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Relation Between f-Vectors and d-Vectors in Cluster Algebras of Finite Type or Rank 2

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Abstract. We study f-vectors, which are the maximal degree vectors of F-polynomials in cluster algebra theory. For a cluster algebra of finite type, we find that positive f-vectors correspond with d-vectors, which are exponent vectors of denominators of cluster variables. Furthermore, using this correspondence and properties of d-vectors, we prove that cluster variables in a cluster are uniquely determined by their f-vectors when the cluster algebra is of finite type or rank 2.

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1. Introduction and Main Theorems

Cluster algebras are commutative subalgebras of the rational function fields. They are generated by *cluster variables* in *clusters*, and cluster variables are obtained by applying *mutations* repeatedly starting from the initial cluster. They are defined by [12] to study the canonical basis or total positivity. Today, we know that they are related to many mathematical subjects. For example, by regarding a mutation as quiver transformation or triangulation of a marked surface, a structure of cluster algebras appears in representation theory of quivers [1,2] or higher Teichmüller theory [6,11]. Also, by considering a mutation of the Markov quiver, a new combinatorial approach to solve the Unicity Conjecture about Markov numbers was given in number theory [5,25].

Cluster algebras of *finite type* and *rank* 2 are important classes in cluster algebra theory. Cluster algebras of finite type have finitely many cluster variables. They are introduced by [12] and are classified completely by [13]. They have connections with Dynkin diagrams or (real) root systems in Lie algebras, and they are applied to the logarithm identities, T, Y-systems and so on [17–19,22].

Cluster algebras of rank 2 have two cluster variables in every cluster. Since they have the simplest structures in cluster algebras with infinitely many cluster variables, they are studied to understand other classes [20,21,27].

The main topic of this paper is a relation between *f-vectors* and *d-vectors* in cluster algebras of finite type. Here, f-vectors are introduced in [10] as the maximal degree vectors of F-polynomials, and F-polynomials was introduced in [14]. On the other hand, d-vectors are exponent vectors of monomials of denominators of cluster variables. They are introduced by [13,14]. Though definitions of these two vectors are independent of each other, previous works [8,14,26] suggested that they have some similar properties in cluster algebras of finite type or rank 2. In this paper, we give the following simple relation between f-vectors and d-vectors (Theorem 1.8):

$$\mathbf{f}_{i;t} = [\mathbf{d}_{i;t}]_+.$$

This relation means that they are the same vectors in almost every situation. By this identification, we can study properties of f-vectors, which are not well known yet, by properties of d-vectors. In this paper, we give a partial solution of the Uniqueness Conjecture [16, Conjecture 4.4], that is, cluster variables in a cluster are uniquely determined by their f-vectors in cluster algebras of finite type or rank 2.

The organization of the paper is as follows: in the rest of this section, we define mutations, cluster algebras, d-vectors and f-vectors. After that, we describe the main theorem (Theorem 1.8) in the paper, that is, a simple relation between d-vectors and f-vectors in cluster algebras of finite type. Furthermore, we describe an application of the main theorem to the Uniqueness Conjecture (Theorem 1.11). In Sect. 2, we give a proof of Theorem 1.8. In Sect. 3, we give a proof of Theorem 1.11 (1) using Theorem 1.8 and some properties of d-vectors. In Sect. 4, we give a proof of Theorem 1.11 (2) using a description of entries of d-vectors. In Sect. 5, we generalize the cluster expansion formula given by [20] to the principal coefficients version along [21], and we give the restoration formula of the F-polynomials from the f-vectors.

1.1. Seed Mutations and Cluster Algebras

We start by recalling definitions of seed mutations and cluster patterns according to [14]. A *semifield* $\mathbb{P}$ is an abelian multiplicative group equipped with an addition $\oplus$ which is distributive over the multiplication. We particularly make use of the following two semifields.

Let $\mathbb{Q}_{\mathrm{sf}}(u_1,\dots,u_\ell)$ be the set of rational functions in $u_1,\dots,u_\ell$ which have subtraction-free expressions. Then, $\mathbb{Q}_{\mathrm{sf}}(u_1,\dots,u_\ell)$ is a semifield by the usual multiplication and addition. We call it the *universal semifield* of $u_1,\dots,u_\ell$ [14, Definition 2.1].

Let $\mathrm{Trop}(u_1,\dots,u_\ell)$ be the abelian multiplicative group freely generated by the elements $u_1,\dots,u_\ell$. Then, $\mathrm{Trop}(u_1,$

$\dots, u_\ell)$ is a semifield by the following addition:

$$\prod_{j=1}^{\ell} u_j^{a_j} \oplus \prod_{j=1}^{\ell} u_j^{b_j} = \prod_{j=1}^{\ell} u_j^{\min(a_j,b_j)}. \tag{1.1}$$

We call it the *tropical semifield* of $u_1, \dots, u_\ell$ [14, Definition 2.2]. For any semifield $\mathbb{P}$ and $p_1, \dots, p_\ell \in \mathbb{P}$, there exists a unique semifield homomorphism π such that

$$\begin{aligned} \pi : \mathbb{Q}_{\mathrm{sf}}(y_1, \dots, y_\ell) &\longrightarrow \mathbb{P} \\ y_i &\longmapsto p_i. \end{aligned} \tag{1.2}$$

For $F(y_1, \dots, y_\ell) \in \mathbb{Q}_{\mathrm{sf}}(y_1, \dots, y_\ell)$, we denote

$$F|_{\mathbb{P}}(p_1, \dots, p_\ell) := \pi(F(y_1, \dots, y_\ell)). \tag{1.3}$$

and we call it the *evaluation* of F at $p_1, \dots, p_\ell$. We fix a positive integer n and a semifield $\mathbb{P}$. Let $\mathbb{Z}\mathbb{P}$ be the group ring of $\mathbb{P}$ as a multiplicative group. Since $\mathbb{Z}\mathbb{P}$ is a domain [12, Section 5], its total quotient ring is a field $\mathbb{Q}(\mathbb{P})$. Let $\mathcal{F}$ be the field of the rational functions in n indeterminates with coefficients in $\mathbb{Q}(\mathbb{P})$.

A *labeled seed with coefficients in* $\mathbb{P}$ is a triplet $(\mathbf{x}, \mathbf{y}, B)$, where

- $\mathbf{x} = (x_1, \dots, x_n)$ is an n-tuple of elements of $\mathcal{F}$ forming a free generating set of $\mathcal{F}$.
- $\mathbf{y} = (y_1, \dots, y_n)$ is an n-tuple of elements of $\mathbb{P}$.
- $B = (b_{ij})$ is an $n \times n$ integer matrix which is *skew-symmetrizable*, that is, there exists a positive integer diagonal matrix S such that SB is skew-symmetric. Also, we call S a *skew-symmetrizer* of B.

We say that $\mathbf{x}$ is a *cluster* and refer to x_i, y_i and B as the *cluster variables*, the *coefficients* and the *exchange matrix*, respectively.

Throughout the paper, for an integer b, we use the notation $[b]_+ = \max(b, 0)$. We note that

$$b = [b]_+ - [-b]_+. \tag{1.4}$$

Let $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed with coefficients in $\mathbb{P}$, and let $k \in \{1, \dots, n\}$. The *seed mutation* μ_k *in direction* k transforms $(\mathbf{x}, \mathbf{y}, B)$ into another labeled seed $\mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}', \mathbf{y}', B')$ defined as follows:

- The entries of $B' = (b'_{ij})$ are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik} [-b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{1.5}$$

- The coefficients $\mathbf{y}' = (y'_1, \dots, y'_n)$ are given by

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{otherwise.} \end{cases} \tag{1.6}$$

- The cluster variables $\mathbf{x}' = (x'_1, \dots, x'_n)$ are given by

$$x'_j = \begin{cases} \dfrac{y_k \prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod_{i=1}^{n} x_i^{[-b_{ik}]_+}}{(y_k \oplus 1)x_k} & \text{if } j = k, \\ x_j & \text{otherwise.} \end{cases} \tag{1.7}$$

Let $\mathbb{T}_n$ be the *n-regular tree* whose edges are labeled by the numbers $1, \dots, n$ such that the n edges emanating from each vertex have different labels. We write $t \overset{k}{\text{———}} t'$ to indicate that vertices $t, t' \in \mathbb{T}_n$ are joined by an edge labeled by k. We fix a vertex $t_0 \in \mathbb{T}_n$, which is called the *rooted vertex*. A *cluster pattern with coefficients in* $\mathbb{P}$ is an assignment of every labeled seed $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ with coefficients in $\mathbb{P}$ to every vertex $t \in \mathbb{T}_n$ such that the labeled seeds Σ_t and $\Sigma_{t'}$ assigned to the endpoints of any edge $t \overset{k}{\text{———}} t'$ are obtained from each other by a seed mutation in direction k. Elements of Σ_t are denoted as follows:

$$\mathbf{x}_t = (x_{1;t}, \dots, x_{n;t}),\ \mathbf{y}_t = (y_{1;t}, \dots, y_{n;t}),\ B_t = (b_{ij;t}). \tag{1.8}$$

In particular, at t_0, we denote

$$\begin{aligned} \mathbf{x} = \mathbf{x}_{t_0} = (x_1, \dots, x_n),\ \mathbf{y} = \mathbf{y}_{t_0} = (y_1, \dots, y_n),& \\ B = B_{t_0} = (b_{ij}).& \end{aligned} \tag{1.9}$$

When we want to emphasize that the initial matrix is B, we denote by Σ_t^B a labeled seed associated with a vertex t. For seeds Σ_t and Σ_s in a cluster pattern, if there exists mutation sequence μ such that $\Sigma_s = \mu(\Sigma_t)$, then we say that Σ_t is *mutation equivalent* to Σ_s.

Definition 1.1. A *cluster algebra* $\mathcal{A}$ associated with a cluster pattern $v \mapsto \Sigma_v$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by $\{x_{i;t}\}_{1 \le i \le n, t \in \mathbb{T}_n}$.

The degree n of the regular tree $\mathbb{T}_n$ is called the *rank* of $\mathcal{A}$, and $\mathcal{F}$ is the *ambient field* of $\mathcal{A}$.

We also denote by $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ a cluster algebra with the initial seed $(\mathbf{x}, \mathbf{y}, B)$.

Example 1.2. We give an example of cluster algebras. This example is based on [14, Example 2.10] (but it is different from [14] with respect to the way of labeling edges). Let $n = 2$, and we consider a tree $\mathbb{T}_2$ whose edges are labeled as follows:

$$\cdots \overset{2}{\text{——}} t_0 \overset{1}{\text{——}} t_1 \overset{2}{\text{——}} t_2 \overset{1}{\text{——}} t_3 \overset{2}{\text{——}} t_4 \overset{1}{\text{——}} t_5 \overset{2}{\text{——}} \cdots. \tag{1.10}$$

Let $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ be the initial exchange matrix at t_0. Then coefficients and cluster variables are given by Table 1.

TABLE 1. Coefficients and cluster variables in type A_2

t	$\mathbf{y}_t$		$\mathbf{x}_t$	
0	y_1	y_2	x_1	x_2
1	$\frac{1}{y_1}$	$\frac{y_1 y_2}{y_1 \oplus 1}$	$\frac{x_2 + y_1}{(y_1 \oplus 1)x_1}$	x_2
2	$\frac{y_2}{y_1 y_2 \oplus y_1 \oplus 1}$	$\frac{y_1 \oplus 1}{y_1 y_2}$	$\frac{x_2 + y_1}{(y_1 \oplus 1)x_1}$	$\frac{x_1 y_1 y_2 + y_1 + x_2}{(y_1 y_2 \oplus y_1 \oplus 1)x_1 x_2}$
3	$\frac{y_1 y_2 \oplus y_1 \oplus 1}{y_2}$	$\frac{1}{y_1(y_2 \oplus 1)}$	$\frac{x_1 y_2 + 1}{(y_2 \oplus 1)x_2}$	$\frac{x_1 y_1 y_2 + y_1 + x_2}{(y_1 y_2 \oplus y_1 \oplus 1)x_1 x_2}$
4	$\frac{1}{y_2}$	$y_1(y_2 \oplus 1)$	$\frac{x_1 y_2 + 1}{(y_2 \oplus 1)x_2}$	x_1
5	y_2	y_1	x_2	x_1

Therefore, we have

$$\begin{aligned}&\mathcal{A}(\mathbf{x},\mathbf{y},B)\\&=\mathbb{ZP}\left[x_1,x_2,\frac{x_2+y_1}{(y_1\oplus 1)x_1},\frac{x_1y_1y_2+y_1+x_2}{(y_1y_2\oplus y_1\oplus 1)x_1x_2},\frac{x_1y_2+1}{(y_2\oplus 1)x_2}\right].\end{aligned}$$

Next, to define classes of cluster algebras which we deal with in this paper, we define non-labeled seeds according to [14]. For a cluster pattern $v \mapsto \Sigma_v$, we introduce the following equivalence relations of labeled seeds: we say that

$$\begin{aligned}\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t), \quad \mathbf{x}_t &= (x_{1;t}, \ldots, x_{n;t}),\\ \mathbf{y}_t &= (y_{1;t}, \ldots, y_{n;t}), \quad B_t = (b_{ij;t})\end{aligned}$$

and

$$\begin{aligned}\Sigma_s = (\mathbf{x}_s, \mathbf{y}_s, B_s), \quad \mathbf{x}_s &= (x_{1;s}, \ldots, x_{n;s}),\\ \mathbf{y}_s &= (y_{1;s}, \ldots, y_{n;s}), \quad B_s = (b_{ij;s})\end{aligned}$$

are *equivalent* if there exists a permutation σ of indices 1, $\ldots, n$ such that

$$\begin{aligned}x_{i;s} &= x_{\sigma(i);t}, \quad y_{j;s} = y_{\sigma(j);t},\\ b_{ij;s} &= b_{\sigma(i),\sigma(j);t}\end{aligned}$$

for all i and j. We denote by $[\Sigma]$ the equivalent classes represented by a labeled seed Σ and call it *non-labeled seed*. Also, We define a *(non-labeled) clusters* $[\mathbf{x}]$ as the set ignored the order of a labeled cluster. Abusing notation, we abbreviate $[\mathbf{x}]$ to $\mathbf{x}$.

Definition 1.3. The *exchange graph* of a cluster algebra is the regular connected graph whose vertices are non-labeled seeds of the cluster pattern and whose edges connect non-labeled seeds related by a single mutation.

Using the exchange graph, we define cluster algebras of finite type.

Definition 1.4. A cluster algebra $\mathcal{A}$ is of *finite type* if the exchange graph of $\mathcal{A}$ is a finite graph.

We say that B is *bipartite* if there is a function $\varepsilon : \{1, \ldots, n\} \to \{1, -1\}$ such that for all i and j,

$$b'_{ij} > 0 \Rightarrow \begin{cases}\varepsilon(i) = 1,\\ \varepsilon(j) = -1.\end{cases} \tag{1.11}$$

.

For an exchange matrix B, we define $A(B) = (a_{ij})$ as

$$a_{ij} = \begin{cases}2 & \text{if } i = j;\\ -|b'_{ij}| & \text{if } i \neq j.\end{cases}$$

If $A(B)$ is a Cartan matrix, then we say that B is of *finite Cartan type*.

Remark 1.5. If $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ is of finite type, then the initial matrix B is mutation equivalent to a bipartite matrix B'. Furthermore, by permuting their indices appropriately, we can choose B' as one of finite Cartan type (see [13, Theorem 1.8,Theorem 7.1]). If the initial matrix B of $\mathcal{A}$ is mutation equivalent to B' which is finite Cartan X_n type, then there exists a bijection between almost positive roots of X_n type and cluster variables of $\mathcal{A}$ (see [13, Theorem 1.9]).

1.2. *d*-Vectors and *f*-Vectors

In this section, we define d-vectors and f-vectors. First, we define d-vectors according to [13,14].

Let $\mathcal{A}$ be a cluster algebra. By the *Laurent phenomenon* [14, Theorem 3.5], every cluster variable $x_{i;t} \in \mathcal{A}$ can be uniquely written as

$$x_{i;t} = \frac{N_{i;t}(x_1, \dots, x_n)}{x_1^{d_{1i;t}} \cdots x_n^{d_{ni;t}}}, \quad d_{ki;t} \in \mathbb{Z}, \tag{1.12}$$

where $N_{i;t}(x_1, \dots, x_n)$ is a polynomial with coefficients in $\mathbb{ZP}$ which is not divisible by any initial cluster variable $x_i \in \mathbf{x}$. We define the *d-vector* $\mathbf{d}_{j;t}$ as the degree vector of $x_{j;t}$, that is,

$$\mathbf{d}_{i;t}^{B;t_0} = \mathbf{d}_{i;t} = \begin{bmatrix} d_{1i;t} \\ \vdots \\ d_{ni;t} \end{bmatrix}. \tag{1.13}$$

We define a *D-matrix* $D_t^{B;t_0}$ as

$$D_t^{B;t_0} {:=} (\mathbf{d}_{1;t}, \dots, \mathbf{d}_{n;t}). \tag{1.14}$$

We remark that $\mathbf{d}_{i;t}$ is independent of the choice of $\mathbb{P}$ (see [14, Section 7]). Therefore, in a cluster algebra $\mathcal{A}$, if $x_{i;t} = x_{j;s}$, then we have $\mathbf{d}_{i;t} = \mathbf{d}_{j;s}$. Moreover, d-vectors are also given by the following recursion: for any $i \in \{1, \dots, n\}$,

$$\mathbf{d}_{i;t_0} = -\mathbf{e}_i,$$

and for any $t \overset{k}{\frac{\quad\quad}{}} t'$,

$$\mathbf{d}_{i;t'} = \begin{cases} \mathbf{d}_{i;t} & \text{if } i \neq k; \\ -\mathbf{d}_{k;t} + \max\left(\sum_{j=1}^{n} [b_{jk;t}]_+ \mathbf{d}_{j;t}, \right. & \\ \qquad\qquad \left. + \sum_{j=1}^{n} [-b_{jk;t}]_+ \mathbf{d}_{j;t} \right) & \text{if } i = k, \end{cases} \tag{1.15}$$

where $\mathbf{e}_i$ is a standard basis element and the operation max on vectors are performed component-wise. By this way of definition, since d-vectors depend only on exchange matrices, we can regard d-vectors as vectors associated with vertices of $\mathbb{T}_n$.

Next, we define f-vectors according to [8]. We will give some preparations.

We say that a cluster pattern $v \mapsto \Sigma_v$ (or a cluster algebra $\mathcal{A}$) has the *principal coefficients* at the initial vertex t_0 if $\mathbb{P} = \mathrm{Trop}(y_1, \dots, y_n)$ and $\mathbf{y}_{t_0} =$

$(y_1,\dots,y_n)$. In this case, we denote $\mathcal{A}=\mathcal{A}_\bullet(B)$. For any $\mathcal{A}_\bullet(B)$ whose rank is n, any $t\in\mathbb{T}_n$ and $i\in\{1,\dots,n\}$, we define the F-*polynomial* $F_{i;t}^{B;t_0}(\mathbf{y})$ as

$$F_{i;t}^{B;t_0}(\mathbf{y})=x_{i;t}(x_1,\dots,x_n)|_{x_1=\cdots=x_n=1}, \tag{1.16}$$

where $x_{i;t}(x_1,\dots,x_n)$ means the expression of $x_{i;t}$ by $x_1,\dots,x_n$.

Using F-polynomials, we define f-vectors. Let $\mathcal{A}_\bullet(B)$ be a cluster algebra with principal coefficients at t_0. We denote by $f_{ij;t}$ the maximal degree of y_i in $F_{j;t}^{B;t_0}(\mathbf{y})$. Then we define the f-*vector* $\mathbf{f}_{i;t}$ as

$$\mathbf{f}_{i;t}^{B;t_0}=\mathbf{f}_{i;t}=\begin{bmatrix}f_{1i;t}\\ \vdots\\ f_{ni;t}\end{bmatrix}. \tag{1.17}$$

We define the F-*matrix* $F_t^{B;t_0}$ as

$$F_t^{B;t_0}:=(\mathbf{f}_{1;t},\dots,\mathbf{f}_{n;t}). \tag{1.18}$$

Remark 1.6. For $\mathbf{b}=\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}$, we denote $[\mathbf{b}]_+=\begin{bmatrix}[b_1]_+\\ \vdots\\ [b_n]_+\end{bmatrix}$. By [8, Proposition 2.7], f-vectors are the same as those defined by the following recursion: for any $i\in\{1,\dots,n\}$,

$$\mathbf{f}_{i;t_0}=\mathbf{0},$$

and for any $t \overset{k}{\text{———}} t'$,

$$\mathbf{f}_{i;t'}=\begin{cases}\mathbf{f}_{i;t} & \text{if } i\neq k;\\ -\mathbf{f}_{k;t}+\max\Bigg([\mathbf{c}_{k;t}]_+ +\sum_{j=1}^n[b_{jk;t}]_+\mathbf{f}_{j;t}, & \\ \qquad [-\mathbf{c}_{k;t}]_+ +\sum_{j=1}^n[-b_{jk;t}]_+\mathbf{f}_{j;t}\Bigg) & \text{if } i=k,\end{cases} \tag{1.19}$$

where $\mathbf{c}_{i;t}$ is a c-*vector*, which is defined by the following recursion: for any $i\in\{1,\dots,n\}$

$$\mathbf{c}_{i;t_0}=\mathbf{e}_i,$$

and for any $t \overset{k}{\text{———}} t'$,

$$\mathbf{c}_{i;t'}=\begin{cases}-\mathbf{c}_{i;t} & \text{if } i\neq k;\\ \mathbf{c}_{i;t}+[b_{ki;t}]_+\ \mathbf{c}_{k;t}+b_{ki;t}[-\mathbf{c}_{k;t}]_+ & \text{if } i=k.\end{cases}$$

By this way of definition, since f-vectors depend only on exchange matrices, we can regard f-vectors as vectors associated with vertices of $\mathbb{T}_n$. In this case, we remark that f-vectors are independent of the choice of coefficient system. So do F-matrices. Furthermore, when we define f-vectors as these recursions, we have the following property: in a cluster algebra $\mathcal{A}$, if $x_{i;t}=x_{j;s}$, then we have $\mathbf{f}_{i;t}=\mathbf{f}_{j;s}$. It follows from [3, Proposition 3 (i)] immediately.

Since we know that d-vectors and f-vectors depend only on B by above discussion, we abbreviate a cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ to $\mathcal{A}(B)$ when we discuss properties of d-vectors or f-vectors.

Example 1.7. Let $\mathcal{A}(B)$ be a cluster algebra given in Example 1.2. Then F-polynomials, F-matrices, and D-matrices are given by Table 2.

We are ready to describe the main results in this paper.

1.3. Main Results

The main result of this paper is the following theorem:

Theorem 1.8. *In a cluster algebra $\mathcal{A}(B)$ of finite type, for any $i \in \{1, \ldots, n\}$ and $t \in \mathbb{T}_n$, we have the following relation:*

$$\mathbf{f}_{i;t} = [\mathbf{d}_{i;t}]_+. \tag{1.20}$$

It is known that Theorem 1.8 holds under the condition that the initial matrix B is bipartite by combining Corollary 10.10 and Proposition 11.1 (1) in [14]. When B is a skew-symmetric matrix, Theorem 1.8 has already proved using 2-Calabi–Yau categories (see [10, Proposition 6.6]). We remove these conditions.

Remark 1.9. In the case that $\mathcal{A}(B)$ is of rank 2, we have (1.20) by combining Corollary 10.10 and Proposition 11.1 (1) in [14]. If $\mathcal{A}$ is of neither finite type nor rank 2, Theorem 1.8 does not hold generally. A counterexample is given by [10, Section 6.4] .

We give an application of Theorem 1.8. Let us introduce the *Uniqueness Conjecture* in [16]:

Conjecture 1.10. [16, Conjecture 4.4] *In a cluster algebra $\mathcal{A}(B)$, for $t, s \in \mathbb{T}_n$, $F_t^{B;t_0} = F_s^{B;t_0}$ implies that $\boldsymbol{x}_t$ and $\boldsymbol{x}_s$ are the same non-labeled cluster.*

This conjecture is also studied in viewpoint of representation theory of algebras. An f-vectors are a dimension vector of the corresponding indecomposable τ-rigid module over an appropriate 2-Calabi–Yau tilted algebras in additive categorification by 2-Calabi–Yau categories. Using the correspondences, Conjecture 1.10 is equivalent to the following problem: support τ-tilting modules are uniquely determined by the set of dimension vectors of these indecomposable direct summands. This problem was solved in the case of skew-symmetric cluster algebras of finite type [15,24], skew-symmetric cluster algebras of affine type [7], and cluster algebras of C_n Dynkin type [9].

In the case that $\mathcal{A}$ is of (skew-symmetrizable) finite type or rank 2, we prove Conjecture 1.10 by showing the following statement:

Theorem 1.11.
1. *In a cluster algebra of finite type of rank n, for any $t, s \in \mathbb{T}_n$, if $(\mathbf{f}_{1;t}, \ldots, \mathbf{f}_{n;t})$ coincides with $(\mathbf{f}_{1;s}, \ldots, \mathbf{f}_{n;s})$ up to order, then $\boldsymbol{x}_t$ and $\boldsymbol{x}_s$ are the same non-labeled cluster.*
2. *In a cluster algebra of rank 2, for $t, s \in \mathbb{T}_2$, if $(\mathbf{f}_{1;t}, \mathbf{f}_{2;t})$ coincides with $(\mathbf{f}_{1;s}, \mathbf{f}_{2;s})$ up to order, then $\boldsymbol{x}_t$ and $\boldsymbol{x}_s$ are the same non-labeled cluster.*

TABLE 2. F-polynomials, F- and D-matrices in type A_2

t	$F_{1;t}^{B;t_0}(\mathbf{y})$	$F_{2;t}^{B;t_0}(\mathbf{y})$	$F_t^{B;t_0}$	$D_t^{B;t_0}$
0	1	1	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
1	y_1+1	1	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
2	y_1+1	$y_1y_2+y_1+1$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
3	y_2+1	$y_1y_2+y_1+1$	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
4	y_2+1	1	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
5	1	1	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Theorem 1.11 is a theorem of slightly stronger form than Conjecture 1.10. In Conjecture 1.10, the order of the f-vectors is fixed, but in Theorem 1.11, it is not.

Remark 1.12. In the case of cluster algebras of A_n or D_n type, Theorem 1.11 has already been proved using marked surfaces [16, Corollary 4.8].

2. Proof of Theorem 1.8

In this section, we will prove Theorem 1.8. We start with proving the special case. For any cluster pattern $v \mapsto \Sigma_v$, we fix a seed Σ_s such that B_s is bipartite. We define the *source mutation* μ_+ and the *sink mutation* μ_- as

$$\mu_+ = \prod_{\varepsilon(k)=1} \mu_k, \quad \mu_- = \prod_{\varepsilon(k)=-1} \mu_k, \tag{2.1}$$

where ε is the sign induced by the bipartite matrix B_s (see (1.11)). The *bipartite belt* induced by Σ_s consists of seeds Σ_t satisfying the following condition: there exists a mutation sequence μ consisting of μ_+ and μ_- such that $\Sigma_t = \mu(\Sigma_s)$.

Remark 2.1. Definition of a bipartite belt in this paper is a generalised version of [14, Definition 8.2]. We do not assume that the initial exchange matrix B is bipartite. A bipartite belt in [14] corresponds with that induced by the initial bipartite seed Σ_{t_0} in this paper.

Lemma 2.2. [14, Corollary 10.10] *In any cluster algebra, if the initial matrix B is bipartite and Σ_t belongs to the bipartite belt induced by Σ_{t_0}, then we have* (1.20).

By Remark 1.5, if $\mathcal{A}$ is of finite type, then $\mathcal{A}$ has a seed whose exchange matrix is bipartite. We prove the case that the initial matrix B is bipartite.

Lemma 2.3. [14, Proposition 11.1 (1)] *In a cluster algebra of finite type, for a bipartite seed Σ_s, every cluster variable belongs to a seed lying on the bipartite belt induced by Σ_s.*

Proposition 2.4. *We fix a cluster algebra of finite type whose initial matrix B is bipartite. For any $i \in \{1, \ldots, n\}$ and $t \in \mathbb{T}_n$, we have (1.20).*

Proof. It follows from Lemmas 2.2 and 2.3. □

Let us generalize Proposition 2.4 to the case that the initial matrix B is non-bipartite. The next lemma is a generalization of Lemma 2.3.

Lemma 2.5. *In a cluster algebra of finite type, for a seed Σ_s, every cluster variable belongs to seeds lying on the bipartite belt induced by Σ_s.*

Proof. Let Σ_t^B be a seed and $\Sigma_s^{B'}$ a bipartite seed. By regarding a change of the initial seed from Σ_t^B to $\Sigma_s^{B'}$ as a change from the expression of cluster variables and coefficients by Σ_t^B to that by $\Sigma_s^{B'}$, the general cases follows from the bipartite cases. □

We introduce a key lemma.

Lemma 2.6. [26, Theorem 2.2], [8, Theorem 3.10]

1. *In a cluster algebra $\mathcal{A}(B)$ of finite type, for $t \in \mathbb{T}_n$, we have*
$$D_t^{B;t_0} = (D_{t_0}^{B_t^T;t})^T. \tag{2.2}$$
2. *In any cluster algebra $\mathcal{A}(B)$, for $t \in \mathbb{T}_n$, we have*
$$F_t^{B;t_0} = (F_{t_0}^{B_t^T;t})^T. \tag{2.3}$$

Remark 2.7. In [26, Theorem 2.2], the duality for D-matrices is given by
$$D_t^{B;t_0} = (D_{t_0}^{-B_t^T;t})^T. \tag{2.4}$$
The Eq. (2.2) derives from (2.4). In fact, by symmetry of the recursion (1.15) of d-vectors, we have $D_{t_0}^{-B_t^T;t} = D_{t_0}^{B_t^T;t}$.

We are ready to prove the main theorem in this paper.

Proof of Theorem 1.8. We fix a bipartite seed Σ in $\mathcal{A}(B)$. Note that $\mathcal{A}(B)$ is of finite type if and only if $\mathcal{A}(B^T)$ is also. Moreover, B_t^T is bipartite if and only if B_t is bipartite. Therefore, $\mathcal{A}(B^T)$ is of finite type, and for any t in a bipartite belt induced by Σ, B_t^T is bipartite. Thus, we have
$$F_{t_0}^{B_t^T;t} = \left[D_{t_0}^{B_t^T;t}\right]_+, \tag{2.5}$$
by Proposition 2.4 (the operation $[\ ]_+$ on matrices are performed componentwise). Therefore, we have
$$F_t^{B;t_0} = \left[D_t^{B;t_0}\right]_+, \tag{2.6}$$
by Proposition 2.6. By Lemma 2.5, for a cluster variable $x_{j;s}$, there exist $i \in \{1,\dots,n\}$ and a vertex t of the bipartite belt induced by a seed Σ such that $x_{j;s} = x_{i;t}$. Thus, $\mathbf{f}_{j;s} = \mathbf{f}_{i;t} = \mathbf{d}_{i;t} = \mathbf{d}_{j;s}$ by (2.6), and we have (1.20) for any initial vertex t_0.

3. Proof of Theorem 1.11 (1)

In this section, we prove Theorem 1.11 (1). We fix any $\mathcal{A}(B)$ of finite type. Through this section, unless otherwise noted, we assume that seeds, cluster variables, clusters, f-vectors, d-vectors, F-matrices, and D-matrices are those of $\mathcal{A}(B)$. We start with proving the special case. We say that a vector $\mathbf{b}$ is *positive* (resp. *negative*) if $\mathbf{b} \neq \mathbf{0}$ and all entries of $\mathbf{b}$ is non-negative (resp. non-positive). Due to Theorem 1.8, we can use the properties of d-vectors to prove Theorem 1.11 (1).

Lemma 3.1. [4, Corollary 3.5] *A cluster variable $x_{i;t}$ is not in the initial cluster if and only if $\mathbf{d}_{i;t}$ is positive.*

By this lemma, we have the following corollary:

Corollary 3.2. *An f-vector* $\mathbf{f}_{i;t}$ *is the zero-vector if and only if* $x_{i;t}$ *is in the initial cluster.*

Proof. The "if" part is clear. We prove the "only if" part. By Theorem 1.8, $\mathbf{f}_{i;t} = \mathbf{0}$ implies that $\mathbf{d}_{i;t}$ is negative or $\mathbf{0}$. By Lemma 3.1, $x_{i;t}$ is in the initial cluster. $\square$

The following propositions and corollary are essential for proving Theorem 1.11:

Proposition 3.3. [14, Theorem 11.1 (2)]) *We fix a cluster algebra* $\mathcal{A}(B)$ *of finite type such that* B *is bipartite and Cartan finite* X_n *type. The d-vectors establish a bijection between cluster variables and the set of all almost positive roots* $\Phi_{\geq -1} = \Phi_+ \cup -\Delta$ *of* X_n *Dynkin type, where* Φ_+ *is the set of all positive roots and* $-\Delta$ *is the set of negative simple roots.*

Let $\mathcal{D}(B)$ be the set of all d-vectors in $\mathcal{A}(B)$.

Proposition 3.4. [23, Theorem 1.3.3] *We fix a cluster algebra* $\mathcal{A}(B)$ *of finite type. Then the cardinality* $|\mathcal{D}(B)|$ *depends only on the Dynkin type* X_n *of* $\mathcal{A}(B)$.

Corollary 3.5. *If* $\mathbf{d}_{i;t} = \mathbf{d}_{j;s}$ *holds, then we have* $x_{i;t} = x_{j;s}$.

Proof. Let B' be a bipartite matrix of finite Cartan X_n type which is mutation equivalent to B. Then by Propositions 3.3 and 3.4, we have

$$|\mathcal{D}(B)| = |\mathcal{D}(B')| = |\Phi_{\geq -1}|. \tag{3.1}$$

Let $\mathcal{X}(B)$ be the set of all cluster variables of $\mathcal{A}(B)$. By Remark 1.5 and Proposition 3.3, we have

$$|\mathcal{X}(B)| = |\mathcal{X}(B')| = |\Phi_{\geq -1}|. \tag{3.2}$$

Therefore, we have

$$|\mathcal{D}(B)| = |\mathcal{X}(B)|. \tag{3.3}$$

If there exist d-vectors $d_{i;t}$ and $d_{j;s}$ such that $d_{i;t} = d_{j;s}$ and $x_{i;t} \neq x_{j;s}$, then we have $|\mathcal{D}(B)| < |\mathcal{X}(B)|$. This conflicts with (3.3). $\square$

By Corollaries 3.2 and 3.5, we have the following proposition:

Proposition 3.6. *If* $\mathbf{f}_{i;t} = \mathbf{f}_{j;s} \neq \mathbf{0}$, *then we have* $x_{i;t} = x_{j;s}$.

Proof. Let $\mathbf{f}$ be an f-vector which is not equal to $\mathbf{0}$. We assume that $\mathbf{f} = \mathbf{f}_{i;t} = \mathbf{f}_{j;s}$. Since all entries of $\mathbf{f}$ are non-negative, and the f-vector is not equal to $\mathbf{0}$, we have $\mathbf{f} = \mathbf{d}_{i;t} = \mathbf{d}_{j;s}$ by Theorem 1.8 and Lemma 3.1. By Proposition 3.5, we have $x_{i;t} = x_{j;s}$. $\square$

While d-vectors can distinguish the initial clusters, f-vectors cannot. Thus, we cannot detect the initial cluster variables contained in a cluster by their f-vectors directly. However, using the property of d-vectors, we can detect them.

Proposition 3.7. *For a D-matrix* $D_t^{B;t_0}$, *negative column vectors of* $D_t^{B;t_0}$ *are uniquely determined by positive column vectors of* $D_t^{B;t_0}$.

Proof. By (2.2), the transposition of a D-matrix in a cluster algebra of finite type is another D-matrix in a cluster algebra of finite type because $\mathcal{A}(B)$ is of finite type if and only if $\mathcal{A}(B_t^T)$ is of finite type. Since negative d-vectors have the form of $-\mathbf{e}_i$, if the (i,j) entry of $D_t^{B;t_0}$ is -1, then entries of the ith row and the jth column of $D_t^{B;t_0}$ are all 0 except for the (i,j)-entry. Since $D_t^{B;t_0}$ do not have the zero column vector by Lemma 3.1, if a D-matrix has just m positive columns, then we have just $n-m$ indices $i_1,\dots,i_{n-m}$ such that the $i_k(k\in\{1,\dots,n-m\})$th entry of all positive d-vectors is 0, and $D_t^{B;t_0}$ has column vectors $-\mathbf{e}_{i_k}(k\in\{1,\dots,n-m\})$. This finishes the proof. □

We are ready to prove Theorem 1.11 (1).

Proof of Theorem 1.11 (1). If $\mathbf{f}_{i;t}=\mathbf{f}_{j;s}\neq\mathbf{0}$, then we have $x_{i;t}=x_{j;s}$ by Proposition 3.6. We assume that there are m zero-vectors in $(\mathbf{f}_{1;t},\dots,\mathbf{f}_{n;t})$ (or $(\mathbf{f}_{1;s},\dots,\mathbf{f}_{n;s})$). By regarding positive f-vectors as d-vectors by Theorem 1.8, we detect the rest of d-vectors in $\mathbf{x}_t$ and $\mathbf{x}_s$ by Proposition 3.7. Since positive d-vectors in $\mathbf{x}_t$ corresponds with that of $\mathbf{x}_s$, we have $\mathbf{x}_t=\mathbf{x}_s$ by Corollary 3.5.

4. Proof of Theorem 1.11 (2)

We prove Theorem 1.11 (2). The strategy of this proof is almost the same as Theorem 1.11 (1), but we sometimes use the special properties of cluster algebras of rank 2.

For a cluster algebra of rank 2, we may assume that the initial matrix B has the following form without loss of generality:

$$B=\begin{bmatrix}0 & b\\ -c & 0\end{bmatrix},\quad b,c\in\mathbb{Z}_{\geq 0},\quad bc\geq 4, \tag{4.1}$$

because when $bc\leq 3$, this cluster algebra is of finite type. We name vertices of $\mathbb{T}_2$ by the rule of (1.10) and consider a cluster pattern $t_n\mapsto(\mathbf{x}_{t_n},\mathbf{y}_{t_n},B_{t_n})$. We abbreviate $\mathbf{x}_{t_n}$ (resp., $\mathbf{y}_{t_n}$, B_{t_n},Σ_{t_n}) to $\mathbf{x}_n$ (resp., $\mathbf{y}_n$, B_n, Σ_n). We also abbreviate d-vectors, D-matrices, f-vectors, and F-matrices in the same way.

We consider a description of D-matrices in the case $n\geq 0$. First, we have

$$D_0^{B;t_0}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},\quad D_1^{B;t_0}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{4.2}$$

by direct calculation. By [20, (1.13)], if $n>0$ is even, then we can denote

$$D_n^{B;t_0}=\begin{bmatrix}S_{\frac{n-2}{2}}(u)+S_{\frac{n-4}{2}}(u) & bS_{\frac{n-2}{2}}(u)\\ cS_{\frac{n-4}{2}}(u) & S_{\frac{n-2}{2}}(u)+S_{\frac{n-4}{2}}(u)\end{bmatrix}, \tag{4.3}$$

and if $n>1$ is odd, then we can denote

$$D_n^{B;t_0}=\begin{bmatrix}S_{\frac{n-1}{2}}(u)+S_{\frac{n-3}{2}}(u) & bS_{\frac{n-3}{2}}(u)\\ cS_{\frac{n-3}{2}}(u) & S_{\frac{n-3}{2}}(u)+S_{\frac{n-5}{2}}(u)\end{bmatrix}, \tag{4.4}$$

where $u = bc - 2$ and $S_p(u)$ is a (normalized) Chebyshev polynomial of the second kind, that is,

$$S_{-1}(u) = 0, \quad S_0(u) = 1, \quad S_p(u) = uS_{p-1}(t) - S_{p-2}(u) \ (p \in \mathbb{N}). \tag{4.5}$$

When $n < 0$, $D_n^{B;t_0}$ is the following matrix:

$$D_n^{B;t_0} = \begin{bmatrix} d_{22;-n}^{-B^T} & d_{21;-n}^{-B^T} \\ d_{12;-n}^{-B^T} & d_{11;-n}^{-B^T} \end{bmatrix}, \tag{4.6}$$

where $d_{ij;-n}^{-B^T}$ is the (i,j) entry of $D_{-n}^{-B^T;t_0}$.

We fix any $\mathcal{A}(B)$ of rank 2. Through the rest of this section, unless otherwise noted, we assume that seeds, cluster variables, clusters, f-vectors, d-vectors, F-matrices, and D-matrices are those of $\mathcal{A}(B)$. Using the above descriptions, we prove some properties for d-vectors.

Lemma 4.1. *The initial cluster variables belong to Σ_0 or $\Sigma_{\pm 1}$. Furthermore, $x_{i;t}$ is not in the initial cluster if and only if $\mathbf{d}_{i;t}$ is positive.*

Proof. We prove it in the case $n > 0$. It suffices to show that for any $u \geq 2$ and $p \geq -1$, $S_p(u) \geq 0$ holds and $S_p(u) = 0$ if and only if $p = -1$. The general term of $S_p(u)$ is

$$S_p(u) = \begin{cases} p+1 & \text{if } u = 2; \\ \dfrac{1}{\sqrt{u^2-4}}\left(\left(\dfrac{u+\sqrt{u^2-4}}{2}\right)^{p+1} - \left(\dfrac{u-\sqrt{u^2-4}}{2}\right)^{p+1}\right) & \text{if } u \neq 2. \end{cases} \tag{4.7}$$

By direct calculation, we have $S_p(u) \geq 0$. Also, $S_p(u) = 0$ holds if and only if $p = -1$ holds. In the case $n < 0$, we can use the result of the case $n > 0$ by (4.6). $\square$

The following corollary is analogous to Corollary 3.2:

Corollary 4.2. *An f-vector $\mathbf{f}_{i;t}$ is the zero-vector if and only if $x_{i;t}$ is in the initial cluster.*

Proof. We can prove it in the same way as Corollary 3.2: we use Lemma 4.1 instead of Lemma 3.1. $\square$

The following lemma is analogous to Corollary 3.5:

Lemma 4.3. *If $\mathbf{d}_{i;t} = \mathbf{d}_{j;s}$, then we have $x_{i;t} = x_{j;s}$.*

Proof. When $\mathbb{P} = \{1\}$, using [20, (1.15)] (cf. Sect. 5), we have the expressions of cluster variables induced by d-vectors. For the general case, the use of [3, Proposition 3 (i)] leads to the case where $\mathbb{P} = \{1\}$. $\square$

The following proposition is analogous to Corollary 3.6:

Proposition 4.4. *If* $\mathbf{f}_{i;t} = \mathbf{f}_{j;s} \neq \mathbf{0}$*, then we have* $x_{i;t} = x_{j;s}$*.*

Proof. We can prove it in the same way as Corollary 3.6: we use Corollary 4.2 and Lemma 4.3 instead of Corollaries 3.2 and 3.5 respectively. □

The following proposition is analogous to Proposition 3.7. Unlike Proposition 3.7, we do not need to use the duality for D-matrices.

Proposition 4.5. *For a D-matrix $D_n^{B;t_0}$, negative column vectors of $D_n^{B;t_0}$ are uniquely determined by positive column vectors of $D_n^{B;t_0}$.*

Proof. When both d-vectors in $D_n^{B;t_0}$ are negative vectors, it is clear. Therefore, we can assume that only one d-vector is negative. By Lemma 4.1, the initial cluster variables only appear in Σ_0 or $\Sigma_{\pm 1}$. Therefore, if $\mathbf{d}_{1;0} = \mathbf{d}_{1;-1} = -\mathbf{e}_1$ is contained in two d-vectors associated with a cluster, then the other is always $\mathbf{d}_{2;-1}$. Similarly, if $\mathbf{d}_{2;0} = \mathbf{d}_{2;1} = -\mathbf{e}_2$ is contained in two d-vectors, then the other is always $\mathbf{d}_{1;1}$. By this observation, it suffices to show $\mathbf{d}_{2;-1} \neq \mathbf{d}_{1;1}$. We have $\mathbf{d}_{2;-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{d}_{1;1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by direct calculation. This finishes the proof. □

We are ready to prove Theorem 1.11 (2).

Proof of Theorem 1.11 (2). We can prove it in the same way as Theorem 1.11 (1): we use Lemma 4.3, Propositions 4.4, and 4.5 instead of Corollary 3.5, Propositions 3.6, and 3.7 respectively.

5. Restoration Formula of Cluster Algebras of Rank 2

We proved that cluster variables are uniquely determined by their f-vectors for cluster algebras of rank 2 in the previous section. In this section, we describe these cluster variables explicitly in the case that coefficients are the principal ones. By this description, we establish a way to restore F-polynomials from f-vectors. Throughout this section, we assume that $\mathcal{A}(B)$ has the following initial matrix:

$$B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}, \quad b, c \in \mathbb{Z}_{\geq 0}. \tag{5.1}$$

We do not assume $bc \geq 4$, thus cluster algebras of finite type and rank 2 (A_2, B_2, G_2 Dynkin types) are contained. Unless otherwise noted, we assume that seeds, cluster variables, clusters, f-vectors, d-vectors, F-matrices, and D-matrices are those of $\mathcal{A}(B)$.

A previous work [20] has given a cluster expansions formula in the case that $\mathbb{P} = \{1\}$. This formula restores the expressions of cluster variables by the initial ones from their d-vectors. We start with an explanation of this formula.

We define Dyck Paths and some notations along [20, Section 1]. Let (a_1, a_2) be a pair of non-negative integers. A *Dyck path* of type $a_1 \times a_2$ is a lattice path from $(0, 0)$ to (a_1, a_2) and it does not go above the diagonal combining $(0, 0)$ with (a_1, a_2). For the Dyck paths of $a_1 \times a_2$ type, there is the *maximal* one $\mathcal{D}^{a_1\times a_2}$. It is defined by the following property: for any lattice point A on $\mathcal{D}$, there is no lattice points between A and the crosspoint of a vertical line including A and the diagonal combining $(0,0)$ with (a_1,a_2).

For $\mathcal{D}=\mathcal{D}^{a_1\times a_2}$, let $\mathcal{D}_1=\{u_1,\dots,u_{a_1}\}$ be the set of horizontal edges of $\mathcal{D}$ indexed from left to right, and $\mathcal{D}_2=\{v_1,\dots,v_{a_2}\}$ be the set of vertical edges of $\mathcal{D}$ indexed from bottom to top.

For any A and B on $\mathcal{D}$, let AB be the subpath of $\mathcal{D}$ starting from A and going in the upper right direction along $\mathcal{D}$ until it reaches B. If we reach (a_1,a_2) before reaching B, we restart from $(0,0)$. If A and B are the same lattice point, then AA is the subpath which starts from A, then passes (a_1,a_2) and ends at A. Here $(0,0)$ and (a_1,a_2) are regarded as the same point, thus if $A=(a_1,a_2)$, then AA corresponds with the maximal Dyck path. We denote by $(AB)_1$ the set of horizontal edges in AB, and by $(AB)_2$ the set of vertical edges in AB. Let AB° be the set of lattice points on the subpath AB except for the endpoints A and B.

Example 5.1. We fix $(a_1,a_2)=(5,3)$, and let $A=(2,1)$, $B=(4,2)$. Then

$$(AB)_1=\{u_3,u_4\},\ (AB)_2=\{v_2\},$$
$$(BA)_1=\{u_5,u_1,u_2\},\ (BA)_2=\{v_3,v_1\},$$

and the subpath AA has length 8 (see Fig. 1).

Next, we define the compatibility in $\mathcal{D}$:

Definition 5.2. [20, Definition 1.10] For $S_1\subseteq\mathcal{D}_1$, $S_2\subseteq\mathcal{D}_2$, we say that the pair (S_1,S_2) is *compatible* if for every $u\in S_1$ and $v\in S_2$, denoting by E the left endpoint of u and F the upper endpoint of v, there exists a lattice point $A\in EF^\circ$ such that

$$|(AF)_1|=b|(AF)_2\cap S_2| \text{ or } |(EA)_2|=c|(EA)_1\cap S_1|. \tag{5.2}$$

We are ready to describe a cluster expansion formula for cluster algebras of rank 2.

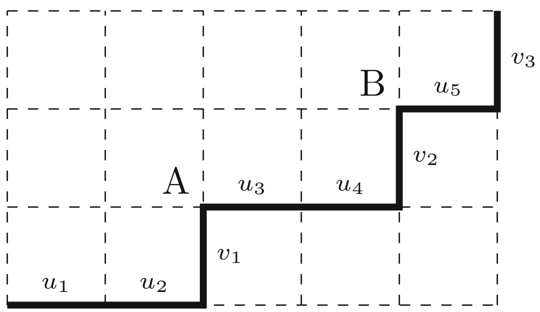

FIGURE 1. A maximal Dyck path ($(a_1,a_2)=(5,3)$)

Theorem 5.3. [20, Theorem 1.11] *For every d-vector* $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$*, the cluster variable* $x_{\mathbf{d}}$ *corresponding to* $\mathbf{d}$ *is given by the following equation:*

$$x_{\mathbf{d}} = x_1^{-d_1} x_2^{-d_2} \sum_{(S_1,S_2)} x_1^{b|S_2|} x_2^{c|S_1|}, \tag{5.3}$$

where the sum is over all compatible pairs (S_1, S_2) *in* $\mathcal{D}^{[d_1]_+ \times [d_2]_+}$.

Remark 5.4. In [20, Theorem 1.11], (5.3) is defined for any $(a_1, a_2) \in \mathbb{Z}^2$ and is called a *greedy element*.

We generalize this formula to the principal coefficients version in a way which is analogous to [21]. First, we define the g-vectors according to [14]. Cluster variables with the principal coefficients are homogeneous by the following $\mathbb{Z}^n$-grading: for any $i \in \{1, \dots, n\}$,

$$\deg(x_i) = \mathbf{e}_i, \quad \deg(y_i) = -\mathbf{b}_i, \tag{5.4}$$

where $\mathbf{b}_i$ is the ith column vector of B (see [14, Proposition 6.1]). We define the g-vector $\mathbf{g}_{i;t} = \begin{bmatrix} g_{1i;t} \\ \vdots \\ g_{ni;t} \end{bmatrix}$ as the degree vector of a cluster variable $x_{i;t}$. Like f-vectors, they are independent of the choice of $\mathbb{P}$ by defining them in the following way: for any $i \in \{1, \dots, n\}$,

$$\mathbf{g}_{i;t_0} = \mathbf{e}_i, \tag{5.5}$$

and for any $t \overset{k}{\text{———}} t'$,

$$g_{ij;t'} = \begin{cases} g_{ij;t} & \text{if } j \neq k; \\ -g_{ik;t} + \sum_{\ell=1}^{n} g_{i\ell;t}[b_{\ell k;t}]_+ & \\ \qquad - \sum_{\ell=1}^{n} b_{i\ell}[c_{\ell k;t}]_+ & \text{if } j = k, \end{cases} \tag{5.6}$$

where $c_{\ell k;t}$ is the ℓth entry of $\mathbf{c}_{k;t}$ (cf. Remark 1.6).

When a cluster algebra is of rank 2, g-vectors are obtained by d-vectors:

Theorem 5.5. *For a g-vector* $\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ *and a d-vector* $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ *of a cluster variable, we have the following equation:*

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} -d_1 \\ cd_1 - d_2 \end{bmatrix}. \tag{5.7}$$

Proof. This is the spacial case of [14, Theorem 10.12]. □

Using g-vectors, we have the following generalization of Theorem 5.3:

Theorem 5.6. *For a d-vector $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, the cluster variable $x_{\mathbf{d}}$ with the principal coefficients corresponding to $\mathbf{d}$ is given by the following equation:*

$$x_{\mathbf{d}} = x_1^{-d_1} x_2^{-d_2} \sum_{(S_1,S_2)} y_1^{[d_1]_+ - |S_1|} y_2^{|S_2|} x_1^{b|S_2|} x_2^{c|S_1|}, \tag{5.8}$$

where the sum is over all compatible pairs (S_1, S_2) in $\mathcal{D}^{[d_1]_+ \times [d_2]_+}$.

Proof. When a d-vector is the negative, we have (5.8) by direct calculation. We assume that a d-vector is positive. For any compatible pair $(S_1, S_2) \in \mathcal{D}^{d_1 \times d_2}$, let $a_1(S_1, S_2)$ and $a_2(S_1, S_2)$ be integers satisfying

$$x_{\mathbf{d}} = x_1^{-d_1} x_2^{-d_2} \sum_{(S_1,S_2)} y_1^{a_1(S_1,S_2)} y_2^{a_2(S_1,S_2)} x_1^{b|S_2|} x_2^{c|S_1|}. \tag{5.9}$$

Since $x_{\mathbf{d}}$ is homogeneous by the grading (5.4), and its degree is $\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} -d_1 \\ cd_1 - d_2 \end{bmatrix}$ by Theorem 5.5, the following equation holds for any compatible pair (S_1, S_2):

$$\begin{aligned} \begin{bmatrix} -d_1 \\ cd_1 - d_2 \end{bmatrix} = -\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + a_1(S_1, S_2) \begin{bmatrix} 0 \\ c \end{bmatrix} \\ + a_2(S_1, S_2) \begin{bmatrix} -b \\ 0 \end{bmatrix} + \begin{bmatrix} b|S_2| \\ c|S_1| \end{bmatrix}. \end{aligned} \tag{5.10}$$

By solving the equation, we have

$$a_1(S_1, S_2) = d_1 - |S_1|, \quad a_2(S_1, S_2) = |S_2|. \tag{5.11}$$

□

By Theorem 5.6, definition of the F-polynomials, and Remark 1.9, we have the following restoration formula of F-polynomials from f-vectors:

Corollary 5.7. *For a f-vector $\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, the F-polynomial $F_{\mathbf{f}}(\boldsymbol{y})$ whose maximal degree vector is $\mathbf{f}$ is given by the following formula:*

$$F_{\mathbf{f}}(y_1, y_2) = \sum_{(S_1,S_2)} y_1^{f_1 - |S_1|} y_2^{|S_2|}, \tag{5.12}$$

where the sum is over all compatible pairs (S_1, S_2) in $\mathcal{D}^{f_1 \times f_2}$.

Example 5.8. Let $B = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{d} = \mathbf{f} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. If $(S_1, S_2) \in \mathcal{D}^{3 \times 2}$ is compatible, then at least one of the sets S_1 and S_2 is empty, or (S_1, S_2) is one of pairs in the following list:

$$(\{u_1\}, \{v_2\}), (\{u_2\}, \{v_2\}), (\{u_3\}, \{v_1\}). \tag{5.13}$$

Then we have an expression of the cluster variable $x_{\mathbf{d}}$ corresponding to d-vector $\mathbf{d}$ in $\mathcal{A}_\bullet(B)$ as follows:

$$x_{\mathbf{d}} = \frac{x_1^8y_1^3y_2^2 + 2x_1^4y_1^3y_2 + y_1^3 + 3x_1^4x_2y_1^2y_2 + 3x_2y_1^2 + 3x_2^2y_1 + x_2^3}{x_1^3x_2^2}. \tag{5.14}$$

Also we have the F-polynomial $F_{\mathbf{f}}(\mathbf{y})$ corresponding to the f-vector $\mathbf{f}$ as follows:

$$F_{\mathbf{f}}(\mathbf{y}) = y_1^3y_2^2 + 2y_1^3y_2 + y_1^3 + 3y_1^2y_2 + 3y_1^2 + 3y_1 + 1. \tag{5.15}$$

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