Ann. Comb. 23 (2019) 285–293

Published online June 3, 2019
https://doi.org/10.1007/s00026-019-00431-0

Annals of Combinatorics

Minimal Polygons with Fixed Lattice Width

Filip Cools and Alexander Lemmens

Abstract. We classify the unimodular equivalence classes of inclusion-minimal polygons with a certain fixed lattice width. As a corollary, we find a sharp upper bound on the number of lattice points of these minimal polygons.

Mathematics Subject Classification. Primary 52B20, 52C05; Secondary 05E18.

Keywords. Convex lattice, Polytopes, Convex lattice polygons, Lattice width.

1. Introduction and Definitions

Let $\Delta \subset \mathbb{R}^2$ be a non-empty lattice polygon, i.e., the convex hull of a finite number of lattice points in $\mathbb{Z}^2$, and consider a lattice direction $v \in \mathbb{Z}^2$, i.e., a non-zero primitive vector. The lattice width of Δ in the direction v is

$$\mathrm{lw}_v(\Delta) = \max_{P\in\Delta} \langle P, v\rangle - \min_{P\in\Delta} \langle P, v\rangle .$$

The *lattice width* of Δ is defined as $\mathrm{lw}(\Delta) = \min_v \mathrm{lw}_v(\Delta)$. Throughout this paper we will assume that Δ is two-dimensional, hence $\mathrm{lw}(\Delta) > 0$. A lattice direction v that satisfies $\mathrm{lw}_v(\Delta) = \mathrm{lw}(\Delta)$ is called a lattice width direction of Δ.

Two lattice polygons Δ and Δ' are called (unimodularly) equivalent if and only if there exists a unimodular transformation φ, i.e., a map of the form

$$\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2,$$
$$x \mapsto Ax + b, \quad \text{where} \quad A \in \mathrm{GL}_2(\mathbb{Z}),\ b \in \mathbb{Z}^2,$$

such that $\varphi(\Delta) = \Delta'$. Equivalent lattice polygons have the same lattice width.

The lattice width of a polygon is a classical notion with connections to algebraic geometry, see for instance [8]. Its study goes back at least to 1974 [6], although the terminology is not uniform.

The lattice width of a polygon can be seen as a specific instance of the more general notion of *lattice size*, which was introduced in [3].

Definition 1.1. Let $X \subset \mathbb{R}^2$ be a subset with positive Jordan measure. Then the lattice size $\mathrm{ls}_X(\Delta)$ of a non-empty lattice polygon Δ is the smallest $d \in \mathbb{Z}_{\geq 0}$ for which there exists a unimodular transformation φ such that $\varphi(\Delta) \subset \mathrm{d}X$.

Note that $\mathrm{lw}(\Delta) = \mathrm{ls}_X(\Delta)$, where $X = \mathbb{R} \times [0, 1]$.

This paper is concerned with polygons Δ that are *minimal* in the following sense: $\mathrm{lw}(\Delta') < \mathrm{lw}(\Delta)$ for each lattice polygon $\Delta' \subsetneq \Delta$. Equivalently, a two-dimensional polygon Δ is minimal if and only if for each vertex P of Δ, we have that $\mathrm{lw}(\Delta_P) < \mathrm{lw}(\Delta)$, where

$$\Delta_P := \mathrm{conv}\left((\Delta \cap \mathbb{Z}^2)\setminus\{P\}\right).$$

This means that removing any vertex and then taking the convex hull of the remaining lattice points always produces a polygon of smaller lattice width.

Our main result is a complete classification of minimal polygons up to unimodular equivalence, see Theorem 2.4. As a corollary, we provide a sharp upper bound on the number of lattice points of these minimal polygons. First, we show in Lemma 2.3 that each minimal polygon Δ satisfies $\mathrm{ls}_\square(\Delta) = \mathrm{lw}(\Delta)$, where

$$\square = \mathrm{conv}\left\{(0,0),(1,0),(1,1),(0,1)\right\}.$$

The latter can also be proven using results on lattice width directions of interior lattice polygons (see [4, Lemma 5.3]), but we choose to keep the paper self-contained and have provided a different proof. Moreover, we use the technical Lemma 2.2 in the proofs of both Lemmas 2.3 and Theorem 2.4.

In the joint paper [4] with Castryck and Demeyer, we study the Betti table of the toric surface $\mathrm{Tor}(\Delta) \subset \mathbb{P}^{\sharp(\Delta\cap\mathbb{Z}^2)-1}$ for lattice polygons Δ. In particular, we present a lower bound for the length of the linear strand of this Betti table in terms of $\mathrm{lw}(\Delta)$, which we conjecture to be sharp. To show this conjecture for polygons of a fixed lattice width, it essentially suffices to prove it for the minimal polygons (see [4, Corollary 5.2]). Hence, Theorem 2.4 allows us to check the conjecture using a computer algebra system.

Remark 1.2. Of course, the question of classifying minimal polytopes can also be asked in higher dimensions. For instance, it can be shown that each three-dimensional minimal polytope $\Delta \subset \mathbb{R}^3$ with $\mathrm{lw}(\Delta) = 1$ is equivalent to a tetrahedron of the form

$$\mathrm{conv}\left\{(0,0,0),(1,0,0),(0,1,0),(1,y,z)\right\}$$

with $1 \leq y \leq z$ and $\gcd(y,z) = 1$. These include the Reeve tetrahedrons (where $y = 1$). For comparison, there is only one minimal polygon with lattice width one up to equivalence, namely the standard simplex $\mathrm{conv}\{(0,0),(1,0),(0,1)\}$.

In all dimensions $k \geq 2$, among the minimal polytopes, we find the so-called *empty lattice simplices* $\Delta \subset \mathbb{R}^k$, i.e., convex hulls of $k+1$ lattice points without interior lattice points. If $k \geq 4$, not all empty lattice simplices have lattice width 1. For more information, see [1,7,9].

2. The Classification of Minimal Polygons

We use the notation from Sect. 1. The following result appears already in [2, Remark following Lemma 5.2], but can be proven in a shorter way.

Lemma 2.1. *Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon with $\mathrm{lw}(\Delta) = d$. If Δ has two linearly independent lattice width directions $v, w \in \mathbb{Z}^2$, then $ls_\square(\Delta) = d$.*

Proof. If v and w do not form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, we take a primitive vector $u \in \mathrm{conv}\{(0,0), v, w\}$ such that v and u form a $\mathbb{Z}$-basis. Let Q, Q' be lattice points of Δ such that

$$\langle Q', u\rangle - \langle Q, u\rangle = \mathrm{lw}_u(\Delta).$$

Write $u = \lambda v + \mu w$ with $0 < \lambda, \mu$ and $\lambda + \mu \leq 1$. Now

$$d \leq \mathrm{lw}_u(\Delta) = \langle Q', (\lambda v + \mu w)\rangle - \langle Q, (\lambda v + \mu w)\rangle \leq \lambda\, \mathrm{lw}_v(\Delta) + \mu\, \mathrm{lw}_w(\Delta) \leq d,$$

so $\mathrm{lw}_u(\Delta) = d$. After applying a unimodular transformation, we may assume that $u = (0,1)$ and $v = (1,0)$, and that Δ fits into $d\square$, hence $ls_\square(\Delta) = d$. $\square$

Lemma 2.2. *Let Δ be a lattice polygon with $\mathrm{lw}(\Delta) = d > 0$. Let P be a vertex of Δ and $v \in \mathbb{Z}^2$ be a primitive vector. If $\mathrm{lw}_v(\Delta_P) < d$ and $\mathrm{lw}_v(\Delta_P) < \mathrm{lw}_v(\Delta) - 1$, then Δ is equivalent to $\Upsilon_{d-1} := \mathrm{conv}\{(0,0), (1,d), (d,1)\}$.*

Proof. Since $\mathrm{lw}_v(\Delta_P) < \mathrm{lw}_v(\Delta) - 1$, we have that either

$$\min_{Q \in \Delta_P} \langle v, Q\rangle > \langle v, P\rangle + 1 \quad \text{or} \quad \max_{Q \in \Delta_P} \langle v, Q\rangle < \langle v, P\rangle - 1.$$

By replacing v by $-v$, we may assume that we are in the first case. Moreover, we may choose v such that the difference $\min_{Q \in \Delta_P} \langle v, Q\rangle - \langle v, P\rangle$ is minimal but greater than one, and such that $\mathrm{lw}_v(\Delta_P) < d$.

We apply a unimodular transformation so that $P = (0,0)$ and $v = (0,1)$. Let y_m (resp. y_M) be the smallest (resp. greatest) y-coordinate occurring in Δ_P. Note that $y_m = \min_{Q \in \Delta_P} \langle v, Q\rangle$ and $y_M = \max_{Q \in \Delta_P} \langle v, Q\rangle$, hence $y_m > 1$ and $y_M - y_m < d$.

Define the cone

$$C_k := \{\lambda(k,1) + \mu(k+1,1) | \lambda, \mu \geq 0\}.$$

Since

$$\Delta \subset (\mathbb{R} \times \mathbb{R}_{>0}) \cup \{P\} = \cup_{k \in \mathbb{Z}} C_k$$

and $y_m > 1$, the polygon Δ is contained in a cone C_k for some $k \in \mathbb{Z}$. Using the unimodular transformation $(x,y) \mapsto (x - ky, y)$, we may assume that $k = 0$, i.e.,

$$\Delta \subseteq C_0 = \{\lambda(0,1) + \mu(1,1) | \lambda, \mu \geq 0\}.$$

In fact, we then have that

$$\Delta \subseteq \mathrm{conv}\{(0,0), (1, y_M), (y_M - 1, y_M)\}.$$

If $y_m = 2$, we have

$$y_M = (y_M - y_m) + 2 \leq d + 1.$$

The strict inequality $y_M < d+1$ is impossible as the horizontal width $\operatorname{lw}_{(1,0)}(\Delta)$ would be less than d. So we have that $y_M = d+1$ and

$$\Delta \subseteq \Delta' = \operatorname{conv}\{(0,0),(1,d+1),(d,d+1)\}.$$

Since $\operatorname{lw}((\Delta')_Q) < d$ for

$$Q \in \{(1,d+1),(d,d+1)\},$$

we must have $\Delta = \Delta'$. This is equivalent to Υ_{d-1} via $(x,y) \mapsto (x, y-x)$.

From now on, assume that $y_m > 2$. Then $(1,2) \notin \Delta$ which means that either

$$\Delta \subseteq \{\lambda(0,1)+\mu(1,2)|\lambda,\mu \geq 0\} \quad \text{or} \quad \Delta \subseteq \{\lambda(1,2)+\mu(1,1)|\lambda,\mu \geq 0\}.$$

We can reduce to the latter case using the transformation $(x,y) \mapsto (y-x,y)$. In fact, we can keep subdividing this cone until we find a cone C containing Δ that does not contain any lattice point with y-coordinate in $\{1,\ldots,y_m-1\}$. Let $\ell \in \mathbb{Z}$ be such that C passes in between $(\ell-1, y_m-1)$ and (ℓ, y_m-1). Then

$$\Delta_P \subseteq \operatorname{conv}\{(\ell, y_m-1),(\ell, y_M),(\ell+y_M-y_m+1, y_M)\}.$$

If x_m (resp. x_M) is the smallest (resp. greatest) x-coordinate occurring in a lattice point of Δ_P, then $2 \leq \ell \leq x_m < y_m$ and $x_M \leq \ell + y_M - y_m$, so $x_M - x_m \leq y_M - y_m < d$. But this means that $\operatorname{lw}_{(1,0)}(\Delta_P) < d$ and

$$1 < \min_{Q\in\Delta_P} \langle (1,0), Q\rangle < y_m = \min_{Q\in\Delta_P} \langle v, Q\rangle,$$

contradicting the minimality of v. □

Lemma 2.3. *If $\Delta \subset \mathbb{R}^2$ is a non-empty minimal lattice polygon with $\operatorname{lw}(\Delta) = d > 0$, then $\operatorname{ls}_\square(\Delta) = d$.*

Proof. By Lemma 2.1, we only have to show that there are two linearly independent lattice width directions. Suppose that v is a lattice width direction and that $Q, Q' \in \Delta \cap \mathbb{Z}^2$ such that $\langle Q, v\rangle - \langle Q', v\rangle = d$. Now let P be a vertex of Δ different from Q, Q'. By minimality of Δ, we have that $\operatorname{lw}(\Delta_P) < d$. That means there exists a direction w such that $\operatorname{lw}_w(\Delta_P) < d$. Because Q and Q' are still in Δ_P, w cannot be v or $-v$, so w must be linearly independent of v. If $\operatorname{lw}_w(\Delta) = d$, we are done. If $\operatorname{lw}_w(\Delta) > d$, then by Lemma 2.2, Δ is equivalent to $\Upsilon_{d-1} \subseteq d\square$. □

Theorem 2.4. *Let $\Delta \subset \mathbb{R}^2$ be a non-empty minimal lattice polygon with $\operatorname{lw}(\Delta) = d$. Then Δ is equivalent to a minimal polygon of one of the following forms:*

- (T1) $\operatorname{conv}\{(0,0),(d,y),(x,d)\}$, *where* $x, y \in \{0,\ldots,d\}$ *satisfy* $x+y \leq d$*;*
- (T2) $\operatorname{conv}\{(x_1,0),(d,y_2),(x_2,d),(0,y_1)\}$, *where* $x_1,x_2,y_1,y_2 \in \{1,\ldots,d-1\}$ *satisfy* $\max(x_2,y_2) \geq \min(x_1,y_1)$ *and* $\max(d-x_2,y_1) \geq \min(d-x_1,y_2)$*;*
- (T3) $\operatorname{conv}\{(0,0),(\ell,0),(d,y+d-\ell),(x+\ell,d),(z,z+d-\ell)\}$ *with* $\ell \in \{2,\ldots,d-2\}$, $x \in \{1,\ldots,d-\ell-1\}$, $y,z \in \{1,\ldots,\ell-1\}$*;*
- (T4) $\operatorname{conv}\{(0,0),(z'+\ell,z'),(d,y+d-\ell),(x+\ell,d),(z,z+d-\ell)\}$ *with* $\ell \in \{2,\ldots,d-2\}$, $y,z \in \{1,\ldots,\ell-1\}$, $x,z' \in \{1,\ldots,d-\ell-1\}$*;*

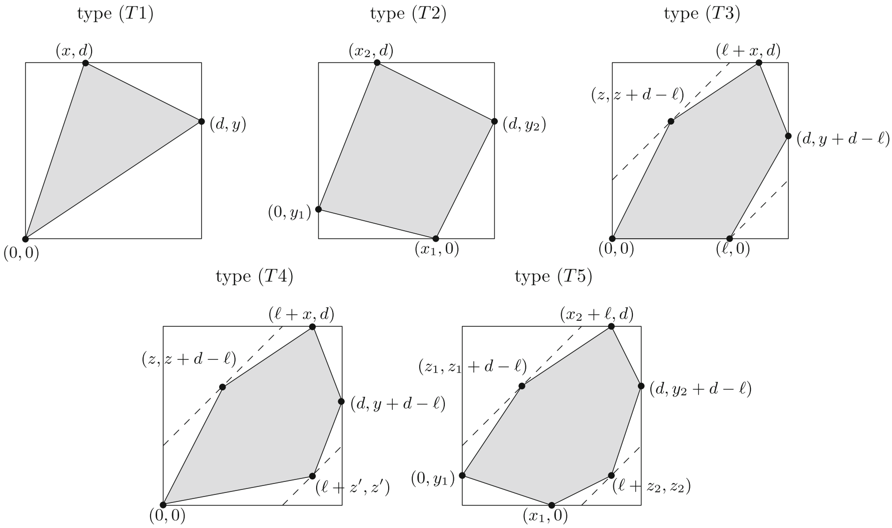

FIGURE 1. The five types in the classification

(T5) $\operatorname{conv}\{(x_1, 0), (z_2 + \ell, z_2), (d, d - \ell + y_2), (x_2 + \ell, d), (z_1, z_1 + d - \ell), (0, y_1)\}$ *with* $\ell \in \{2, \dots, d-2\}$, $x_1, y_2, z_1 \in \{1, \dots, \ell - 1\}$, $x_2, y_1, z_2 \in \{1, \dots, d - \ell - 1\}$.

Remark 2.5. See Fig. 1 for a picture of the five types. The minimal polygons appearing in the types (T3), (T4) and (T5) are inscribed in the hexagon

$$H_\ell := \operatorname{conv}\{(0,0), (\ell, 0), (d, d-\ell), (d, d), (\ell, d), (0, d-\ell)\}.$$

This is also the case for the triangles of type (T1) with $(x, y) \in \{(d, 0), (0, d)\}$ (where we allow $\ell \in \{0, d\}$) and for the quadrangles of type (T2) with $\max(d - x_2, y_1) = \min(d - x_1, y_2)$.

Proof of Theroem 2.4. If $d = 0$, then Δ consists of a single point and it is of shape (T1). So assume $d \geq 1$. Because of Lemma 2.3, we may assume that $\Delta \subset d\square = [0, d] \times [0, d]$. Moreover, we may assume that $\Delta \not\cong \Upsilon_{d-1}$ since Υ_{d-1} is of type $(T1)$. Let P be any vertex of Δ. By Lemma 2.2, if $\operatorname{lw}_v(\Delta_P) < d$ for some primitive vector $v \in \mathbb{Z}^2$, then $\operatorname{lw}_v(\Delta_P) \geq \operatorname{lw}_v(\Delta) - 1$, which together with $\operatorname{lw}_v(\delta) \geq d$ implies $\operatorname{lw}_v(\Delta_P) = d - 1$ and $\operatorname{lw}_v(\Delta) = d$, hence v is a lattice width direction.

By minimality, we know that there always exists a lattice direction v satisfying $\operatorname{lw}_v(\Delta_P) < d$. We claim that we can always take

$$v \in \{(0,1), (1,0), (1,1), (1,-1)\}.$$

Indeed, suppose that $v = (v_x, v_y) \in \mathbb{Z}^2$ satisfies

$$\{v, -v\} \cap \{(0,1), (1,0), (1,1), (1,-1)\} = \emptyset \quad \text{and} \quad \operatorname{lw}_v(\Delta_P) < d.$$

After a unimodular transformation, we may assume that $0 < v_x < v_y$, hence $(1,1) \in \text{conv}\{(0,0),(1,0),v\}$. Using a similar trick as in Lemma 2.1, we get that $\text{lw}_{(1,1)}(\Delta_P) < d$, which proves the claim.

Let $\mathcal{V}$ be the set consisting of vectors $v \in \{(1,1),(1,-1)\}$ for which there exists a vertex P of Δ with $\text{lw}_v(\Delta_P) < d$. If $\mathcal{V} = \{(1,1),(1,-1)\}$, then Δ has 4 different lattice width directions, namely $(1,0),(0,1),(1,1)$ and $(1,-1)$. By [2, Lemma 5.2(v)] or [5], this means that

$$\Delta \cong \text{conv}\,\{(d/2,0),(0,d/2),(d/2,d),(d,d/2)\}$$

for some even d, hence it is of type (T2). If $\mathcal{V} = \emptyset$, we claim that Δ is of type (T1) or (T2). Indeed, for every vertex P of Δ, we have that either $\text{lw}_{(1,0)}(\Delta_P)$ or $\text{lw}_{(0,1)}(\Delta_P)$ is smaller than d. In particular, this means that there has to be a side of $d\Box$ with P as its only point in Δ. One then easily checks the claim: if Δ is a triangle, then it will be of type (T1); if it is a quadrangle, then it is of type (T2).

From now on, suppose that $\mathcal{V}$ is not equal to $\emptyset$ or $\{(1,1),(1,-1)\}$, hence $\mathcal{V} = \{(1,1)\}$ or $\mathcal{V} = \{(1,-1)\}$. We can suppose that $\mathcal{V} = \{(1,-1)\}$ using the transformation $(x,y) \mapsto (x,-y)$ if necessary. Hence, for each vertex P of Δ, there is a vector $v \in \{(1,0),(0,1),(1,-1)\}$ with $\text{lw}_v(\Delta_P) < d$. Since $\text{lw}_{(1,-1)}(\Delta) = d$, there exists an integer $\ell \in \{0,\ldots,d\}$ such that

$$\langle Q,(1,-1)\rangle \in [\ell - d, \ell]$$

for all $Q \in \Delta$. If $\ell \in \{0,d\}$, then Δ is a triangle whose vertices are vertices of $d\Box$, so it is of the form $(T1)$. Now assume that $\ell \in \{1,\ldots,d-1\}$, hence Δ is contained in the hexagon H_ℓ from Remark 2.5. Each side of H_ℓ contains at least one lattice point of Δ, and if it contains more than one point, it is also an edge of Δ. Otherwise, there would be a vertex P lying on exactly one side of H_ℓ, while not being the only point of Δ on that side of H_ℓ. But then there is no $v \in \{(0,1),(1,0),(1,-1)\}$ with $\text{lw}_v(\Delta_P) < d$ (as every side of H_ℓ contains a point of Δ_P), a contradiction.

Denote by $\mathcal{S}$ the set of sides that Δ and H_ℓ have in common. Then $\mathcal{S}$ cannot contain two adjacent sides S_1, S_2: otherwise for the vertex $P = S_1 \cap S_2$, each side of H_ℓ would have a non-empty intersection with Δ_P, contradicting the fact that there is a $v \in \{(0,1),(1,0),(1,-1)\}$ with $\text{lw}_w(\Delta_P) < d$.

Assume that $\sharp\mathcal{S} \geq 2$ and take $S_1 = [Q_1,Q_2] \in \mathcal{S}$. Its adjacent sides of H_ℓ contain no points of Δ except from Q_1 and Q_2. This implies that $\mathcal{S} = \{S_1, S_2\}$, where S_1, S_2 are opposite edges of H_ℓ, and that Δ is the convex hull of these two edges. Hence Δ is equivalent to the quadrangle

$$\text{conv}\,\{(\ell,0),(d,d-\ell),(\ell,d),(0,d-\ell)\} \subset H_\ell,$$

which is of type $(T2)$.

If $\mathcal{S}$ consists of a single side S, we may assume that $S = [Q_1,Q_2]$ is the bottom edge of H_ℓ. Let P_1 (resp. P_2) be the vertex of Δ on the upper left diagonal side (resp. the right vertical edge) of H_ℓ. If P_1 is also on the top edge of H_ℓ (i.e., $P_1 = (\ell,d)$), then Δ has only four vertices, namely Q_1, Q_2, P_1, P_2. Applying the transformation $(x,y) \mapsto (x,-x+y+\ell)$, we end up with a quadrangle of type (T2). By a similar reasoning, if P_2 is on the top edge of

H_ℓ (i.e., $P_2 = (d, d)$), we end up with type (T2). If neither P_1 nor P_2 is on the top edge of H_ℓ, then there is a fifth vertex P_3 on that top edge, and we are in case (T3).

The only remaining case is when $\mathcal{S} = \emptyset$, hence each edge of H_ℓ contains only one point of Δ. If H_ℓ and Δ have no common vertex, then Δ is of type (T5). If they share one vertex, we can reduce to type (T4) using a transformation if necessary. Note that two common vertices of H_ℓ and Δ can never be connected by an edge of H_ℓ as that edge would be in $\mathcal{S}$, so there are at most three common vertices. If there are three shared vertices, then Δ is a triangle of type (T1), again using a transformation if necessary. So assume H_ℓ and Δ share two vertices. Together these two points occupy four edges of H_ℓ and each of the other two edges of H_ℓ (call them A and B) contains exactly one vertex of Δ. Take two pairs of opposite sides of H_ℓ (so four sides in total) that together contain A and B, then they contain all vertices of Δ: since any common vertex of H_ℓ and Δ lies on two sides of H_ℓ, they cannot lie both on the sides we did not choose, as they are parallel. We can find a unimodular transformation mapping these sides into the four sides of $d\square$, hence Δ is of type (T2). $\square$

Remark 2.6. From the classification in Theorem 2.4, one can easily deduce the following result from [6]: $\operatorname{vol}(\Delta) \geq \frac{3}{8} \operatorname{lw}(\Delta)^2$ for each lattice polygon $\Delta \subset \mathbb{R}^2$, and equality holds for minimal polygons of type (T1) with d even and $x = y = \frac{d}{2}$. For odd d, this inequality can be sharpened to

$$\operatorname{vol}(\Delta) \geq \frac{3}{8} \operatorname{lw}(\Delta)^2 + \frac{1}{8},$$

and equality holds for minimal polygons of type (T1) with $x = \frac{d-1}{2}$ and $y = \frac{d+1}{2}$.

Corollary 2.7. *If* $\Delta \subset \mathbb{R}^2$ *is a non-empty minimal lattice polygon with* $\operatorname{lw}(\Delta) = d > 1$, *then*

$$\sharp(\Delta \cap \mathbb{Z}^2) \leq \max\left(\left(d-1\right)^2 + 4, \left(d+1\right)\left(d+2\right)/2\right).$$

Moreover, this bound is sharp.

Note that from $d = 6$ onwards $(d-1)^2 + 4$ starts winning.

Proof of Corollary 2.7. Note that there exist minimal polygons attaining the bound (see Fig. 2): the simplex $\operatorname{conv}\{(0,0),(d,0),(0,d)\}$ is of type (T1) and has $(d+1)(d+2)/2$ lattice points, and the quadrangle $\operatorname{conv}\{(1,0),(d,1),(d-1,d),(0,d-1)\}$ is of type (T2) and has $(d-1)^2 + 4$ lattice points.

Now let us show that we indeed have an upper bound. If Δ is minimal of type (T2), (T4) or (T5), then

$$\sharp(\Delta \cap \mathbb{Z}^2) \leq (d-1)^2 + 4,$$

since there are at most 4 lattice points of Δ on the boundary of $\mathrm{d}\square$ and all the others are in

$$(\mathrm{d}\square)^\circ \cap \mathbb{Z}^2 = \{1, \ldots, d-1\} \times \{1, \ldots, d-1\}.$$

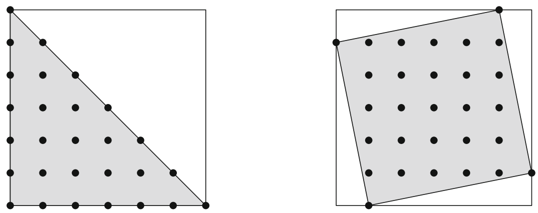

Figure 2. Minimal polygons attaining the upper bound

This also holds for triangles of type (T1) with x and y non-zero. If Δ is of type $(T3)$, we obtain the same upper bound $(d-1)^2+4$ after applying a unimodular transformation that maps the bottom edge of Δ to the left upper diagonal edge of H_ℓ. We are left with triangles of type (T1) with either x or y zero. Assume that $y = 0$ (the case $x = 0$ is similar). Then Δ has the edge $[(0,0),(d,0)]$ in common with d□ and its other vertex is (x,d). For each $k \in \{0,\ldots,d\}$, the intersection of Δ with the horizontal line on height k is a line segment of length $d-k$, hence it contains at most $d-k+1$ lattice points. So in total, Δ has at most

$$\sum_{k=0}^{d}(d-k+1) = (d+1)(d+2)/2$$

lattice points. □

Acknowledgements

The second author is supported by the Flemish Research Council (FWO). We also want to thank the anonymous referees for their useful remarks.

References

[1] Barile, M., Bernardi, D., Borisov, A., Kantor, J.-M.: On empty lattice simplices in dimension 4. Proc. Amer. Math. Soc. 139(12), 4247–4253 (2011)

[2] Castryck, W., Cools, F.: Linear pencils encoded in the Newton polygon. Int. Math. Res. Not. IMRN 2017(10), 2998–3049 (2017)

[3] Castryck, W. Cools, F.: The lattice size of a lattice polygon. J. Combin. Theory Ser. A 136, 64–95 (2015)

[4] Castryck, W., Cools, F., Demeyer, J., Lemmens, A.: Computing graded Betti tables of toric surfaces. Accepted in Trans. Amer. Math. Soc., arXiv:1606.08181 (2016)

[5] Draisma, J., McAllister, T.B., Nill, B.: Lattice-width directions and Minkowski's 3^d-theorem. SIAM J. Discrete Math. 26(3), 1104–1107 (2012)

[6] Fejes Tóth, L., Makai Jr., E.: On the thinnest non-separable lattice of convex plates. Stud. Sci. Math. Hungar. 9(1974), 191–193 (1975)

[7] Haase, C., Ziegler, G.M.: On the maximal width of empty lattice simplices, European J. Combin. 21(1), 111–119 (2000)

[8] Lubbes, N., Schicho, J.: Lattice polygons and families of curves on rational surfaces. J. Algebraic Combin. 34(2), 213–236 (2012)

[9] Sebő, A.: An introduction to empty lattice-simplices. In: Cornuéjols, G., Burkard, R.E., Woeginger, G.E. (eds.) Integer Programming and Combinatorial Optimization (Graz, 1999), pp. 400–414. Lecture Notes in Comput. Sci., 1610, Springer, Berlin (1999)

Filip Cools and Alexander Lemmens
Department of Mathematics
The Katholieke Universiteit Leuven
Oude Markt 13
3000 Leuven
Belgium
e-mail: alexanderlemmens@hotmail.com

Filip Cools
e-mail: f.cools@kuleuven.be

Received: 4 February 2017.
Accepted: 1 November 2018.