Ann. Comb. 28 (2024) 1021–1033
© 2023 The Author(s), under exclusive licence to
Springer Nature Switzerland AG
Published online November 11, 2023
https://doi.org/10.1007/s00026-023-00672-0

Annals of Combinatorics

Localised Graph Maclaurin Inequalities

Lucas Aragão and Victor Souza

Abstract. The Maclaurin inequalities for graphs are a broad generalisation of the classical theorems of Turán and Zykov. In a nutshell they provide an asymptotically sharp answer to the following question: what is the maximum number of cliques of size q in a graph with a given number of cliques of size s and a given clique number? We prove an extension of the graph Maclaurin inequalities with a weight function that captures the local structure of the graph. As a corollary, we settle a recent conjecture of Kirsch and Nir, which simultaneously encompass the previous localised results of Bradač, Malec and Tompkins and of Kirsch and Nir.

1. Introduction

One of the foundational results in extremal graph theory is Turán's theorem [15], which states that a graph G that is K_{r+1}-free cannot have more edges than a balanced complete r-partite graph. Zykov [16] later showed that these graphs also maximise the number of copies of K_q amongst K_{r+1}-free graphs.

Denote by $K_s(G)$ the set of s-cliques in G and $k_s(G) = |K_s(G)|$. If G is K_{r+1}-free, Zykov's theorem gives

$$k_q(G) \le \binom{r}{q} \left(\frac{k_1(G)}{r} \right)^q . \tag{1}$$

The *graph Maclaurin inequalities* are a broad extension of (1). Indeed, they state that if G is K_{r+1}-free, then

$$\frac{k_1(G)}{\binom{r}{1}} \ge \left(\frac{k_2(G)}{\binom{r}{2}} \right)^{1/2} \ge \cdots \ge \left(\frac{k_r(G)}{\binom{r}{r}} \right)^{1/r} . \tag{2}$$

Whilst Khadzhiivanov [6] was the first to prove this result, his original proof had a gap, later filled by Nikiforov [11]. This inequality was also rediscovered and reproved by Sós and Straus [14], by Fisher and Ryan [3] and by Petingi and Rodriguez [12].

Using (2), one can address the following question: for $s < q \le r$, what is the maximum number of copies of K_q that a K_{r+1}-free graph can have assuming that it has a given number of copies of K_s? Turán's theorem gives the exact answer for $s = 1$ and $q = 2$ and Zykov's theorem for $s = 1$ and $q \ge 2$. Eckhoff [2] and Frohmander [4] gave further exact results. Inequality (2) is asymptotically sharp and gives the precise answer under certain divisibility conditions. Our main result is a strengthening of (2).

Theorem 1.1. *Given a graph G and integers $1 \le s \le q$, we have*

$$\sum_{I \in K_q(G)} \binom{\sigma(I)}{s}^{q/s} \binom{\sigma(I)}{q}^{-1} \le k_s(G)^{q/s}, \tag{3}$$

where $\sigma(I)$ is the size of the largest clique in G containing I. Moreover, equality holds only when the subgraph of G induced on the set of vertices that belong to an s-clique is a complete multipartite graph with equal parts.

Indeed, Theorem 1.1 generalises (2). The function $t \mapsto \binom{t}{s}^{q/s} / \binom{t}{q}$ is decreasing for $t \ge q$ for any fixed $1 \le s \le q$. Therefore, when G is K_{r+1}-free, inequality (3) implies

$$k_q(G) \binom{r}{s}^{q/s} \binom{r}{q}^{-1} \le k_s(G)^{q/s}$$

which is precisely the Maclaurin inequalities in (2).

An important feature Theorem 1.1 is the local nature of the function $\sigma(I)$. This result fits in an ongoing enterprise to show similarly *localised* versions of results in extremal combinatorics. The case $s = 1$ and $q = 2$, a localised version of Turán's theorem, was proposed in 2022 by Balogh and Lidický in an Oberwolfach [8] problem session. Soon after, this case was settled independently by Bradač [1] and by Malec and Tompkins [9]. The full case $s = 1$, a localised version of Zykov's theorem, was then proven analytically by Kirsch and Nir [7]. They also conjectured in [7, Conjecture 6.1] the case $s = 2$, which we settle in greater generality.

To prove Theorem 1.1, we follow the strategy of Nikiforov and Khadzhiivanov, building upon the Motzkin–Straus [10] analytical proof of Turán's theorem. We now review some aspects of this analytical approach.

For $\mathbf{x} = (\mathbf{x}_v)_{v \in V} \in \mathbb{R}^V$, write $\mathbf{x} \ge 0$ (or $\mathbf{x} > 0$) if $\mathbf{x}_v \ge 0$ (or $\mathbf{x}_v > 0$) for all $v \in V$. For a set $I \subseteq V$, denote the product $\mathbf{x}_I := \prod_{v \in I} \mathbf{x}_v$. Given a graph G and an integer s, define the following homogeneous polynomial

$$h_{s,G}(\mathbf{x}) := \sum_{J \in K_s(G)} \mathbf{x}_J.$$

The following inequalities appear in the work of Khadzhiivanov [6]. If G is a K_{r+1}-free graph and $\mathbf{x} \ge 0$, then

$$\frac{h_{1,G}(\mathbf{x})}{\binom{r}{1}} \ge \left(\frac{h_{2,G}(\mathbf{x})}{\binom{r}{2}} \right)^{1/2} \ge \cdots \ge \left(\frac{h_{r,G}(\mathbf{x})}{\binom{r}{r}} \right)^{1/r}. \tag{4}$$

Applying (4) with $\mathbf{x} = \mathbf{1}$ (i.e. $\mathbf{x}_v = 1$ for all $v \in V$), we recover (2). In the case that $G = K_n$, the functions h_{s,K_n} are the elementary symmetric polynomials, and for $r = n$, the inequalities (4) are the classical Maclaurin inequalities (see [5, p. 52]). For this reason, we refer to (4) (and (2)) as a Maclaurin inequality for graphs. Our main technical result is the following.

Theorem 1.2. *Given a graph G and $1 \le s \le q$, define*

$$f_{s,q,G}(\mathbf{x}) := \sum_{I \in K_q(G)} \binom{\sigma(I)}{s}^{q/s} \binom{\sigma(I)}{q}^{-1} \mathbf{x}_I. \tag{5}$$

Then, for every $\mathbf{x} \ge 0$, we have

$$f_{s,q,G}(\mathbf{x}) \le h_{s,G}(\mathbf{x})^{q/s}. \tag{6}$$

Moreover, equality holds for $\mathbf{x} > 0$ only when the subgraph of G induced on the set of vertices that belong to an s-clique is a complete ℓ-partite graph with parts $V_1, \dots, V_\ell$, for some $\ell \ge q$, and $\sum_{v \in V_i} \mathbf{x}_v = \sum_{u \in V_j} \mathbf{x}_u$ for all $1 \le i, j \le \ell$.

To obtain Theorem 1.1 from Theorem 1.2, just take $\mathbf{x} = 1$. At first glance (6) seems stronger than (3), but they are in fact equivalent, as we will see in Proposition 2.3.

Whilst it is not evident at first glance why we take the weight as we do in (3) and (5), if we want an inequality of the form

$$\sum_{I \in K_q(G)} \rho\left(\sigma(I)\right) \mathbf{x}_I \le h_{s,G}(\mathbf{x})^{\beta},$$

that attains equality when $G = K_n$ and $\mathbf{x} = \mathbf{1}$, then we must take $\rho(t) = \binom{t}{s}^{q/s}\binom{t}{q}^{-1}$. Homogeneity forces us to take $\beta = q/s$.

2. Localised Inequalities for Clique Counts

For a graph $G = (V, E)$, the *clique number* $\omega(G)$ is the size of its largest clique. For a subset $S \subseteq V$, denote by $G[S]$ the subgraph of G spanned by S. If a subset $I \subseteq V$ spans a clique, we denote by $\sigma_G(I)$ the size of the largest clique in G containing I. We omit the subscript and write $\sigma(I)$ whenever G is clear from context.

Recall that for $\mathbf{x} = (\mathbf{x}_v)_{v \in V} \in \mathbb{R}^V$, we write $\mathbf{x} \ge 0$ if $\mathbf{x}_v \ge 0$ for all $v \in V$ and $\mathbf{x} > 0$ analogously. The support of $\mathbf{x}$ is the set $\operatorname{supp} \mathbf{x} := \{v \in V : \mathbf{x}_v \ne 0\}$. For a set $I \subseteq V$, recall that $\mathbf{x}_I := \prod_{v \in I} \mathbf{x}_v$. For integers $1 \le s \le q$, consider the function

$$f_{s,q,G}(\mathbf{x}) := \sum_{I \in K_q(G)} \rho_{s,q}(\sigma(I)) \mathbf{x}_I, \tag{7}$$

where $\rho_{s,q}$ is defined, for $t \ge s$, as

$$\rho_{s,q}(t) := \binom{t}{s}^{q/s} \binom{t}{q}^{-1}.$$

Note that in (7), $\rho_{s,q}(t)$ is only evaluated for $t \geq q$. It is important to note that in this range, $\rho_{s,q}(t)$ is a decreasing function of t. Indeed, we have

$$\rho_{s,q}^{s}(t) = \frac{(q!)^s}{(s!)^q} \prod_{i=0}^{qs-1} \frac{t - \lfloor i/q \rfloor}{t - \lfloor i/s \rfloor},$$

so $\rho_{s,q}^{s}$ is the product of sq terms, each of which is non-increasing in $t \geq q$. Recall from the introduction that

$$h_{s,G}(\mathbf{x}) := \sum_{J \in K_s(G)} \mathbf{x}_J.$$

Note that $f_{s,q,G}$ is homogeneous of degree q and $h_{s,G}$ homogeneous of degree s. Moreover, for $\mathbf{x} > 0$, if $h_{s,G}(\mathbf{x}) = 0$, then G has no s-clique, and thus $f_{s,q,G}(\mathbf{x}) = 0$ as well. Therefore, to show that $f_{s,q,G}(\mathbf{x}) \leq h_{s,G}(\mathbf{x})^{q/s}$ for all $\mathbf{x} \geq 0$, it is enough to do so restricted to

$$\mathcal{S}_{s,G} = \left\{ \mathbf{x} \in \mathbb{R}^V \,:\, \mathbf{x} \geq 0,\, h_{s,G}(\mathbf{x}) = 1 \right\}.$$

The inequality (6) in Theorem 1.2 is then equivalent to the following proposition.

Proposition 2.1. *Given a graph G and $1 \leq s \leq q \leq \omega(G)$, we have*

$$f_{s,q,G}(\mathbf{x}) \leq 1, \tag{8}$$

for every $\mathbf{x} \in S_{s,G}$.

We defer the discussion of the case of equality in Theorem 1.2 to Sect. 3. The goal of this section is to prove Proposition 2.1. For convenience, we denote

$$M_{s,q,G} := \sup_{\mathbf{x} \in S_{s,G}} f_{s,q,G}(\mathbf{x}). \tag{9}$$

We note that if $k_q(G) > 0$, then $M_{s,q,G} > 0$. For every graph G, the set $\mathcal{S}_{1,G}$ is the standard simplex, which is compact. For $s \geq 2$, the set is $\mathcal{S}_{s,G}$ is closed and unbounded. The following proposition gives the crucial structural information about the optimisation problem.

Proposition 2.2. *Given a graph G and $1 \leq s \leq q \leq \omega(G)$, the function $f_{s,q,G}(\mathbf{x})$ attains its maximum at a point $\mathbf{x} \in \mathcal{S}_{s,G}$ with* $\operatorname{supp} \mathbf{x}$ *being a clique in G.*

We will now see that Proposition 2.2 quickly gives us Proposition 2.1.

Proof of Proposition 2.1. By Proposition 2.2, there is $\mathbf{x}^* \in \mathcal{S}_{s,G}$ such that $f_{s,q,G}(\mathbf{x}^*) = M_{s,q,G}$ and $\operatorname{supp} \mathbf{x}^*$ is a clique, let's say, a K_R. Recall that $\rho_{s,q}$ is decreasing, so

$$\begin{aligned} f_{s,q,G}(\mathbf{x}^*) &\leq \sum_{I \in K_q(K_R)} \rho_{s,q}(\sigma_G(I)) \mathbf{x}_I^* \leq \rho_{s,q}(R) \sum_{I \in K_q(K_R)} \mathbf{x}_I^* \\ &= \rho_{s,q}(R) h_{q,K_R}(\mathbf{x}^*). \end{aligned}$$

By Maclaurin's inequality (4), we have

$$h_{q,K_R}(\mathbf{x}^*) \leq \binom{R}{q} \left(\frac{h_{s,K_R}(\mathbf{x}^*)}{\binom{R}{s}} \right)^{q/s} = 1/\rho_{s,q}(R).$$

Therefore, $f_{s,q,G}(\mathbf{x}^*) \leq 1$ as we wanted. □

With Proposition 2.1, we can easily get the inequality (3) in Theorem 1.1 by setting $\mathbf{x} = K_s(G)^{-1/s}\mathbf{1}$. What is less clear to see is that Theorem 1.1 if actually equivalent to Theorem 1.2. The idea of using combinatorial means to prove analytical inequalities can be traced back to Sidorenko [13].

Proposition 2.3. *Inequality* (3) *implies inequality* (6).

Proof. The inequality (3) in Theorem 1.1 says that $f_{s,q,G}(\mathbf{x}) \leq h_{s,G}(\mathbf{x})^{q/s}$ for $\mathbf{x} = \mathbf{1}$. To get the inequality for all $\mathbf{x} \geq 0$ note that since both $f_{s,q,G}$ and $h_{s,G}$ are continuous, it is enough to prove it for $\mathbf{x}$ with all coordinate rationals. If some coordinate is 0, we can remove the associated vertex. Moreover, $f_{s,q,G}$ and $h_{s,G}^{q/s}$ are both homogeneous of the same degree, so we can rescale the coordinates of $\mathbf{x}$ to be all integers.

To go from integer coordinates to all 1's coordinates, we can consider the blowup of G. Denote by $G_{\mathbf{x}}$ the graph obtained from G by replacing each vertex v with an independent set U_v of size $\mathbf{x}_v$, and for every edge $uv \in E(G)$, replace the edge uv with a complete bipartite graph with vertex classes U_u and U_v. By construction, the vertices of a clique in $G_{\mathbf{x}}$ belongs to distinct classes U_v. Moreover, every $J \in K_s(G)$ leads to the creation of $\mathbf{x}_J$ cliques in $G_{\mathbf{x}}$. Therefore, we have

$$k_s(G_{\mathbf{x}}) = \sum_{J \in K_s(G)} \mathbf{x}_J = h_{s,G}(\mathbf{x}).$$

Similarly, every $I \in K_q(G)$ leads to the creation of $\mathbf{x}_I$ cliques in $G_{\mathbf{x}}$ with the same value of σ, therefore

$$f_{s,q,G_{\mathbf{x}}}(\mathbf{1}) = \sum_{I \in K_q(G)} \rho_{s,q}(\sigma_G(I))\mathbf{x}_I = f_{s,q,G}(\mathbf{x}).$$

Finally, this gives

$$f_{s,q,G}(\mathbf{x}) = f_{s,q,G_{\mathbf{x}}}(\mathbf{1}) \leq k_s(G_{\mathbf{x}})^{q/s} = h_{s,G}(\mathbf{x})^{q/s},$$

as claimed. □

The proof of Proposition 2.2 is divided in two steps. The first step is a quite technical one, we must show that the supremum (9) is actually attained. That is, we must show that there is $\mathbf{x}^* \in \mathcal{S}_{s,G}$ such that $f_{s,q,G}(\mathbf{x}) \leq f_{s,q,G}(\mathbf{x}^*)$ for all $\mathbf{x} \in \mathcal{S}_{s,G}$. As pointed out before, the set $\mathcal{S}_{1,G}$ is compact and the existence of $\mathbf{x}^*$ is trivial. For $s \geq 2$, however, $\mathcal{S}_{s,G}$ is closed and unbounded, so we must deal with this issue. This is the content of Lemma 2.4, the proof which we postpone to Sect. 4 as it is quite lengthy and not the highlight of the proof.

Lemma 2.4. *Given a graph G and $1 \leq s \leq q \leq \omega(G)$, the function $f_{s,q,G}(\mathbf{x})$ attains a maximum with $\mathbf{x} \in \mathcal{S}_{s,G}$.*

The last ingredient of the proof of Proposition 2.2 is a symmetrisation argument.

Lemma 2.5. *Let G be a graph and $1 \leq s \leq q \leq \omega(G)$. Suppose that every vertex of G is in an s-clique and that $f_{s,q,G}$ attains a maximum, restricted to $\mathcal{S}_{s,G}$, at some point $\mathbf{x} > 0$. If u and v are not adjacent, then there is $\mathbf{y} \in \mathcal{S}_{s,G}$ such that $f_{s,q,G}(\mathbf{y}) = f_{s,q,G}(\mathbf{x})$ and $\mathbf{y}_u = 0$.*

Proof. Define $\mathbf{y}$ as

$$\mathbf{y}_z := \begin{cases} \mathbf{x}_z & \text{if } z \neq u, v, \\ \mathbf{x}_u + \xi_u & \text{if } z = u, \\ \mathbf{x}_v + \xi_v & \text{if } z = v, \end{cases}$$

where ξ_u and ξ_v will be chosen later. Observe that for any $w \in V$, we have

$$\frac{\partial h_{s,G}(\mathbf{x})}{\partial \mathbf{x}_w} = \sum_{\substack{J \in K_s(G) \\ w \in J}} \mathbf{x}_{J \setminus \{w\}}.$$

Thus, as u and v are not neighbours, we obtain

$$h_{s,G}(\mathbf{y}) = \xi_u \frac{\partial h_{s,G}(\mathbf{x})}{\partial \mathbf{x}_u} + \xi_v \frac{\partial h_{s,G}(\mathbf{x})}{\partial \mathbf{x}_v} + h_{s,G}(\mathbf{x}), \tag{10}$$

and similarly,

$$f_{s,q,G}(\mathbf{y}) = \xi_u \frac{\partial f_{s,q,G}(\mathbf{x})}{\partial \mathbf{x}_u} + \xi_v \frac{\partial f_{s,q,G}(\mathbf{x})}{\partial \mathbf{x}_v} + f_{s,q,G}(\mathbf{x}). \tag{11}$$

Note that $\partial h_{s,G}(\mathbf{x})/\partial \mathbf{x}_w > 0$ for all $w \in V$ as $\mathbf{x} > 0$ and all vertices are in some s-clique. We set

$$\xi_u = -\mathbf{x}_u, \qquad \xi_v = \mathbf{x}_u \frac{\partial h_{s,G}(\mathbf{x})/\partial \mathbf{x}_u}{\partial h_{s,G}(\mathbf{x})/\partial \mathbf{x}_v},$$

so $h_{s,G}(\mathbf{y}) = h_{s,G}(\mathbf{x}) = 1$ by (10). In particular, $\mathbf{y} \in \mathcal{S}_{s,G}$.

By the Lagrange's method, as $f_{s,q,G}$ is maximised at $\mathbf{x}$ subject to $h_{s,G}(\mathbf{x}) = 1$, there is $\lambda \in \mathbb{R}$ such that,

$$\frac{\partial f_{s,q,G}(\mathbf{x})}{\partial \mathbf{x}_w} = \lambda \frac{\partial h_{s,G}(\mathbf{x})}{\partial \mathbf{x}_w},$$

for all $w \in V$. Together with (11), this implies that

$$f_{s,q,G}(\mathbf{y}) = \lambda \left(\xi_u \frac{\partial h_{s,G}(\mathbf{x})}{\partial \mathbf{x}_u} + \xi_v \frac{\partial h_{s,G}(\mathbf{x})}{\partial \mathbf{x}_v} \right) + f_{s,q,G}(\mathbf{x}) = f_{s,q,G}(\mathbf{x}).$$

We are done as $\mathbf{y}_u = 0$. □

We are now ready to prove Proposition 2.2, that is, to show that there is a maximum point $\mathbf{x} \in \mathcal{S}_{s,G}$ whose support is a clique.

Proof of Proposition 2.2. Amongst the points $\mathbf{x} \in \mathcal{S}_{s,G}$ with $f_{s,q,G}(\mathbf{x}) = M_{s,q,G}$, choose one with $|\operatorname{supp} \mathbf{x}|$ being minimal. Such $\mathbf{x}$ exists from Lemma 2.4. Moreover, $\mathbf{x}$ is also a maxima restricted to the support $R = \operatorname{supp} \mathbf{x}$, that is, $f_{s,q,G[R]}(\mathbf{x}) = M_{s,q,G[R]}$. Suppose that $G[R]$ is not a clique and let $u, v \in R$ be distinct vertices such that $uv \notin E(G)$. Applying Lemma 2.5 to $G[R]$, there is $\mathbf{y} \in \mathcal{S}_{s,G[R]}$ such that $f_{s,q,G[R]}(\mathbf{y}) = M_{s,q,G[R]}$ and moreover, $\mathbf{y}_u = 0$. This contradicts the minimality of $|\operatorname{supp} \mathbf{x}|$. □

As previously established, the inequalities in Theorems 1.1 and 1.2 follow from Proposition 2.2.

3. When Equality Holds

Having established the inequalities (3) in Theorem 1.1 and (6) in Theorem 1.2, we now determine when equality holds. To do so, we must apply the symmetrisation argument of Lemma 2.5 is a more careful way. Moreover, we must use the fact that equality holds in the original Maclaurin's inequality (4) for cliques only when all the coordinates are equal.

Proposition 3.1. *Let G be a graph with clique number ω and $1 \leq s < q \leq \omega$. Let $U_s \subseteq V(G)$ be the set of vertices that are contained in an s-clique in G. The equality*

$$f_{s,q,G}(\mathbf{x}) = h_{s,G}(\mathbf{x})^{q/s}, \tag{12}$$

holds for $\mathbf{x} > 0$ if and only if the graph G induced on U_s is a complete ω-partite graph with parts $V_1, \dots, V_\omega$ and $\sum_{v \in V_i} \mathbf{x}_v = \sum_{u \in V_j} \mathbf{x}_u$ for all $1 \leq i, j \leq \omega$.

Proof. Let $\mathbf{x} > 0$ be such that (12) holds. We may assume that $G = G[U_s]$, the values of $\mathbf{x}_v$ for $v \notin U_s$ do not interfere with the values of neither $f_{s,q,G}(\mathbf{x})$ nor $h_{s,G}(\mathbf{x})$. As $\mathbf{x} > 0$, we also have $h_{s,G}(\mathbf{x}) > 0$, so by homogeneity of both sides of (12), we may assume that $h_{s,G}(\mathbf{x}) = 1$. By Theorem 1.2, we then have $f_{s,q,G}(\mathbf{x}) = M_{s,q,G} = 1$.

Let $V_1, \dots, V_\ell$ be the vertices of the connected components of the complement of G. We say that a clique $I \subseteq V(G)$ is *canonical* if $|I \cap V_j| \leq 1$ for all $1 \leq j \leq \ell$. Our goal is to show that $\sigma(I) = \ell$ for every canonical clique in G.

Let U be a canonical K_ℓ in G. Repeatedly applying Lemma 2.5 over the non-edges in V_i, we find $\mathbf{y} \in \mathcal{S}_{s,G}$ with $f_{s,q,G}(y) = f_{s,q,G}(x) = M_{s,q,G}$ and $\operatorname{supp} y = U$. Indeed, we may do so as each V_i is connected in the complement. We obtain

$$1 = f_{s,q,G}(\mathbf{y}) = \sum_{I \in K_q(U)} \rho_{s,q}(\sigma_G(I))\mathbf{y}_I \leq \rho_{s,q}(\ell) h_{q,G[U]}(\mathbf{y}). \tag{13}$$

By Maclaurin's inequality (4), we have

$$h_{q,G[U]}(\mathbf{y}) \leq \left(h_{s,G[U]}(\mathbf{y})\right)^{q/s} \binom{\ell}{q} \binom{\ell}{s}^{-q/s} = \rho_{s,q}(\ell)^{-1}.$$

Therefore, we have equality in (13), which means that $\sigma_G(I) = \ell$ for all canonical copies of K_q in $G[U]$. Since a canonical K_q in G can be extended to a canonical K_ℓ, we have that $\sigma_G(I) = \ell$ for every canonical K_q in G.

We now claim that each V_i is an independent set. Indeed, suppose that there is an edge $uv \in G[V_i]$. There must be a canonical clique U in G of size ℓ with $u \in U$. Then $U \cup \{v\}$ must span a $K_{\ell+1}$ clique in G, which contradicts the fact that $\sigma_G(I) = \ell$ for every K_q in U. Therefore, G is a complete multipartite graph, and thus, $\ell = \omega$.

Consider now the reduced graph R, which is a clique with vertex set $\{1, \ldots, \omega\}$. Let $\mathbf{z} \in \mathbb{R}^R$ be defined as $\mathbf{z}_i = \sum_{v \in V_i} \mathbf{x}_v$. For $W \subseteq V(R)$ denote by $V_W := \bigcup_{i \in W} V_i$. Thus, if $W = \{w_1, \ldots, w_t\}$, we have

$$\mathbf{z}_W = \prod_{w \in W} \left(\sum_{v \in V_w} \mathbf{x}_v \right) = \sum_{\substack{v_i \in V_{w_i} \\ i=1,\ldots,t}} \prod_{i=1}^{t} \mathbf{x}_{v_i} = h_{t,V_W}(\mathbf{x}).$$

Since $\sigma(I) = \omega$ for every clique in G, we have

$$h_{s,G}(\mathbf{x}) = \sum_{J \in K_s(G)} \mathbf{x}_J = \sum_{W \in K_s(R)} \sum_{J \in K_s(V_W)} \mathbf{x}_J = \sum_{W \in K_s(R)} h_{s,V_W}(\mathbf{x}) = h_{s,R}(\mathbf{z}).$$

In particular $h_{s,R}(\mathbf{z}) = 1$. Similarly, we have

$$1 = f_{s,q,G}(\mathbf{x}) = \rho_{s,q}(\omega) h_{q,G}(\mathbf{x}) = \rho_{s,q}(\omega) h_{q,R}(\mathbf{z}).$$

By Maclaurin's inequality (4),

$$1 = \rho_{s,q}(\omega) h_{q,R}(\mathbf{z}) \leq \rho_{s,q}(\omega) \left(h_{s,R}(\mathbf{z}) \right)^{q/s} \binom{\omega}{q} \binom{\omega}{s}^{-q/s} = 1.$$

As equality holds for Maclaurin inequality only when $\mathbf{z}_i$ are all equal, we are done. The converse follows in the same way. □

Now, a proof of Lemma 2.4 is the only step missing for a complete proof of Theorems 1.1 and 1.2.

4. Attaining the Maxima

In this section, we give a proof of Lemma 2.4. Nikiforov [11] noticed that Khadzhiivanov's [6] proof of (4) was incomplete as it assumed without proof that the supremum (9) was actually attained, which is not a triviality when $s \geq 2$. We deal with this issue in essentially the same way that Nikiforov did. Unfortunately, our proof is lengthy, for which we apologise.

Proof of Lemma 2.4. As $\mathcal{S}_{1,G}$ is compact, $f_{1,q,G}$ attains a maximum in $\mathcal{S}_{1,G}$, so assume $s \geq 2$. If $s = q$, then $f_{s,q,G} = h_{s,G}$, so $f_{s,q,G}$ attains the maximum at any point $\mathbf{x} \in \mathcal{S}_{s,G}$. Assume $s < q$.

First note that $f_{s,q,G}$ is bounded on $\mathcal{S}_{s,G}$. Indeed, let $\mathbf{x} \in \mathcal{S}_{s,G}$ and we give an uniform bound on $f_{s,q,G}(\mathbf{x})$. First observe that for $J \in K_s(G)$, we have

$\mathbf{x}_J \le h_{s,G}(\mathbf{x}) = 1$. For any $t \ge s$, if $I \in K_t(G)$, the AM-GM inequality gives

$$\mathbf{x}_I = \left(\prod_{J\in K_s(I)} \mathbf{x}_J\right)^{1/\binom{t-1}{s-1}} \le \left(\frac{\sum_{J\in K_s(I)} \mathbf{x}_J}{\binom{t}{s}}\right)^{\binom{t}{s}/\binom{t-1}{s-1}} \le 1. \tag{14}$$

Applying this bound with $t = q$, and recalling that $\rho_{s,q}$ is decreasing, we obtain the bound $f_{s,q,G}(\mathbf{x}) \le \rho_{s,q}(q)k_q(G) < \infty$ for all $\mathbf{x} \in \mathcal{S}_{s,G}$. In particular, $M_{s,q,G} < \infty$.

We prove this lemma by induction on n, the number of vertices of G. Since $\omega(G) \ge q$, we may assume $n \ge q$. For $n = q$, we can assume $G = K_q$. For every $\mathbf{x} \in \mathcal{S}_{s,G}$, the AM-GM inequality gives

$$f_{s,q,G}(\mathbf{x}) = \rho_{s,q}(q)\mathbf{x}_V \le \rho_{s,q}(q)\left(\frac{\sum_{I\in K_s(K_q)} \mathbf{x}_I}{\binom{q}{s}}\right)^{\binom{q}{s}/\binom{q-1}{s-1}} = \rho_{s,q}(q)\binom{q}{s}^{-q/s} = 1.$$

On the other hand, if $\mathbf{y}$ is defined as $\mathbf{y}_v = \binom{q}{s}^{-1/s}$ for all $v \in V$, then $\mathbf{y} \in \mathcal{S}_{s,G}$ and

$$f_{s,q,G}(\mathbf{y}) = \rho_{s,q}(q)\mathbf{y}_V = \rho_{s,q}(q)\binom{q}{s}^{-q/s} = 1,$$

so the maximum is of $f_{s,q,G}$ is indeed attained in $\mathcal{S}_{s,G}$, and moreover $M_{s,q,K_q} = 1$.

Now, assume that the assertion holds for all graphs with $n-1$ vertices or fewer. If G contains a vertex v not in any K_s of G, then x_v does not occur in $f_{s,q,G}$ nor in $h_{s,G}$. That is to say, we have $f_{s,g,G}(\mathbf{x}) = f_{s,q,G-v}(\mathbf{x}')$ and $h_{s,G}(\mathbf{x}) = h_{s,G-v}(\mathbf{x}')$, where $\mathbf{x}' = (\mathbf{x}_u)_{u\in V(G-v)}$. Therefore, the assertion holds for G as it holds for $G - v$ by induction. We now assume that every vertex of G is contained in a copy of K_s.

Our goal is to show that there is $\mathbf{y} \in \mathcal{S}_{s,G}$ with $f_{s,q,G}(\mathbf{y}) = M_{s,q,G}$. Consider a sequence $\mathbf{x}^{(i)}$ in $\mathcal{S}_{s,G}$ with $\lim_{i\to\infty} f_{s,q,G}(\mathbf{x}^{(i)}) = M_{s,q,G}$. If for all $v \in V$, the sequence $\mathbf{x}_v^{(i)}$ is bounded, then $\mathbf{x}^{(i)}$ has an accumulation point $\mathbf{y} \in \mathcal{S}_{s,G}$ and by continuity, $f_{s,q,G}(\mathbf{y}) = M_{s,q,G}$ as desired.

The remaining case is when there is a vertex $v \in V$, for which $\mathbf{x}_v^{(i)}$ is unbounded. By our previous assumption, there is a clique $W \in K_s(G)$ with $v \in W$. If there is some $c > 0$ such that $\mathbf{x}_u^{(i)} > c$ for all $u \in W$, $u \ne v$ and all $i \ge 1$, then we have

$$1 \ge \mathbf{x}_J^{(i)} > c^{s-1}\mathbf{x}_v^{(i)},$$

which contradicts the fact that $\mathbf{x}_v^{(i)}$ is unbounded.

Therefore, there is $u \in W$, $u \ne v$ for which $\liminf_{i\to\infty} \mathbf{x}_u^{(i)} = 0$. We pass to a subsequence where $\mathbf{x}_u^{(i)} \to 0$, $\mathbf{x}_v^{(i)} \to \infty$ and $f_{s,q,G}(\mathbf{x}^{(i)}) \to M_{s,q,G}$ as

$i \to \infty$. Observe that

$$\begin{aligned} f_{s,q,G}(\mathbf{x}^{(i)}) &= \sum_{I \in K_q(G), u \in I} \rho_{s,q}(\sigma_G(I))\mathbf{x}_I^{(i)} + \sum_{I \in K_q(G), u \notin I} \rho_{s,q}(\sigma_G(I))\mathbf{x}_I^{(i)} \\ &= \sum_{I \in K_q(G)} \rho_{s,q}(\sigma_G(I))\mathbf{x}_{I\setminus\{u\}}^{(i)}\mathbf{x}_u^{(i)} + \sum_{I \in K_q(G-u)} \rho_{s,q}(\sigma_G(I))\mathbf{x}_I^{(i)}. \end{aligned} \tag{15}$$

To bound the first sum in (15), recall from (14) that $\mathbf{x}_{I\setminus\{u\}}^{(i)} \le 1$, and so

$$\sum_{I \in K_q(G)} \rho_{s,q}(\sigma_G(I))\mathbf{x}_{I\setminus\{u\}}^{(i)}\mathbf{x}_u^{(i)} \le \sum_{I \in K_q(G)} \rho_{s,q}(\sigma_G(I))\mathbf{x}_u^{(i)} \le \rho_{s,q}(q)k_q(G)\mathbf{x}_u^{(i)}. \tag{16}$$

For the second sum in (15), observe that

$$\sum_{I \in K_q(G-u)} \rho_{s,q}(\sigma_G(I))\mathbf{x}_I^{(i)} \le \sum_{I \in K_q(G-u)} \rho_{s,q}(\sigma_{G-u}(I))\mathbf{x}_I^{(i)} = f_{s,q,G-u}(\tilde{\mathbf{x}}^{(i)}),$$

where $\tilde{\mathbf{x}}^{(i)} = (\mathbf{x}_z^{(i)})_{z \in V(G-u)}$. The point $\tilde{\mathbf{x}}^{(i)}$ may not be in $\mathcal{S}_{s,G-u}$, so we may need a rescaling. To be precise, we need to rule out that $h_{s,G-u}(\tilde{\mathbf{x}}^{(i)}) = 0$. Indeed, if that is the case, then $\mathbf{x}_J^{(i)} = 0$ for all $J \in K_s(G-u)$, thus $\mathbf{x}_I^{(i)} = 0$ for all $I \in K_q(G-u)$. In particular, $f_{s,q,G-u}$ is identically zero, so combining (16) with (15) and taking the limit $i \to \infty$, we have $M_{s,q,G} = 0$. This implies that $k_q(G) = 0$, a contradiction.

We now assume that $h_{s,G-u}(\tilde{\mathbf{x}}^{(i)}) > 0$ and let $\alpha_i := 1/(h_{s,G-u}(\tilde{\mathbf{x}}^{(i)}))^{1/s}$, so that we have $\alpha_i\tilde{\mathbf{x}}^{(i)} \in \mathcal{S}_{s,G-u}$. Also notice that $h_{s,G-u}(\tilde{\mathbf{x}}^{(i)}) \le h_{s,G}(\mathbf{x}^{(i)}) = 1$. Therefore, we have

$$f_{s,q,G-u}(\tilde{\mathbf{x}}^{(i)}) = \Big(h_{s,G-u}(\tilde{\mathbf{x}}^{(i)})\Big)^{q/s} f_{s,q,G-u}(\alpha_i\tilde{\mathbf{x}}^{(i)}) \le f_{s,q,G-u}(\alpha_i\tilde{\mathbf{x}}^{(i)}). \tag{17}$$

By induction, there is a point $\tilde{\mathbf{y}} \in \mathcal{S}_{s,G-u}$ at which $f_{s,q,G-u}$ attains the maximum, that is $f_{s,q,G-u}(\tilde{\mathbf{y}}) = M_{s,q,G-u}$. Combining (16) and (17) back in (15), we obtain

$$f_{s,q,G}(\mathbf{x}^{(i)}) \le \rho_{s,q}(q)k_q(G)\mathbf{x}_u^{(i)} + f_{s,q,G-u}(\tilde{\mathbf{y}}). \tag{18}$$

Define $\mathbf{y} \in \mathbb{R}^V$ as $\mathbf{y}_z = \tilde{\mathbf{y}}_z$ for $z \in V(G-u)$ and $\mathbf{y}_u = 0$. Note that $\mathbf{y} \in S_{s,G}$ as $h_{s,G}(\mathbf{y}) = h_{s,G-u}(\tilde{\mathbf{y}})$. Similarly, $f_{s,q,G-u}(\tilde{\mathbf{y}}) = f_{s,q,G}(\mathbf{y}) \le M_{s,q,G}$. Therefore, as $i \to \infty$, (18) implies

$$M_{s,q,G} \le f_{s,q,G}(\mathbf{y}) \le M_{s,q,G}.$$

Hence, the supremum is indeed attained in $\mathbf{y} \in \mathcal{S}_{s,G}$.

All the proofs are then complete.

5. Further Directions

In this paper, we have provided a so called localised version of the graph Maclaurin inequalities. It would be of great interest to explore further which combinatorial results can be extended in this way. Malec and Tompkins [9] have provided, for instance, localised versions of Erdős–Gallai theorem, the Lubell–Yamamoto–Meshalkin–Bollobás inequality, the Erdős–Ko–Rado theorem and the Erdős–Szekeres theorem on monotone sequences. Kirsch and Nir [7] extended this list with several results in generalised Turán problems. More importantly, we believe that these versions should have interesting applications that are yet to be discovered.

Acknowledgements

The authors are grateful for the continued support of their respective supervisors, Rob Morris and Béla Bollobás. We also would like to thank the referees for their valuable suggestions. L. Aragão was supported by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior-Brasil (CAPES)-Finance Code 001, and V. Souza was supported by Trinity College, Cambridge.

Data Availability The authors declare that the manuscript has no associated data.

Declarations

Conflict of Interest On behalf of all authors, Lucas Aragão states that there is no conflict of interest.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

[1] D Bradač, *A generalization of Turán's theorem* (2022), available at arXiv:2205.08923

[2] J Eckhoff, *A new Turán-type theorem for cliques in graphs*, Discrete Mathematics **282** (2004), 113–122.

[3] D C Fisher and J Ryan, *Bounds on the number of complete subgraphs*, Discrete Mathematics **103** (1992), 313–320.

[4] A Frohmader, *Face vectors of flag complexes*, Israel Journal of Mathematics **164** (2008), 153–164.

[5] G H Hardy, J E Littlewood, and G Pólya, *Inequalities*, 2nd edition, Cambridge Mathematical Library, Cambridge University Press, 1988.

[6] N Khadzhiivanov, *Inequalities for graphs*, Comptes rendus de l'Academie Bulgare des sciences **30** (1977), 793–796 (in Russian).

[7] R Kirsch and J D Nir, *A localized approach to generalized Turán problems* (2023), available at arXiv:2301.05678.

[8] M Krivelevich, R Morris, O Riordan, and A Steger, *Combinatorics, Probability and Computing*, Technical Report 22/2022, Mathematisches Forschungsinstitut Oberwolfach, 2022.

[9] D Malec and C Tompkins, *Localized versions of extremal problems*, European Journal of Combinatorics **112** (2023), 103715.

[10] T S Motzkin and E G Straus, *Maxima for graphs and a new proof of a theorem of Turán*, Canadian Journal of Mathematics **17** (1965), 533–540.

[11] V Nikiforov, *An extension of Maclaurin's inequalities* (2006), available at arXiv:math/0608199.

[12] L Petingi and J Rodriguez, *A new proof of the Fisher-Ryan bounds for the number of cliques of a graph*, Congressus Numerantium **146** (2000), 143–146.

[13] A F Sidorenko, *The maximal number of edges in a homogeneous hypergraph containing no prohibited subgraphs*, Mathematical notes of the Academy of Sciences of the USSR **41** (1987), no. 3, 247–259.

[14] V T Sós and E G Straus, *Extremals of functions on graphs with applications to graphs and hypergraphs*, Journal of Combinatorial Theory. Series B **32** (1982), 246–257.

[15] P Turán, *On an extremal problem in graph theory*, Mat. Fiz. Lapok **48** (1941), 436–452 (in Hungarian).

[16] A A Zykov, *On some properties of linear complexes*, Matematicheskii Sbornik, Novaya Seriy **24(66)** (1949), 163–188 (in Russian).

Lucas Aragão
IMPA
Estrada Dona Castorina 110, Jardim Botânico
Rio de Janeiro
RJ
Brazil
e-mail: l.aragao@impa.br

Victor Souza
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Cambridge
UK
e-mail: vss28@cam.ac.uk

Communicated by Benjamin Sudakov
Received: 13 May 2023.
Accepted: 17 October 2023.