



Complexity of Ice Quiver Mutation Equivalence

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Abstract. We prove NP-hardness results for determining whether ice quivers are mutation equivalent to quivers with given properties, specifically, determining whether an ice quiver is mutation equivalent to an ice quiver with exactly k arrows between any two of its vertices is NP-hard. Also, determining whether an ice quiver is mutation equivalent to a quiver with no edges between frozen vertices is strongly NP-hard. Finally, we present a characterization of mutation classes of ice quivers with two mutable vertices.

1. Introduction

Quivers and their mutations (defined in Sect. 2) were introduced by Fomin and Zelevinsky in [11] and [12] in the context of cluster algebras. They are widely used in algebraic combinatorics (see [16] for a survey or [10] for an introductory book). However, many combinatorial questions about these objects remain unresolved.

The question of whether a given quiver Q is equivalent to only finitely many other quivers was addressed in [6], where a list of such quivers is given. Also, Fomin and Neville show in [8] that there are long cycles in the graph of quivers. Recently, Fomin asked in [7] for algorithmic solutions to the following questions:

1. Given quivers Q_1 and Q_2 , determine whether Q_1 and Q_2 are mutation equivalent.
2. Given a quiver Q and a nonnegative integer k , determine whether there exists a quiver $Q' \in [Q]$ such that Q' has two vertices with exactly k arrows between them.

However, Fomin also proposed that these problems may be computationally difficult or even undecidable:

“We don’t have any algorithm that would detect if two quivers are mutation equivalent or not ... of course it would be absurd if this were algorithmically undecidable - there must be an algorithm - well, who knows? Maybe not.”

–Sergey Fomin, [7, approx. 19:00], May 16, 2022

Formally, Fomin’s problem asks whether these questions are decidable for general quivers (cf. Problem 2.6.14 and Remark 4.1.13 in [10]). We approach the problem from both ends. First, we present a couple of NP-hardness results about the second of Fomin’s questions. Past results such as [3] have shown that certain determinants are preserved by quiver mutation. Since determinants can be computed in polynomial time, however, these results show that it is unlikely that a determinantal formula can capture everything that is going on in a quiver. Next, we will show that quivers with only two mutable vertices can only have a very limited set of equivalent quivers and derive asymptotics of the quivers in such mutation classes.

Note that we use a slight generalization of quivers, namely ice quivers, in which we do not ignore edges between frozen vertices. Such quivers have been studied in the literature (in e.g. [18]). For brevity, we will use the term ‘quivers’ throughout the paper.

1.1. Hardness Results

We begin by stating our main results. Both concern complexity of questions related to quiver mutation equivalence, specifically Fomin’s second question. We note that these results do assume that frozen vertices and arrows between them are permitted in quivers.

Theorem 1.1. (NP-hardness) *Let Q be a quiver, and let $k > 1$ be an integer. The following problem is NP-hard: Determine whether there exists a quiver Q' which is mutation equivalent to Q such that Q' contains two vertices with exactly k arrows between them.*

In the context of quivers, it is natural to be interested in **strong NP-hardness**. In ordinary NP-hardness, the inputs to the problem are assumed to be in binary. Specifically, when there are k arrows between two vertices in a quiver, this is assumed to take $\log_2 k$ bits of input. However, the arrows in a quiver may each carry algebraic information and thus have independent meaning. When inputs to a decision problem are given in unary instead of binary, then the corresponding notion is **strong NP-hardness**. Problems such as KNAPSACK or SUBSET SUM do not meet this stronger criterion. See [14] for background on this topic.

Let an arrow in a quiver be *icebound* if it goes between two frozen vertices.

Theorem 1.2. (Strong NP-hardness) *Let Q be a quiver. The following problem is strongly NP-hard: Determine whether there exists a sequence of mutations which takes Q to a quiver with no icebound arrows.*

See Sect. 4.3 for implications of these results.

1.2. Asymptotic Result

Let us limit the number of vertices at which we are allowed to mutate the quiver, then the set of mutation-equivalent quivers becomes quite limited. If there is only one mutable vertex, then, since mutation is an involution, there can only be two quivers in a mutation class.

Our theorem describes the mutation classes of quivers with exactly two mutable vertices. Again, since mutations are involutions, the only way to get new quivers is to alternate mutating at the two vertices.

Theorem 1.3. *Let Q be a quiver with exactly two mutable vertices called C and D . Define α to be the number of arrows between C and D . Then:*

If $\alpha = 0$, we have $|[Q]| \leq 4$.

If $\alpha = 1$, we have $|[Q]| \leq 10$.

If $\alpha = 2$, then in any nontrivial case the number of arrows in $(\mu_D \mu_C)^n(Q)$ grows linearly.

If $\alpha \geq 3$, then in any nontrivial case the number of arrows in $(\mu_D \mu_C)^n(Q)$ grows exponentially.

Furthermore, if $\alpha \geq 2$, let $\delta_{I,J}(n)$ be the number of arrows between vertices I and J in $(\mu_D \mu_C)^n(Q)$. For any vertex $A \neq C, D$ we have:

$$\lim_{n \rightarrow \infty} \frac{\delta_{A,C}(n)}{\delta_{A,D}(n)} = \frac{1}{2} \left(\alpha \pm \sqrt{\alpha^2 - 4} \right).$$

See 4.6 for possible extensions of this result.

1.3. Structure of the paper

We will proceed as follows. In Sect. 2 we begin with notation, definitions, and examples. Next, we prove our theorem in Sect. 3. We conclude with final remarks in Sect. 4.

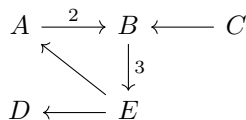
2. Notation, Definitions, and Examples

2.1. Basic Definitions

For positive integers n , define $[n]$ to be the set $\{1, 2, \dots, n\}$. Also, let \mathbb{N} be the set $\{0, 1, 2, \dots\}$.

2.2. Quivers

A *quiver* is a directed multigraph with no loops or 2-cycles, the edges of which are called *arrows*. We will indicate multiple arrows between vertices by labeling edges with numbers. For example, the following graph is a quiver on five vertices with eight arrows:

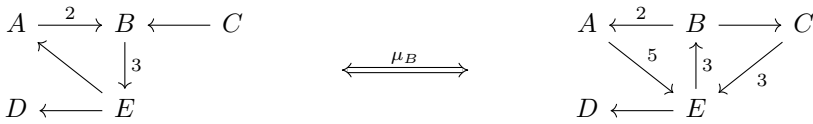


2.3. Quiver Mutation

In a quiver, we assign a subset of the vertices to be *mutable*; the remaining vertices are *frozen*. While Fomin and Zelevinsky’s original definition ignored any arrows between frozen vertices, we will follow [18] and allow them. To each mutable vertex in the quiver, we associate an operation called *mutation*. For a vertex X , mutation at X , denoted by μ_X , proceeds in the following three steps:

1. for every two-step path $Y \rightarrow X \rightarrow Z$, add an arrow from Y to Z ,
2. reverse the direction of every arrow incident to X ,
3. remove 2-cycles one by one.

For example, applying the mutation μ_B will turn the quiver on the left into the quiver on the right and vice versa in the picture below:



It is easily seen that every mutation is an involution. That is, $\mu_X(\mu_X(Q)) = Q$ for every quiver Q with vertex X . It is also easy to see that mutations at nonadjacent vertices commute. Two quivers are said to be *mutation equivalent* if one can be obtained from the other by a finite sequence of mutations. Mutation equivalence is an equivalence relation, so we can define the *mutation class* of a quiver Q , denoted $[Q]$, to be the equivalence class of Q under this relation.

3. Proofs

Proof of Theorem 1.1

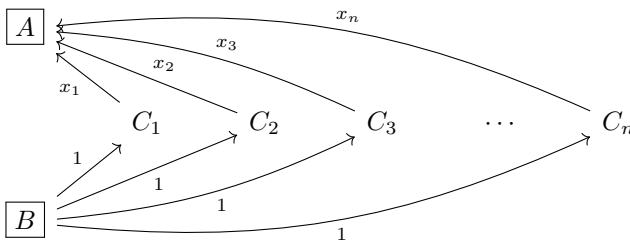
We reduce the problem to SUBSET SUM, which is defined as follows:

SUBSET SUM

Input: $X \subset \mathbb{N}$ a finite set, and $k \in \mathbb{N}$.

Decide: $\exists A \subseteq X$ such that $\sum_{a \in A} a = k$?

This problem is known to be NP-hard (see e.g., [15, §A3.2]). Let $X = \{x_1, \dots, x_n\}$ be a set of positive integers, and let $k > 1$ be another integer. Let Q be the following quiver:



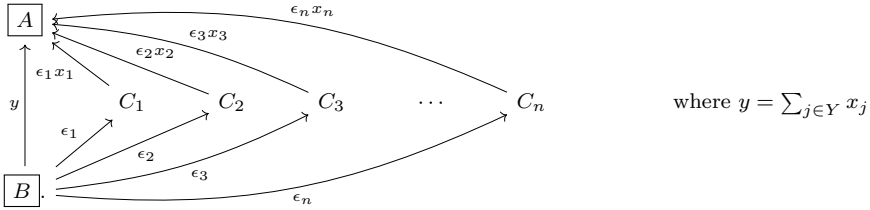
For each $i \in [n]$, let μ_i be μ_{C_i} . Suppose we apply the sequence of mutations $\mu = \mu_{i_1} \cdots \mu_{i_k}$. Define $Y \subseteq [n]$ by

$$Y = \{j \in [n] \mid \mu_{C_j} \text{ is used an odd number of times}\}.$$

Then, for each $j \in [n]$ let

$$\epsilon_j = \begin{cases} 1 & \text{if } j \notin Y \\ -1 & \text{if } j \in Y. \end{cases}$$

An easy induction shows that $\mu(Q)$ is given by



That means that if $k \notin \{0, 1\} \cup X$, the only way for $\mu(Q)$ to contain an arrow with weight k is for k to be the weight of the arrow between B and A . That means that k is present in some quiver equivalent to Q if and only if k is a subset sum of X . The result follows from NP-hardness of SUBSET SUM. \square

Proof of Theorem 1.2

We use the following formulation of the 3-PARTITION problem:

3-PARTITION

Input: $n \geq 3$, and $\mathcal{X} \subseteq \binom{[n]}{3}$.

Decide: $\exists \mathcal{A} \subseteq \mathcal{X}$ such that every $i \in [n]$ is contained in exactly one $A \in \mathcal{A}$?

Given a positive integer n and a subset $\mathcal{X} \subseteq \binom{[n]}{3}$, does there exist a partition of $[n]$ into elements of \mathcal{X} ? This is strongly NP-hard (see e.g., [GJ], §A3.1). Without loss of generality, we may assume that each element of $[n]$ is in at least one of the elements of \mathcal{X} .

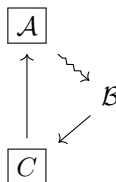
Given n and X , we construct a quiver with vertices:

$$\underbrace{A_1, \dots, A_n}_{\text{frozen}}, \{B_X\}_{X \in \mathcal{X}}, \underbrace{C}_{\text{frozen}}.$$

Take the following edges:

- One edge from A_i to B_X if $i \in X$.
- One edge from B_X to C for each X .
- One edge from C to A_i for each i .

The resulting quiver has this shape:



Here \mathcal{A} and \mathcal{B} represent the sets of vertices of the form A_i and B_X , respectively. There is only one vertex labeled C . The solid arrows represent one arrow between every pair of vertices from the respective sets. The squiggly arrow between \mathcal{A} and \mathcal{B} represents arrows between an A_i and a B_X if and only if $i \in X$.

If a partition $\mathcal{P} \subset \mathcal{X}$ exists, we can apply the mutations $\{\mu_{B_P} : P \in \mathcal{P}\}$, which will eliminate all icebound edges.

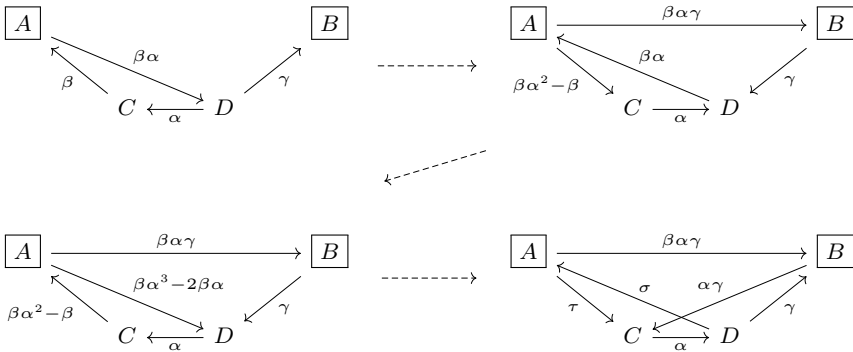
More generally, note that all mutations commute. Moreover, they are all involutions. So we need only consider the effect of using mutations at most once. In that case, we eliminate the icebound edges if and only if the mutations we use correspond to a partition of $[n]$. We have therefore reduced the problem to 3-PARTITION. Because 3-PARTITION is strong NP-hard, the result follows.

Proof of Theorem 1.3

First, we note that it suffices to prove the case where Q has exactly four vertices. This is because, for any subset $Q' \subset V(Q)$ of size 4 containing both C and D , the action of μ_C and μ_D commutes with restriction to Q' .

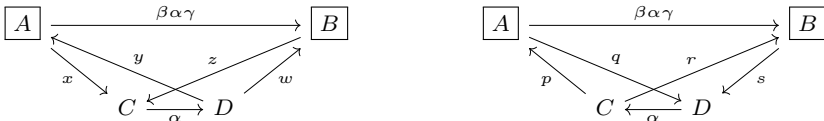
Let the other two vertices in Q' be A and B . It also suffices to consider the case where A and B start with no arrows between them. If C and D have no arrows between them to start, then the statement is trivial. If C and D have one arrow between them, then it is an easy computation to check that $(\mu_D \mu_C)^{10} Q = Q$.

So assume there are $\alpha \geq 2$ arrows from C to D . One possible case consists of arrows from A to C and from D to B . Then we can write down the first few quivers that we get:



where $\sigma = \beta\alpha^3 - 2\beta\alpha$ and $\tau = \beta\alpha^4 - 2\beta\alpha^2 - \beta\alpha^3 + \beta$. Note that these are both positive since $\alpha \geq 2$.

Consider the quivers $Q_1(x, y, z, w)$ and $Q_2(p, q, r, s)$:



We claim that all future quivers will be of one of these two forms and thus $\beta\alpha\gamma$ is the only thing that appears on top. We can compute that

$$\mu_C(Q_1(x, y, z, w)) = Q_2(x, \alpha x - y, z, \alpha z - w),$$

so long as $\alpha x > y$ and $\alpha z > w$. Next, we apply μ_D to find

$$\begin{aligned} \mu_D(\mu_C(Q_1(x, y, z, w))) &= Q_1(\alpha(\alpha x - y) - x, \alpha x - y, \alpha(\alpha z - w) - z, \alpha z - w) \\ &:= Q_1(x', y', z', w'), \end{aligned}$$

this time assuming $\alpha(\alpha x - y) > x$ and $\alpha(\alpha z - w) > w$. This is a stronger condition than the previous one. Note that our conditions are equivalent to

$$\frac{\alpha}{\alpha^2 - 1} < \min\left(\frac{x}{y}, \frac{z}{w}\right),$$

which is satisfied by our original picture.

However, we have computed

$$\frac{x'}{y'} = \frac{\alpha(\alpha x - y) - x}{\alpha x - y} = \alpha - \frac{x}{\alpha x - y} = \alpha - \frac{x/y}{\alpha(x/y) - 1}.$$

So the problem reduces to iteratively applying the function

$$f(t) = \alpha - \frac{t}{\alpha t - 1},$$

and it is easy to see that this converges to a limit of

$$\frac{x}{y} = t = \frac{1}{2} \left(\alpha \pm \sqrt{\alpha^2 - 4} \right).$$

A similar picture holds for the other three starting positions. □

4. Final Remarks and Open Problems

4.1. Undecidability

The paper [8] does show the existence of small quivers which are nonetheless polynomially far apart with respect to mutation. Of course, undecidability is far stronger. Suppose, for example, that it is undecidable whether or not two quivers are mutation equivalent. Then, there would exist quivers Q_1 and Q_2 with a_1 and a_2 arrows, respectively, such that the shortest sequence of mutations taking one to the other has length

$$\ell \geq (\text{Tow}(\text{Tow}(a_1 + a_2))),$$

where $\text{Tow}(k)$ is a tower of 2's of length k . Put fancifully, this means there is no limit as to how far into the sky one has to go to show that two quivers are mutation equivalent. Note that Theorem 1.3 shows that more than two mutable vertices are needed for this to happen.

4.2. Knots and Plabic Graphs

Deep connections exist between quiver mutation equivalence and knot theory including via *plabic graphs* (see e.g., [13], [2], [19] or [9]). For knots and links, upper bounds exist for the number of Reidemeister moves needed to show equivalence.

Here is the best known bound due to [5]. Suppose D_1 and D_2 are diagrams of the same link or knot. Let their crossing numbers be c_1 and c_2 , respectively. Then there exists a sequence of Reidemeister moves taking D_1 to D_2 of length at most

$$\text{Totw}(C)^{(c_1+c_2)} \quad \text{where } C = \left(10^{10^6}\right)^{(c_1+c_2)}.$$

Again, $\text{Totw}(k)$ is a tower of 2s of length k . It may well be the case that such a bound exists for mutation equivalence of quivers as well.

4.3. Quiver Invariants

Fix a quiver Q and some $k \in \mathbb{N}$. Fomin's question in the introduction asks for an algorithm to determine whether there exists a quiver $Q' \in [Q]$ such that two vertices in Q' have exactly k arrows between them.

One hope is that determinantal invariants would be able to answer these questions. Our results suggest that one should investigate the quivers used in the construction of Theorems 1.1 and 1.2.

4.4. Mutable and Immutable Vertices

Frozen vertices and arrows between them are essential for the proofs of Theorems 1.1 and 1.2. It would be interesting to see whether the number of frozen vertices can be reduced.

4.5. Other Properties of Quivers

There are many other questions about quivers for which an algorithmic test would be of interest. For instance, one could ask whether a quiver is *mutation acyclic*, that is, mutation equivalent to an acyclic quiver. Much work remains to be done in this area.

4.6. Quiver Gadgets

Embedding difficult problems into quiver mutation equivalence requires the construction of quivers whose mutations can be controlled. Many questions even about simple quivers remain unanswered. In particular, one method would be to embed *Hilbert's tenth problem* or the *post-correspondence problem* into quivers (see e.g., [17]). We are still far away from this.

To illustrate, we give a natural possible generalization of Theorem 1.3. Let Q be a quiver of the following form:

$$\boxed{A} \xrightarrow{x_0} C_1 \xrightarrow{x_1} \cdots \xrightarrow{x_{k-1}} C_k \xrightarrow{x_k} \boxed{B}$$

Then we conjecture that for all Q' which is mutation equivalent to Q , the number of arrows between A and B is always 0 or $x_0x_1 \cdots x_k$. The cases $k = 0$ and $k = 1$ are trivial, and the case $k = 2$ is proven in Theorem 1.3. However, the general case is open.

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Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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