dots, C_d$ *the chains of* P *associated with* C. *For* $g \in T_K^{(e)}$, *we have*

$$(\Psi^{(e)}(g))(v_i) = g(v_i) + \sum_{v \in C_i \setminus \{v_i\}} \varepsilon(v) g(v).$$

Proof. Since $T_K^{(e)} \subset V_\varepsilon$ by Lemma 3.13 (b), we have $|g(v)| = \varepsilon(v)g(v)$ for $g \in T_K^{(e)}$ and $v \in P$; thus, $|g| \in T_C$. By Proposition 3.15, we see that

$$\max\{S(|g|; B) : B \in \mathrm{MC}(P_{\le v_j})\} = \sum_{v \in C_j} |g(v)| = \sum_{v \in C_j} \varepsilon(v) g(v),$$

where we recall $S(f; C) = \sum_{v \in C} |f(v)|$ for $f \in \mathbb{R}^P$ and a chain C of P, and $\mathrm{MC}(P_{\le v_j})$ is the set of maximal chains of the subposet $P_{\le v_j}$. Hence, we obtain the desired identity. □

Proof of Theorem 1.4 (a). If K is a maximal chain in $\mathcal{F}^{(e)}(P)$, then it follows from Lemma 3.17 that $\Psi^{(e)}$ is a unimodular linear map on $T_K^{(e)}$. Hence, if L is a chain in $\mathcal{F}^{(e)}(P)$ contained in K, then we see that $S_L^{(e)} = \Psi^{(e)}(T_L^{(e)})$ is a unimodular simplex, because $T_L^{(e)}$ is a unimodular simplex (Theorem 1.4 (b)). □

We can use Lemma 3.17 to give a set of defining inequalities of a facet $S_K^{(e)}$, where K is a maximal chain in $\mathcal{F}^{(e)}(P)$.

Proposition 3.18. *Let* K *be a maximal chain in* $\mathcal{F}^{(e)}(P)$ *and put* $C = \operatorname{supp}(K)$, $\varepsilon = \operatorname{sgn}(K)$. *Let* $(v_1, \dots, v_d)$ *be the linear extension and* $C_1, \dots, C_d$ *the chains of* P *associated to* C. *If we put*

$$\widetilde{M}_i^K(f) = \sum_{l=1}^{r} \varepsilon(u_l) \prod_{j=l+1}^{r} (1 - \varepsilon(u_j)) f(u_l) \quad (f \in \mathbb{R}^P),$$

where $C_i = \{u_1 \lessdot u_2 \lessdot \cdots \lessdot u_r = v_i\}$, *then the face* $S_K^{(e)}$ *of the triangulation* $\mathcal{S}_P^{(e)}$ *of* $\mathcal{O}^{(e)}(P)$ *is given by*

$$S_K^{(e)} = \{f \in \mathbb{R}^P : 0 \le \widetilde{M}_1^K(f) \le \widetilde{M}_2^K(f) \le \cdots \le \widetilde{M}_d^K(f) \le 1\}. \tag{24}$$

Proof. It is easy to prove by induction on k that

$$f(u_k) = g(u_k) + \sum_{i=1}^{k} \varepsilon(u_i) g(u_i) \quad (k = 1, \ldots, r)$$

if and only if

$$g(u_k) = f(u_k) - \sum_{i=1}^{k-1} \varepsilon(u_i) \prod_{j=i+1}^{k-1} (1 - \varepsilon(u_j)) f(u_i) \quad (k = 1, \ldots, r).$$

Hence, we have

$$\widetilde{M}_i^K(f) = \widetilde{L}_i^K(\Phi^{(e)}(f)).$$

On the other hand, by Theorem 1.4 (b) and Corollary 3.16, we see that $f \in S_K^{(e)}$ if and only if

$$0 \leq \widetilde{L}_1^K(\Phi^{(e)}(f)) \leq \cdots \leq \widetilde{L}_d^K(\Phi^{(e)}(f)) \leq 1.$$

Hence, we obtain (24). □

4. Identification With Ohsugi–Tsuchiya's Triangulations

Ohsugi–Tsuchiya [7,8] computed the squarefree initial ideals of the toric ideals of $\mathcal{O}^{(e)}(P)$ and $\mathcal{C}^{(e)}(P)$ with respect to certain monomial orderings. This gives regular unimodular triangulations of $\mathcal{O}^{(e)}(P)$ and $\mathcal{C}^{(e)}(P)$. In this section, we show that these triangulations coincide with the triangulations given in Theorem 1.4.

Let $\mathbb{K}[\boldsymbol{x}] = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables $x_1, \ldots, x_n$ over a field $\mathbb{K}$ and Δ a simplicial complex on $[n] := \{1, 2 \ldots, n\}$. To a subset $F \subset [n]$, we associate a monomial

$$\boldsymbol{x}_F = \prod_{i \in F} x_i.$$

The *Stanley–Reisner ideal* of Δ is the ideal I_Δ of $\mathbb{K}[\boldsymbol{x}]$ which is generated by those squarefree monomials $\boldsymbol{x}_F$ with $F \notin \Delta$. On the other hand, given an arbitrary squarefree monomial ideal I of $\mathbb{K}[\boldsymbol{x}]$, there is a unique simplicial complex $\Delta(I)$ such that $I = I_{\Delta(I)}$.

4.1. Triangulation of Enriched Order Polytope

In this subsection, we prove that the triangulation $\mathcal{S}_P^{(e)}$ of $\mathcal{O}^{(e)}(P)$ given in Theorem 1.4 (a) coincides with the one algebraically defined in [8].

Let P be a finite poset with d elements and let $R[\mathcal{O}^{(e)}] = \mathbb{K}[\{x_f : f \in \mathcal{F}^{(e)}(P)\}]$ be the polynomial ring in the variables x_f ($f \in \mathcal{F}^{(e)}(P)$).

Proposition 4.1. ([8, Theorem 5.2]) *Let $I_{\mathcal{O}^{(e)}(P)}$ be the ideal of $R[\mathcal{O}^{(e)}]$ generated by all squarefree monomials $x_f x_g$ satisfying either of the following conditions:*

(i) *there exists $v \in \min(\mathrm{supp}(f)) \cap \min(\mathrm{supp}(g))$ such that $f(v) \neq g(v)$;*

(ii) $\mathrm{supp}(f) \not\sim \mathrm{supp}(g)$ *and* $f(v) = g(v)$ *for each* $v \in \min(\mathrm{supp}(f)) \cap \min(\mathrm{supp}(g))$, *where the symbol* $A \not\sim B$ *means that* $A \not\subseteq B$ *and* $A \not\supseteq B$.

Then $\Delta(I_{\mathcal{O}^{(e)}(P)})$ *is a regular unimodular triangulation of* $\mathcal{O}^{(e)}(P)$.

Now, we can show that this triangulation $\Delta(I_{\mathcal{O}^{(e)}(P)})$ coincides with the triangulation given in Theorem 1.4 (a).

Proposition 4.2. *With the notations above, we have* $\Delta(I_{\mathcal{O}^{(e)}(P)}) = \mathcal{S}_P^{(e)}$.

Proof. Since both of $\Delta(I_{\mathcal{O}^{(e)}(P)})$ and $\mathcal{S}_P^{(e)}$ are unimodular triangulations of $\mathcal{O}^{(e)}(P)$, the numbers of maximal simplices are same. Hence, it is enough to show that $x_{f_0} \cdots x_{f_d} \notin I_{\mathcal{O}^{(e)}(P)}$ for any maximal chain $K = \{f_0 > \cdots > f_d\}$ of $\mathcal{F}^{(e)}(P)$.

Let $K = \{f_0 > \cdots > f_d\}$ be a maximal chain of $\mathcal{F}^{(e)}(P)$, and assume to the contrary that $x_{f_0} \cdots x_{f_d} \in I_{\mathcal{O}^{(e)}(P)}$. Then there exists a pair of indices $i < j$ such that f_i and f_j satisfy the condition (i) or (ii) in Proposition 4.1. Since $f_i > f_j$ in $\mathcal{F}^{(e)}(P)$, one has $\mathrm{supp}(f_i) \supsetneq \mathrm{supp}(f_j)$ and f_i and f_j do not satisfy the condition (ii). Hence, there exists $v \in \min(\mathrm{supp}(f_i)) \cap \min(\mathrm{supp}(f_j))$ with $f_i(v) \neq f_j(v)$. However, since $f_i > f_j$ in $\mathcal{F}^{(e)}(P)$ and $v \in \mathrm{supp}(f_j)$ and v is minimal in $\mathrm{supp}(f_i)$, we obtain $f_i(v) = f_j(v)$, which is a contradiction. Thus, it follows that $x_{f_0} \cdots x_{f_d} \notin I_{\mathcal{O}^{(e)}(P)}$. □

4.2. Triangulation of Enriched Chain Polytope

In this subsection, we prove that the triangulation $\mathcal{T}_P^{(e)}$ of $\mathcal{C}^{(e)}(P)$ given in Theorem 1.4 (b) coincides with the one algebraically defined in [7]. Let $R[\mathcal{C}^{(e)}]$ be the polynomial ring in variables y_g $(g \in \mathcal{A}^{(e)}(P))$.

Proposition 4.3. ([7, Theorem 1.4]) *Let* $I_{\mathcal{C}^{(e)}(P)}$ *be the ideal of* $R[\mathcal{C}^{(e)}]$ *generated by all squarefree monomials* $y_g y_h$ *satisfying either of the following conditions:*

(i) *there exists* $v \in \mathrm{supp}(g) \cap \mathrm{supp}(h)$, *such that* $g(v) \neq h(v)$;
(ii) $\langle \mathrm{supp}(g) \rangle \not\sim \langle \mathrm{supp}(h) \rangle$ *and* $g(v) = h(v)$ *for any* $v \in \mathrm{supp}(g) \cap \mathrm{supp}(h)$.

Then $\Delta(I_{\mathcal{C}^{(e)}(P)})$ *is a regular unimodular triangulation of* $\mathcal{C}^{(e)}(P)$.

Finally, we show that this triangulation $\Delta(I_{\mathcal{C}^{(e)}(P)})$ coincides with the triangulation given in Theorem 1.4 (b).

Proposition 4.4. *With the same notation as above, we have* $\Delta(I_{\mathcal{C}^{(e)}(P)}) = \mathcal{T}_P^{(e)}$.

Proof. This follows from the fact that the map $x_f \mapsto y_{\Phi^{(e)}(f)}$ induces the ring isomorphism:

$$\frac{R[\mathcal{O}^{(e)}(P)]}{I_{\mathcal{O}^{(e)}(P)}} \cong \frac{R[\mathcal{C}^{(e)}(P)]}{I_{\mathcal{C}^{(e)}(P)}}.$$

□

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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