Annals of Combinatorics

Approximate Sampling of Graphs with Near-P-Stable Degree Intervals

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Abstract. The approximate uniform sampling of graph realizations with a given degree sequence is an everyday task in several social science, computer science, engineering etc. projects. One approach is using Markov chains. The best available current result about the well-studied switch Markov chain is that it is rapidly mixing on P-stable degree sequences (see `DOI:10.1016/j.ejc.2021.103421`). The switch Markov chain does not change any degree sequence. However, there are cases where degree intervals are specified rather than a single degree sequence. (A natural scenario where this problem arises is in hypothesis testing on social networks that are only partially observed.) Rechner, Strowick, and Müller–Hannemann introduced in 2018 the notion of *degree interval Markov chain* which uses three (separately well studied) local operations (switch, hinge-flip and toggle), and employing on degree sequence realizations where any two sequences under scrutiny have very small coordinate-wise distance. Recently, Amanatidis and Kleer published a beautiful paper (`DOI:10.4230/LIPIcs.STACS.2023.7`), showing that the degree interval Markov chain is rapidly mixing if the sequences are coming from a system of very thin intervals which are centered not far from a regular degree sequence. In this paper, we substantially extend their result, showing that the degree interval Markov chain is rapidly mixing if the intervals are centered at P-stable degree sequences.

Keywords. Degree sequences, Realizations, Switch Markov chain, Rapidly mixing, Sinclair's multi-commodity flow method, P-stability, Weak P-stability.

1. Introduction

In this relatively short, highly technical paper, we prove a substantial extension of a recent result of Amanatidis and Kleer [1, Theorem 1.3]. Our proof is based on the unified approach that was developed in Erdős et al. [4] for P-stable

degree sequences. For sake of brevity in this section, we concisely describe the problem itself, but we will not give a detailed description of the background. For further details, the diligent reader is referred to Amanatidis and Kleer and Erdős et al. [1,4].

Approximate sampling graphs with given degree sequences play increasingly important role in modeling different real-life dynamics. One basic way to study them is the switch Markov chain method, made popular by Kannan et al. [9]. The currently best result via this method is Erdős et al. [4] where it is proved that the switch Markov chain is rapidly mixing on P-stable degree sequences. The notion of P-stability was introduced by Jerrum and Sinclair [8] and studied for its own sake at first by Jerrum et al. [6].

In real-life applications, it is not always possible to know the exact degree sequence of the targeted network. For example, a natural scenario where this problem arises is in hypothesis testing on social networks that are only partially observed. Therefore, it can happen that we have to sample networks with slightly different degree sequences. It is possible to study the situation via Markov chain decompositions, where there is another Markov chain to move among the component chains. A good example for this approach is the proof of Amanatidis and Kleer [1, Theorem 1.1].

Another possibility is to introduce further local operations, since the switch operation itself does not change the degree sequence. Such operations are the *hinge flip* and the *toggle* (the deletion–insertion) operations. These two latter operations were introduced by Jerrum and Sinclair in their seminal work about approximate 0-1 permanents [7]. (The number of perfect matchings of a bipartite graph is equal to the permanent of the bipartite adjacency matrix.) These three operations together are often applied in network building applications in *practice* (as it was pointed out in Coolen et al. [2]) but without any theoretical insurance for the correct result.

Rechner et al. [10] defined a Markov chain with these three local operations for bipartite graphs. Amanatidis and Kleer recognized in their important, recent paper [1] the following very interesting fact: assume that the inconsistencies in the degree sequences are never bigger than one (the degrees can be i or $i+1$) coordinate-wise, and the degree intervals are placed close to a given constant r (the interval placements can vary between $[r - r^\alpha, r + r^\alpha]$ where alpha is at most $1/2$). The authors coined the name *near-regular degree intervals* for this degree sequence property and the name *degree interval Markov chain* for this whole setup. Their result is that the degree interval Markov chain for near-regular degree intervals is rapidly mixing.

Our main result (Theorem 2.20) is that this Markov chain is rapidly mixing for such tight degree intervals where they are placed at P-stable degree sequences. Since all degree sequences close to some constant are P-stable, but P-stable degree sequences can be very far from regular sequences, our result is clearly a very extensive generalization of the theorem of Amanatidis and Kleer.

To our great surprise, it turned out that this result can be derived from the proof of the main theorem of Erdős et al. [4]. For that end, we had to

analyze in detail the auxiliary structures of the proof and to extend to cover this setup. The result of this analysis is the notion of *precursor* (Sect. 3.3). In turn, this notion is conducive to a rather short proof of the rapidly mixing property. Therefore, the main task in this paper is to define the appropriate precursor.

2. Definitions and Notation

Many of the definitions in this section are extensions or generalizations of notions introduced in Erdős et al. [4]. We will alert the reader whenever this is the case.

We consider $\mathbb{N}$ the set of non-negative integers. Let $[n] = \{1, \ldots, n\}$ denote the integers from 1 to n, and let $\binom{[n]}{k}$ denote the set of k-element subsets of $[n]$. Given a subset $S \subseteq [n]$, let $\mathbb{1}_S : [n] \to \{0, 1\}$ be the characteristic function of S, that is, $\mathbb{1}_S(s) = 1 \Leftrightarrow s \in S$. We often use $\uplus$ to emphasize that a union of pairwise disjoint sets is taken. The graphs in this paper are vertex-labeled and finite. Parallel edges and loops are forbidden, and unless otherwise stated, the labeled vertex set of an n-vertex graph is $[n]$. The line graph $L(G)$ of a graph G is a graph on the vertex set $E(G)$ (so the vertices of $L(G)$ are taken from $\binom{[n]}{2}$), where any two edges $e, f \in E(G)$ that are adjacent are joined (by an edge). The line graph is also free of parallel edges and loops. A trail is a walk that does not visit any edge twice. An open trail starts and ends on two distinct vertices. A closed trail does not have a start nor an end vertex. Given a matrix $M \in \mathbb{Z}^{n \times n}$, its ℓ_1-norm is $\|M\|_1 = \sum_{ij} |M_{ij}|$.

Definition 2.1. Given two graphs on $[n]$ as vertices, say, $X = ([n], E(X))$ and $Y = ([n], E(Y))$, we define their symmetric difference graph

$$X \triangle Y = ([n], E(X) \triangle E(Y)).$$

Definition 2.2. Given a set of edges $R \subseteq \binom{[n]}{2}$, we may treat R as a graph. If X is a graph on the vertex set $[n]$, let

$$X \triangle R = ([n], E(X) \triangle R).$$

Definition 2.3. A *degree sequence* on n vertices is a vector $d \in \mathbb{N}^n$ which is coordinate-wise at most $n-1$. The set of *realizations* of d denotes the following set of graphs:

$$\mathcal{G}(d) = \left\{ G \middle| V(G) = [n], \deg_G(i) = d_i \forall i \in [n] \right\},$$

where $\deg_G(i)$ is the degree of the ith vertex in G. The degree sequence d is *graphic* if $\mathcal{G}(d)$ is non-empty. A set of degree sequences $\mathcal{D}$ may contain graphic as well as non-graphic degree sequences.

The degree sequence of a graph G on $[n]$ as vertices is the vector $\deg_G = (\deg_G(1), \deg_G(2), \ldots, \deg_G(n))$.

Definition 2.4. For a pair of vectors $\ell, u \in \mathbb{N}^n$ we write $\ell \leq u$ if and only if ℓ is coordinate-wise less than or equal to u, that is, $\ell_i \leq u_i$ for all $i \in [n]$. Furthermore, let

$$[\ell, u] = \{d \in \mathbb{N}^n \mid \ell \leq d \leq u\}.$$

Definition 2.5. If $\ell \leq u$ are both degree sequences of length n, then $[\ell, u]$ is a *degree sequence interval*. A degree sequence interval $[\ell, u]$ is called *thin* if $u_i \leq \ell_i + 1$ for all $i \in [n]$. We denote the set of *realizations* of the degree sequence interval $[\ell, u]$ by

$$\mathcal{G}(\ell, u) = \bigcup_{d \in [\ell, u]} \mathcal{G}(d).$$

Remark 2.6. Not every degree sequence in $[\ell, u]$ is necessarily graphic, even if both ℓ and u are graphic.

Definition 2.7. Given a polynomial $p \in \mathbb{R}[x]$, we say that a degree sequence $d \in \mathbb{N}^n$ is *p-stable* if

$$\left| \mathcal{G}(d) \cup \bigcup_{\{i,j\} \in \binom{[n]}{2}} \mathcal{G}(d + \mathbb{1}_{\{i,j\}}) \right| \leq p(n) \cdot |\mathcal{G}(d)|.$$

Definition 2.8. A set of degree sequences $\mathcal{D}$ is p-stable if every degree sequence $d \in \mathcal{D}$ is p-stable.

Definition 2.9. A set of degree sequences $\mathcal{D}$ is *P-stable* if there exists $p \in \mathbb{R}[x]$ such that $\mathcal{D}$ is p-stable.

In Erdős et al. [4], only P-stability is defined, but in this paper, it is more convenient to also define p-stability.

Remark 2.10. A finite set of degree sequences $\mathcal{D}$ is always P-stable.

Let us introduce a weaker stability notion for degree sequence intervals.

Definition 2.11. Given $p \in \mathbb{R}[x]$, we say that a degree sequence interval $[\ell, u] \subseteq \mathbb{N}^n$ is *weakly p-stable* if

$$\left| \bigcup_{\{i,j\} \in \binom{[n]}{2}} \mathcal{G}(\ell, u + \mathbb{1}_{\{i,j\}}) \right| \leq p(n) \cdot |\mathcal{G}(\ell, u)|. \tag{1}$$

Definition 2.12. A set $\mathcal{I}$ of degree sequence intervals is weakly P-stable if there exists $p \in \mathbb{R}[x]$ such that every $[\ell, u] \in \mathcal{I}$ is weakly p-stable. (Any finite $\mathcal{I}$ is weakly P-stable.)

Remark 2.13. If the set of degree sequences $[\ell, u]$ is p-stable, then $[\ell, u]$ is weakly p-stable.

Remark 2.14. It is possible indeed that $[\ell, u]$ is weakly p-stable, but $[\ell, u]$ (as a set of degree sequences) is not p-stable. For example, take $\ell = (0)_{i=1}^n$ and $u = (n-1)_{i=1}^n$: the interval $[\ell, u]$ is clearly 1-stable, but most of the degree sequences on n vertices are not 1-stable.
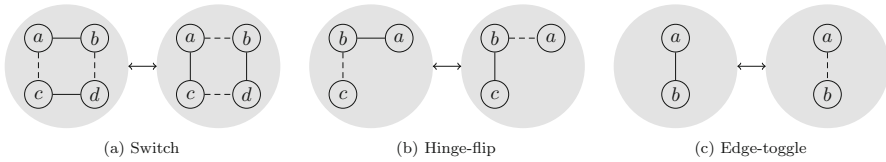
(a) Switch   (b) Hinge-flip   (c) Edge-toggle

FIGURE 1. The three types of operations employed by the degree interval Markov chain. Solid (——) and dashed (- - -) line segments represent edges and non-edges, respectively

Definition 2.15. (Degree interval Markov chain) Let us define the degree interval Markov chain $\mathbb{G}(\ell, u)$. The state space of the Markov chain is $\mathcal{G}(\ell, u)$. In the following, we define three types of transitions: *switches*, *hinge-flips*, and *edge-toggles* (Fig. 1). If the current state of the Markov chain is $G \in \mathcal{G}(\ell, u)$, then

- with probability $1/2$, the chain stays in G (the Markov chain is lazy),
- with probability $1/6$, pick 4 vertices a, b, c, d (uniformly and randomly), and the Markov chain changes its state to $G' = G \triangle \{ab, cd, ac, bd\}$ if $\deg_{G'} = \deg_G$ (in which case the transition is a *switch*), otherwise the chain stays in G,
- with probability $1/6$, pick 3 vertices a, b, c (uniformly and randomly), and the Markov chain changes its state to $G'' = G \triangle \{ab, bc\}$ if $e(G'') = e(G)$ and $G'' \in \mathcal{G}(\ell, u)$ (a *hinge-flip*), otherwise the chain stays in G,
- with probability $1/6$, pick a pair of vertices a, b (uniformly and randomly), and the Markov chain changes its state to $G''' = G \triangle \{ab\}$ if $G''' \in \mathcal{G}(\ell, u)$ (an *edge-toggle*), otherwise the chain stays in G.

We will use the following seminal result of Sinclair. Let $\Pr_{\mathbb{G}}(x \to y)$ denote the transition probability from state x to y in the Markov chain $\mathbb{G}$. Let $\sigma \equiv |V(\mathbb{G})|^{-1}$ be the unique stationary distribution on $\mathbb{G}$. Given a multicommodity flow f on $\mathbb{G}$, let $\ell(f)$ be the length of the longest path with positive flow, and let $\rho(f)$ be the maximum loading through an oriented edge of the Markov graph, that is,

$$\rho(f) = \max_{xy \in E(\mathbb{G})} \frac{1}{\sigma(x) \Pr_{\mathbb{G}}(x \to y)} \sum_{xy \in \gamma \in \Gamma_{X,Y}} f(\gamma). \tag{2}$$

where $\Gamma_{X,Y}$ is the set of all simple directed paths from X to Y in $\mathbb{G}$.

Theorem 2.16. (adapted from Sinclair [11, Proposition 1 and Corollary 6']) *Let $\mathbb{G}$ be an irreducible, symmetric, reversible, and lazy Markov chain. Let f be a multicommodity flow on $\mathbb{G}$ which sends $\sigma(X)\sigma(Y)$ commodity between any ordered pair $X, Y \in V(\mathbb{G})$. Then, the mixing time of the Markov chain in which it converges ε close in ℓ_1-norm to σ started from any element in $V(\mathbb{G})$ is*

$$\tau_{\mathbb{G}}(\varepsilon) \leq \rho(f) \cdot \ell(f) \cdot (\log |V(\mathbb{G})| - \log \varepsilon), \tag{3}$$

One of the most famous applications of this idea is the result of Jerrum and Sinclair [7] providing a probabilistic approximation of the permanent. The following result also relies on Theorem 2.16, and it describes the largest known class of degree sequences where the switch Markov chain is rapidly mixing (that is, the rate of convergence of the Markov chain is bounded by a polynomial of the length of the degree sequence).

Theorem 2.17. [4] *The switch Markov chain is rapidly mixing on the realizations of any degree sequence in a set of P-stable degree sequences (the rate of convergence depends on the set).*

There are several known P-stable regions, one of the earliest and most well-known ones is the following.

Theorem 2.18. (Jerrum, McKay, and Sinclair [6]) *Let* $\delta = \min(d)$ *and* $\Delta = \max(d)$ *be the minimum and maximum value elements in d. The set of degree sequences d satisfying*

$$(\Delta - \delta + 1)^2 \le 4\delta(n - \Delta - 1), d \in \mathbb{N}^n \tag{4}$$

for any n are P-stable. (See Fig. 2.)

Amanatidis and Kleer [1] recently published a surprising new type of result, a clever *approximate uniform sampler* (see, for e.g., [7]) for $\mathcal{G}(\ell, u)$ where elements of $[\ell, u]$ are near regular. They achieve this using a composite Markov chain. They also provide the first step in the direction of sampling $\mathcal{G}(\ell, u)$ directly using the degree interval Markov chain.

Let us reiterate that Amanatidis and Kleer [1] apply the Markov chain suggested by Rechner, Strowick, and Müller–Hannemann [10], which is routinely used in practice.

Theorem 2.19. (Theorem 1.3 in Amanatidis and Kleer [1]) *Let* $0 < \alpha < \frac{1}{2}$ *and* $0 < \rho < 1$ *be fixed. Let* $r = r(n)$ *with* $2 \le r \le (1-\rho)n$. *If* $[\ell_i, u_i] \subseteq [r-r^\alpha, r+r^\alpha]$ *and* $u_i - 1 \le \ell_i \le u_i$ *for all* $i \in [n]$, *then the degree interval Markov chain* $\mathbb{G}(\ell, u)$ *is rapidly mixing.*

Let w_m be the number of realizations in $\mathcal{G}(\ell, u)$ with m edges. The conditions $u_i - 1 \le \ell_i \le u_i$ for all $i \in [n]$ are sufficient to prove that w_m is log-concave, i.e., $w_{m-1}w_{m+1} \le w_m^2$, see Amanatidis and Kleer [1, Theorem 5.4]. The main idea for that proof is a symmetric difference decomposition, which we also characterize in our key decomposition lemma, Lemma 3.21.

Our contribution. The main objective of this paper is to prove the following theorem.

Theorem 2.20. *Suppose* $\mathcal{I}$ *is a set of weakly P-stable and thin degree sequence intervals. Then, the degree interval Markov chain* $\mathbb{G}(\ell, u)$ *is rapidly mixing for any* $(\ell, u) \in \mathcal{I}$.

It is not hard to see that Theorem 2.19 is a special case of Theorem 2.20: substituting into Eq. (4), we get
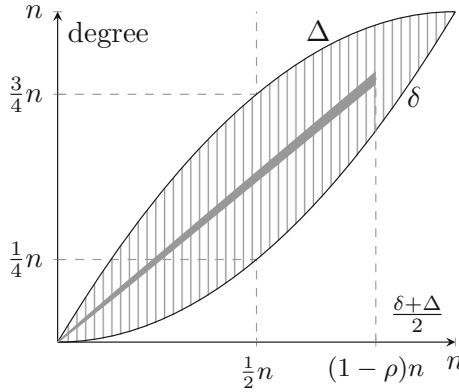
FIGURE 2. Theorem 2.18 defines pairs of lower and upper bounds (δ and Δ), such that any degree sequence which obeys these bounds is P-stable; the area between these functions is filled with vertical lines. The pairs (δ, Δ) of most distant bounds allowed by Eq. (4) are given by intersections with vertical lines. For example, any degree sequence which is (element-wise) between $\delta = \frac{1}{4}n$ and $\Delta = \frac{3}{4}n$ is P-stable. In comparison, the solid gray region represents a $\sqrt{r}$-wide region around the regular degree sequences, which corresponds to the domain of Theorem 2.19

$$(2r^\alpha + 1)^2 \leq 4(r - r^\alpha)(n - r - r^\alpha - 1),$$

which holds for any r and α if n is large enough; see Fig. 2.

The switch Markov chain can be embedded into the degree interval Markov chain (the transition probabilities differ by constant factors). Actually, we will use the proof of Theorem 2.17 as a plug-in in the proof of Theorem 2.20, so this paper does not provide a new proof for the switch Markov chain. We will not consider bipartite and direct degree sequences in this paper, but note that Theorem 2.17 applies to those as well. It is easy to check that the proof of Theorem 2.20 works verbatim for bipartite graphs, because the edge-toggles and hinge-flips are applied on vertices that are joined by paths of odd length (hence in different classes). In all likelihood, the proof of Theorem 2.20 can be probably extended to directed graphs, because directed graphs can be represented as bipartite graphs endowed with a forbidden 1-factor.

3. Constructing and Bounding the Multicommodity Flow

We will define a number of auxiliary structures. Via these structures, we will define a multicommodity flow on the degree interval Markov chain $\mathbb{G}(\ell, u)$ and measure its load.

3.1. Constructing and Counting the Auxiliary Matrices

Kannan et al. [9] already introduced an auxiliary matrix to examine the load of a multicommodity flow. Our auxiliary matrices will be a little different. We start with some definitions, then prove two easy statements.

Definition 3.1. Let the adjacency matrix of a graph X on vertex set $[n]$ be $A_X \in \{0,1\}^{n \times n}$. Let $A_{(vw)}$ be the adjacency matrix of the graph $([n], \{vw\})$ with exactly one edge. Let us define

$$\widehat{M}(X, Y, Z) = A_X + A_Y - A_Z.$$

Remark 3.2. If X, Y, Z are graphs on $[n]$, then $\widehat{M}(X, Y, Z) \in \{-1, 0, 1, 2\}^{n \times n}$.

Let us define the *matrix switch* operation. (In a previous paper [4], this operation was called a generalized switch.)

Definition 3.3. (Switch on a matrix) The switch operation on a matrix M on vertices (a, b, c, d) produces the matrix

$$M - A_{(ab)} - A_{(cd)} + A_{(ac)} + A_{(bd)}.$$

Remark 3.4. A switch on a graph X corresponds to a switch on its adjacency matrix A_X.

Definition 3.5. Let $M \in \{-1, 0, 1, 2\}^{n \times n}$ and let $\deg_M \in \mathbb{Z}^n$ with $(\deg_M)_i = \sum_{j=1}^{n} M_{ij}$ be the sequence of its row-sums. We say that M is *c-tight* (for some $c \in \mathbb{N}$) if M is a symmetric matrix with zero diagonal and there exists a graph $W \in \mathcal{G}(\deg_M, \deg_M + \mathbb{1}_{\{i,j\}})$ for some $\{i, j\} \in \binom{[n]}{2}$ such that $\|M - A_W\|_1 \leq 2c$.

Recall the definition of weak p-stability and Eq. (1). We will use the number of c-tight matrices to bound the number of auxiliary matrices.

Lemma 3.6. *The number of matrices $M \in \{-1, 0, 1, 2\}^{n \times n}$ that are c-tight and $\deg_M \in [\ell, u]$ for a weakly p-stable $[\ell, u]$ is at most*

$$\left| \left\{ M \in \{-1, 0, 1, 2\}^{n \times n} \mid \deg_M \in [\ell, u] \, and \, M \, is \, c-tight \right\} \right| \leq n^{2c} \cdot p(n) \cdot |\mathcal{G}(\ell, u)|.$$

Proof. We can obtain any c-tight M we want to enumerate as follows. First, select, an appropriate $\{i, j\} \in \binom{n}{2}$ and a realization $W \in \mathcal{G}(\ell, u + \mathbb{1}_{\{i,j\}})$: by weak p-stability, there are at most $p(n) \cdot |\mathcal{G}(\ell, u)|$ such choices. Then, select c symmetric pairs of positions where the adjacency matrix A_W is changed to -1 or $+2$ (while preserving symmetry). The latter selection can be made in at most $\binom{n}{2}^c \cdot 2^c$ different ways. □

The following lemma is crucial for proving the tightness of the auxiliary matrices arising in the multicommodity flow. A switch operation on a matrix takes a 2×2 submatrix and adds $+1$'s and -1's to the two–two diagonally opposed entries so that the row- and column-sums are preserved.

Lemma 3.7. (Based on Lemma 7.2 of Erdős et al. [4]) *Suppose $M \in \{-1, 0, 1, 2\}^{n \times n}$ is a symmetric matrix whose diagonal is zero. Suppose further, that*
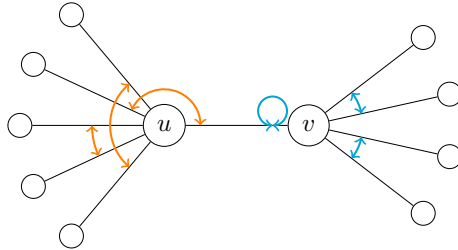
FIGURE 3. The functions $s(u, \bullet)$ and $s(v, \bullet)$ pair the edges incident on u and v, respectively. The orange arcs (↜↝) join edges that are pairs in $s(u, \bullet)$. The cyan arcs (↜↝) join edges that are pairs in $s(v, \bullet)$. The cyan loop (◯) corresponds to the relation $s(v, uv) = uv$

(i) the number of $+2$ entries of M is at most 4,
(ii) the number of -1 entries of M is at most 2,
(iii) there exists $V \subseteq [n]$ where $M|_{V \times V}$ contains every $+2$ and -1 entries of M,
(iv) there exists $v \in V$ such that the $+2$ and -1 entries of M are all located in the row and column corresponding to v,
(v) the row-sum of v in $M|_{V \times V}$ is minimal, and finally,
(vi) every row- and column sum in $M|_{V \times V}$ is at least 1 and at most $|V| - 2$.

Then, M is 5-tight.

Proof. By Lemma 7.2 of Erdős et al. [4], there exist at most two matrix switches that turn M into a $\{0, 1\}$ matrix with the possible exception of a symmetric pair of -1 entries. The -1's remaining after the two matrix switches can be removed by adding $+1$ to the pairs of negative entries.   □

3.2. The Alternating-Trail Decomposition

We consider the set $\binom{[n]}{2}$ in lexicographic order, which induces an order on the set of edges of any graph defined on $[n]$.

Definition 3.8. Given a set of edges $\nabla \subseteq \binom{[n]}{2}$ on $[n]$ as vertices, let $\nabla_v = \{e \mid v \in e \in \nabla\}$. We call $s : \{(v, e) \mid v \in e \in \nabla\} \to \nabla$ a *pairing function* on ∇ if $s(v, \bullet) : \nabla_v \to \nabla_v$ defined as $s(v, \bullet) : e \mapsto s(v, e)$ is an involution, i.e., $s(v, \bullet)$ is it own inverse for any $v \in [n]$. (The bullet $\bullet$ is the placeholder for the variable e which is the second argument of s.) The set of all pairing functions on ∇ is denoted by $\Pi(\nabla)$ (Fig. 3).

Definition 3.9. Let $L(\nabla, s)$ be the following subgraph of the line graph of $([n], \nabla)$: join $e, f \in \nabla$ if and only if $e \neq f$ and there exists a vertex $v \in e \cap f$ such that $s(v, e) = f$ (or equivalently, $s(v, f) = e$).

Lemma 3.10. *Each connected component of $L(\nabla, s)$ is a path or a cycle.*

Proof. Every edge $e = ij \in \nabla$ has at most two neighbors in $L(\nabla, s)$, the edges $s(i, e)$ and $s(j, e)$, thus the maximum degree in $L(\nabla, s)$ is 2.   □

Remark 3.11. A cycle in a line graph corresponds to a closed trail in the original graph. A path in a line graph corresponds to an open trail in the original graph (which may in theory start and end at the same vertex, but this will never be the case in our applications, see Lemma 3.21). Definition 3.9 generalizes a concept of Kannan et al. [9], where all of the components are cycles.

Definition 3.12. Suppose $\nabla \subseteq \binom{[n]}{2}$ and s is a pairing function on ∇. Denote by p_s the number of connected components of $L(\nabla, s)$, and let us define the unique partition

$$\nabla = W_1^s \uplus W_2^s \uplus \cdots \uplus W_{p_s}^s, \tag{5}$$

where each W_k^s is the vertex set of a component of $L(\nabla, s)$, and the sets $(W_k^s)_{k=1}^{p_s}$ are listed in the order induced by their lexicographically first edges.

Definition 3.13. For any set of edges W and $s \in \Pi(\nabla)$, let

$$s|_W = \{(v, e) \mapsto s(v, e) \mid v \in e \in W \text{ and } s(v, e) \in W\}.$$

Subsequently, we also define

$$s - e = s|_{\nabla \setminus \{e\}}.$$

Remark 3.14. If W is the vertex set of a component of $L(\nabla, s)$, then $s|_W \in \Pi(W)$.

Definition 3.15. If $\nabla = \{u_{i-1} u_i \mid i = 1, \ldots, r\}$ is a set of r distinct edges, let $s = u_0 u_1 \ldots u_{r-1} u_r \in \Pi(\nabla)$ denote

$$s = \bigcup_{1 \leq i \leq r-1} \left\{ (u_i, u_{i-1} u_i) \mapsto u_i u_{i+1}, \ (u_i, u_i u_{i+1}) \mapsto u_{i-1} u_i \right\} \cup$$

$$\cup \begin{cases} \{(u_0, u_0 u_1) \mapsto u_0 u_1, (u_r, u_{r-1} u_r) \mapsto u_{r-1} u_r\} & \text{if } u_0 \neq u_r, \\ \emptyset & \text{if } u_0 = u_r. \end{cases}$$

Lemma 3.16. *Let* $\nabla = \{u_{i-1} u_i \mid i = 1, \ldots, r\}$ *and* $s = u_0 u_1 \ldots u_r$. *Then,*

- *the walk* $u_0 u_1 \ldots u_r$ *is a closed trail if and only if* $L(\nabla, s)$ *is a cycle, and*
- *the walk* $u_0 u_1 \ldots u_r$ *is an open trail if and only if* $L(\nabla, s)$ *is a path.*

In other words, the Eulerian trails on ∇ *can be naturally identified with those pairing functions* $s \in \Pi(\nabla)$ *for which* $L(\nabla, s)$ *is connected.*

Proof. Trivial. □

Figure 4 shows a closed trail defined by a pairing function.

From now on, by slight abuse of notation, we will not distinguish between $s = u_0 u_1 \ldots u_r$ as a pairing function and the trail it describes.

Definition 3.17. Let Z be an arbitrary graph on n-vertices and let $\nabla \subseteq \binom{[n]}{2}$ be an arbitrary subset of pairs of vertices. A pairing function $s \in \Pi(\nabla)$ is said to be *Z-alternating* or *alternating in Z* if for every $v \in e \in \nabla$ either

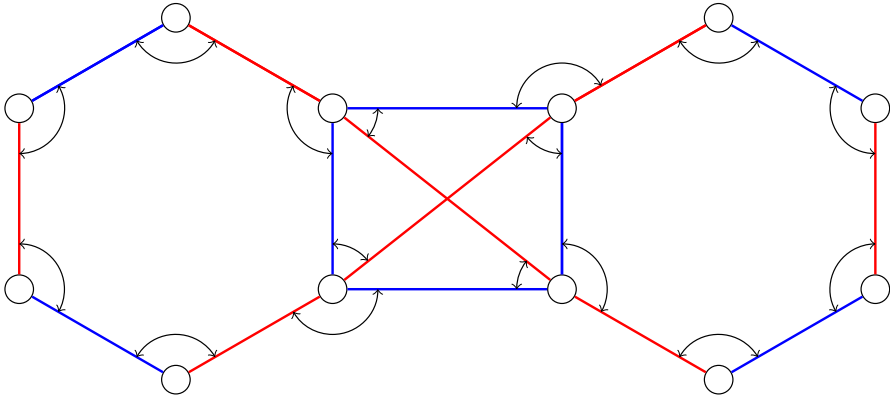- e is a unique solution to $s(v, e) = e$ (the function $s(v, \bullet)$ has at most one fixpoint), or

FIGURE 4. An example for $\nabla = \nabla_{X,Y}$ and an (X,Y)-alternating $s \in \Pi(\nabla)$ (see Definition 3.18). Red edges belong to X and blue edges belong to Y. There are 2^4 different $\pi \in \Pi(\nabla)$ that are (X,Y)-alternating. There is one such π where $L(\nabla, \pi)$ has 3 components (the two C_6's and a C_4 in the middle), and there are 6 cases where $L(\nabla, \pi)$ has 2 components. The black arcs represent an s such that $L(\nabla, s)$ has exactly one component, or, in other words, s defines a closed Eulerian trail on ∇

- $e \in \nabla \cap E(Z)$ and $s(v,e) \in \nabla \backslash E(Z)$, or
- $e \in \nabla \backslash E(Z)$ and $s(v,e) \in \nabla \cap E(Z)$.

In other words, the trail $s|_{W_k^s}$ traverses edges in Z and $\overline{Z}$ in turn for any $k = 1, \ldots, p_s$; furthermore, at any vertex $v \in [n]$, there is at most one trail $s|_{W_k^s}$ which starts or ends at v. For example, if $ij \notin E(Z)$ and $s(i,ij) = ij = s(j,ij)$, then the trail $s|_{\{ij\}}$ consists of one non-edge of Z.

Furthermore, we say that $s \in \Pi(\nabla)$ is *Z-alternating with at most c exceptions* if

$$\Big|\{\{e, s(v,e)\} \; v \in e \in \nabla, \; s(v,e) \neq e \text{ and}$$
$$\Big(\{e, s(v,e)\} \subseteq \nabla \cap E(Z) \text{ or } \{e, s(v,e)\} \subseteq \nabla \setminus E(Z)\Big)\}\Big| \leq c \qquad (6)$$

and $s(v, \bullet)$ has at most one fixpoint for every $v \in [n]$. We say that v is a site of non-alternation of s in Z if $\{e, s(v,e)\}$ is set of size 2 which is a subset of either $\nabla \cap E(Z)$ or $\nabla \setminus E(Z)$.

Definition 3.18. Let $X, Y \in \mathcal{G}(\ell, u)$, where $[\ell, u]$ is a thin degree sequence interval. Denote $\nabla_{X,Y} = E(X) \triangle E(Y)$. An $s \in \Pi(\nabla_{X,Y})$ which is both X-alternating and Y-alternating is called (X,Y)-*alternating*.

Lemma 3.19. *Any pairing function $s \in \Pi(\nabla_{X,Y})$ is X-alternating if and only if s is Y-alternating.*

Proof. Trivial, since $\nabla_{X,Y} \backslash E(X) = E(Y) \backslash E(X)$ and $\nabla_{X,Y} \cap E(X) = E(X) \backslash E(Y)$. $\qquad \square$

Definition 3.20. Given a degree sequence interval $[\ell, u]$, for any $X, Y \in \mathcal{G}(\ell, u)$, define

$$S_{X,Y} = \Big\{ s \in \Pi(\nabla_{X,Y}) \mid s \text{ is } (X,Y) - \text{alternating} \Big\}.$$

Recall Definition 3.13. The following **key decomposition lemma** (KD-lemma) will be referred to repeatedly in this paper.

Lemma 3.21. (Key decomposition lemma) *Let $[\ell, u]$ be a thin degree sequence interval, and let $X, Y \in \mathcal{G}(\ell, u)$, $s \in S_{X,Y}$. Then, $s|_{W_k^s}$ is (X,Y)-alternating, and $s|_{W_k^s}$ describes an Eulerian trail on W_k^s for any $1 \leq k \leq p_s$. If $s|_{W_k^s}$ describes an open trail, then its end-vertices are (by definition) distinct, and the end-vertices of the trail $s|_{W_k^s}$ are disjoint from the end-vertices of any other open trail $s|_{W_j^s}$ ($j \neq k$).*

Proof. We have $|\deg_X(v) - \deg_Y(v)| \leq 1$ for any $v \in V$. Thus, the involution $s(v, \bullet)$ pairs the X-edges of $\nabla_{X,Y}$ incident to v to the Y-edges of $\nabla_{X,Y}$ incident to v, with the exception of the at most one fixpoint of $s(v, \bullet)$. The closed trails must have even length, because $s(v, \bullet)$ pairs X-edges to Y-edges at any v.

Clearly, if an open trail $s|_{W_i^s}$ both starts and ends at v, then $s(v, \bullet)$ has at least two fixpoints, which is a contradiction. Similarly, we have a contradiction if more than one trail terminates at some vertex v. Lastly, if $s|_{W_k^s}$ is an open trail, then the degree $\deg_{W_k^s}(v)$ is even, except if v is one of the two end-vertices of $s|_{W_k^s}$, in which case $\deg_{W_k^s}(v)$ is odd. $\square$

Lemma 3.22. *For any thin degree sequence interval $[\ell, u]$ on n vertices and any two graphs $X, Y \in \mathcal{G}(\ell, u)$*

$$|S_{X,Y}| = \prod_{v \in [n]} \lceil \frac{\deg_{\nabla_{X,Y}}(v)}{2} \rceil!, \tag{7}$$

where the right hand side is the product of factorials.

Proof. We have

$$|\deg_{E(X) \setminus E(Y)}(v) - \deg_{E(Y) \setminus E(X)}(v)| = |\deg_X(v) - \deg_Y(v)| \leq 1.$$

If $\deg_X(v) = \deg_Y(v)$, then we have $(\deg_X(v) - \deg_{X \cap Y}(v))! = (\frac{1}{2} \deg_{\nabla_{X,Y}}(v))!$ ways to choose $s(v, \bullet)$ such that it is an involution which maps edges of X to edges of Y: if $s(v, \bullet)$ had a fixpoint, then by parity it must have had another, too, which contradicts Definition 3.17.

If $\deg_X(v) = \deg_Y(v) + 1$ and $s(v, e) = e$, then $e \in E(X) \setminus E(Y)$ and e is the only fixpoint of $s(v, \bullet)$. Therefore, there are $\deg_X(v) - \deg_{X \cap Y}(v) = \frac{1}{2}(\deg_{\nabla_{X,Y}}(v) + 1)$ ways to choose the fixpoint, and $(\deg_X(v) - \deg_{X \cap Y}(v) - 1)!$ ways to choose the rest of the map $s(v, \bullet)$. $\square$

Lemma 3.23. *For any graph $Z \in \mathcal{G}(\ell, u)$ for a thin degree sequence interval $[\ell, u]$ and any $\nabla \subseteq \binom{[n]}{2}$, we have*

$$\left| \left\{ s \in \Pi(\nabla) \mid s \ isZ - alternating \ with \ at \ mostcexceptions \right\} \right|$$

$$\leq n^{3c} \cdot \prod_{v \in [n]} \lceil \frac{\deg_\nabla(v)}{2} \rceil! \tag{8}$$

Proof. There are at most n^{3c} different choices for the set on the left hand side of Eq. (6). If we fix the non-alternating pairs, then the number of remaining choices at $s(v, \bullet)$ are still upper bounded by $\lceil \frac{1}{2} \deg_\nabla(v) \rceil!$, thus Eq. (8) holds. □

3.3. The Precursor

So far, every proof of rapid mixing for the switch Markov chain which is based on Sinclair's method contains at its core a counting lemma (Greenhill [5]). The purpose of the counting lemma is to enumerate the possible auxiliary structures and parameter sets from which the source and sink of any commodity passing through a realization Z can be recovered from. The difficult technical parts of the proofs are concerned with the maintenance and upkeep associated to these structures. To our surprise, for thin degree sequence intervals, by slightly tweaking these structures, the arising technicalities can almost entirely be reduced to Erdős et al. [4], and a major shortcut is taken by this paper by reusing these parts. A relatively long, but mostly elementary Definition 3.25 will specify the properties that we expect from the auxiliary structures and parameter sets borrowed from Erdős et al. [4]. In Sect. 4, we will use this framework to recombine the borrowed parts into a proof for thin degree sequence intervals.

The decomposition in Definition 3.12 is formally very similar to the decomposition in Erdős et al. [4, Section 4.1]. Whenever the degree sequences of X and Y are identical ($\nabla = \nabla_{X,Y}$ and $s \in S_{X,Y}$), the two decompositions are actually identical. In any other case, for every two unit differences between the degree sequences of X and Y we will utilize a hinge-flip or an edge-toggle in the multicommodity flow between X and Y.

Let us now turn to defining the framework for the reduction to Erdős et al. [4]. We need the following structure and in particular the matrix M to be able to find an appropriate reduction which is compatible with the processes of Erdős et al. [4].

Definition 3.24. Let $M \in \{0, 1, 2\}^{n \times n}$ be a symmetric matrix with zero diagonal. For technical purposes, let us define the following set of triples:

$$\mathfrak{D}_M = \left\{ (X, Y, s) \ \middle| \ \text{where} \ \begin{cases} X, Y \text{ graphs on } [n], \ s \in S_{X,Y}, \\ \{vw \mid v \neq w, \ M_{vw} = 2\} \subseteq E(X) \cap E(Y), \\ \{vw \mid v \neq w, \ M_{vw} = 0\} \subseteq E(\overline{X}) \cap E(\overline{Y}). \end{cases} \right\}. \tag{9}$$

The next definition collects a number of properties (of the multicommodity flow and the auxiliary structures designed for the switch Markov chain) that we want to preserve from Erdős et al. [4].

Definition 3.25. We call the ordered triple (Υ, B, π) a *precursor with parameter* $c \in \mathbb{N}$, if the following properties hold. The objects Υ_M, B_M, and π_M are functions for any symmetric matrix $M \in \{0, 1, 2\}^{n \times n}$ with zero diagonal, where $n \in \mathbb{N}$. We require that the domain of Υ_M satisfies

$$\operatorname{dom}(\Upsilon_M) \subseteq \mathfrak{D}_M, \tag{10}$$

Furthermore, for any $(X, Y, s) \in \operatorname{dom}(\Upsilon_M)$, let us define two degree sequences:

$$\ell_{X,Y} = \Big(\min\{\deg_X(i), \deg_Y(i)\} \Big)_{i=1}^{n},$$
$$u_{X,Y} = \Big(\max\{\deg_X(i), \deg_Y(i)\} \Big)_{i=1}^{n}.$$

We require that $\Upsilon_M(X, Y, s)$ is a sequence of graphs that forms a path connecting X and Y in the Markov graph $\mathbb{G}(\ell_{X,Y}, u_{X,Y})$. We require that π_M and B_M is defined on

$$\operatorname{dom}(\pi_M) = \operatorname{dom}(B_M) = \{(X, Y, s, Z) \mid Z \in \Upsilon_M(X, Y, s), \ (X, Y, s) \in \operatorname{dom}(\Upsilon_M)\}.$$

Moreover,

a. The length of $\Upsilon_M(X, Y, s)$ is at most $c \cdot |\nabla_{X,Y}|$.
b. The size $|(E(X) \triangle E(Z)) \setminus \nabla_{X,Y}| \leq c$ for any $Z \in \Upsilon_M(X, Y, s)$.
c. The matrix $M - A_Z$ is c-tight for any $Z \in \Upsilon_M(X, Y, s)$.
d. The pairing function $\pi_M(X, Y, s, Z)$ is a member of $\Pi(\nabla_{X,Y})$ and it is alternating in Z with at most c exceptions.
e. $\pi_M(X, Y, s, X) = \pi_M(X, Y, s, Y) = s$.
f. If $L(\nabla_{X,Y}, s)$ is connected, then $L(\nabla_{X,Y}, \pi_M(X, Y, s, Z))$ is also connected.
g. The cardinality of

$$\mathfrak{B}_n = \Big\{ B_M(X, Y, s, Z) \ \mid \ Z \in \Upsilon_M(X, Y, s), \ M \text{ arbitrary},$$
$$(X, Y, s) \in \operatorname{dom}(\Upsilon_M), \ |V(X)| = n \Big\}$$

is at most a constant times n^c, i.e., $|\mathfrak{B}_n| = \mathcal{O}(n^c)$.
h. The function

$$\Psi = \{(Z, \nabla_{X,Y}, \pi_M(X, Y, s, Z), B_M(X, Y, s, Z))$$
$$\mapsto (X, Y, s) \mid M \text{ is arbitrary}, \ Z \in \Upsilon_M(X, Y, s)\}$$

is well defined, i.e., two different images in the co-domain are not assigned to the same element from the domain of Ψ.   □

Typically, the value of $B_M(X, Y, s, Z)$ will be a long tuple (an ordered set of parameters). The exact value of c is not important here, the requirements only impose a lower bound on its value. However, it is important to note that c is a constant, independent even from the number of vertices n. Note also that in applications of Definition 3.25, the matrix M will not be completely arbitrary.

Definition 3.26. A subset $\mathfrak{P}$ is a *precursor domain* if it is a set of triples (X, Y, s) such that X and Y have the same vertex set $[n]$ for some $n \in \mathbb{N}$ (where n may vary) and $s \in S_{X,Y}$. We say that a precursor (Υ, B, π) is defined on a *precursor domain* $\mathfrak{P}$ if and only if for any $n \in \mathbb{N}$ and symmetric matrix $M \in \{0, 1, 2\}^{n \times n}$ with zero diagonal, we have

$$\mathrm{dom}(\Upsilon_M) \supseteq \mathfrak{P} \cap \mathfrak{D}_M.$$

Let us define two precursor domains:

$$\mathfrak{C}_{\mathrm{thin}} = \Big\{ (X, Y, s) \ \Big| \ s \in S_{X,Y}, \ L(\nabla_{X,Y}, s) \text{ is connected, and}$$

$$\| \deg_X - \deg_Y \|_\infty \leq 1 \Big\}, \tag{11}$$

$$\mathfrak{R}_{\mathrm{thin}} = \Big\{ (X, Y, s) \ \Big| \ s \in S_{X,Y} \text{ and } \| \deg_X - \deg_Y \|_\infty \leq 1 \Big\}. \tag{12}$$

The set $\mathfrak{C}_{\mathrm{thin}}$ describes the identifiers of the small parts from which the whole multicommodity flow will be built from. In contrast, the multicommodity flow was built in Erdős et al. [4] for each triple in $\mathfrak{R}_{\mathrm{thin}}$ directly.

Lemma 3.27. *If there exists a precursor with parameter c which is defined on $\mathfrak{C}_{\mathrm{thin}}$, then there exists a precursor on $\mathfrak{R}_{\mathrm{thin}}$ with parameter $3c$.*

Proof. We will show that the precursor can be extended so that it is also defined on $\mathfrak{R}_{\mathrm{thin}}$ without violating Definition 3.25. For any $(X, Y, s) \in \mathfrak{R}_{\mathrm{thin}} \cap \mathfrak{D}_M$, we construct a path in the Markov graph of $\mathbb{G}(\ell_{X,Y}, u_{X,Y})$, where $[\ell_{X,Y}, u_{X,Y}]$ is the smallest degree sequence interval that contains both $\deg_X$ and $\deg_Y$. By the thinness of $\mathcal{I}$, we have $|\ell(i) - u(i)| \leq 1$ for every $i \in [n]$. According to Definition 3.12 and the KD-lemma (Lemma 3.21), any $s \in S_{X,Y}$ partitions $\nabla_{X,Y} = E(X) \triangle E(Y)$ into edge sets of (X, Y)-alternating trails, let that decomposition be

$$\nabla_{X,Y} = W_1^s \uplus W_2^s \uplus \cdots \uplus W_{p_s}^s.$$

Let

$$G_k^{X,Y} = X \triangle \bigcup_{i=1}^{k} W_i^s,$$

so that $G_0^{X,Y} = X$ and $G_{p_s}^{X,Y} = Y$. By definition, $s|_{W_k^s}$ is connected, so $(G_{k-1}^{X,Y}, G_k^{X,Y}, s|_{W_k^s}) \in \mathfrak{C}_{\mathrm{thin}}$ for $k = 1, \ldots, p_s$. Let us confirm that $G_k^{X,Y} \in \mathcal{G}(\ell, u)$. If s_k is a closed trail, then the degree sequences of $G_k^{X,Y}$ and $G_{k-1}^{X,Y}$ are identical. If s_k is an open trail whose end-vertices are v and w, then the degree sequences of $G_k^{X,Y}$ and $G_{k-1}^{X,Y}$ differ by 1 precisely on v and w; since these end-vertices are distinct from any other end-vertices of another open trail s_j, such a change of the degree of v and w not occur for any other k. Thus, the degree v satisfies

$$\deg_{G_i^{X,Y}}(v) \in \{\deg_X(v), \deg_Y(v)\} \text{ for any } i = 1, \ldots, p_s \text{ and any } v \in [n],$$

$$\tag{13}$$

and so $\deg_{G_i^{X,Y}} \in [\ell_{X,Y}, u_{X,Y}]$.

It is easy to see that $(G_{k-1}^{X,Y}, G_k^{X,Y}, s|_{W_k^s}) \in \mathfrak{D}_M$. If $e \in E(X) \cap E(Y)$, then $e \notin W_i^s$ for any i. Similarly, if $e \in E(\overline{X}) \cap E(\overline{Y})$, then $e \notin W_i^s$ for any i.

We may now define Υ_M on $\mathfrak{R}_{\text{thin}}$ recursively: concatenate the sequences $\Upsilon_M(G_{k-1}^{X,Y}, G_k^{X,Y}, s|_{W_k^s})$ in increasing order of k to obtain

$$\Upsilon_M(X, Y, s) = \left(\Upsilon_M \left(G_{k-1}^{X,Y}, G_k^{X,Y}, s|_{W_k^s} \right) \right)_{k=1}^{p_s}, \tag{14}$$

where the concatenation keeps only one of the last and first element of consecutive sequences. For $Z \in \Upsilon_M \left(G_{k-1}^{X,Y}, G_k^{X,Y}, s|_{W_k^s} \right)$ (take the maximal k such that the relation holds) let

$$\pi_M(X, Y, s, Z) = \left(\bigcup_{i=1}^{k-1} s|_{W_i^s} \right) \cup \pi_M \left(G_{k-1}^{X,Y}, G_k^{X,Y}, s|_{W_k^s}, Z \right) \cup \left(\bigcup_{i=k+1}^{p_s} s|_{W_i^s} \right) \tag{15}$$

$$B_M(X, Y, s, Z) = \left(k - 1, B_M \left(G_{k-1}^{X,Y}, G_k^{X,Y}, s|_{W_k^s}, Z \right) \right) \tag{16}$$

We claim that the extended functions provide a precursor on $\mathfrak{R}_{\text{thin}}$. Let us check the non-trivial properties of Definition 3.25. Suppose that $Z \in \Upsilon_M \left(G_{k-1}^{X,Y}, G_k^{X,Y}, s|_{W_k^s} \right)$. Then, $\deg_Z \in [\ell_{X,Y}, u_{X,Y}]$, since $\deg_{G_{k-1}^{X,Y}}, \deg_{G_k^{X,Y}} \in [\ell_{X,Y}, u_{X,Y}]$.

Checking Definition 3.25(b). By Definition 3.25(b), $\left| \left(E\left(G_{k-1}^{X,Y}\right) \triangle E(Z) \right) \setminus W_k^s \right| \leq c$, and

$$(E(X) \triangle E(Z)) \setminus \nabla_{X,Y} = \left(E\left(G_{k-1}^{X,Y}\right) \triangle E(Z) \triangle \bigcup_{i=1}^{k-1} W_i^s \right) \setminus \nabla_{X,Y}$$

$$\subseteq \left(E\left(G_{k-1}^{X,Y}\right) \triangle E(Z) \right) \setminus W_k^s, \tag{17}$$

therefore, the LHS has cardinality at most c as well.

Checking Definition 3.25(b). The precursor property holds for $\mathfrak{C}_{\text{thin}}$ and $\left(G_{k-1}^{X,Y}, G_k^{X,Y}, s_k \right) \in \mathfrak{C}_{\text{thin}} \cap \mathfrak{D}_M$, therefore, $M - A_Z$ is c-tight.

Checking Definition 3.25(d). By Eq. (17), $\pi_M(X, Y, s, Z)$ alternates in Z with at most $2c$ extra exceptions on top of the c non-alternations of $\pi_M \left(G_{k-1}^{X,Y}, G_k^{X,Y}, s|_{W_k^s}, Z \right)$.

Checking Definition 3.25(g).

The cardinality of the range of $B_M(X, Y, s, Z)$ grows by a factor of at most n^2, due to the one extra integer $k - 1$.

Checking Definition 3.25(h). The last missing piece to proving that the extended functions are a precursor on $\mathfrak{R}_{\text{thin}}$ is showing that Ψ is well defined on the larger domain. By Definition 3.25(f), the connected components of $L(\nabla_{X,Y}, \pi_M(X, Y, s, Z))$ determine W_i^s and $s|_{W_i^s}$ for any $i \neq k$, and also $\pi_M(G_{k-1}^{X,Y}, G_k^{X,Y}, s|_{W_k^s}, Z)$ and W_k^s.

The number $k-1$ is recorded in $B_M(X, Y, s, Z)$ (see Eq. (16)), which is an argument of Ψ. Knowing k, we can select $\pi_M(G^{X,Y}_{k-1}, G^{X,Y}_k, s|_{W^s_k}, Z)$ from the components of $L(\nabla_{X,Y}, \pi_M(X, Y, s, Z))$, see Eq. (15). Because the original functions provide a precursor on $\mathfrak{C}_{\mathrm{thin}}$, the original Ψ function is well defined, so the following value:

$$\Psi\left(Z, W^s_k, \pi_M\left(G^{X,Y}_{k-1}, G^{X,Y}_k, s|_{W^s_k}, Z\right), B_M\left(G^{X,Y}_{k-1}, G^{X,Y}_k, s|_{W^s_k}, Z\right)\right)$$

$$= \left(G^{X,Y}_{k-1}, G^{X,Y}_k, s|_{W^s_k}\right).$$

is determined. Since

$$E(X) = E\left(G^{X,Y}_{k-1}\right) \triangle \bigcup_{i=1}^{k-1} W^s_i, \qquad E(Y) = E(X) \triangle \nabla_{X,Y}, \qquad s = \bigcup_{i=1}^{p_s} s|_{W^s_i},$$

we have shown that Ψ is well defined even on the extended domain. $\square$

In the proof of Lemma 3.27, we extensively used the fact that the degree sequence intervals in $\mathcal{I}$ are thin.

Theorem 3.28. *Let $\mathcal{I}$ be a set of weakly P-stable degree sequence intervals. If there exists a precursor on $\mathfrak{R}_{\mathrm{thin}}$ with parameter c then the degree interval Markov chain $\mathcal{G}(\ell, u)$ is rapidly mixing for any $[\ell, u] \in \mathcal{I}$.*

Proof. This proof is not new and fairly straightforward, but it is presented for the sake of completeness. The core of this approach had already appeared in the paper of Kannan et al. [9]. We will practically repeat the skeleton of the proof of Erdős et al. [4] using the definitions of the precursor, which hides the majority of the technical difficulties. We will take $M = A_X + A_Y$, but we want to be explicit about the dependence on X and Y even when M appears as an index, so let $X + Y$ denote the matrix $A_X + A_Y$ in this proof.

Let $[\ell, u] \in \mathcal{I}$, where ℓ and u are degree sequences on $[n]$. Let us define the multicommodity flow f on the Markov graph of $\mathbb{G}(\ell, u)$: for every $X, Y \in \mathcal{G}(\ell, u)$ and $s \in S_{X,Y}$, send $\sigma(X)\sigma(Y)/|S_{X,Y}|$ amount of flow on $\Upsilon_{X+Y}(X, Y, s)$. The total flow in f from X to Y sums to $\sigma(X)\sigma(Y)$.

Let us recall Eq. (3):

$$\tau_{\mathbb{G}(\ell,u)}(\varepsilon) \leq \rho(f)\,\ell(f)\,(\log|\mathbb{G}(\ell, u)| - \log \varepsilon) \leq \rho(f)\,\ell(f)\left(\binom{n}{2} - \log \varepsilon\right). \tag{18}$$

By Definition 3.25(a), $\ell(f) \leq c \cdot \binom{n}{2}$. It only remains to show that $\rho(f)$ is polynomial in n. Continuing Eq. (2) with the substitution $\mathbb{G} = \mathbb{G}(\ell, u)$:

$$\rho(f) = \max_{ZW \in E(\mathbb{G})} \frac{1}{\sigma(Z)\Pr_{\mathbb{G}}(Z \to W)} \sum_{\substack{X,Y \in \mathcal{G}(\ell,u),\ s \in S_{X,Y} \\ ZW \in E(\Upsilon_{X+Y}(X,Y,s))}} f(\Upsilon_{X+Y}(X, Y, s))$$

$$\rho(f) \leq \max_{ZW \in E(\mathbb{G})} \frac{1}{\sigma(Z) \cdot 6/n^4} \sum_{\substack{X,Y \in \mathcal{G}(\ell,u),\ s \in S_{X,Y} \\ ZW \in E(\Upsilon_{X+Y}(X,Y,s))}} \frac{\sigma(X)\sigma(Y)}{|S_{X,Y}|}$$

$$\rho(f) \le \frac{n^4}{6|\mathcal{G}(\ell,u)|} \cdot \max_{\substack{ZW \in E(\mathbb{G})}} \sum_{\substack{X,Y \in \mathcal{G}(\ell,u),\ s \in S_{X,Y} \\ ZW \in E(\Upsilon_{X+Y}(X,Y,s))}} \frac{1}{|S_{X,Y}|}$$

$$\rho(f) \le \frac{n^4}{6|\mathcal{G}(\ell,u)|} \cdot \max_{\substack{Z \in V(\mathbb{G})}} \sum_{\substack{X,Y \in \mathcal{G}(\ell,u),\ s \in S_{X,Y} \\ Z \in \Upsilon_{X+Y}(X,Y,s)}} \frac{1}{|S_{X,Y}|} \tag{19}$$

According to Definition 3.25(h), given Z, $\nabla_{X,Y}$, $\pi_{X+Y}(X,Y,s,Z)$, and $B_{X+Y}(X,Y,s,Z)$, the function Ψ determines (X,Y,s). Therefore, the relation $Z \in \Upsilon_{X+Y}(X,Y,s)$ is equivalent to saying that there exists a triple (∇, π, B) such that $(Z, \nabla, \pi, B) \in \Psi^{-1}(X,Y,s)$:

$$\rho(f) \le \frac{n^4}{6|\mathcal{G}(\ell,u)|} \cdot \max_{Z \in V(\mathbb{G})} \sum_{(Z,\nabla,\pi,B) \in \Psi^{-1}(X,Y,s)} \frac{1}{|S_{X,Y}|} \tag{20}$$

Next, we use Lemma 3.22, which shows that $|S_{X,Y}|$ is determined by $\nabla_{X,Y}$, and its value does not depend directly on X or Y:

$$\rho(f) \le \frac{n^4}{6|\mathcal{G}(\ell,u)|} \cdot \max_{Z \in V(\mathbb{G})} \sum_{(Z,\nabla,\pi,B) \in \Psi^{-1}(X,Y,s)} \prod_{v \in [n]} \left(\lceil \frac{\deg_\nabla(v)}{2} \rceil ! \right)^{-1} \tag{21}$$

Given Z, the matrix $\widehat{M}(X,Y,Z)$ (Definition 3.1) determines $\nabla_{X,Y} = E(X) \triangle E(Y)$: the edges that belong to $\nabla_{X,Y}$ are precisely those where the sum of the adjacency matrices $A_X + A_Y = \widehat{M}(X,Y,Z) + A_Z$ takes 1. Furthermore, by a property of the precursor, for any $Z \in \Upsilon_{X+Y}(X,Y,s)$, we have $\deg_Z \in [\ell_{X,Y}, u_{X,Y}]$, therefore,

$$\deg_{\widehat{M}(X,Y,Z)} = \deg_{A_X + A_Y - A_Z} = \deg_X + \deg_Y - \deg_Z \in [\ell_{X,Y}, u_{X,Y}] \subseteq [\ell, u].$$

Now using that $\widehat{M}(X,Y,Z)$ is c-tight, it follows from Lemmas 3.6 and 3.23 that

$$\rho(f) \le \frac{n^4}{6|\mathcal{G}(\ell,u)|} \cdot \max_{Z \in V(\mathbb{G})} \sum_{B \in \mathfrak{B}_n} n^{5c} \cdot p(n) \cdot |\mathcal{G}(\ell,u)| \le \frac{1}{6} n^{5c+4} \cdot p(n) \cdot |\mathfrak{B}_n|, \tag{22}$$

where the right hand side is dominated by a polynomial of n (according to Definition 3.25(g)). In conclusion, the mixing time in Eq. (18) is polynomial.
$\square$

To prove Theorem 2.20, it only remains to construct a precursor on $\mathfrak{C}_{\text{thin}}$. The next section proceeds with the construction in two separate stages.

4. Constructing the Precursor

We will construct a precursor on $\mathfrak{C}_{\text{thin}}$ for any weakly P-stable thin set of degree sequence intervals $\mathcal{I}$ in two stages. In the first stage, we show that there exists a precursor on $\mathfrak{C}_{\text{id}}$ (see Definition 4.1), and then we will extend

this precursor to $\mathfrak{C}_{\text{thin}}$ in the second stage. Then, we will apply Lemma 3.27 and Theorem 3.28 to prove Theorem 2.20.

4.1. Stage 1: Closed trails

Definition 4.1. Let us define

$$\mathfrak{C}_{\text{id}} = \left\{ (X, Y, s) \ \middle| \ s \in S_{X,Y}, \ L(\nabla_{X,Y}, s) \text{ is connected, and } \deg_X = \deg_Y \right\},$$

$$\mathfrak{R}_{\text{id}} = \left\{ (X, Y, s) \ \middle| \ s \in S_{X,Y} \text{ and } \deg_X = \deg_Y \right\}.$$

The graph $L(\nabla_{X,Y}, s)$ is a cycle for any $(X, Y, s) \in \mathfrak{C}_{\text{id}}$, because the degree sequences of X and Y are identical. To handle this case, a large machinery was developed in Erdős et al. [4]. However, there the range of auxiliary matrices M was much smaller. Because of the larger range of auxiliary matrices in the current paper, we had to introduce and explicitly define the precursor. Therefore, we unfortunately need to repeat some parts of the proof of Erdős et al. [4] to obtain those claims in the desired generality. The following lemma collects the necessary technical lemmas proved in Erdős et al. [4].

Lemma 4.2. *There exists a precursor on $\mathfrak{C}_{\text{id}}$ with parameter $c = 12$.*

Proof. Let $(X, Y, s) \in \mathfrak{C}_{\text{id}}$ be arbitrary with $X, Y \in \mathcal{G}(d)$. Since s is an (X, Y)-alternating closed trail, $|\nabla_{X,Y}|$ is even. In Erdős et al. [4], the path $\Upsilon(X, Y, s)$ in the switch Markov graph is defined exactly when the degree sequences of X and Y are identical and $s \in S_{X,Y}$. We use the definition of $\Upsilon(X, Y, s)$ from Erdős et al. [4] only when $s \in S_{X,Y}$ and $L(\nabla_{X,Y}, s)$ is a cycle, so when $p_s = 1$.

First, let us recall that $\Upsilon(X, Y, s)$ in Erdős et al. [4] describes a sequence of graphs such that each two consecutive graphs can be obtained from each other by a switch. In Erdős et al. [4, Definition 4.2], for any $s \in S_{X,Y}$, the path $\Upsilon(X, Y, s)$ is composed by concatenating a number of SWEEP sequences:

$$\Upsilon(X, Y, s) = \left(\left(\text{SWEEP}(G_r^k, C_r^k) \right)_{r=1}^{\mu_k + 1} \right)_{k=1}^{p_s},$$

where C_r^k are circuits and $G_{r+1}^k = G_r^k \triangle C_r^k$, where $G_0^k = X \triangle \uplus_{i=1}^{k-1} W_i^s$ and $G_{\mu_k+1}^k = X \triangle \uplus_{i=1}^{k} W_i^s$, and $W_k^s = \uplus_{r=1}^{\mu_k+1} E(C_r^k)$ and $\nabla_{X,Y} = \uplus_{k=1}^{p_s} W_k^s$. It is easy to check in Erdős et al. [4, Algorithm 2.1], that $\text{SWEEP}(G_r^k, C_r^k)$ is a sequence of switches such that each switch is incident with all four vertices on $V(C_r^k)$.

When $(X, Y, s) \in \mathfrak{C}_{\text{id}} \cap \mathfrak{D}_M$, by definition $L(\nabla_{X,Y}, s)$ is connected and $p_s = 1$, thus we may define

$$\Upsilon_M(X, Y, s) = \left(\text{SWEEP}(X \triangle \uplus_{i=1}^{r-1} C_i, C_r) \right)_{r=1}^{\mu+1}, \tag{23}$$

where s decomposes $\nabla_{X,Y}$ into (primitive) circuits $(C_r)_{r=1}^{\mu}$ such that $\nabla = \uplus_{r=1}^{\mu} E(C_r)$, see Erdős et al. [4, Lemma 5.13]. The circuit C_r defines a cyclical order on its vertices, but SWEEP takes a linear order, so we still need to select the **cornerstone**, where the linear order starts to enumerate the vertices in the given cyclical order. The choice of the cornerstone ([4, eq. (5.11)]) only plays a role in proving that $\widehat{M}(X, Y, Z)$ is close to the adjacency matrix of an

appropriate graph in ℓ_1-norm. From the rest of Erdős et al. [4]'s point of view, the cornerstone is arbitrarily chosen.

In this adaptation of the proof in Erdős et al. [4], the index M of $\Upsilon_M(X, Y, s)$ matters only in the choice of the cornerstones. The current proof is slightly more general than that of Erdős et al. [4], because we not only consider $M = X + Y$, but also any other M such that $(X, Y, s) \in \mathfrak{D}_M$ (recall Eq. (9)). In the path $\Upsilon_M(X, Y, s)$ incorporating $\text{SWEEP}(G_r, C_r)$ (see Eq.)23)) choose the cornerstone v_r of the $\text{SWEEP}(G_r, C_r)$ as follows:

Let $v_r \in V(C_r)$ be the vertex which minimizes the row-sum in

$$\left(M - A_{X \triangle \uplus_{i=1}^{r-1} C_i} \right) \Big|_{V(C_r) \times V(C_r)} \tag{24}$$

and$v_r$is lexicographically minimal with respect to this condition.

Since $X, Y \in \mathcal{G}(d)$ for some d, Lemma 2.6 of Erdős et al. [4] applies, which claims that $\text{SWEEP}(X \triangle \uplus_{i=1}^{r-1} C_i, C_r)$ is a sequence of at most $\frac{1}{2} |E(C_r)| - 1$ switches that connect $X \triangle \uplus_{i=1}^{r-1} C_i$ to $X \triangle \uplus_{i=1}^{r} C_i$. Thus, the total length of the switch sequence $\Upsilon_M(X, Y, s)$ is at most $\frac{1}{2} |\nabla_{X,Y}| - 1$. For any $Z \in \Upsilon_M(X, Y, s)$, the degree sequences of X, Y and Z are identical, because switches preserve the degree sequence. Note that for any $j \neq r$, the sequence $\text{SWEEP}(X \uplus_{i=1}^{j-1} C_i, C_j)$ does not depend on the cornerstone v_r.

For any $Z \in \Upsilon_M(X, Y, s)$, the matrix $M - A_Z$ belongs to $\{-1, 0, 1, 2\}^{n \times n}$. Recall Eq. (9). If $(M - A_Z)_{vw} = 2$, then vw is an edge in both X and Y, but vw is not present in Z as an edge. If, however, $(M - A_Z)_{vw} = -1$, then $vw \notin E(X), E(Y)$ and $vw \in E(Z)$. With formulae,

$$\{vw \mid (M - A_Z)_{vw} = +2\} \subseteq E(X) \setminus E(Z) \setminus \nabla_{X,Y} \tag{25}$$

$$\{vw \mid (M - A_Z)_{vw} = -1\} \subseteq E(Z) \setminus E(X) \setminus \nabla_{X,Y} \tag{26}$$

respectively. In Erdős et al. [4, Lemma 2.7], the set $R = R_Z$ is defined, and it has cardinality at most 4. By its definition, the set of edges in R is a superset of $(E(X) \triangle E(Z)) \setminus \nabla_{X,Y}$, which is the union of the right hands sides of Eqs. (25) and (26). In short, every $+2$ and -1 entry of $M - A_Z$ is in a position which is associated to an edge in R.

We will show that $M - A_Z$ is 7-tight. Lemma 8.2 in Erdős et al. [4] is the analogue of this tightness statement, and its proof can be repeated for this case with little to no modification. Suppose first that every edge in R is incident on v_r from Eq. (24): then, Erdős et al. [4, Lemma 7.1] claim that the entries in $A_X + A_Y - A_Z$ associated to edges in R consist of at most two pairs of symmetric $+2$ entries, and at most one pair of symmetric -1 entries. By Eqs. (25) and (26), $M - A_Z$ also contains at most two pairs of symmetric $+2$ entries, and at most one pair of symmetric -1 entries.

Recall that Z is obtained from $X \triangle \uplus_{i=1}^{r-1} C_i$ through a series of switches that only touch edges whose vertices are contained in $V(C_r)$. Thus, the row- and columns-sums of the submatrices

$$(M - A_Z)|_{V(C_r) \times V(C_r)} \quad \text{and} \quad \left(M - A_{X \triangle \uplus_{i=1}^{r-1} C_i} \right) \Big|_{V(C_r) \times V(C_r)}$$

are identical. Let v and w be two distinct vertices in $V(C_r)$.

- If $M_{vw} = 2$, then by Eq. (25), $vw \in E(X)$ and $vw \notin \nabla_{X,Y}$, thus

$$\left(M - A_{X \triangle \uplus_{i=1}^{r-1} C_i} \right)_{vw} = (M - A_X)_{vw} = 2 - 1 = 1.$$

- If $M_{vw} = 0$, then by Eq. (26), $vw \notin E(X)$ and $vw \notin \nabla_{X,Y}$, thus

$$\left(M - A_{X \triangle \uplus_{i=1}^{r-1} C_i} \right)_{vw} = (M - A_X)_{vw} = 0 - 0 = 0.$$

- If $M_{vw} = 1$ and $vw \in E\left(X \triangle \uplus_{i=1}^{r-1} C_i \right)$, then $\left(M - A_{X \triangle \uplus_{i=1}^{r-1} C_i} \right)_{vw} = 0$.

- If $M_{vw} = 1$ and $vw \notin E\left(X \triangle \uplus_{i=1}^{r-1} C_i \right)$, then $\left(M - A_{X \triangle \uplus_{i=1}^{r-1} C_i} \right)_{vw} = 1$.

Every entry of $\left(M - A_{X \triangle \uplus_{i=1}^{r-1} C_i} \right)\big|_{V(C_r) \times V(C_r)}$ is either a 0 or a 1, and the diagonal is identically zero. Since C_r is alternating in $X \triangle \uplus_{i=1}^{r-1} C_i$, there is at least one 0 entry and one 1 entry in every row and every column. Therefore, the row- and column-sums of $(M - A_Z)|_{V(C_r) \times V(C_r)}$ are at least 1 and at most $|V(C_r)| - 2$. Moreover, Eq. (24) ensures that the row-sum corresponding to v_r in $(M - A_Z)_{V(C_r) \times V(C_r)}$ is minimal. By Lemma 3.7, $M - A_Z$ is 5-tight.

We will again use Erdős et al. [4, Lemma 2.7] to understand the more detailed structure of R_Z. If there is an edge in R_Z which is not incident on v_r, then R falls under case (e) of Erdős et al. [4, Lemma 2.7]. Let $Z \triangle F$ be the next graph in the SWEEP sequence, where F is a C_4. By Erdős et al. [4, Lemma 2.7(d)], every edge in the set $R_{Z \triangle F}$ is incident on v_r. As previously, Lemma 3.7 implies that $M - A_{Z \triangle F}$ is 5-tight, and thus $M - A_Z$ is 7-tight. $\qquad \square$

Next, we will cite 3 lemmas from Erdős et al. [4]. The first of these lemmas refers to the graph $Z' = Z \triangle R$, which is defined in Erdős et al. [4, eq. (13)]. Note that the graph Z' is just a slight perturbation of Z.

Lemma 4.3. (Adapted from Lemma 5.15 in Erdős et al. [4]) *For any $Z \in \Upsilon(X, Y, s)$ for $s \in S_{X,Y}$, there exists $\pi_{Z'} \in \Pi(\nabla)$ which defines a closed Eulerian trail on $\nabla_{X,Y}$ which is alternating in Z' with at most 4 exceptions.*

Lemma 4.4. (Lemma 5.21 in Erdős et al. [4]) *For a fixed number n of vertices of X and Y, the cardinality of the set of possible tuples $B(X, Y, Z, s)$ is $\mathcal{O}(n^8)$, where $s \in S_{X,Y}$ and $Z \in \Upsilon(X, Y, s)$ are arbitrary.*

Lemma 4.5. (Lemma 5.22 in Erdős et al. [4]) *The quadruplet composed of the graphs Z, ∇, $\pi_{Z'}$, and $B(X, Y, Z, s)$ uniquely determines the triplet (X, Y, s).*

We define $\pi_M(X, Y, s, Z) = \pi_{Z'}$. Lemma 4.3 implies that $\pi_{Z'}$ is alternating in Z with at most $4 + 2|R_Z| \leq 12$ exceptions. Let $B_M(X, Y, s, Z)$ be identical with the parameter set $B(X, Y, Z, s)$ defined in Erdős et al. [4]. Lemmas 4.3 to 4.5 ensure that every itemized requirement of Definition 3.25 holds, similarly to the situation in Erdős et al. [4]. $\qquad \square$**Lemma**4.2

Now we are at the point where Theorem 2.17 is reproved by the generalized machinery: the Markov chain $\mathbb{G}(d)$ (using switches only) is rapidly mixing for any d from a P-stable set. By Lemmas 4.2 and 3.27, there exists a precursor on $\mathfrak{R}_{id}$ with parameter $3c$, and the theorem follows from Remark 2.13 and Theorem 3.28.

4.2. Stage 2: Open trails

Until now, the degree sequences of X and Y in $(X, Y, s) \in \mathfrak{C}_{\text{thin}}$ were identical, that is, s was a closed trail. In the second stage we deal with the case when $\| \deg_X - \deg_Y \|_1 = 2$ (while $\| \deg_X - \deg_Y \|_\infty = 1$). The following lemma is actually a framework for reducing the construction of the precursor on $\mathfrak{C}_{\text{thin}}$ to Lemma 4.2. Note that we do not aim to optimize our estimate of the mixing time, we are merely interested in bounding it polynomially. Surprisingly, to construct the precursor on $\mathfrak{C}_{\text{thin}}$, it is sufficient to consider only those open trails s that have odd length.

Informally, the forthcoming Lemma 4.6 states that if any open (X, Y)-alternating trail of odd length can be cut up into a constant number of segments that can be reassembled into at most two (X, Y)-alternating trails that are either closed or can be closed by including $v_0 v_\lambda$ or $v_1 v_{\lambda-1}$ to join the two ends (alternation is not required there), then we can reduce the precursor construction on $\mathfrak{C}_{\text{thin}}$ to a precursor construction on $\mathfrak{C}_{\text{id}}$.

Lemma 4.6. *Suppose there exists a precursor on $\mathfrak{C}_{\text{id}}$ with parameter c, and let c' be a fixed integer. Suppose, moreover, that for any $(X, Y, s) \in \mathfrak{C}_{\text{thin}}$ where $s = v_0 v_1 \ldots v_\lambda \in \Pi(\nabla_{X,Y})$ is an open trail with $v_0 <_{\text{lex}} v_\lambda$ for some odd integer λ, there exist ∇_1, ∇_2 and $s_1 \in \Pi(\nabla_1), s_2 \in \Pi(\nabla_2)$ (where $\nabla_2 = \emptyset$ is allowed) such that*

1. *$\nabla_{X,Y} \backslash \{v_0 v_\lambda\} \subseteq \nabla_1 \cup \nabla_2 \subseteq \begin{cases} \nabla_{X,Y} \cup \{v_0 v_\lambda\} & \text{if } v_1 = v_{\lambda-1}, \\ \nabla_{X,Y} \cup \{v_0 v_\lambda, v_1 v_{\lambda-1}\} & \text{if } v_1 \neq v_{\lambda-1} \end{cases}$,*
2. *$\nabla_{X,Y} \triangle \nabla_1 \triangle \nabla_2 \subseteq \{v_0 v_\lambda\}$,*
3. *if $v_1 v_{\lambda-1} \in (\nabla_1 \cup \nabla_2) \backslash \nabla_{X,Y}$, then $v_0 v_\lambda \in \nabla_{X,Y}$ and s_1 or s_2 is equal to $v_0 v_1 v_{\lambda-1} v_\lambda v_0$.*

Moreover, for both $i = 1, 2$:

  (4) *the line graph $L(\nabla_i, s_i)$ is an even cycle (or an empty graph),*
  (5) *$s_i - v_0 v_\lambda - v_1 v_{\lambda-1}$ is (X, Y)-alternating,*
  (6) *$s_i - v_0 v_\lambda$ is (X, Y)-alternating with 0 or 2 exceptions,*
  (7) *$s_i - v_1 v_{\lambda-1}$ is (X, Y)-alternating with 0 or 2 exceptions,*
  (8) *the number of components of $L(\nabla_i, s_i) \cap L(\nabla, s)$ is at most c'.*

Then, there exists a precursor on $\mathfrak{C}_{\text{thin}}$ with parameter $3c + 60c' + 300$.

We are aware that such a huge parameter is nowhere near a practical bound. We made virtually zero effort to optimize the parameter.

Proof. Let $(X, Y, s) \in \mathfrak{C}_{\text{thin}}$ be such that $s = v_0 v_1 \ldots v_\lambda$ and $v_0 <_{\text{lex}} v_\lambda$. We will now consider the case when λ is odd. As discussed earlier, the case of even λ will be handled by a reduction to the odd case. For an odd λ, we must have either $\deg_Y = \deg_X + \mathbb{1}_{\{v_0, v_\lambda\}}$ or $\deg_Y = \deg_X - \mathbb{1}_{\{v_0, v_\lambda\}}$, because s is (X, Y)-alternating and its length λ is odd.

Let ∇_i and $s_i \in \Pi(\nabla_i)$ for $i = 1, 2$ be the set of edges and pairing function assumed to exist in the statement of this lemma. Let M be such that $(X, Y, s) \in \mathfrak{D}_M$, and we will first define $\Upsilon_M(X, Y, s)$, then we will also define $\pi_M(X, Y, s, Z)$ and $B_M(X, Y, s, Z)$ for any $Z \in \Upsilon_M(X, Y, s)$.

Let us modify the auxiliary matrix M. Recall from Definition 3.24 that if $ab \in \nabla_{X,Y}$, then $M_{ab} = 1$. By Assumption 3 of this lemma, if $M_{v_1 v_{\lambda-1}} \neq 1$ and $v_1 v_{\lambda-1} \in \nabla_i$, then $v_0 v_\lambda \in \nabla_{X,Y}$ and $M_{v_0 v_\lambda} = 1$. Let us define

$$M' = \begin{cases} M + A_{(v_0 v_\lambda)} & \text{if } M_{v_0 v_\lambda} = 0, \\ M - A_{(v_0 v_\lambda)} & \text{if } M_{v_0 v_\lambda} = 2, \\ M + A_{(v_1 v_{\lambda-1})} - A_{(v_0 v_1)} - A_{(v_{\lambda-1} v_\lambda)} & \text{if } M_{v_0 v_\lambda} = 1 \text{ and } M_{v_1 v_{\lambda-1}} = 0 \text{ and } v_1 \neq v_{\lambda-1}, \\ M - A_{(v_1 v_{\lambda-1})} + A_{(v_0 v_1)} + A_{(v_{\lambda-1} v_\lambda)} & \text{if } M_{v_0 v_\lambda} = 1 \text{ and } M_{v_1 v_{\lambda-1}} = 2 \text{ and } v_1 \neq v_{\lambda-1}, \\ M & \text{if } M_{v_0 v_\lambda} = 1 \text{ and } M_{v_1 v_{\lambda-1}} = 1 \text{ or } v_1 = v_{\lambda-1}, \end{cases}$$

(27)

so that $M'_{v_0 v_\lambda} = 1$. Also, $M'_{v_1 v_{\lambda-1}} = 1$ if $v_1 v_{\lambda-1} \in \nabla_1 \cup \nabla_2$. The row-sums of M and M' are equal on every vertex except possibly on v_0 and v_λ.

By assumption (3), $|\nabla_i|$ is even. From Assumptions 1 and 2 it follows that any $v_j v_{j+1}$ is contained in either ∇_1 or ∇_2, but not both, except if $\{v_j, v_{j+1}\} = \{v_0, v_\lambda\}$ or $\{v_j, v_{j+1}\} = \{v_1, v_{\lambda-1}\}$. Therefore, $\nabla_1 \cap \nabla_2 \subseteq \{v_0 v_\lambda, v_1 v_{\lambda-1}\}$. Let us start to extend the precursor. Without loss of generality, we may assume that $\nabla_1 \neq \emptyset$.

Case A1: $\nabla_2 = \emptyset$. Note that $|\nabla_{X,Y}| = \lambda$ is odd. Since $|\nabla_1|$ is even and $\nabla_1 \setminus \nabla_{X,Y} \subseteq \{v_0 v_\lambda\}$, we must have $v_0 v_\lambda \notin \nabla_{X,Y} \Leftrightarrow v_0 v_\lambda \in \nabla_1$. Also, $v_1 v_{\lambda-1} \in \nabla_1 \Leftrightarrow v_1 v_{\lambda-1} \in \nabla_{X,Y}$. Let us slightly change X and Y, so that the symmetric difference of the modified graphs X_1, Y_1 is exactly ∇_1:

$$X_1 = \begin{cases} X \triangle v_0 v_\lambda, & \text{if } s_1 \text{ is not alternating in } X; \\ X, & \text{if } s_1 \text{ is alternating in } X; \end{cases}$$

$$Y_1 = \begin{cases} Y, & \text{if } s_1 \text{ is not alternating in } X; \\ Y \triangle v_0 v_\lambda, & \text{if } s_1 \text{ is alternating in } X; \end{cases}$$

Suppose $s_1 - v_0 v_\lambda$ is not alternating in X: then $v_1 v_{\lambda-1} \in \nabla_{X,Y}$ and the two non-alternations of $s_1 - v_0 v_\lambda$ are located at v_1 and $v_{\lambda-1}$. But because $s_1 - v_0 v_\lambda - v_1 v_\lambda$ is alternating in X, we have $|\deg_{E(X) \cap \nabla_{X,Y}}(v_1) - \deg_{E(Y) \cap \nabla_{X,Y}}(v_1)| = 2$, so s cannot possibly be (X, Y)-alternating, a contradiction. It follows that s_1 is alternating in X_1 (and thus Y_1): indeed, if s_1 is not alternating in X, then the two exceptions are the endpoints of $v_0 v_\lambda$. Therefore, $(X_1, Y_1, s_1) \in \mathfrak{C}_{\mathrm{id}} \cap \mathfrak{D}_{M'}$.

We extend the precursor to (X, Y, s) as follows.

$$\Upsilon_M(X, Y, s) = \begin{cases} X \xrightarrow{\text{toggle} v_0 v_\lambda} \Upsilon_{M'}(X_1, Y_1, s_1) & \text{if } s_1 \text{ is not alternating in } X \\ \Upsilon_{M'}(X_1, Y_1, s_1) \xrightarrow{\text{toggle} v_0 v_\lambda} Y & \text{if } s_1 \text{ is alternating in } X, \end{cases}$$

(28)

$$\pi_M(X, Y, s, Z) = \begin{cases} s & \text{if } Z = X, Y \\ \pi_{M'}(X_1, Y_1, s_1, Z)|_{\nabla_{X,Y}} & \text{if } Z \in \Upsilon_{M'}(X_1, Y_1, s_1) \setminus \{X, Y\} \end{cases}$$

(29)

$$B_M(X,Y,s,Z) = \begin{cases} (0, \text{true}) & \text{if } Z = X \\[2mm] (0, \text{false}) & \text{if } Z = Y \\[2mm] \big(1, \lambda, v_0 v_\lambda, v_0 v_\lambda \in E(X), \pi_M(X,Y,s,Z) \triangle \pi_{M'}(X_1, Y_1, \\ \qquad\qquad\qquad s_1, Z), B_{M'}(X_1, Y_1, s_1, Z)\big) \\ \qquad\qquad \text{if } Z \in \Upsilon_{M'}(X_1, Y_1, s_1) \setminus \{X, Y\} \end{cases}$$

$$(30)$$

Let us verify that Definition 3.25 holds for the extension. The defined path $\Upsilon_M(X,Y,s)$ in the Markov graph utilizes one edge-toggle, while the rest of the steps are switches. When the edge-toggle occurs, the degree sequence of the then current graph changes from $\deg_X$ to $\deg_Y$, because the rest of the steps do not change the degree sequence.

If $Z = X, Y$, then $M - A_Z$ is 0-tight, because $(X \triangle Z) \setminus \nabla = (Y \triangle Z) \setminus \nabla = \emptyset$. Suppose next, that $Z \in \Upsilon_{M'}(X_1, Y_1, s_1)$. If $M' = M$, then $M - A_Z$ is c-tight by induction. If $M' = M \pm A_{v_0 v_\lambda}$, then note that the row-sums of $M' - A_X$ are equal to the row-sums of $M - A_Y$, and the row-sums of $M' - A_Y$ are equal to the row-sums of $M - A_X$. The degree sequence of Z is equal to $\deg_X$ or $\deg_Y$, so $M' - A_Z$ is c-tight, and therefore, $M - A_Z$ is $c + 1$-tight.

The length of $\Upsilon_M(X,Y,s)$ is at most $1 + c|\nabla_1| = 1 + c|\nabla_{X,Y}| + c$, still linear. The symmetric difference of X and Z outside $\nabla_{X,Y}$ may also include $v_0 v_\lambda$, so the upper bound in Definition 3.25(b) increases by at most one.

The maximum number of exceptions to alternation of $\pi_M(X,Y,s,Z)$ in Z is no more than the number of exceptions to alternation of $\pi_{M'}(X_1, Y_1, s_1, Z)$ in Z, because $v_0 v_\lambda \notin \nabla_{X,Y}$. Since $\pi_{M'}(X_1, Y_1, s_1, Z)$ is a closed trail, even if we restrict its domain from ∇_1 to $\nabla_{X,Y}$, it remains connected. The range of $B_M(X,Y,s,Z)$ increases by a polynomial multiplicative factor (of at most $4n^4$, but this will be dwarfed by the bound in the next case).

Lastly, Ψ is still well defined. Trivially, if $B_M(X,Y,s,Z) = (0, \text{true})$ (alternatively $(0, \text{false})$), then $X = Z$ ($Y = Z$) and $Y = Z \triangle \nabla_{X,Y}$ ($X = Z \triangle \nabla_{X,Y}$). If $B_M(X,Y,s,Z) = (1, \cdots)$, then we can recover $\pi_{M'}(X_1, Y_1, s_1, Z)$ from $\pi_M(X,Y,s,Z)$ using their symmetric difference, and subsequently, we can recover X_1 and Y_1 via Ψ, because we have a precursor on $\mathfrak{C}_{\text{id}}$. From these graphs we can easily recover both X and Y, as $B_M(X,Y,s,Z)$ describes whether $v_0 v_\lambda$ is in $E(X)$ or not (and the same containment relation holds for $E(Y)$ because $v_0 v_\lambda \notin \nabla_{X,Y}$).

Case A2: $\nabla_1 \neq \emptyset$ and $\nabla_2 \neq \emptyset$. *Task 1: constructing $\Upsilon_M(X,Y,s)$.* Obviously, $|\nabla_i| \geq 4$ for $i = 1, 2$ (s_i an even length closed trail), and $|\nabla_1 \cap \nabla_2| \leq 2$, so $\lambda \geq 5$. The reduction is similar to the previous case, however, the construction of the precursor on (X,Y,s) will be reduced to not one, but two elements of $\mathfrak{C}_{\text{id}}$. Recall, that any $v_j v_{j+1} \neq v_0 v_\lambda, v_1 v_{\lambda-1}$ appears in exactly one of ∇_1 and ∇_2. If $v_1 v_{\lambda-1} \in \nabla_{X,Y} \cup \nabla_1 \cup \nabla_2$, then the edge $v_1 v_{\lambda-1}$ appears in exactly two of $\nabla_{X,Y}, \nabla_1, \nabla_2$. Observe, that for any vertex $v \in [n]$, we have

$$\deg_X(v) - \deg_{E(X) \cap \nabla_{X,Y}} + \deg_{E(Y) \cap \nabla_{X,Y}} = \deg_Y(v). \qquad (31)$$

Thus, for any $v \neq v_0, v_\lambda$, we have

$$\deg_{E(X) \cap \nabla_{X,Y}} = \deg_{E(Y) \cap \nabla_{X,Y}}. \tag{32}$$

Suppose that $s_i - v_0 v_\lambda$ is not alternating in X for $i = 1$ and $i = 2$. Then, $s_i - v_0 v_\lambda$ is not alternating at v_1 and $v_{\lambda-1}$, which implies that $v_1 v_{\lambda-1} \in \nabla_1, \nabla_2$ and $v_1 v_{\lambda-1} \notin \nabla_{X,Y}$. If, say, $v_1 v_{\lambda-1} \in E(X)$, then $s_i(v_1, v_1 v_{\lambda-1}) \in E(X)$, but $s_i - v_0 v_\lambda - v_1 v_{\lambda-1}$ is alternating (for $i = 1, 2$); thus, we have $\deg_{E(X) \cap \nabla_{X,Y}}(v_1) = \deg_{E(Y) \cap \nabla_{X,Y}}(v_1) + 2$, so s cannot possibly be (X, Y)-alternating, a contradiction. The case $v_1 v_{\lambda-1} \notin E(X)$ similarly leads to a contradiction, therefore, at least one of $s_1 - v_0 v_\lambda$ and $s_2 - v_0 v_\lambda$ must be alternating in X.

By swapping ∇_1 with ∇_2 and s_1 with s_2, we may assume that $s_1 - v_0 v_\lambda$ is alternating in X. We claim that

$$s_2 - v_0 v_\lambda \text{ is not alternating in } X \iff v_1 v_{\lambda-1} \in \nabla_1 \cap \nabla_2. \tag{33}$$

If $v_1 v_{\lambda-1} \in \nabla_1, \nabla_2$, then $v_1 v_{\lambda-1} \notin \nabla_{X,Y}$, and as before, we get a contradiction if $s_i - v_0 v_\lambda$ is alternating in X for both $i = 1, 2$, so $s_2 - v_0 v_\lambda$ must not alternate in X. If $s_2 - v_0 v_\lambda$ is not alternating in X, then $v_1 v_{\lambda-1} \in \nabla_2$. Thus, if $s_2 - v_0 v_\lambda$ is not alternating in X and $v_1 v_{\lambda-1} \notin \nabla_1$, then $v_1 v_{\lambda-1} \in \nabla_{X,Y}$, and so $|\deg_{E(X) \cap \nabla_{X,Y}}(v_1) - \deg_{E(Y) \cap \nabla_{X,Y}}(v_1)| = 2$, a contradiction.

Let us define now 4 auxiliary graphs.

$$X_1 = \begin{cases} X \triangle v_0 v_\lambda, & \text{if } s_1 \text{ is not alternating in} X \\ X, & \text{if } s_1 \text{ is alternating in} X \end{cases}$$

$$Y_1 = X_1 \triangle \nabla_1$$

$$X_2 = \begin{cases} Y_1 \triangle v_0 v_\lambda, & \text{if } s_2 \text{ is not alternating in} Y_1 \\ Y_1, & \text{if } s_2 \text{ is alternating in } Y_1 \end{cases}$$

$$Y_2 = X_2 \triangle \nabla_2.$$

By our assumptions, s_1 is alternating in X_1. Furthermore, from (33) it follows that s_2 is alternating in X_2. Because s_i defines an alternating trail in X_i, we have $\deg_{X_i} = \deg_{Y_i}$. Trivially, $E(X_1) \triangle E(X) \subseteq \{v_0 v_\lambda\}$, and by the assumptions of the lemma,

$$E(Y_2) \triangle E(Y) \subseteq \{v_0 v_\lambda\} \cup \big(E(Y_1) \triangle \nabla_2 \triangle E(Y)\big)$$
$$\subseteq \{v_0 v_\lambda\} \cup \big(E(X) \triangle \nabla_1 \triangle \nabla_2 \triangle E(Y)\big)$$
$$E(Y_2) \triangle E(Y) \subseteq \{v_0 v_\lambda\} \cup \big(\nabla_{X,Y} \triangle \nabla_1 \triangle \nabla_2\big)$$
$$E(Y_2) \triangle E(Y) \subseteq \{v_0 v_\lambda\}$$

We claim that

$$s_1 \text{ is alternating in} X \text{ or} s_2 \text{ is alternating in} Y_1 \text{ (or both)}. \tag{34}$$

Suppose that s_1 is not alternating in X and s_2 is not alternating in Y_1. Then, $X_1 = X \triangle v_0 v_\lambda$ and $X_2 = Y_1 \triangle v_0 v_\lambda$. Because s_1 is not alternating in X, we have $v_0 v_\lambda \in \nabla_1$. Also, because $s_2 - v_1 v_{\lambda-1}$ is not alternating in $Y_1 = X_1 \triangle \nabla_1 = X \triangle (\nabla_1 \backslash \{v_0 v_\lambda\})$, $s_2 - v_1 v_{\lambda-1}$ is not alternating in X either. But this implies

that $|\deg_{X \cap \nabla_{X,Y}}(v_0) - \deg_{Y \cap \nabla_{X,Y}}(v_0)| \in \{2, 3\}$ (depends on whether $v_0 v_\lambda$ is in $\nabla_{X,Y}$ or not), which is a contradiction.

From now on, we assume that $X_1 = X$ or $X_2 = Y_1$. In other words, at least one of the following three symmetric differences is an empty set:

$$E(X) \triangle E(X_1), E(Y_1) \triangle E(X_2), E(Y_2) \triangle E(Y) \subseteq \{v_0 v_\lambda\}. \tag{35}$$

If exactly one of them is an empty set, then observe that

$$2 = \| \deg_X - \deg_Y \|_1 \equiv \| \deg_X - \deg_{X_1} \|_1 + \| \deg_{Y_1} - \deg_{X_2} \|_1 + \| \deg_{Y_2} - \deg_Y \|_1$$
$$\equiv 2 + 2 \pmod 4,$$

which is a contradiction. Thus, there are exactly two empty sets on the left hand side of Eq. (35). From $\deg_{X_i} = \deg_{Y_i}$ for $i = 1, 2$, it follows that

$$\deg_{X_i}, \deg_{Y_i} \in \{\deg_X, \deg_Y\} \text{ for } i = 1, 2. \tag{36}$$

In other words, we have shown that $(X_i, Y_i, s_i) \in \mathfrak{C}_{\mathrm{id}} \cap \mathfrak{D}_{M'}$ for $i = 1, 2$, and we may proceed with the reduction. By (34), we have three cases:

$\Upsilon_M(X, Y, s)$

$$= \begin{cases} X \xrightarrow{\text{togglev}_0 v_\lambda} \Upsilon_{M'}(X_1, Y_1, s_1) \to \Upsilon_{M'}(X_2, Y_2, s_2) \to Y & \text{if } s_1 \text{ is not alternating in } X, \\ X \to \Upsilon_{M'}(X_1, Y_1, s_1) \xrightarrow{\text{togglev}_0 v_\lambda} \Upsilon_{M'}(X_2, Y_2, s_2) \to Y & \text{if } s_2 \text{ is not alternating in } Y_1, \\ X \to \Upsilon_{M'}(X_1, Y_1, s_1) \to \Upsilon_{M'}(X_2, Y_2, s_2) \xrightarrow{\text{togglev}_0 v_\lambda} Y & \text{otherwise,} \end{cases}$$

where the $\to$ signs simply represent joining two sequences (repeated graphs are dropped from the sequence). By the above observations about the symmetric differences and (36), $\Upsilon_M(X, Y, s)$ is indeed a path in the desired Markov graph.

$M' - A_Z$ is c-tight by the properties of the precursor on $\mathfrak{C}_{\mathrm{id}}$. Therefore, $M - A_Z$ is $(c + 3)$-tight.

Task 2: Constructing $\pi_M(X, Y, s)$. We have to construct a connected $\pi_M(X, Y, s, Z)$ from the current $\pi_{M'}(X_i, Y_i, s_i, Z)$ (where $Z \in \Upsilon_{M'}(X_i, Y_i, s_i)$). Notice that $L(\nabla_{X,Y}, s) - \nabla_i$ (delete ∇_i from the vertex set of the line graph) has at most $c' + 1$ components, since $L(\nabla_{X,Y}, s)$ is a path. Furthermore, $L(\nabla_i, \pi_{M'}(X_i, Y_i, s_i)) - (\nabla_i \backslash \nabla_{X,Y})$ has at most 2 components (since $|\nabla_i \backslash \nabla_{X,Y}| \leq 2$). For $Z \in \Upsilon_{M'}(X_i, Y_i, s_i)$, let

$$\sigma_Z = \pi_{M'}(X_i, Y_i, s_i, Z)|_{\nabla_{X,Y}} \cup s|_{\nabla_{X,Y} \backslash \nabla_i}.$$

The graph $L(\nabla_{X,Y}, \sigma_Z)$ has at most $c' + 3$ components because $\pi_{M'}(X_i, Y_i, s_i)|_{\nabla_{X,Y}}$ and $s|_{\nabla_{X,Y} \backslash \nabla_i}$ are composed of at most 2 and $c' + 1$ trails, respectively. Note, that

$$\sigma_{X_i} = s_i|_{\nabla_{X,Y}} \cup s|_{\nabla_{X,Y} \backslash \nabla_i},$$

and thus, $|\sigma_{X_i} \triangle s| \leq 2(c' + 3)$.

We claim that there exists $\sigma_Z' \in \Pi(\nabla_{X,Y})$ such that $\sigma_Z' \supseteq \sigma_Z$ (extends σ_Z) and $|\sigma_Z' \triangle \sigma_Z| \leq 2(c' + 3)$. Let $U_x = \{xy \in \nabla_{X,Y} \mid (x, xy) \notin \mathrm{dom}(\sigma_Z)\}$ be the set of unpaired edges incident to x. In total, we have $\sum_{x \in [n]} |U_x| \leq 2(c' + 3)$.

It is sufficient now to define $\sigma'_Z(x, \bullet)$ on U_x for every $x \in [n]$. To do so, observe that:

$$|U_x| = \deg_{\nabla_{X,Y}}(x) - |\{(x, xy) \in \text{dom}(\pi_{M'}(X_i, Y_i, s_i)|_{\nabla_{X,Y}})\}|$$
$$-|\{(x, xy) \in \text{dom}(s|_{\nabla_{X,Y} \setminus \nabla_i})\}|.$$

Then, the parity of $|U_x|$ satisfies:

$$|U_x| \equiv \deg_{\nabla_{X,Y}}(x) + |\{(x, xy) \in \text{dom}(s|_{\nabla_{X,Y} \setminus \nabla_i})\}| \pmod 2$$
$$|U_x| \equiv \deg_{\nabla_{X,Y}}(x) + |\{xy \in \nabla_{X,Y} \setminus \nabla_i \mid s(x, xy) \in \nabla_{X,Y} \setminus \nabla_i\}| \pmod 2$$
$$|U_x| \equiv \deg_{\nabla_{X,Y}}(x) + I_{x=v_0} \cdot I_{v_0v_1 \notin \nabla_i} + I_{x=v_\lambda} \cdot I_{v_{\lambda-1}v_\lambda \notin \nabla_i} \pmod 2$$

From the last congruence it follows that $|U_x|$ is even for $x \neq v_0, v_\lambda$, so we may choose $\sigma'_Z(x, \bullet)$ such that it pairs the edges in U_x. If $v_0v_1 \notin \nabla_i$, then $|U_{v_0}|$ is even, and we may choose $\sigma'_Z(v_0, \bullet)$ such that it pairs the edges in U_x (note that $\sigma'_Z(v_0, v_0v_1) = \sigma_Z(v_0, v_0v_1) = v_0v_1$). If $v_0v_1 \in \nabla_i$, then $|U_{v_0}|$ is odd and by definition $\pi_{M'}(X_i, Y_i, s_i, Z)$ cannot map (v_0, v_0v_1) to v_0v_1; thus, $\sigma'_Z(v_0, \bullet)$ can pair all edges of U_{v_0} except one, which $\sigma'_Z(v_0, \bullet)$ will map to itself. Define $\sigma'_Z(v_\lambda, \bullet)$ on U_{v_λ} analogously. In any case, $L(\nabla_{X,Y}, \sigma'_Z)$ is composed of a path and a certain number of cycles, in total still no more than $c' + 3$ components.

Furthermore, we claim that there exists $\pi_Z \in \Pi(\nabla_{X,Y})$ such that $|\pi_Z \triangle \sigma'_Z| \leq 4(c'+3)$ and $L(\nabla_{X,Y}, \pi_Z)$ is connected. The pairing function σ'_Z defines one open trail and at most $2(c' + 3) - 1$ closed trails in $\nabla_{X,Y}$, and these trails partition the edge set of the connected trail s. Any closed trail intersecting the open trail can be incorporated into the open trail by changing the pairing function such that the symmetric difference increases by 4.

Let the pairing function associated to Z be

$$\pi_M(X, Y, s, Z) = \pi_Z.$$

We know that σ_Z alternates with at most $3c$ exceptions in Z, since $|(E(X_i) \triangle E(Z)) \setminus \nabla_i| \leq c$ and $\pi_{M'}(X_i, Y_i, s_i)$ alternates in Z with at most c exceptions. Since $|\pi_Z \triangle \sigma_Z| \leq 6(c'+3)$, we get that π_Z alternates in Z with at most $9(c'+3)$ exceptions.

Task 3: Constructing $B_M(X, Y, s)$. Let us identify the ends of intervals of ∇_i edges in a pairing function ϑ:

$$T_Z(\vartheta) = \{(x, xy) \mid xy \in \nabla_i \text{ and } ((x, xy) \notin \text{dom}(\vartheta) \text{ or } \vartheta(x, xy) \notin \nabla_i)\}$$
$$C_Z(\vartheta) = \{\min_{\text{lex}} V(L) \mid L \text{ is a component in } L(\nabla_i \cap \nabla_{X,Y}, \vartheta)\},$$

where $\min_{\text{lex}} V(L)$ stands for the lexicographically minimal edge in $V(L)$. Retracing the steps by which π_Z is obtained, we have

$$|T_Z(\pi_Z)| \leq |T_Z(\sigma_Z)| + 6(c' + 3) \leq \left|T_Z(\pi_{M'}(X_i, Y_i, s_i)|_{\nabla_{X,Y}}\right| + 6(c' + 3)$$
$$\leq 8 + 6(c' + 3), \tag{37}$$
$$|C_Z(\pi_Z)| \leq |C_Z(\sigma_Z)| + 6(c' + 3) \leq \left|C_Z(\pi_{M'}(X_i, Y_i, s_i)|_{\nabla_{X,Y}}\right| + 6(c' + 3)$$
$$\leq 2 + 6(c' + 3). \tag{38}$$

Let

$$
B_M(X, Y, s, Z) = \begin{cases} (0, Z \equiv X) & \text{if } Z = X, Y \\ (2, \lambda, v_0 v_\lambda, v_0 v_\lambda \in E(X), v_1, v_{\lambda-1}, (s_i|_{\nabla_{X,Y}} \cup \\ s|_{\nabla_{X,Y} \setminus \nabla_i}) \triangle s, T_Z(\pi_Z), C_Z(\pi_Z), i, \pi_Z \triangle \sigma_Z, \\ \pi_M(X, Y, s, Z)|_{\nabla_i} \triangle \pi_{M'}(X_i, Y_i, s_i, Z), B_{M'}(X_i, Y_i, s_i, Z)) \\ \text{if } Z \in \Upsilon_{M'}(X_i, Y_i, s_i) \setminus \{X, Y\} \end{cases}
$$

Every set listed in $B_M(X, Y, s, Z)$ has at most a constant size, so the size of the range of B_M increases by a polynomial factor of n (of at most $n^{60c'+240}$). It remains to show that Ψ is still well defined. This is trivial if $Z = X, Y$. Suppose from now on, that $Z \in \Upsilon_{M'}(X_i, Y_i, s_i)$. Since $L(\nabla_{X,Y}, \pi_Z)$ is composed of paths and cycles, T_Z determines the ends of intervals of consecutive ∇_i edges in the trails determined by π_Z, and C_Z determines those $L(\nabla_{X,Y}, \pi_Z)$ components whose vertex set is a subset of ∇_i. Therefore, $\nabla_{X,Y}, T_Z, C_Z$ and π_Z determine ∇_i. Thus, $\pi_Z|_{\nabla_i} = \pi_M(X, Y, s, Z)|_{\nabla_i}$ can be determined, and in turn $\pi_{M'}(X_i, Y_i, s_i, Z)$ can be reconstructed too. Since we have a precursor on $\mathfrak{C}_{\mathrm{id}}$, we get

$$(X_i, Y_i, s_i) = \Psi(Z, \nabla_i, \pi_{M'}(X_i, Y_i, s_i, Z), B_{M'}(X_i, Y_i, s_i, Z)).$$

Notice, that $X - v_0 v_\lambda = X_1 - v_0 v_\lambda$ and $Y - v_0 v_\lambda = Y_2 - v_0 v_\lambda$. Since $v_0 v_\lambda \in E(Y)$ if and only if $v_0 v_\lambda \in E(X) \triangle \nabla_{X,Y}$, both X and Y are determined by (X_i, Y_i). Furthermore, $\sigma_Z|_{\nabla_{X,Y} \setminus \nabla_i} = s|_{\nabla_{X,Y} \setminus \nabla_i}$ is already determined, and together with $s_i|_{\nabla_{X,Y}}$ and the auxiliary parameters, they determine s.

We have now defined the precursor on any $(X, Y, s) \in \mathfrak{C}_{\mathrm{thin}}$ where s is an open trail of odd length. Suppose from now on that $(X, Y, s) \in \mathfrak{C}_{\mathrm{thin}}$ where s is an open trail of even length.

Case B: $s = v_0 v_1 \ldots v_{\lambda-1} v_\lambda$ is an open trail of even length and $v_0 = v_{\lambda-1}$. We will perform exactly one hinge-flip $\{v_{\lambda-2} v_0, v_{\lambda-2} v_\lambda\}$. This case is very similar to when s is an open trail of odd length and $\nabla_2 = \emptyset$, so we will give the construction, but checking the precursor properties is left to the diligent reader. Let

$$
\begin{aligned}
s_1 &= v_0 v_1 \ldots v_{\lambda-2} v_\lambda v_0 \\
\nabla_1 &= \nabla_{X,Y} \triangle \{v_{\lambda-2} v_0, v_{\lambda-2} v_\lambda\} \\
M' &= \begin{cases} M + A_{(v_{\lambda-2} v_\lambda)} & \text{if } M_{v_{\lambda-2} v_\lambda} = 0 \\ M - A_{(v_{\lambda-2} v_\lambda)} & \text{if } M_{v_{\lambda-2} v_\lambda} = 2 \\ M & \text{if } M_{v_{\lambda-2} v_\lambda} = 1 \end{cases} \\
X_1 &= \begin{cases} X \triangle \{v_{\lambda-2} v_0, v_{\lambda-2} v_\lambda\}, & \text{if } s_1 \text{ is not alternating in } X \\ X, & \text{if } s_1 \text{ is alternating in } X \end{cases} \\
Y_1 &= \begin{cases} Y, & \text{if } s_1 \text{ is not alternating in } X \\ Y \triangle \{v_{\lambda-2} v_0, v_{\lambda-2} v_\lambda\}, & \text{if } s_1 \text{ is alternating in } X \end{cases} \\
\Upsilon_M(X, Y, s) &= \begin{cases} X \xrightarrow{\text{hinge-flip}} \Upsilon_{M'}(X_1, Y_1, s_1) & \text{if } s_1 \text{ is not alternating in } X \\ \Upsilon_{M'}(X_1, Y_1, s_1) \xrightarrow{\text{hinge-flip}} Y & \text{if } s_1 \text{ is alternating in } X \end{cases}
\end{aligned}
$$

We define $\pi_M(X, Y, s, Z)$ simply by replacing the $(v_{\lambda-2}, v_{\lambda-2}v_\lambda)$ with $(v_{\lambda-2}, v_{\lambda-2}v_0)$ in the pairing $\Upsilon_{M'}(X_1, Y_1, s_1)$, remove $(v_\lambda, v_{\lambda-2}v_\lambda)$ from the pairing (and create the self-paired edges at v_0 and v_λ). Defining a suitable $B_M(X, Y, s, Z)$ is straightforward and it is also left to the reader.

Case C: $s = v_0 v_1 \ldots v_{\lambda-1} v_\lambda$ **is an open trail of even length and** $v_0 \neq v_{\lambda-1}$. Let $s' = s - v_{\lambda-1}v_\lambda$ and observe that we have already defined $\Upsilon_M(X, Y \triangle v_{\lambda-1}v_\lambda, s')$ in the previous subsection, since $L(\nabla - v_{\lambda-1}v_\lambda, s')$ is a path of odd length.

However, choosing $\Upsilon_M(X, Y, s) = \Upsilon_M(X, Y \triangle v_{\lambda-1}v_\lambda, s') \to Y$ violates the precursor property because the degree at $v_{\lambda-1}$ may become too small or too large when the edge-toggle is performed on $v_0 v_{\lambda-1}$ (the rest of the steps are switches). Fortunately, this is very easy to fix: simply replace the edge-toggle on $v_0 v_{\lambda-1}$ in the previous definitions of Υ_M with the hinge-flip between $v_0 v_{\lambda-1}$ and $v_{\lambda-1}v_\lambda$ to obtain $\Upsilon_M(X, Y \triangle v_{\lambda-1}v_\lambda, s')$. Since every other step in $\Upsilon_M(X, Y \triangle v_{\lambda-1}v_\lambda, s')$ is a switch, this ensures that for any $Z \in \Upsilon_M(X, Y, s)$ we have $\deg_Z \in \{\deg_X, \deg_Y\}$.

We also need to define π_M and B_M. Since the odd length case already describes a trail from v_0 to $v_{\lambda-1}$, we can join $v_{\lambda-1}v_\lambda$ to the edge ending the trail at $v_{\lambda-1}$ to obtain a suitable $\pi_M(X, Y, s, Z)$ (in the derived bounds, this essentially increases c' by 1). Furthermore, we also need to store in $B_M(X, Y, s, Z)$ that the identify of $v_{\lambda-1}$ and v_λ. Since for any $Z \in \Upsilon_M(X, Y, s)$. As a result, the range of $B_M(X, Y, s, Z)$ increases by a polynomial factor (also note that the parameter of the precursor has to be increased by a constant to accommodate $v_{\lambda-1}v_\lambda$).

The well definedness of Ψ follows, because the constant number of differences compared to the previous case can be all stored in $B_M(X, Y, s, Z)$ without violating Definition 3.25(g).

One can say that this is proof is not very detailed, but we think it is not worth describing the details, because it would be an almost verbatim repetition of the first two cases. $\qquad\square$

5. Proof of Theorem 2.20.

Let $\mathcal{I}$ be a set of weakly P-stable thin degree sequence intervals. By Lemma 4.2, there exists a precursor with parameter $c = 12$ on $\mathfrak{C}_{\mathrm{id}}$. We want to apply Lemma 4.6 to prove that there exists a precursor on $\mathfrak{C}_{\mathrm{thin}}$ with some fixed parameter. Showing this, Theorem 2.20 follows: the precursor can be extended to $\mathfrak{R}_{\mathrm{thin}}$ by Lemma 3.27, which is sufficient for proving rapid maxing of $\mathbb{G}(\ell, u)$ on every $(\ell, u) \in \mathcal{I}$ by Theorem 3.28. Suppose $(X, Y, s) \in \mathfrak{C}_{\mathrm{thin}}$. If s is a closed trail, then $(X, Y, s) \in \mathfrak{C}_{\mathrm{id}}$, on which we have already defined a precursor.

Suppose from now on that $s = v_0 v_1 \ldots v_\lambda$ is an open trail of odd length (possibly 1). By the KD-lemma (Lemma 3.21), $v_0 \neq v_\lambda$. To apply Lemma 4.6, it is enough to define s_1 and s_2, since their domains determine ∇_1, ∇_2. The premises of Lemma 4.6 are elementary and trivial to check once $s_1 \in \Pi(\nabla_1)$ and $s_2 \in \Pi(\nabla_2)$ are given. We will finish the proof by a complete case analysis, where we provide a suitable s_1 and s_2 for each case.

We will prove that Lemma 4.6 holds for $\mathfrak{C}_{\text{thin}}$ with $c' = 2$. We will distinguish between 8 main cases, 3 of these have 2 subcases. The cases will be distinguished based on the relationship between $v_0, v_1, v_{\lambda-1}, v_\lambda$ and s. Recall that $v_0 \neq v_\lambda$. On the corresponding figures, by exchanging X and Y, we may suppose that $v_0 v_1 \in E(X)$. Thus, the edges of X are drawn with solid lines, edges of Y with dashed lines, and edges not contained in $\nabla_{X,Y}$ are dotted. Those pairs that are contained in $\nabla_{X,Y}$ are joined by thick solid or dashed lines. The similarly thick dash-dotted lines represent (X,Y)-alternating segments of the trail s. Recall, that a trail may visit a vertex multiple times, but it can only traverse an edge at most once. The trails traversed by s_1 and s_2 are colored in blue and red, respectively.

Case 1. First, we assume that $v_0 v_\lambda \notin \nabla_{X,Y}$.

$$s_1 = v_0 v_1 \ldots v_{\lambda-1} v_\lambda v_0,$$
$$s_2 = \emptyset.$$
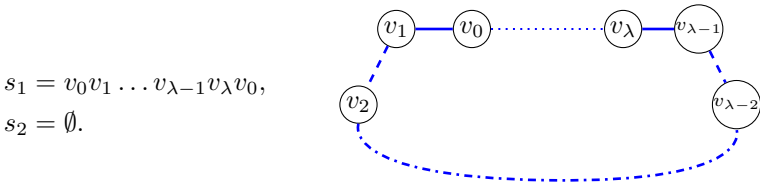
From now on, we assume that $v_0 v_\lambda \in \nabla_{X,Y}$. In other words, the open trail $s \in \Pi(\nabla_{X,Y})$ traverses $v_0 v_\lambda$, that is, there exists $2 \leq j \leq \lambda - 2$ such that $\{v_0, v_\lambda\} = \{v_j, v_{j+1}\}$.

Case 2. We assume in this case that j is even.

Case 2a. If $v_j = v_0$ and $v_{j+1} = v_\lambda$, then let

$$s_1 = v_0 v_1 \ldots v_{j-1} v_j,$$
$$s_2 = v_{j+1} v_{j+2} \ldots v_{\lambda-1} v_\lambda.$$
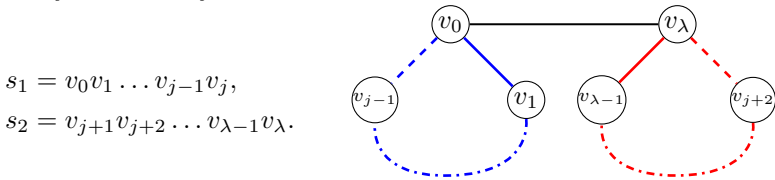
Case 2b. If $v_{j+1} = v_0$ and $v_j = v_\lambda$, then let

$$s_1 = v_0 v_1 \ldots v_{j-1} v_j v_{\lambda-1} v_{\lambda-2}$$
$$\ldots v_{j+2} v_{j+1},$$
$$s_2 = \emptyset.$$
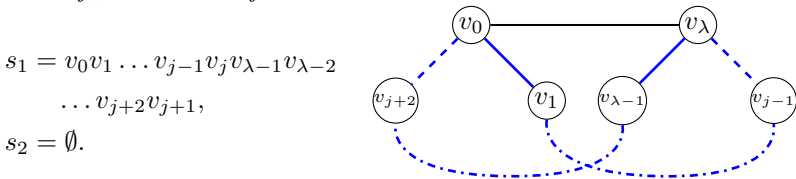
From now on, we assume that j is odd.

Case 3. If $v_j = v_\lambda$ and $v_{j+1} = v_0$, then let

$$s_1 = v_0 v_1 \ldots v_j v_{j+1},$$
$$s_2 = v_j v_{j+1} \ldots v_{\lambda-1} v_\lambda.$$
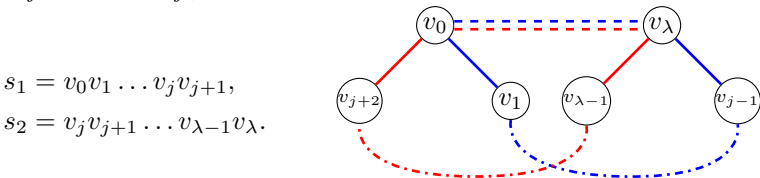
From now on, we assume that $v_j = v_0$ and $v_{j+1} = v_\lambda$.

Case 4. If $v_1 = v_{\lambda-1}$, then let

$$s_1 = v_1 v_2 \ldots v_j v_{j+1} v_1,$$
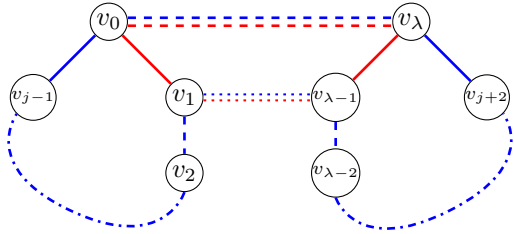$$s_2 = v_j v_{j+1} \ldots v_{\lambda-2} v_{\lambda-1} v_0.$$
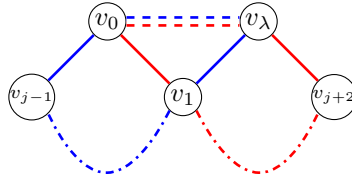
From now on, we assume that $v_1 \neq v_{\lambda-1}$.

Case 5. If $v_1 v_{\lambda-1} \notin \nabla_{X,Y}$.

$$s_1 = v_j v_{j-1} \ldots v_2 v_1 v_{\lambda-1}$$
$$\qquad v_{\lambda-2} \ldots v_{j+1} v_j,$$
$$s_2 = v_0 v_1 v_{\lambda-1} v_\lambda v_0.$$

From now on, we assume that $v_1 v_{\lambda-1} \in \nabla_{X,Y}$. In other words, the open trail $s \in \Pi(\nabla_{X,Y})$ traverses $v_1 v_{\lambda-1}$, that is, there exists $1 \leq k \leq \lambda - 1$ such that $\{v_1, v_{\lambda-1}\} = \{v_k, v_{k+1}\}$.

First, we assume that $k < j$; the case $k > j$ will follow easily by symmetry.

Case 6. Suppose that k is even.

Case 6a. If $v_k = v_1$ and $v_{k+1} = v_{\lambda-1}$, then let

$$s_1 = v_{k+1} v_{k+2} \ldots v_{j-1} v_j$$
$$\qquad v_{j+1} v_{\lambda-1},$$
$$s_2 = v_0 v_1 \ldots v_{k-1} v_k v_{\lambda-1}$$
$$\qquad v_{\lambda-2} \ldots v_{j+2} v_{j+1} v_j.$$
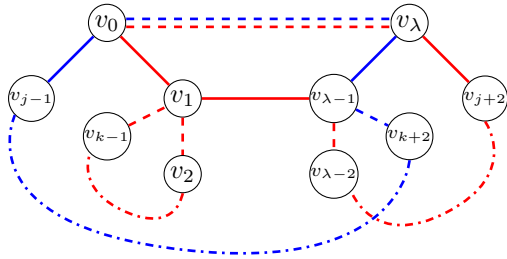
Case 6b. If $v_{k+1} = v_1$ and $v_k = v_{\lambda-1}$, then let

$$s_1 = v_0 v_1 \ldots v_{k-1} v_k v_\lambda v_0,$$
$$s_2 = v_k v_{k+1} v_{k+2} \ldots v_{j-1} v_0$$
$$\qquad v_\lambda v_{j+2} \ldots v_{\lambda-2} v_{\lambda-1}.$$
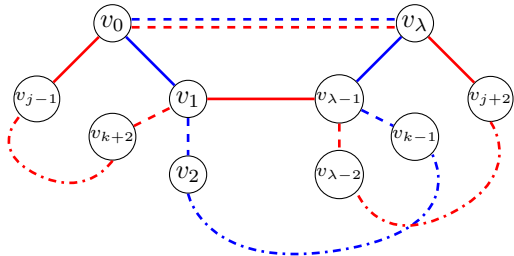
Case 7. Suppose that k is odd. **Case 7a.** If $v_k = v_1$ and $v_{k+1} = v_{\lambda-1}$, then let

$$s_1 = v_{k+1}v_{k+2}\ldots v_{j-1}v_0 v_\lambda v_{j+2}\ldots v_{\lambda-2}v_{\lambda-1},$$

$$s_2 = \begin{cases} v_0 v_1 v_2 \ldots v_{k-1} v_k v_{k+1} v_\lambda v_0 & \text{if } k > 1, \\ v_0 v_1 v_2 v_\lambda v_0 & \text{if } k = 1. \end{cases}$$
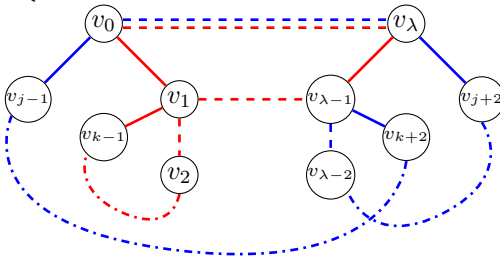
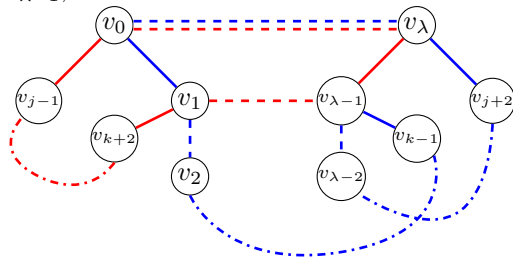**Case 7b.** If $v_{k+1} = v_1$ and $v_k = v_{\lambda-1}$, then let

$$s_1 = v_0 v_1 v_2 \ldots v_{k-2} v_{k-1}$$
$$v_{\lambda-1} v_{\lambda-2}$$
$$\ldots v_{j+2} v_{j+1} v_j,$$
$$s_2 = v_k v_{k+1} v_{k+2} \ldots$$
$$v_{j-1} v_j v_{j+1} v_{\lambda-1}.$$

Case 8. The remaining case is when $k > j$. By taking the reverse order $v_i' = v_{\lambda-i}$ for $i = 0, \ldots, \lambda$, we have $\lambda - k - 1 < \lambda - j - 1$, so one of the previous subcases of Case 6 or Case 7 applies to $s' = v_0' v_1' \ldots v_{\lambda-1}' v_\lambda'$. Clearly, the relevant properties of s_i are preserved by reversing the order of the indices.

Acknowledgements

Declarations

Conflict of interest On behalf of all the authors, the corresponding author states that there is no conflict of interest.

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References

[1] G. Amanatidis and P. Kleer. Approximate Sampling and Counting of Graphs with Near-Regular Degree Intervals. In: 40th International Symposium on Theoretical Aspects of Computer Science (STACS 2023). Ed. by P. Berenbrink, P. Bouyer, A. Dawar, and M. M. Kanté. Vol. 254. Leibniz International Proceedings in Informatics (LIPIcs). ISSN: 1868-8969. Dagstuhl, Germany: Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2023, 7:1–7:23. isbn: 978-3-95977-266-2. https://doi.org/10.4230/LIPIcs.STACS.2023.7.https://drops.dagstuhl.de/opus/volltexte/2023/17659 (visited on 10/13/2023).

[2] T. Coolen, A. Annibale, and E. Roberts. Generating Random Networks and Graphs. Oxford University Press, May 26, 2017. 325 pp. isbn: 978-0-19-101981-4.

[3] C. Cooper, M. Dyer, and C. Greenhill. Sampling Regular Graphs and a Peer-to-Peer Network. In: Combinatorics, Probability and Computing 16.4 (July 2007), pp. 557–593. issn: 1469-2163, 0963-5483. https://doi.org/10.1017/S0963548306007978.https://www.cambridge.org/core/journals/combinatoricsprobability-and-computing/article/sampling-regular-graphs-and-a-peertopeer-network/3C6DABD887139971589C69A0F0B52688 (visited on 03/14/2019).23

[4] P. L. Erdős, C. Greenhill, T. R. Mezei, I. Miklós, D. Soltész, and L. Soukup. The mixing time of switch Markov chains: A unified approach. In: European Journal of Combinatorics 99 (Jan. 1, 2022), pp. 99–146. issn: 0195-6698. https://doi.org/10.1016/j.ejc.2021.103421.https://www.sciencedirect.com/science/article/pii/S0195669821001141 (visited on 12/08/2021).

[5] C. Greenhill. The switch Markov chain for sampling irregular graphs (Extended Abstract). In: Proceedings of the 2015 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA). Proceedings. Society for Industrial and Applied Mathematics, Dec. 22, 2014, pp. 1564–1572. isbn: 978-1-61197-374-7. https://doi.org/10.1137/1.9781611973730.103.https://epubs.siam.org/doi/10.1137/1.9781611973730.103 (visited on 02/25/2022).

[6] M. Jerrum, B. D. McKay, and A. Sinclair. When is a Graphical Sequence Stable? In: Random Graphs: Volume 2 (eds. A. Frieze and T. Luczak). Wiley, Aug. 4, 1992, pp. 101–116. isbn: 978-0-471-57292-3.

[7] M. Jerrum and A. Sinclair. Approximating the Permanent. In: SIAM J. Comput. 18.6 (Dec. 1, 1989), pp. 1149–1178. issn: 0097-5397. https://doi.org/

10.1137/0218077.https://epubs.siam.org/doi/abs/10.1137/0218077 (visited on 02/24/2019).

[8] M. Jerrum and A. Sinclair. Fast uniform generation of regular graphs. In: Theoretical Computer Science 73.1 (June 8, 1990), pp. 91–100. issn: 0304-3975. https://doi.org/10.1016/0304-3975(90)90164-D.http://www.sciencedirect.com/science/article/pii/030439759090164D (visited on 02/08/2019).

[9] R. Kannan, P. Tetali, and S. Vempala. Simple Markov-chain algorithms for generating bipartite graphs and tournaments. In: Random Structures & Algorithms 14.4 (1999), pp. 293–308. issn: 1098-2418. https://doi.org/10.1002/(SICI)1098-2418(199907)14:4⟨293::AID-RSA1⟩3.0.CO;2-G. https://onlinelibrary.wiley.com/doi/abs/10.1002/%28SICI%291098-2418%28199907%2914%3A4%3C293%3A%3AAID-RSA1%3E3.0.CO%3B2-G (visited on 03/14/2019).

[10] S. Rechner, L. Strowick, and M. Müller-Hannemann. Uniform sampling of bipartite graphs with degrees in prescribed intervals. In: Journal of Complex Networks 6.6 (Dec. 1, 2018), pp. 833–858. issn: 2051-1329. https://doi.org/10.1093/comnet/cnx059. (visited on 02/13/2022).

[11] A. Sinclair. Improved Bounds for Mixing Rates of Markov Chains and Multicommodity Flow. In: Combinatorics, Probability and Computing 1.4 (Dec. 1992), pp. 351–370. issn: 1469-2163, 0963-5483. https://doi.org/10.1017/S0963548300000390.url: https://www.cambridge.org/core/journals/combinatoricsprobability-and-computing/article/improved-bounds-for-mixing-rates-of-markov-chains-andmulticommodity-flow/0AD89F9FA25567A5CBBF672C371FBC2B# (visited on 03/02/2019). 24

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