



# Lagrangian-Perfect Hypergraphs

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**Abstract.** Hypergraph Lagrangian function has been a helpful tool in several celebrated results in extremal combinatorics. Let  $G$  be an  $r$ -uniform graph on  $[n]$  and let  $\vec{x} = (x_1, \dots, x_n) \in [0, \infty)^n$ . The graph Lagrangian function is defined to be  $\lambda(G, \vec{x}) = \sum_{e \in E(G)} \prod_{i \in e} x_i$ . The graph Lagrangian is defined as  $\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in \Delta\}$ , where  $\Delta = \{\vec{x} = (x_1, x_2, \dots, x_n) \in [0, 1]^n : x_1 + x_2 + \dots + x_n = 1\}$ . The Lagrangian density  $\pi_\lambda(F)$  of an  $r$ -graph  $F$  is defined to be  $\pi_\lambda(F) = \sup\{r!\lambda(G) : G \text{ does not contain } F\}$ . Sidorenko (Combinatorica 9:207–215, 1989) showed that the Lagrangian density of an  $r$ -uniform hypergraph  $F$  is the same as the Turán density of the extension of  $F$ . Therefore, determining the Lagrangian density of a hypergraph will add a result to the very few known results on Turán densities of hypergraphs. For an  $r$ -uniform graph  $H$  with  $t$  vertices,  $\pi_\lambda(H) \geq r!\lambda(K_{t-1}^r)$  since  $K_{t-1}^r$  (the complete  $r$ -uniform graph with  $t-1$  vertices) does not contain a copy of  $H$ . We say that an  $r$ -uniform hypergraph  $H$  with  $t$  vertices is  $\lambda$ -perfect if the equality  $\pi_\lambda(H) = r!\lambda(K_{t-1}^r)$  holds. A fundamental theorem of Motzkin and Straus implies that all 2-uniform graphs are  $\lambda$ -perfect. It is interesting to understand the  $\lambda$ -perfect property for  $r \geq 3$ . Our first result is to show that the disjoint union of a  $\lambda$ -perfect 3-graph and  $S_{2,t} = \{123, 124, 125, 126, \dots, 12(t+2)\}$  is  $\lambda$ -perfect, this result implies several previous results: Taking  $H$  to be the 3-graph spanned by one edge and  $t = 1$ , we obtain the result by Hefetz and Keevash (J Comb Theory Ser A 120:2020–2038, 2013) that a 3-uniform matching of size 2 is  $\lambda$ -perfect. Doing it repeatedly, we obtain the result in Jiang et al. (Eur J Comb 73:20–36, 2018) that any 3-uniform matching is  $\lambda$ -perfect. Taking  $H$  to be the 3-uniform linear path of length 2 or 3 and  $t = 1$  repeatedly, we obtain the results in Hu et al. (J Comb Des 28:207–223, 2020). Earlier results indicate that  $K_4^{3-} = \{123, 124, 134\}$  and  $F_5 = \{123, 124, 345\}$  are not  $\lambda$ -perfect, we show that the disjoint union of  $K_4^{3-}$  (or  $F_5$ ) and  $S_{2,t}$  are  $\lambda$ -perfect. Furthermore, we show the disjoint union of a 3-uniform hypergraph  $H$  and  $S_{2,t}$  is  $\lambda$ -perfect if  $t$  is large. We also give an irrational Lagrangian density of a family of four 3-uniform hypergraphs.

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### 1. Introduction

In 1965, Motzkin–Straus [21] applied the graph Lagrangian function to give a new proof of the theorem on Turán densities of complete graphs [33]. This aroused the great interests in exploring hypergraph Lagrangian method in extremal combinatorics. For example, in the celebrated result of Frankl–Rödl [9], they disproved the long-standing jumping constant conjecture of Erdős by applying hypergraph Lagrangian method. Frankl–Füredi [8] and Sidorenko [29] also applied the graph Lagrangian function in evaluating Turán densities of hypergraphs in 1980s. Recently, the connection between Lagrangian densities of hypergraphs and Turán densities of their extensions has been developed further. We refer the reader to Keevash’s survey paper ‘Hypergraph Turán Problems’ [17] for other interesting applications of hypergraph Lagrangian.

For a set  $V$  and a positive integer  $r$ , let  $V^r$  denote the family of all  $r$ -subsets of  $V$ . An  $r$ -uniform graph or  $r$ -graph  $G$  consists of a set  $V(G)$  of vertices and a set  $E(G) \subseteq V(G)^r$  of edges. An edge  $e = \{a_1, a_2, \dots, a_r\}$  will be simply denoted by  $a_1 a_2 \dots a_r$ . An  $r$ -graph  $H$  is a subgraph of an  $r$ -graph  $G$ , denoted by  $H \subseteq G$ , if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . A subgraph of  $G$  induced by  $V' \subseteq V$ , denoted as  $G[V']$ , is the  $r$ -graph with vertex set  $V'$  and edge set  $E' = \{e \in E(G) : e \subseteq V'\}$ . For  $S \subseteq V(G)$ , let  $G - S$  denote the subgraph of  $G$  induced by  $V(G) \setminus S$ . Let  $K_t^r$  denote the complete  $r$ -graph on  $t$  vertices. Let  $K_t^{r-}$  be obtained by removing one edge from the complete  $r$ -graph on  $t$  vertices. For a positive integer  $n$ , let  $[n]$  denote  $\{1, 2, 3, \dots, n\}$ .

**Definition 1.1.** Let  $G$  be an  $r$ -graph on  $[n]$  and let  $\vec{x} = (x_1, \dots, x_n) \in [0, \infty)^n$ . Define the *Lagrangian* function

$$\lambda(G, \vec{x}) = \sum_{e \in E(G)} \prod_{i \in e} x_i.$$

The *Lagrangian* of  $G$ , denoted by  $\lambda(G)$ , is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in \Delta\},$$

where

$$\Delta = \{\vec{x} = (x_1, x_2, \dots, x_n) \in [0, 1]^n : x_1 + x_2 + \dots + x_n = 1\}.$$

The value  $x_i$  is called the *weight* of the vertex  $i$  and a vector  $\vec{x} \in \Delta$  is called a *feasible weight vector* on  $G$ . A feasible weight vector  $\vec{y} \in \Delta$  is called an *optimum weight vector* for  $G$  if  $\lambda(G, \vec{y}) = \lambda(G)$ .

In [21], Motzkin and Straus established a connection between the Lagrangian of a 2-graph and its maximum complete subgraphs.

**Theorem 1.2.** [21] *If  $G$  is a 2-graph in which a maximum complete subgraph has  $t$  vertices, then  $\lambda(G) = \lambda(K_t^2) = \frac{1}{2}(1 - \frac{1}{t})$ .*

However, determining the Lagrangian of an  $r$ -graph for  $r \geq 3$  is much more difficult than graphs and there is no conclusion similar to Theorem 1.2 for hypergraphs. It is of great interests to estimate the Lagrangian of  $r$ -graphs that have some certain properties. An interesting conjecture of Frankl–Füredi [8] states that the maximum Lagrangian among all  $r$ -graphs with  $m$  edges is achieved on the  $r$ -graph whose edges are the first  $r$ -tuples in colex order. Talbot [30] made a first breakthrough in confirming this conjecture for some cases. Subsequent progress were made in [18–20, 22, 31, 32, 34]. Recently, Gruslys–Letzter–Morrison [11] confirmed this conjecture for  $r = 3$  and sufficiently large  $m$ , and showed that the conjecture is not always true for  $r \geq 4$ . As remarked in [11], it would be interesting to find the maximizers for other values of  $m$ .

Given an  $r$ -graph  $F$ , an  $r$ -graph  $G$  is said to be  $F$ -free if it does not contain an isomorphic copy of  $F$ . The *Lagrangian density*  $\pi_\lambda(F)$  of an  $r$ -graph  $F$  is defined to be

$$\pi_\lambda(F) = \sup\{r!\lambda(G) : G \text{ is } F\text{-free}\}.$$

The Lagrangian density is closely related to the Turán density. Determining the Turán density is one of the central problems in extremal combinatorics. For a fixed positive integer  $n$  and an  $r$ -graph  $F$ , the *Turán number* of  $F$ , denoted by  $ex(n, F)$ , is the maximum number of edges in an  $F$ -free  $r$ -graph on  $n$  vertices. An averaging argument of Katona et al. [16] shows that the sequence  $\frac{ex(n, F)}{\binom{n}{r}}$  is non-increasing. Hence  $\lim_{n \rightarrow \infty} \frac{ex(n, F)}{\binom{n}{r}}$  exists. The *Turán density* of  $F$  is defined as

$$\pi(F) = \lim_{n \rightarrow \infty} \frac{ex(n, F)}{\binom{n}{r}}.$$

Denote

$$\Pi_r = \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a family of } r\text{-uniform graphs}\}.$$

For 2-graphs, Erdős–Stone–Simonovits determined the Turán numbers of all non-bipartite graphs asymptotically. Very few results are known for  $r \geq 3$  and finding good estimation for Turán densities of hypergraphs is believed to be one of the most challenging problems in extremal combinatorics. The following proposition implies that when we get the Lagrangian density of an  $r$ -graph  $H$ , then we get the Turán density of a corresponding hypergraph.

A pair of vertices  $\{i, j\}$  is *covered* in a hypergraph  $F$  if there exists an edge  $e$  in  $F$  such that  $\{i, j\} \subseteq e$ . We say that  $F$  *covers pairs* if every pair of vertices in  $F$  is covered. Let  $r \geq 3$  and  $F$  be an  $r$ -graph. The *extension* of  $F$ , denoted by  $H^F$  is obtained as follows: For each pair of vertices  $v_i$  and  $v_j$  not covered in  $F$ , we add a set  $B_{ij}$  of  $r - 2$  new vertices and the edge  $\{v_i, v_j\} \cup B_{ij}$ , where the  $B_{ij}$ s are pairwise disjoint over all such pairs  $\{i, j\}$ .

**Proposition 1.3.** [4, 28, 29] *Let  $F$  be an  $r$ -graph. Then,*

- (i)  $\pi(F) \leq \pi_\lambda(F)$ ;
- (ii)  $\pi(H^F) = \pi_\lambda(F)$ . *In particular, if  $F$  covers pairs, then  $\pi(F) = \pi_\lambda(F)$ .*

For an  $r$ -graph  $H$  on  $t$  vertices, it is clear that  $\pi_\lambda(H) \geq r!\lambda(K_{t-1}^r)$ . An  $r$ -graph  $H$  on  $t$  vertices is  $\lambda$ -perfect if  $\pi_\lambda(H) = r!\lambda(K_{t-1}^r)$ . Theorem 1.2 implies that all 2-graphs are  $\lambda$ -perfect. It is interesting to understand what kind of hypergraphs are  $\lambda$ -perfect. Sidorenko [29] and Brandt–Irwin–Jiang [4] showed that the enlargement of a tree satisfying Erdős–Sos’s conjecture is  $\lambda$ -perfect. This first implied the Turán densities of infinitely many hypergraphs. Pikhurko [24] showed that a 4-uniform tight path of length 2 (2 edges intersecting at 3 vertices) is  $\lambda$ -perfect, and this led to confirm the conjecture of Frankl–Füredi [7] on the Turán number of its extension, the so-called  $r$ -uniform generalized triangle for the case  $r = 4$ . Norin and Yepremyan [23] determined for  $r = 5$  or 6 by extending the earlier result of Frankl–Füredi in [8]. Jenssen [14] showed that a path of length 2 formed by two edges intersecting at  $r - 2$  vertices for  $r = 3, 4, 5, 6, 7$  is  $\lambda$ -perfect. Hefetz and Keevash [12] showed that the Lagrangian density of a 3-uniform matching of size 2 is  $\lambda$ -perfect, Jiang–Peng–Wu [15] determined the Lagrangian density for any 3-uniform matching. Hefetz and Keevash [12] conjectured that an  $r$ -uniform matching of size 2 is not  $\lambda$ -perfect for  $r \geq 4$ . This conjecture was confirmed in [3] by Bene Watts, Norin and Yepremyan (independently, in [36] for  $r = 4$ ). Wu [35] also gave the Lagrangian density of the 4-uniform matching of size 3. In [5, 13, 37, 39], the authors showed a 3-uniform linear path of length 3 or 4,  $\{123, 234, 456\}$ ,  $\{123, 345, 561\}$  and  $\{123, 124, 345\}$ , the disjoint union of a 3-uniform linear path of length 2 or 3 and a 3-uniform matching, and the disjoint union of a 3-uniform tight path of length 2 and a 3-uniform matching are  $\lambda$ -perfect. It is shown in [38] that the 5-uniform linear path of length 2 is  $\lambda$ -perfect. These were all the previously known results on Lagrangian densities. For 3-uniform graphs spanned by 3 edges, there is one remaining unsolved case:  $K_4^{3-} = \{123, 124, 134\}$ . By Proposition 1.3, the Lagrangian density and the Turán density of  $K_4^{3-}$  are equal. It would be very interesting to obtain the Turán density of  $K_4^{3-}$  by determining the Lagrangian density of  $K_4^{3-}$ . A well-known and long-standing conjecture of Frankl and Füredi [7] is that  $\pi(K_4^{3-}) = \frac{2}{7}$ , one of the major open hypergraph Turán problems.

An  $r$ -uniform hypergraph is *linear* if any two edges have at most 1 vertex in common. Let  $G \cup H$  denote the disjoint union of  $G$  and  $H$ . The following conjectures were proposed in [39].

**Conjecture 1.4.** [39] For  $r \geq 3$ , there exists  $n$  such that a linear  $r$ -graph with at least  $n$  vertices is  $\lambda$ -perfect.

**Conjecture 1.5.** [39] For  $r \geq 3$ , there exists  $n$  such that if  $G$  and  $H$  are  $\lambda$ -perfect  $r$ -graphs with at least  $n$  vertices, then  $G \cup H$  is  $\lambda$ -perfect.

Let  $S_{2,t}$  denote the 3-graph with vertex set  $\{v_1, v_2, u_1, u_2, \dots, u_t\}$  and edge set  $\{v_1 v_2 u_1, v_1 v_2 u_2, \dots, v_1 v_2 u_t\}$ . A result of Sidorenko in [29] implies that  $S_{2,t}$  is  $\lambda$ -perfect. In [39], we proved that  $S_{2,t} \cup H$  is  $\lambda$ -perfect if  $H$  is  $\lambda$ -perfect and  $t \geq 3$ . Our first result in this paper removes the condition that  $t \geq 3$ .

**Theorem 1.6.** *If  $H$  is  $\lambda$ -perfect, then  $H \cup S_{2,t}$  is  $\lambda$ -perfect for any  $t \geq 1$ .*

Taking  $H$  to be the 3-graph spanned by one edge and  $t = 1$  in the theorem, we obtain the result by Heftz and Keevash [12] that a 3-uniform matching of size 2 is  $\lambda$ -perfect. Doing it repeatedly, we obtain the result in [15] that a 3-uniform matching is  $\lambda$ -perfect. Taking  $H$  to be the 3-uniform linear path of length 2 or 3, and  $t = 1$ , we obtain the results in [13].

The condition that  $H$  is  $\lambda$ -perfect in the above theorem is not necessary. Let  $F_5$  denote the 3-graph with vertex set  $\{v_1, v_2, v_3, v_4, v_5\}$  and edge set  $\{v_1v_2v_3, v_1v_2v_4, v_3v_4v_5\}$ . In [39], we proved that  $\pi_\lambda(F_5) = \frac{4}{9}$  and this implies that  $F_5$  is not  $\lambda$ -perfect. We show that  $F_5 \cup S_{2,t}$  is however  $\lambda$ -perfect. Let  $H \cup \{e\}$  denote the disjoint union of  $H$  and a single edge throughout the paper.

**Theorem 1.7.**  $F_5 \cup \{e\}$  is  $\lambda$ -perfect.

**Theorem 1.8.**  $F_5 \cup S_{2,t}$  is  $\lambda$ -perfect for  $t \geq 2$ .

Frankl and Füredi [7] showed that  $\pi(K_4^{3-}) \geq \frac{2}{7}$  which implies that  $K_4^{3-}$  is not  $\lambda$ -perfect. We show however that  $K_4^{3-} \cup \{e\}$  is  $\lambda$ -perfect.

**Theorem 1.9.**  $K_4^{3-} \cup S_{2,t}$  is  $\lambda$ -perfect for any  $t \geq 1$ .

We also discuss  $H \cup S_{2,t}$  for any 3-graph  $H$ .

**Theorem 1.10.** Let  $H$  be a 3-graph with  $s$  vertices. Then,  $H \cup S_{2,t}$  is  $\lambda$ -perfect if  $t \geq \frac{3}{2}s^2 - \frac{1}{2}s + 4$ .

In contrast to the case  $r = 2$ , we know very few about the set  $\Pi_r$  for  $r \geq 3$ . In 1998, Chung and Graham [6] proposed the conjecture that every element in  $\Pi_r$  is a rational number. Recently, Baber and Talbot [2] applied Razborov’s flag algebra method [26] to show that the Turán density of a family of three 3-graphs is the Lagrangian of a corresponding 3-graph which is an irrational number. Independently, Pikhurko [25] showed that there is a finite family of  $r$ -graphs with irrational Turán density for every  $r \geq 3$  by applying the Strong Removal Lemma of Rödl and Schacht [27]. In the proof, Pikhurko also used the theorem that the Lagrangian of an  $r$ -graph is the Turán density of a finite family of  $r$ -graphs. In [10], Grosu constructed explicitly some finite families of  $r$ -graphs with irrational densities for every  $r \geq 3$ . Baber and Talbot [2] asked whether there exists a single  $r$ -graph  $F$  such that  $\pi(F)$  is irrational. We give an irrational Lagrangian density as in the following result.

**Theorem 1.11.** Let  $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$  where  $P_1 = \{123, 124, 134, 234, 567\}$ ,  $P_2 = \{123, 124, 134, 234, 561, 562, 783\}$ ,  $P_3 = \{123, 124, 134, 234, 561, 562, 734\}$  and  $P_4 = \{123, 124, 134, 234, 561, 562, 357\}$ . Then,  $\pi_\lambda(\mathcal{P}) = \frac{\sqrt{3}}{3}$ .

To strengthen this result, we show that the Lagrangian density of  $K_4^3 \cup \{e\}$  is  $\frac{\sqrt{3}}{3}$ . Since that proof is quite involving, it is written in a separate paper [40]. This implies that extension of  $K_4^3 \cup \{e\}$  has an irrational Turán density and answers the question of Baber and Talbot [2].

Although  $K_4^3 \cup \{e\}$  is not  $\lambda$ -perfect, we can show that  $K_4^3 \cup \{e\} \cup \{e\}$  (indeed  $K_4^3 \cup \{e\} \cup S_{2,t}$  is  $\lambda$ -perfect (similar to the proof of Theorem 1.10, we

omit the details). These results lead us to ask whether a ‘sparse enough’  $r$ -graph must be  $\lambda$ -perfect. Let us be a little bit precise. Note that the property ‘ $\lambda$ -perfect’ is monotone in the sense that an  $r$ -graph obtained by removing an edge from a  $\lambda$ -perfect  $r$ -graph (keep the same vertex set) is  $\lambda$ -perfect. It is interesting to understand the relation between the number of edges in a hypergraph and the ‘ $\lambda$ -perfect’ property. We propose that the number of edges in a hypergraph is no more than the number of edges in a linear hyperpath would guarantee the ‘ $\lambda$ -perfect’ property.

**Conjecture 1.12.** For  $r \geq 3$ , there exists  $m_0$  such that for an  $r$ -graph  $G$  with  $m \geq m_0$  edges, if the number of vertices in  $G$  is at least  $m(r - 1) + 1$ , then  $G$  is  $\lambda$ -perfect.

Theorems 1.6 to 1.10 give some evidence to this conjecture. We remark that a hypergraph is not  $\lambda$ -perfect if it has many edges.

*Remark 1.13.* Let  $H$  be a 3-graph on  $t$  vertices with at least  $\binom{t-1}{3} + \binom{t-2}{2} + 2$  edges. Then,  $H$  is not  $\lambda$ -perfect.

*Proof.* Let  $G$  be a 3-graph with vertex set  $[t]$  and edge set  $\{ijk|i, j, k \in [t - 1]\} \cup \{ijt|i, j \in [t - 2]\} \cup \{1(t - 1)t\}$ . Note that  $G$  has  $\binom{t-1}{3} + \binom{t-2}{2} + 1$  edges, so  $G$  is  $H$ -free. Let  $x_i = \frac{1}{t-1}$  for  $1 \leq i \leq t - 2$ ,  $x_j = \frac{1}{2t-2}$  for  $t - 1 \leq j \leq t$ , and  $\vec{x}$  be a feasible vector such that vertex  $i$  has weight  $x_i$ . Then,  $\lambda(G, \vec{x}) = \frac{1}{(t-1)^3} \left[ \binom{t-1}{3} + \frac{1}{4} \right] > \frac{1}{(t-1)^3} \binom{t-1}{3} = \lambda(K_{t-1}^3)$ . Therefore  $H$  is not  $\lambda$ -perfect.  $\square$

None of the known  $\lambda$ -perfect hypergraphs ( $r \geq 3$ ) covers pairs although a complete graph covers pairs and is  $\lambda$ -perfect. It is interesting to understand whether the property ‘covering pairs’ plays some role for the ‘ $\lambda$ -perfect’ property. Let  $r \geq 3$ . Is it impossible for a  $\lambda$ -perfect  $r$ -graph to cover pairs if  $r \geq 3$ ? As stated in Proposition 1.3, the Lagrangian density and the Turán density are the same for a large class of  $r$ -graphs. The advantage of transferring to the Lagrangian density is that we can assume that  $G$  is dense (so covering pairs) when considering the maximum Lagrangian of an  $H$ -free  $r$ -graph  $G$ . This assumption makes the structural analysis ‘nicer’ in some cases, so we hope that this method helps us to better understand the set  $\Pi_r$ .

We give some properties of Lagrangians of  $r$ -graphs in the next section. The proofs of Theorems 1.6–1.10 will be given in Sect. 3. In Sect. 4, we give an irrational Lagrangian density.

## 2. Preliminaries

The following fact follows immediately from the definition of the Lagrangian.

*Fact 2.1.* Let  $G_1, G_2$  be  $r$ -graphs and  $G_1 \subseteq G_2$ . Then  $\lambda(G_1) \leq \lambda(G_2)$ .

*Fact 2.2.* [9] Let  $G$  be an  $r$ -graph on  $[n]$ . Let  $\vec{x} = (x_1, x_2, \dots, x_n)$  be an optimum weight vector on  $G$ . Then

$$\frac{\partial \lambda(G, \vec{x})}{\partial x_i} = r\lambda(G)$$

for every  $i \in [n]$  satisfying  $x_i > 0$ .

Given an  $r$ -graph  $G$ , and  $i, j \in V(G)$ , define

$$L_G(j \setminus i) = \{e : i \notin e, e \cup \{j\} \in E(G) \text{ and } e \cup \{i\} \notin E(G)\}.$$

*Fact 2.3.* Let  $G$  be an  $r$ -graph on  $[n]$ . Let  $\vec{x} = (x_1, x_2, \dots, x_n)$  be a feasible weight vector on  $G$ . Let  $i, j \in [n], i \neq j$  satisfying  $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$ . Let  $\vec{y} = (y_1, y_2, \dots, y_n)$  be defined by letting  $y_\ell = x_\ell$  for every  $\ell \in [n] \setminus \{i, j\}$  and  $y_i = y_j = \frac{1}{2}(x_i + x_j)$ . Then  $\lambda(G, \vec{y}) \geq \lambda(G, \vec{x})$ . Furthermore, if the pair  $\{i, j\}$  is contained in an edge of  $G$ ,  $x_i > 0$  for each  $1 \leq i \leq n$ , and  $\lambda(G, \vec{y}) = \lambda(G, \vec{x})$ , then  $x_i = x_j$ .

*Proof.* Since  $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$ , then

$$\lambda(G, \vec{y}) - \lambda(G, \vec{x}) = \sum_{\{i, j\} \subseteq e \in G} \left( \frac{(x_i + x_j)^2}{4} - x_i x_j \right) \prod_{k \in e \setminus \{i, j\}} x_k \geq 0.$$

If the pair  $\{i, j\}$  is contained in an edge of  $G$  and  $x_i > 0$  for each  $1 \leq i \leq n$ , then the equality holds only if  $x_i = x_j$ . □

An  $r$ -graph  $G$  is *dense* if  $\lambda(G') < \lambda(G)$  for every proper subgraph  $G'$  of  $G$ .

*Fact 2.4.* [9] Let  $G = (V, E)$  be a dense  $r$ -graph. Then  $G$  covers pairs.

Note that the converse of Fact 2.4 is not true. For example, the Fano plane covers pairs but it is not dense. Indeed, many counterexamples exist by Theorem 2.1 in the paper of Talbot [30]. While considering the Lagrangian density of an  $r$ -graph  $F$ , we can always reduce to consider dense  $F$ -free  $r$ -graphs.

*Remark 2.5.* Let  $F, G$  be  $r$ -graphs and  $G$  be  $F$ -free. Then (a) there exists a dense subgraph  $G'$  of  $G$  such that  $\lambda(G') = \lambda(G)$  and  $G'$  is  $F$ -free. (b) To show  $\pi_\lambda(F) \leq a$ , it is sufficient to show that  $\lambda(G) \leq \frac{a}{r!}$  for any dense  $F$ -free  $r$ -graph. (c) To show that a  $t$ -vertex  $r$ -graph is  $\lambda$ -perfect, it is sufficient to show that  $\lambda(G) \leq \lambda(K_{t-1}^r)$  for any dense  $F$ -free  $r$ -graph.

*Proof.* (a) Let  $G$  be an  $r$ -graph on  $n$  vertices. If  $G$  is dense, then we are fine. If not, then we can find  $G' \subset G$  such that  $\lambda(G') = \lambda(G)$  and  $|V(G')| < |V(G)|$ . If  $G'$  is dense, then we stop. Otherwise, we continue this process until we find a dense subgraph. This process terminates since the number of vertices is reduced by at least one in each step.

(b) and (c) follow immediately from (a). □

### 3. Proof of the Main Results

We will prove Theorems 1.6–1.10 in this section.

**3.1. Preliminaries**

For a 3-graph  $G$  and  $v \in V(G)$ , we define the link graph of  $v$  as  $G_v = \{ab|vab \in E(G)\}$ . Let  $\omega(G)$  denote the order of a maximum clique of  $G$ .

**Claim 3.1.** *Let  $G$  be a 3-graph with  $\lambda(G) > \lambda(K_{k+1}^3)$  and let  $\vec{x}$  be an optimal weight vector. Then for any  $v \in V(G)$ , its weight  $x_v$  satisfies that  $x_v < 1 - \frac{\sqrt{k(k-1)}}{k+1}$ .*

*Proof.* Note that  $\frac{\partial \lambda(G, \vec{x})}{\partial x_v} = \sum_{\{v_1 v_2 v\} \in E(G)} x_{v_1} x_{v_2}$  and the link graph  $G_v$  of  $v$  is a graph with  $\sum_{u \in V(G) \setminus \{v\}} x_u = 1 - x_v$ . By Fact 2.2 and the theorem of Motzkin and Straus (Theorem 1.2), we have

$$3\lambda(K_{k+1}^3) < 3\lambda(G) = \frac{\partial \lambda(G, \vec{x})}{\partial x_v} \leq \frac{1}{2}(1 - x_v)^2.$$

So

$$x_v < 1 - \frac{\sqrt{k(k-1)}}{k+1}.$$

□

**Claim 3.2.** *Let  $G$  be a 3-graph with  $\lambda(G) > \lambda(K_{k+1}^3)$  and let  $\vec{x}$  be an optimal weight vector. Then for  $v \in V(G)$  with  $\omega(G_v) \leq k$ , its weight  $x_v$  satisfies that  $x_v < \frac{1}{k+1}$ .*

*Proof.* Note that  $\frac{\partial \lambda(G, \vec{x})}{\partial x_v} = \sum_{\{v_1 v_2 v\} \in E(G)} x_{v_1} x_{v_2}$  and the link graph  $G_v$  of  $v$  is a graph with  $\sum_{u \in V(G) \setminus \{v\}} x_u = 1 - x_v$ . By Fact 2.2 and the theorem of Motzkin and Straus (Theorem 1.2), we have

$$3\lambda(K_{k+1}^3) < 3\lambda(G) = \frac{\partial \lambda(G, \vec{x})}{\partial x_v} \leq \frac{1}{2} \times \left(1 - \frac{1}{k}\right) (1 - x_v)^2.$$

So

$$x_v < \frac{1}{k+1}.$$

□

**Claim 3.3.** *Let  $v$  be a vertex in a 3-graph  $G$  and  $x_v$  be the weight of  $v$  in an optimal weight vector  $\vec{x}$  of  $G$ . If  $G - \{v\}$  is  $H$ -free, then  $\lambda(G) \leq \frac{\pi_\lambda(H)(1-x_v)^3}{6(1-3x_v)}$ .*

*Proof.* Since  $G - \{v\}$  is  $H$ -free,  $\lambda(G - \{v\}, \vec{x}) \leq \frac{\pi_\lambda(H)}{6}(1 - x_v)^3$ . Therefore

$$\lambda(G) \leq \frac{\pi_\lambda(H)}{6}(1 - x_v)^3 + x_v \frac{\partial \lambda(G, \vec{x})}{\partial x_v}.$$

By Fact 2.2, we have

$$\lambda(G) \leq \frac{\pi_\lambda(H)}{6}(1 - x_v)^3 + 3x_v \lambda(G).$$

Then  $\lambda(G) \leq \frac{\pi_\lambda(H)(1-x_v)^3}{6(1-3x_v)}$ .

□

*Remark 3.4.*  $f(x) = \frac{(1-x)^3}{1-3x}$  is increasing in  $(0, \frac{1}{3})$ .



*Proof.* Since  $f'(x) = \frac{6x(1-x)^2}{(1-3x)^2}$ , then  $f'(x) > 0$  in  $(0, \frac{1}{3})$ . So  $f(x) = \frac{(1-x)^3}{1-3x}$  is increasing in  $(0, \frac{1}{3})$ .  $\square$

**Claim 3.5.** *Let a 3-graph  $G$  be  $H \cup S_{2,t}$ -free, where  $H$  is a 3-graph with  $s$  vertices. Let  $S_{2,s+t} = \{v_1v_2b_1, v_1v_2b_2, \dots, v_1v_2b_{s+t}\} \subseteq G$ . Then  $G - \{v_1, v_2\}$  is  $H$ -free.*

*Proof.* Suppose that  $H \subseteq G - \{v_1, v_2\}$ . Since  $|V(H)| = s$ , then  $|\{b_1, b_2, \dots, b_{s+t}\} \cap V(H)| \leq s$ , and  $|\{b_1, \dots, b_{s+t}\} \setminus V(H)| \geq t$ . So the induced subgraph of  $G$  by  $\{v_1, v_2, b_1, \dots, b_{s+t}\} \setminus V(H)$  contains an  $S_{2,t}$ , and  $G$  contains  $H \cup S_{2,t}$ .  $\square$

**Claim 3.6.** *Let a 3-graph  $G$  be  $H \cup S_{2,t}$ -free, where  $H$  is a 3-graph with  $s$  vertices. Let  $v \in V(H)$ . If  $H \subseteq G - \{v\}$ , then  $\omega(G_v) \leq s + t$ .*

*Proof.* Note that  $G - \{v\}$  contains  $H$ . If  $\omega(G_v) \geq s + t + 1$ , then assume that a maximum clique of  $G_v$  has vertex set  $U_1$ . Since  $|U_1 \cap V(H)| \leq s$ ,  $|U_1 \setminus V(H)| \geq t + 1$  and the induced subgraph of  $G$  by  $U_1 \cup \{v\} \setminus V(H)$  contains an  $S_{2,t}$ . So  $H \cup S_{2,t} \subseteq G$ , a contradiction.  $\square$

**Claim 3.7.** *Let a 3-graph  $G$  be  $H \cup S_{2,t}$ -free, where  $H$  is a 3-graph with  $s$  vertices. If  $H \subseteq G - \{v_1\}$  and  $H \not\subseteq G - \{v_1, v_2\}$ , then  $\omega((G - \{v_2\})_{v_1}) \leq s + t - 1$ .*

*Proof.* Assume that  $\omega((G - \{v_2\})_{v_1}) \geq s + t$  and a maximum clique of  $(G - \{v_2\})_{v_1}$  has vertex set  $U_2$ . Since  $H \subseteq G - \{v_1\}$  and  $H \not\subseteq G - \{v_1, v_2\}$ , then  $v_2 \in V(H)$ . Therefore  $|U_2 \cap V(H)| \leq s - 1$ . So  $|U_2 \setminus V(H)| \geq t + 1$  and the induced subgraph of  $G$  by  $U_2 \cup \{v_1\} \setminus V(H)$  contains an  $S_{2,t}$ . Therefore,  $H \cup S_{2,t} \subseteq G$ .  $\square$

**Theorem 3.8.** [1,26]  $\pi(K_4^{3-}) \leq 0.2871, \pi(K_4^3) \leq 0.5615$ .

**3.2. The Disjoint Union of a  $\lambda$ -Perfect 3-Graph and  $S_{2,t}$**

We show that the disjoint union of a  $\lambda$ -perfect 3-graph and  $S_{2,t}$  is  $\lambda$ -perfect.

*Proof of Theorem 1.6.* Assume that  $H$  is  $\lambda$ -perfect on  $s \geq 3$  vertices. Note that  $H \cup S_{2,t}$  has  $s + t + 2$  vertices. By Remark 2.5(c), it is sufficient to show that if  $G$  is  $H \cup S_{2,t}$ -free dense 3-graph then  $\lambda(G) \leq \lambda(K_{s+t+1}^3)$ . Suppose on the contrary that  $\lambda(G) > \lambda(K_{s+t+1}^3)$ . Let  $\vec{x}$  be an optimal weight vector of  $G$ .

*Case 1.* There exists  $v \in V(G)$  with weight  $x_v$  such that  $G - \{v\}$  is  $H$ -free.

By Claim 3.1,  $x_v < 1 - \frac{\sqrt{(s+t)(s+t-1)}}{s+t+1}$ . By Claim 3.3 and that  $H$  is  $\lambda$ -perfect,

$$\lambda(G) \leq \frac{\lambda(K_{s-1}^3)(1-x_v)^3}{1-3x_v} = f(x_v). \tag{3}$$

By Remark 3.4,  $f(x_v)$  is increasing in  $[0, 1 - \frac{\sqrt{(s+t)(s+t-1)}}{s+t+1})$ , then

$$\begin{aligned} \lambda(G) &\leq f\left(1 - \frac{\sqrt{(s+t)(s+t-1)}}{s+t+1}\right) \\ &= \frac{1}{6} \frac{(s-2)(s-3)}{(s-1)^2} \frac{\binom{(s+t)(s+t-1)\sqrt{(s+t)(s+t-1)}}{(s+t+1)^3}}{\frac{3\sqrt{(s+t)(s+t-1)}}{s+t+1} - 2} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{6} \frac{(s+t)(s+t-1)}{(s+t+1)^2} \times \frac{(s-2)(s-3)}{(s-1)^2} \times \frac{\sqrt{(s+t)(s+t-1)}}{3\sqrt{(s+t)(s+t-1)} - 2(s+t+1)} \\
 &= \lambda(K_{s+t+1}^3) \times \frac{(s-2)(s-3)}{(s-1)^2} \times \frac{\sqrt{(s+t)(s+t-1)}}{3\sqrt{(s+t)(s+t-1)} - 2(s+t+1)}.
 \end{aligned}$$

To prove  $\lambda(G) \leq \lambda(K_{s+t+1}^3)$ , it is sufficient to prove that

$$\frac{(s-2)(s-3)}{(s-1)^2} \times \frac{\sqrt{(s+t)(s+t-1)}}{3\sqrt{(s+t)(s+t-1)} - 2(s+t+1)} \leq 1.$$

This is equivalent to

$$\frac{2(s-1)^2(s+t+1)}{2s^2 - s - 3} \leq \sqrt{(s+t)(s+t-1)}.$$

The above inequality is equivalent to

$$s+t - \frac{1}{2} - \frac{(6t-1)s - 10t - 1}{4s^2 - 2s - 6} \leq \sqrt{(s+t)(s+t-1)}.$$

By direct calculation, it holds for  $s = 3$  or  $4$  and  $t = 1$ . Since

$$s+t - \frac{1}{2} - \frac{(6t-1)s - 10t - 1}{4s^2 - 2s - 6} \leq s+t - \frac{1}{2} - \frac{1}{s+1},$$

and

$$s+t - \frac{1}{2} - \frac{1}{s+1} \leq \sqrt{(s+t)(s+t-1)}$$

holds for any  $s \geq 3$  and  $t \geq 2$  or  $s \geq 5$  and  $t = 1$ . So  $\lambda(G) \leq \lambda(K_{s+t+1}^3)$ , a contradiction.

*Case 2.* For any  $v \in V(G)$ ,  $H \subseteq G - \{v\}$ .

Since  $\lambda(G) > \lambda(K_{s+t+1}^3)$  and  $S_{2,s+t}$  is  $\lambda$ -perfect,  $S_{2,s+t} = \{v_1v_2b_1, v_1v_2b_2, \dots, v_1v_2b_{s+t}\} \subseteq G$ . By Claim 3.5,  $G - \{v_1, v_2\}$  is  $H$ -free. By Claim 3.6, we have

$$\omega(G_{v_1}) \leq s+t \quad \text{and} \quad \omega(G_{v_2}) \leq s+t.$$

By Claim 3.7, we have  $\omega((G - \{v_2\})_{v_1}) \leq s+t-1$  and  $\omega((G - \{v_1\})_{v_2}) \leq s+t-1$ .

Assume the weight of  $v_1$  and  $v_2$  are  $a_1$  and  $a_2$  respectively, and  $a_1 + a_2 = 2a$ . Since  $G - \{v_1, v_2\}$  is  $H$ -free and  $H$  is  $\lambda$ -perfect, the contribution of edges containing neither  $v_1$  nor  $v_2$  to  $\lambda(G, \vec{x})$  is at most  $\lambda(K_{s-1}^3)(1-2a)^3$ . Since  $\omega((G - \{v_2\})_{v_1}) \leq s+t-1$  and  $\omega((G - \{v_1\})_{v_2}) \leq s+t-1$ , by Theorem 1.2, the contribution of edges containing either  $v_1$  or  $v_2$  (but not both) to  $\lambda(G, \vec{x})$  is at most  $2 \times \frac{1}{2}a(1 - \frac{1}{s+t-1})(1-2a)^2$ . The contribution of edges containing both  $v_1$  and  $v_2$  to  $\lambda(G, \vec{x})$  is at most  $a^2(1-2a)$ . Therefore,

$$\begin{aligned}
 \lambda(G) &\leq \lambda(K_{s-1}^3)(1-2a)^3 + a^2(1-2a) + 2 \times \frac{1}{2}a \left(1 - \frac{1}{s+t-1}\right) (1-2a)^2 \\
 &\leq \lambda(K_{s+t-1}^3)(1-2a)^3 + a^2(1-2a) + 2 \times \frac{1}{2}a \left(1 - \frac{1}{s+t-1}\right) (1-2a)^2 \\
 &= \lambda \left( K_{s+t+1}^3, \left( a, a, \frac{1-2a}{s+t-1}, \dots, \frac{1-2a}{s+t-1} \right) \right) \leq \lambda(K_{s+t+1}^3).
 \end{aligned}$$

□

### 3.3. Disjoint Union of $F_5$ and $S_{2,t}$

We show that the disjoint union of  $F_5$  and  $S_{2,t}$  is  $\lambda$ -perfect.

*Proof of Theorem 1.8.* Let  $t \geq 2$ . Let  $G$  be a dense  $F_5 \cup S_{2,t}$ -free 3-graph on  $n$  vertices. By Remark 2.5(c), it is sufficient to show that  $\lambda(G) \leq \lambda(K_{t+6}^3)$ . Suppose on the contrary that  $\lambda(G) > \lambda(K_{t+6}^3)$ . By Claim 3.1, for any  $v \in V(G)$  with weight  $x_v$  we have

$$x_v < 1 - \frac{\sqrt{(t+5)(t+4)}}{t+6}.$$

*Case 1.*  $G - \{v\}$  is  $F_5$ -free for some  $v \in V(G)$ .

By the result in [39] that  $\pi_\lambda(F_5) = \frac{4}{9}$  and Claim 3.3, we have

$$\begin{aligned} \lambda(G) &\leq \frac{2}{27} \frac{(1-x_v)^3}{1-3x_v} = f(x_v) \\ &\leq f\left(1 - \frac{\sqrt{(t+5)(t+4)}}{t+6}\right) \\ &= \frac{1}{6} \frac{(t+4)(t+5)}{(t+6)^2} \times \frac{4}{9} \times \frac{\sqrt{(t+5)(t+4)}}{3\sqrt{(t+5)(t+4)} - 2(t+6)} \\ &= \lambda(K_{t+6}^3) \times \frac{4}{9} \times \frac{\sqrt{(t+5)(t+4)}}{3\sqrt{(t+5)(t+4)} - 2(t+6)}. \end{aligned}$$

It is sufficient to show that

$$\frac{4}{9} \times \frac{\sqrt{(t+5)(t+4)}}{3\sqrt{(t+5)(t+4)} - 2(t+6)} \leq 1.$$

It is equivalent to show that

$$18(t+6) \leq 23\sqrt{(t+5)(t+4)}.$$

It holds for  $t \geq 2$ .

*Case 2.*  $G - \{v\}$  contains  $F_5$  for any  $v \in V(G)$ .

Since  $\lambda(G) > \lambda(K_{t+6}^3)$  and  $S_{2,t+5}$  is  $\lambda$ -perfect,  $S_{2,t+5} = \{v_1v_2u_1, v_1v_2u_2, \dots, v_1v_2u_{t+5}\} \subseteq G$ . Assume the weight of  $v_1$  and  $v_2$  are  $a_1$  and  $a_2$ , respectively, and  $a_1 + a_2 = 2a$ . By Claim 3.5,  $G - \{v_1, v_2\}$  is  $F_5$ -free. So, the contribution of edges containing neither  $v_1$  nor  $v_2$  to  $\lambda(G, \vec{x})$  is at most  $\frac{\pi_\lambda(F_5)}{6}(1-2a)^3 = \frac{2}{27}(1-2a)^3$ . By Claim 3.7, we have  $\omega((G - \{v_2\})_{v_1}) \leq t+4$  and  $\omega((G - \{v_1\})_{v_2}) \leq t+4$ . By Theorem 1.2, the contribution of edges containing either  $v_1$  or  $v_2$  to  $\lambda(G, \vec{x})$  is at most  $2a \times \frac{1}{2}(1 - \frac{1}{t+4})(1-2a)^2$ . The contribution of edges containing both  $v_1$  and  $v_2$  to  $\lambda(G, \vec{x})$  is at most  $a^2(1-2a)$ . Therefore,

$$\begin{aligned} \lambda(G) &\leq \frac{2}{27}(1-2a)^3 + a^2(1-2a) + 2a \times \frac{1}{2} \left(1 - \frac{1}{t+4}\right) (1-2a)^2 \\ &\leq \lambda(K_{t+4}^3)(1-2a)^3 + a^2(1-2a) + 2a \times \frac{1}{2} \left(1 - \frac{1}{t+4}\right) (1-2a)^2 \\ &= \lambda\left(K_{t+6}^3, \left(a, a, \frac{1-2a}{t+4}, \dots, \frac{1-2a}{t+4}\right)\right) \end{aligned}$$

$$\leq \lambda(K_{t+6}^3).$$

□

The following lemma is prepared for the proof of Theorem 1.7.

**Lemma 3.9.** *Let  $G$  be a dense and  $F_5 \cup \{e\}$ -free 3-graph. If  $K_4^{3-} \cup K_4^{3-} \subseteq G$ , then  $\lambda(G) < \frac{5}{49} = \lambda(K_7^3)$ .*

*Proof.* Let  $K = K_4^{3-} \cup K_4^{3-}$  with vertex set  $\{a_1, a_2, b_1, b_2, \dots, b_6\}$  and edge set  $\{a_1b_1b_2, a_1b_1b_3, a_1b_2b_3, a_2b_4b_5, \{a_2b_4b_6, a_2b_5b_6\}\}$ . Let  $\vec{x}$  be an optimal weight vector. To simplify the notation, we simply write the weight of vertex  $a$  in  $\vec{x}$  as  $a$ . In other words, in the proof,  $a$  sometimes means vertex  $a$ , sometimes means the weight of vertex  $a$  in the optimal weight vector  $\vec{x}$ . They are distinguishable from the context. By Fact 2.2,

$$24\lambda(G) = \sum_{i=1}^2 \frac{\partial\lambda(G, \vec{x})}{\partial a_i} + \sum_{i=1}^6 \frac{\partial\lambda(G, \vec{x})}{\partial b_i}. \tag{1}$$

Note that  $\frac{\partial\lambda(G, \vec{x})}{\partial x_v} = \sum_{\{v_1v_2v\} \in E(G)} x_{v_1}x_{v_2}$ . Then we have the following claim.

- Claim 3.10.** (i) *For vertices  $c_i, c_j \in V(G) \setminus V(K)$ , the product  $c_i c_j$  appears at most twice in  $\sum_{i=1}^2 \frac{\partial\lambda(G, \vec{x})}{\partial a_i} + \sum_{i=1}^6 \frac{\partial\lambda(G, \vec{x})}{\partial b_i}$ .*  
 (ii) *For vertex  $c_i \in V(G) \setminus V(K)$ , the product  $c_i b_j$  appears at most twice in  $\sum_{i=1}^2 \frac{\partial\lambda(G, \vec{x})}{\partial a_i} + \sum_{i=1}^6 \frac{\partial\lambda(G, \vec{x})}{\partial b_i}$ .*  
 (iii) *For vertex  $c_i \in V(G) \setminus V(K)$ , the products  $c_i a_1, c_i a_2$  appear at most 7 times in  $\sum_{i=1}^2 \frac{\partial\lambda(G, \vec{x})}{\partial a_i} + \sum_{i=1}^6 \frac{\partial\lambda(G, \vec{x})}{\partial b_i}$ .*  
 (iv) *The product  $b_i b_j$  appears at most 3 times in  $\sum_{i=1}^2 \frac{\partial\lambda(G, \vec{x})}{\partial a_i} + \sum_{i=1}^6 \frac{\partial\lambda(G, \vec{x})}{\partial b_i}$ .*  
 (v) *The products  $a_i b_j, a_1 a_2$  appear at most 6 times in  $\sum_{i=1}^2 \frac{\partial\lambda(G, \vec{x})}{\partial a_i} + \sum_{i=1}^6 \frac{\partial\lambda(G, \vec{x})}{\partial b_i}$ .*

*Proof.* Let  $F$  be a copy of  $F_5 \cup \{e\}$ .

(i) We first show that  $c_i c_j$  can appear at most 1 time in  $\frac{\partial\lambda(G, \vec{x})}{\partial a_1} + \sum_{i=1}^3 \frac{\partial\lambda(G, \vec{x})}{\partial b_i}$ . If not, suppose that  $c_i c_j$  appears at least 2 times in it. If  $c_i c_j b_s, c_i c_j b_t \in E(G)$ , where  $b_s, b_t \in \{b_1, b_2, b_3\}$ , then  $\{a_1, b_s, b_t, c_i, c_j\}$  forms an  $F_5$ . So there is an  $F$  formed by  $F_5 \cup \{a_2 b_4 b_5\}$ . If  $c_i c_j b_s, c_i c_j a_1 \in E(G)$  where  $b_s \in \{b_1, b_2, b_3\}$ , then  $\{a_1, b_t, b_s, c_i, c_j\}$  forms an  $F_5$ , where  $b_t \in \{b_1, b_2, b_3\} \setminus \{b_s\}$ . So  $F_5 \cup \{a_2 b_4 b_5\} \subseteq G$ . Similarly,  $c_i c_j$  can appear at most 1 time in  $\frac{\partial\lambda(G, \vec{x})}{\partial a_2} + \sum_{i=4}^6 \frac{\partial\lambda(G, \vec{x})}{\partial b_i}$ . Therefore,  $c_i c_j$  appears at most twice in  $\sum_{i=1}^2 \frac{\partial\lambda(G, \vec{x})}{\partial a_i} + \sum_{i=1}^6 \frac{\partial\lambda(G, \vec{x})}{\partial b_i}$ .

(ii) Without loss of generality, let  $b_j = b_1$ . We first show that  $c_i b_1$  appears in  $\frac{\partial\lambda(G, \vec{x})}{\partial a_1} + \frac{\partial\lambda(G, \vec{x})}{\partial b_2} + \frac{\partial\lambda(G, \vec{x})}{\partial b_3}$  at most 1 time. Since if  $c_i b_1 b_2, c_i b_1 b_3 \in E(G)$ , then  $\{a_1, b_1, b_2, b_3, c_i\}$  forms an  $F_5$ , then there is an  $F$  formed by  $F_5 \cup \{a_2 b_4 b_5\}$ . If  $c_i b_1 b_2$  (or  $c_i b_1 b_3$ ) and  $c_i b_1 a_1 \in E(G)$ , then  $\{a_1, b_1, b_2, b_3, c_i\}$  forms an  $F_5$ , then there is an  $F$  formed by  $F_5 \cup \{a_2 b_4 b_5\}$ .

Next, we show that  $c_i b_1$  appears in  $\frac{\partial\lambda(G, \vec{x})}{\partial a_2} + \sum_{i=4}^6 \frac{\partial\lambda(G, \vec{x})}{\partial b_i}$  at most 1 time. If  $c_i b_1 b_s, c_i b_1 b_t \in E(G)$ , where  $b_s, b_t \in \{b_4, b_5, b_6\}$ , then  $\{a_2, b_1, b_s, b_t, c_i\}$

forms an  $F_5$ , and there is an  $F$  formed by  $F_5 \cup \{a_1b_2b_3\}$ . If  $c_i b_1 a_2, c_i b_1 b_s \in E(G)$ , where  $b_s \in \{b_4, b_5, b_6\}$ , then  $\{a_2, b_1, b_s, b_t, c_i\}$  forms an  $F_5$  where  $b_s, b_t \in \{b_4, b_5, b_6\}$ . So, there is an  $F$  formed by  $F_5 \cup \{a_1b_2b_3\}$ .

(iii) Let  $j = 1$  or  $2$ . Since  $c_i a_j \notin \frac{\partial \lambda(G, \vec{x})}{\partial a_j}$ , then  $c_i a_j$  appears at most 7 times.

(iv) If  $b_i \in \{b_1, b_2, b_3\}$  and  $b_j \in \{b_4, b_5, b_6\}$ , without loss of generality, let  $i = 1$  and  $j = 4$ . Note that  $b_1 b_2 b_4 \notin E(G)$ . Otherwise,  $\{a_1, b_1, b_2, b_3, b_4\}$  forms an  $F_5$  and there is an  $F$  formed by  $F_5 \cup \{a_2 b_5 b_6\}$ . Similarly,  $b_1 b_3 b_4, b_1 b_4 b_5, b_1 b_4 b_6, b_1 b_2 b_5, b_1 b_2 b_6 \notin E(G)$ . So at most  $b_1 b_4 a_1, b_1 b_4 a_2 \in E(G)$ . If  $b_i, b_j \in \{b_1, b_2, b_3\}$ , without loss of generality, let  $i = 1$  and  $j = 2$ . Then at most  $b_1 b_2 a_1, b_1 b_2 a_2, b_1 b_2 b_3 \in E(G)$ .

(v) Since  $a_i b_j$  can not appear in  $\frac{\partial \lambda(G, \vec{x})}{\partial a_j} + \frac{\partial \lambda(G, \vec{x})}{\partial b_i}$ , then  $a_i b_j$  can appear at most 6 times in  $\sum_{i=1}^2 \frac{\partial \lambda(G, \vec{x})}{\partial a_i} + \sum_{i=1}^6 \frac{\partial \lambda(G, \vec{x})}{\partial b_i}$ . The reason for  $a_1 a_2$  is similar. □

Combining (1) and Claim 3.10, we have

$$24\lambda(G) \leq 2 \sum_{c_i, c_j \in V(G) \setminus V(K)} c_i c_j + 2 \sum_{i \in [6], c_j \in V(G) \setminus V(K)} b_i c_j + 7 \sum_{i \in [2], c_j \in V(G) \setminus V(K)} a_i c_j + 3 \sum_{i \neq j \in [6]} b_i b_j + 6 \sum_{i=1}^2 \sum_{j=1}^6 a_i b_j + 6a_1 a_2.$$

Let  $\sum_i c_i = c, \sum_{i=1}^2 a_i = a, \sum_{i=1}^6 b_i = b$ , then  $a + b + c = 1$ . So

$$\begin{aligned} 24\lambda(G) &\leq 2 \left(\frac{c}{n-8}\right)^2 \binom{n-8}{2} + 2bc + 7ac + 3 \left(\frac{b}{6}\right)^2 \binom{6}{2} + 6ab + 6 \left(\frac{a}{2}\right)^2 \\ &\leq c^2 + 2bc + 7ac + \frac{3}{2}b^2 + 6ab + \frac{3}{2}a^2 \\ &= (1 - a - b)^2 + 2b(1 - a - b) + 7a(1 - a - b) + \frac{3}{2}b^2 + 6ab + \frac{3}{2}a^2 \\ &= -\frac{9}{2}a^2 + \frac{1}{2}b^2 - ab + 5a + 1 \\ &= -\frac{9}{2}a^2 + (5 - b)a + \frac{1}{2}b^2 + 1 \\ &\leq \frac{5}{9}b^2 - \frac{5}{9}b + \frac{43}{18} \quad \left(a = \frac{5 - b}{9}\right) \\ &\leq \frac{43}{18}. \end{aligned}$$

So

$$\lambda(G) \leq \frac{43}{18 \times 24} < \frac{5}{49} = \lambda(K_7^3).$$

□

*Proof of Theorem 1.7.* Let  $G$  be a dense  $F_5 \cup \{e\}$ -free 3-graph. By Remark 2.5(c), it is sufficient to show that  $\lambda(G) \leq \lambda(K_7^3)$ . Suppose on the contrary that

$\lambda(G) > \frac{5}{49} = \lambda(K_7^3)$ . For  $v \in V(G)$  with weight  $x_v$ , if  $G - \{v\}$  is  $K_4^3$ -free, then by Claim 3.3, Proposition 1.3 and Theorem 3.8, we have

$$\lambda(G) \leq \frac{\pi_\lambda(K_4^3)}{6} \frac{(1-x_v)^3}{1-3x_v} = \frac{\pi(K_4^3)}{6} \frac{(1-x_v)^3}{1-3x_v} \leq \frac{1}{18} \frac{(1-x_v)^3}{1-3x_v}.$$

By Claim 3.1, we have  $x_v < 1 - \frac{\sqrt{30}}{7} < \frac{1}{3}$ . By Remark 3.4,  $\frac{1}{18} \frac{(1-x_v)^3}{1-3x_v}$  is increasing in  $(0, \frac{1}{3})$ , so

$$\lambda(G) \leq \frac{1}{18} \frac{(1-x_v)^3}{1-3x_v} \Big|_{x_v=1-\frac{\sqrt{30}}{7}} < \frac{5}{49}.$$

Hence, we may assume that  $G - \{v\}$  still contains  $K_4^3$  for any vertex  $v$ . Note that  $\omega(G_v) \leq 6$ . Since if  $\omega(G_v) \geq 7$ , assume that  $U = \{u_1, u_2, \dots, u_7\}$  is a clique in  $G_v$ . Let  $K_4^3 \subseteq G - \{v\}$  have the vertex set  $W$ . Then  $|U - W| \geq 3$  and we can find a  $K_4^3$  in  $\{v_1\} \cup U - W$ . Therefore  $K_4^3 \cup K_4^3 \subseteq G$ . Note that  $\frac{\partial \lambda(G, \vec{x})}{\partial x_v} = \sum_{\{v_1 v_2 v\} \in E(G)} x_{v_1} x_{v_2}$  and the link graph  $G_v$  of  $v$  is a graph with  $\sum_{u \in V(G) \setminus \{v\}} = 1 - x_v$ . By Fact 2.2 and the theorem of Motzkin and Straus (Theorem 1.2), we have

$$3 \times \frac{5}{49} < 3\lambda(G) = \frac{\partial \lambda(G, \vec{x})}{\partial x_v} \leq \frac{1}{2} \left(1 - \frac{1}{6}\right) (1 - x_v)^2.$$

So  $x_v < \frac{1}{7}$  holds for any  $v$ .

If for  $v \in V(G)$  with weight  $x_v$ ,  $G - \{v\}$  is  $F_5$ -free, then by the result in [39] that  $\pi_\lambda(F_5) = \frac{4}{9}$  and Claim 3.3, we have

$$\lambda(G) \leq \frac{2}{27} \frac{(1-x_v)^3}{1-3x_v} \leq \frac{2}{27} \frac{(1-x_v)^3}{1-3x_v} \Big|_{x_v=\frac{1}{7}} < \frac{5}{49}.$$

So, we may assume that  $G - \{v\}$  still contains both  $F_5$  and  $K_4^3$  for any vertex  $v$ . Since  $\lambda(G) > \frac{5}{49}$  and  $S_{2,t}$  is  $\lambda$ -perfect, then  $S_{2,6} \subseteq G$ . Let  $S_{2,6} = \{v_1 v_2 b_1, v_1 v_2 b_2, \dots, v_1 v_2 b_6\}$ . By Claim 3.5,  $G - \{v_1, v_2\}$  is  $F_5$ -free. By Claim 3.7,  $\omega((G - \{v_2\})_{v_1}) \leq 5$  and  $\omega((G - \{v_1\})_{v_2}) \leq 5$ . Assume the weight of  $v_1$  and  $v_2$  are  $a_1$  and  $a_2$  respectively and  $a_1 + a_2 = 2a$ . By Claim 3.2,  $a_1 < \frac{1}{7}$  and  $a_2 < \frac{1}{7}$ . Therefore  $a < \frac{1}{7}$ . So

$$\begin{aligned} \lambda(G) &\leq a^2(1-2a) + \frac{2}{27}(1-2a)^3 + 2a \times \frac{1}{2} \times \left(1 - \frac{1}{5}\right) (1-2a)^2 \\ &= \frac{82}{135} a^3 - \frac{59}{45} a^2 + \frac{16}{45} a + \frac{2}{27} = f(a) \\ f'(a) &= \frac{82}{45} a^2 - \frac{118}{45} a + \frac{16}{45} > 0 \end{aligned}$$

if  $a \in [0, \frac{1}{7}]$ . So,  $f(a)$  is increasing in  $[0, \frac{1}{7}]$ . Then,

$$\lambda(G) \leq f(a) = f\left(\frac{1}{7}\right) < \frac{5}{49}.$$

□

### 3.4. The Disjoint Union of $K_4^{3-}$ and $S_{2,t}$

We show that the disjoint union of  $K_4^{3-}$  and  $S_{2,t}$  is  $\lambda$ -perfect.

*Proof of Theorem 1.9.* Let  $G$  be a dense  $K_4^{3-} \cup S_{2,t}$ -free 3-graph on  $n$  vertices. By Remark 2.5(c), it is sufficient to show that  $\lambda(G) \leq \lambda(K_{t+5}^3)$ . Suppose on the contrary that  $\lambda(G) > \lambda(K_{t+5}^3)$ . If there exists  $v \in V(G)$  with weight  $x_v$  such that  $G - \{v\}$  is  $S_{2,t}$ -free, then by Claim 3.1, Claim 3.3 and  $S_{2,t}$  is  $\lambda$ -perfect, we have

$$x_v < 1 - \frac{\sqrt{(t+3)(t+4)}}{t+5} \quad \text{and}$$

$$\lambda(G) \leq \frac{\pi_\lambda(S_{2,t})(1-x_v)^3}{6(1-3x_v)} = \frac{\lambda(K_{t+1}^3)(1-x_v)^3}{1-3x_v} = f(x_v).$$

Since  $f(x)$  is increasing in  $(0, \frac{1}{3})$ , then

$$\begin{aligned} \lambda(G) &\leq f\left(1 - \frac{\sqrt{(t+3)(t+4)}}{t+5}\right) \\ &= \frac{1}{6} \frac{t(t-1)}{(t+1)^2} \frac{\frac{(t+4)(t+3)\sqrt{(t+4)(t+3)}}{(t+5)^3}}{\frac{3\sqrt{(t+4)(t+3)}}{t+5} - 2} \\ &= \frac{1}{6} \frac{(t+4)(t+3)}{(t+5)^2} \frac{t(t-1)}{(t+1)^2} \frac{\sqrt{(t+4)(t+3)}}{3\sqrt{(t+4)(t+3)} - 2(t+5)} \\ &= \lambda(K_{t+5}^3) \frac{t(t-1)}{(t+1)^2} \frac{\sqrt{(t+4)(t+3)}}{3\sqrt{(t+4)(t+3)} - 2(t+5)}. \end{aligned}$$

So, it is sufficient to show that

$$\frac{t(t-1)}{(t+1)^2} \times \frac{\sqrt{(t+4)(t+3)}}{3\sqrt{(t+4)(t+3)} - 2(t+5)} \leq 1.$$

This is equivalent to show that

$$t + \frac{7}{2} - \frac{11t+1}{4t^2+14t+6} \leq \sqrt{(t+4)(t+3)}.$$

This is true since

$$\frac{11t+1}{4t^2+14t+6} \geq \frac{1}{t+1}, (t \geq 1)$$

and

$$t + \frac{7}{2} - \sqrt{(t+4)(t+3)} = \frac{(t + \frac{7}{2})^2 - (t+3)(t+4)}{t + \frac{7}{2} + \sqrt{(t+4)(t+3)}} < \frac{1}{4(t+1)} < \frac{1}{t+1}.$$

So, we may assume that  $S_{2,t} \in G - \{v\}$  for any  $v \in V(G)$ . By Claim 3.6,  $\omega(G_v) \leq t+4$ . Let  $x_v$  be the weight of  $v$  in an optimum vector. By Claim 3.2,  $x_v < \frac{1}{t+5}$ .

*Case 1.*  $G - \{v\}$  is  $K_4^{3-}$ -free for some  $v \in V(G)$ .

Then by Claim 3.3, Proposition 1.3 and Theorem 3.8, we have

$$\begin{aligned} \lambda(G) &\leq \frac{1}{18} \frac{(1-x_v)^3}{1-3x_v} = f(x_v) \quad \left(x_v < \frac{1}{t+5}\right) \\ &< f\left(\frac{1}{t+5}\right) \\ &= \frac{1}{18} \frac{(t+4)^3}{(t+2)(t+5)^2} \\ &< \frac{1}{6} \frac{(t+4)(t+3)}{(t+5)^2} = \lambda(K_{t+5}^3). \end{aligned}$$

Case 2.  $G - \{v\}$  contains  $K_4^{3-}$  for any  $v \in V(G)$ .

Since  $\lambda(G) > \lambda(K_{t+5}^3)$  and  $S_{2,t+4}$  is  $\lambda$ -perfect, then  $S_{2,t+4} = \{v_1v_2u_1, v_1v_2u_2, \dots, v_1v_2u_{t+4}\} \subseteq G$ . By Claim 3.5,  $G - \{v_1, v_2\}$  is  $K_4^{3-}$ -free. Since  $G - \{v_1\}$  contains  $K_4^{3-}$  and  $S_{2,t}$ , by Claim 3.7,  $\omega((G - \{v_2\})_{v_1}) \leq t + 3$  and  $\omega((G - \{v_1\})_{v_2}) \leq t + 3$ . Assume the weight of  $v_1$  and  $v_2$  are  $a_1$  and  $a_2$ , respectively. Let  $a_1 + a_2 = 2a$ . Then,

$$\begin{aligned} \lambda(G) &\leq \frac{1}{18}(1-2a)^3 + a^2(1-2a) + 2a \times \frac{1}{2} \left(1 - \frac{1}{t+3}\right) (1-2a)^2 \\ &\leq \lambda(K_{t+3}^3)(1-2a)^3 + a^2(1-2a) + 2a \times \frac{1}{2} \left(1 - \frac{1}{t+3}\right) (1-2a)^2 \\ &< \lambda\left(K_{t+5}^3, \left(a, a, \frac{1-2a}{t+3}, \dots, \frac{1-2a}{t+3}\right)\right) \\ &\leq \lambda(K_{t+5}^3). \end{aligned}$$

□

### 3.5. General Case

We show that  $H \cup S_{2,t}$  is  $\lambda$ -perfect if  $H$  is a 3-graph with  $s$  vertices and  $t \geq \frac{3}{2}s^2 - \frac{1}{2}s + 4$ .

*Proof of Theorem 1.10.* Let  $G$  be a dense  $H \cup S_{2,t}$ -free 3-graph on  $n$  vertices. By Remark 2.5(c), it is sufficient to show that  $\lambda(G) \leq \lambda(K_{s+t+1}^3)$ . Suppose on the contrary that  $\lambda(G) > \lambda(K_{s+t+1}^3)$ . If there exists  $v \in V(G)$  with weight  $x_v$  such that  $G - \{v\}$  is  $H$ -free, then by Claims 3.1 and 3.3, we have

$$\lambda(G) \leq \frac{1}{6} \pi_\lambda(H) \frac{(1-x_v)^3}{1-3x_v}, \quad \text{and} \quad x_v < 1 - \frac{\sqrt{(s+t)(s+t-1)}}{s+t+1}.$$

Since  $\pi_\lambda(H) \leq \pi_\lambda(K_s^3) \leq 1 - \frac{2}{(s-1)(s-2)}$  (see [17]), then

$$\begin{aligned} \lambda(G) &\leq \frac{1}{6} \left(1 - \frac{2}{(s-1)(s-2)}\right) \frac{(1-x_v)^3}{1-3x_v} \Big|_{x_v=1-\frac{\sqrt{(s+t)(s+t-1)}}{s+t+1}} \\ &\leq \lambda(K_{s+t+1}^3) \frac{s^2-3s}{(s-1)(s-2)} \times \frac{\sqrt{(s+t)(s+t-1)}}{3\sqrt{(s+t)(s+t-1)}-2}. \end{aligned}$$



So, it is sufficient to show that

$$\frac{s^2 - 3s}{(s - 1)(s - 2)} \times \frac{\sqrt{(s + t)(s + t - 1)}}{3\sqrt{(s + t)(s + t - 1)} - 2} \leq 1.$$

This is equivalent to show that

$$1 - \frac{1}{s^2 - 3s + 3} \leq \sqrt{(s + t)(s + t - 1)}.$$

It holds for  $s \geq 3$  and  $t \geq 1$ .

So we can assume that  $H \subseteq G - \{v\}$  for any  $v \in V(G)$ . By Claims 3.6 and 3.2, we have  $\omega(G_v) \leq s + t$  and

$$x_v < \frac{1}{s + t + 1}.$$

Since  $\lambda(G) > \lambda(K_{s+t+1}^3)$  and  $S_{2,s+t}$  is  $\lambda$ -perfect, then  $S_{2,s+t} = \{v_1 v_2 u_1, v_1 v_2 u_2, \dots, v_1 v_2 u_{s+t}\} \subseteq G$ . By Claim 3.5,  $G - \{v_1, v_2\}$  is  $H$ -free. Applying Claim 3.7, we have  $\omega((G - \{v_2\})_{v_1}) \leq s + t - 1$  and  $\omega((G - \{v_1\})_{v_2}) \leq s + t - 1$ . Since  $t \geq \frac{3}{2}s^2 - \frac{11}{2}s + 4$ , then

$$6\lambda(K_{s+t-1}^3) = \frac{(s + t - 2)(s + t - 3)}{(s + t - 1)^2} \geq \frac{s^2 - 3s}{s^2 - 3s + 2} > \pi_\lambda(H).$$

Assume the weight of  $v_1$  and  $v_2$  are  $a_1$  and  $a_2$ , respectively, and  $a_1 + a_2 = 2a$ , then

$$\begin{aligned} \lambda(G) &\leq \frac{1}{6}\pi_\lambda(H)(1 - 2a)^3 + a^2(1 - 2a) + 2a \times \frac{1}{2} \left(1 - \frac{1}{s + t - 1}\right) (1 - 2a)^2 \\ &\leq \frac{1}{6} \frac{s^2 - 3s}{(s - 1)(s - 2)} (1 - 2a)^3 + a^2(1 - 2a) + 2a \times \frac{1}{2} \left(1 - \frac{1}{s + t - 1}\right) (1 - 2a)^2 \\ &\leq \lambda(K_{s+t-1}^3)(1 - 2a)^3 + a^2(1 - 2a) + 2a \times \frac{1}{2} \left(1 - \frac{1}{s + t - 1}\right) (1 - 2a)^2 \\ &= \lambda \left( K_{s+t+1}^3, \left( a, a, \frac{1 - 2a}{s + t - 1}, \dots, \frac{1 - 2a}{s + t - 1} \right) \right) \\ &\leq \lambda(K_{s+t+1}^3). \end{aligned}$$

□

### 4. Irrational Lagrangian Densities

In this section, we prove Theorem 1.11.

*Fact 4.1.* Let  $V(S_2(n)) = [n]$  and  $E(S_2(n)) = \{12i | i \in [n] \setminus \{1, 2\}\} \cup \{ijk | i \in [2], j, k \in [n] \setminus \{1, 2\}\}$ , then  $\lambda(S_2(n)) = \frac{\sqrt[3]{3}}{18}$  as  $n \rightarrow \infty$ .

*Proof.* Let  $\vec{x} = \{x_1, x_2, \dots, x_n\}$  be an optimal weight vector on  $S_2(n)$ . By Fact 2.3, we may assume that  $x_1 = x_2, x_3 = x_4 = \dots = x_n$ . Let  $x_1 = a$ , then  $x_3 = \frac{1-2a}{n-2}$  and  $a < \frac{1}{2}$ . So,

$$\lambda(S_2(n)) = \max_{a \in (0,1)} \left\{ a^2(1 - 2a) + 2a \left( \frac{1 - 2a}{n - 2} \right)^2 \binom{n - 2}{2} \right\}$$

$$\begin{aligned} & \stackrel{n \rightarrow \infty}{=} \max_{a \in (0,1)} \{2a^3 - 3a^2 + a\} \\ & = \frac{\sqrt{3}}{18}, \end{aligned}$$

when  $a = \frac{3-\sqrt{3}}{6}$ . □

*Proof of Theorem 1.11.* Since  $S_2(n)$  is  $\mathcal{P}$ -free, then by Fact 4.1 we have  $\pi_\lambda(\mathcal{P}) \geq 3!\lambda(S_2(n)) = \frac{\sqrt{3}}{3}$ . For the upper bound, we assume that  $G$  is a  $\mathcal{P}$ -free dense 3-graph with vertex set  $[n]$ . It is sufficient to show that  $\lambda(G) \leq \frac{\sqrt{3}}{18}$ . Suppose on the contrary that  $\lambda(G) > \frac{\sqrt{3}}{18}$ , then  $n > 6$ , since  $\lambda(K_6^3) = \frac{5}{54} < \frac{\sqrt{3}}{18}$ . By Proposition 1.3 and Theorem 3.8,  $\pi_\lambda(K_4^3) = \pi(K_4^3) < \frac{\sqrt{3}}{3}$ . So  $K_4^3 \subseteq G$ , without loss of generality, let  $\{1, 2, 3, 4\}$  form a  $K_4^3$ . Let  $P = \{123, 124, 134, 234, 561, 562\}$ .

**Lemma 4.2.**  $P \not\subseteq G$ .

*Proof of Lemma 4.2.* Suppose that  $P \subseteq G$ . We claim that for any  $u_1, u_2 \in V(G) \setminus \{1, 2, 3, 4, 5, 6\}$ ,  $N(u_1, u_2) \subseteq \{1, 2\}$ . If there exists  $u_3 \in V(G) \setminus \{1, 2, 3, 4, u_1, u_2\}$  such that  $u_1 u_2 u_3 \in E(G)$ , then  $\{1, 2, 3, 4, u_1, u_2, u_3\}$  forms a  $P_1$ . If there exist  $u_3 \in \{3, 4\}$  such that  $u_1 u_2 u_3 \in E(G)$ , then  $\{1, 2, 3, 4, 5, 6, u_1, u_2, u_3\}$  forms a  $P_2$ . And  $u_1 34 \notin E(G)$  since otherwise  $\{1, 2, 3, 4, 5, 6, u_1\}$  forms a  $P_3$ . And  $u_1 35 \notin E(G)$  since otherwise  $\{1, 2, 3, 4, 5, 6, u_1\}$  forms a  $P_4$ , similarly  $u_1 36, u_1 45, u_1 46 \notin E(G)$ . Since  $G$  is  $P_1$ -free, every edge contains a vertex from  $\{1, 2, 3, 4\}$ . So we have shown that  $G \subseteq S_2(n)$ . By Fact 4.1,  $\lambda(G) \leq \frac{\sqrt{3}}{18}$ , a contradiction. □

For any  $v \in V(G)$  with weight  $x_v$ , we claim that  $\omega(G_v) \geq 3$ . Otherwise by Theorem 1.2 and Fact 2.2, we have

$$3\lambda(G) = \frac{\partial \lambda(G, \vec{x})}{\partial x_v} \leq \frac{1}{2} \left(1 - \frac{1}{2}\right) (1 - x_v)^2 \leq \frac{1}{4} < \frac{\sqrt{3}}{6},$$

a contradiction. Let  $x \in V(G) \setminus \{1, 2, 3, 4\}$ . Since  $\omega(G_x) \geq 3$ , then  $x$  is contained in a  $K_4^{3-}$ , denoted by  $K_x$ , and  $d_{K_x}(x) = 3$ . Since  $G$  is  $P_1$ -free,  $|K_x \cap \{1, 2, 3, 4\}| = 2$  or  $3$ .

*Case 1.* There exists  $x \in V(G) \setminus \{1, 2, 3, 4\}$  such that  $|K_x \cap \{1, 2, 3, 4\}| = 2$ .

Without loss of generality, let  $K_x = \{1, 2, x, y\}$ . Then  $\{1, 2, 3, 4, x, y\}$  forms a  $P$ , by Lemma 4.2, a contradiction.

*Case 2.* For any  $x \in V(G) \setminus \{1, 2, 3, 4\}$ ,  $|K_x \cap \{1, 2, 3, 4\}| = 3$ .

Without loss of generality, let  $K_x = \{1, 2, 3, x\}$  for some  $x \in V(G) \setminus \{1, 2, 3, 4\}$ . Then  $K_x$  forms a  $K_4^3$ . We claim that for any  $y \in V(G) \setminus \{1, 2, 3, 4, x\}$ ,  $K_y = \{1, 2, 3, y\}$ . If not, without loss of generality, let  $K_y = \{2, 3, 4, y\}$ . Then,  $K_x \cup \{y42, y43\}$  forms a  $P$ , by Lemma 4.2, a contradiction. We claim that for any  $v_1, v_2, v_3 \in V(G) \setminus \{1, 2, 3\}$ ,  $v_1 v_2 v_3 \notin E(G)$  since otherwise there exist a  $P_1$ . And for any pair  $v_1, v_2 \in V(G) \setminus \{1, 2, 3\}$ , we claim that  $|N(v_1, v_2) \cap \{1, 2, 3\}| = 1$ . If  $N(v_1, v_2) \cap \{1, 2, 3\} \geq 2$ , take  $v_3 \in V(G) \setminus \{1, 2, 3, v_1, v_2\}$ , then  $\{1, 2, 3, v_1, v_2, v_3\}$  forms a  $P$ , by Lemma 4.2, a contradiction.

Assume the weight of  $i \in V(G)$  is  $x_i$  and  $a = x_1 + x_2 + x_3$ . Consider  $\sum_{k=1}^3 \frac{\partial \lambda}{\partial x_k}$ . From the previous discussion we know that  $x_i x_j$  only appears once in  $\sum_{k=1}^3 \frac{\partial \lambda}{\partial x_k}$  for  $4 \leq i, j \leq n$ .  
 So

$$\begin{aligned} 9\lambda(G) &= \sum_{k=1}^3 \frac{\partial \lambda(G, \vec{x})}{\partial x_k} \\ &\leq \left( \sum_{i,j \in V(G) \setminus \{1,2,3\}} x_i x_j \right) + x_1 x_2 + x_1 x_3 + x_2 x_3 + 2(x_1 + x_2 + x_3) \sum_{j \geq 4} x_j \\ &= \sum_{i,j} x_i x_j + (x_1 + x_2 + x_3) \sum_{j \geq 4} x_j \\ &\leq \frac{1}{2} + a(1 - a) \leq \frac{3}{4}. \end{aligned}$$

Then,  $\lambda(G) \leq \frac{1}{12} < \frac{\sqrt{3}}{18}$ , a contradiction. □

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