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The Generalized Translation Dual of a Semifield*

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Abstract. In this paper, elaborating on the link between semifields of dimension n over their left nucleus and $\mathbb{F}_s$-linear sets of rank en disjoint from the secant variety $\Omega(\mathcal{S}_{n,n})$ of the Segre variety $\mathcal{S}_{n,n}$ of $PG(n^2-1,q)$, $q=s^e$, we extend some operations on semifield whose definition relies on *dualising* the relevant linear set.

Keywords: semifield, linear set, Segre variety

1. Introduction

Linear sets are the natural generalization of the notion of subgeometry of a projective geometry. We briefly recall the definitions.

Let $PG(k-1,q^t)=PG(V,\mathbb{F}_{q^t})$ be the projective geometry of the $\mathbb{F}_{q^t}$-subspaces of V, where V is a vector space of rank k over the Galois field $\mathbb{F}_{q^t}$ of order q^t. A set L of points of $PG(k-1,q^t)$ is an $\mathbb{F}_q$-*linear set* of $PG(k-1,q^t)$ if there exists a subset W of V, which is an $\mathbb{F}_q$-vector subspace of V, such that a point of $PG(k-1,q^t)$ belongs to L if and only if it is defined by a vector of W. We will write $L=L(W)$. If W has rank m as vector space over $\mathbb{F}_q$ we say that $L(W)$ has *rank* m, and we will write $rk_{\mathbb{F}_q}L=m$. If $L(W)$ has rank k and $\langle W\rangle_{q^t}=V$, then $L(W)\simeq PG(W,\mathbb{F}_q)=PG(k-1,q)$ is said to be a *subgeometry* of $PG(V,\mathbb{F}_{q^t})=PG(k-1,q^t)$.

A finite semifield is a finite division algebra in which multiplication is not necessarily associative and throughout this paper the term semifield will be always used to denote a finite semifield (see, e.g., [11, Chapter 6] for definitions and notations on finite semifields). Every field is a semifield and the term *proper semifield* will mean a semifield which is not isotopic to a field. The left nucleus $\mathbb{N}_l$ and the center $\mathbb{K}$ of

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a semifield $\mathbb{S}$ are fields contained in $\mathbb{S}$ as substructures ($\mathbb{K}$ is a subfield of $\mathbb{N}_l$) and $\mathbb{S}$ is a vector space over $\mathbb{N}_l$ and over $\mathbb{K}$. Semifields are studied up to an equivalence relation called *isotopy* and the dimensions of a semifield over its left nucleus and over its center are invariant up to isotopy.

If $\mathbb{S}$ satisfies all the axioms for a semifield except, possibly, the existence of the identity element for the multiplication, then it is a *presemifield*. In such a case the nuclei and the center of $\mathbb{S}$ can be defined as fields of linear maps contained in $End(\mathbb{S}, \mathbb{F}_p)$ (where p is the characteristic of $\mathbb{S}$) (see, e.g., [19, Theorem 2.2]) and all that we stated and defined above for semifields can be applied for presemifields.

Let $\mathbb{S}$ be a presemifield of dimension n over $\mathbb{F}_q$, which is a subfield of the left nucleus of $\mathbb{S}$, and of dimension en over $\mathbb{F}_s$ ($q = s^e$), which is a subfield of the center of $\mathbb{S}$. Let $\mathbb{P} = PG\big(\mathbb{E}, \mathbb{F}_q\big) = PG\big(n^2 - 1, q\big)$, where $\mathbb{E} = End(\mathbb{S}, \mathbb{F}_q)$ is the vector space of all the $\mathbb{F}_q$-endomorphisms of $\mathbb{S}$, we can associate with $\mathbb{S}$ an $\mathbb{F}_s$-linear set of rank en, of the projective space $\mathbb{P}$ disjoint from the variety $\Omega(\mathcal{S}_{n,n})$ of $PG\big(n^2 - 1, q\big)$ defined by the non-invertible elements of $\mathbb{E}$, and conversely (see [15] for $n = 2$ and [10] for $n \geq 2$). We will call such a linear set a *geometric* (*semifield*) *spread set*. For a survey on the theory of finite semifields we refer to [11].

The relationship between linear sets and semifields has been previously studied in [15] in order to investigate the connection between semifield flocks and ovoids of $Q(4, q)$ introduced by Thas in [22]. Such a relation was detailed in [9], where it was shown that the procedure of *dualising* the ovoid links the two sets of semifields related to a semifield flock. In [13] it has been proven that an ovoid of $Q(4, q)$ associated with a semifield flock is a translation ovoid and, conversely, that any translation ovoid of $Q(4, q)$ defines a semifield flock. This relationship has been used in [1] to construct the sporadic semifield flock of order 3^5 from the Penttila-Williams ovoid of $Q\big(4, 3^5\big)$ ([20]). Finally, in [15], it has been proven that translation ovoids of the Klein quadric $Q^+(5, q)$ and semifields of dimension 2 over their left nucleus are equivalent objects and we can represent the translation ovoid as an $\mathbb{F}_s$-linear set Γ of rank $2e$ of $PG(3, q)$ ($q = s^e$) disjoint from a hyperbolic quadric $Q^+(3, q)$. The dual of Γ with respect to the polarity defined by $Q^+(3, q)$ is still a linear set of rank $2e$ disjoint from $Q^+(3, q)$ and defines another semifield of dimension 2 over its left nucleus called the *translation dual* of the starting semifield (for further details see [16]). The dual of Γ gives rise to a semifield flock if and only if Γ is contained in a plane, i.e., if and only if the translation ovoid is contained in $Q(4, q)$ ([15]). Hence, Thas' construction is a particular case of the translation dual.

In [2, Section 6], the authors starting from a semifield S two-dimensional over its left nucleus construct another semifield $S^{\mathbb{T}}$. In [7, Theorem 6.1] (see also [16]), it has been proved that the construction in [2] is invariant under isotopisms. In [9, Remark 3.1] it was stated that such a semifield $S^{\mathbb{T}}$ is, up to isotopisms, one of the six Knuth derivatives of the translation dual of S. This result was also proven in [16, Theorem 2.2].

Another construction can be obtained using similar ideas. A symplectic semifield $\mathbb{S}$ of dimension 3 over its left nucleus is represented by an $\mathbb{F}_s$-linear set Δ of rank $3e$ disjoint from the variety $\mathcal{M}$ of the secants of a Veronese surface of $PG(5, q)$ ($q = s^e$). When q is odd there is a polarity of $PG(5, q)$ interchanging conic and tangent planes of $\mathcal{M}$, and the dual of Δ with respect to such a polarity is still an $\mathbb{F}_s$-linear set of rank

$3e$ disjoint from $\mathcal{M}$. Hence, it defines a symplectic semifield of dimension 3 over its left nucleus, called the *symplectic dual* of $\mathbb{S}$ ([17]).

The aim of this paper is to extend the construction of the translation dual by using the link between semifields of dimension n over their left nucleus and $\mathbb{F}_s$-linear sets of rank en disjoint from the $(n-2)$-secant variety $\Omega(\mathcal{S}_{n,n})$ of the Segre variety $\mathcal{S}_{n,n}$ of $PG(n^2-1,q)$, $q=s^e$.

We will note that $\Omega(\mathcal{S}_{n,n})$ is covered by a family $\mathbb{T}$ of subspaces of dimension n^2-n-1 and we prove that there is a polarity $\perp$ of $PG(n^2-1,q)$ such that

$$\mathbb{T}=\{X^{\perp} \mid X\in\mathcal{R}\},$$

where $\mathcal{R}$ is one of the two systems of $\mathcal{S}_{n,n}$. Hence, if Γ is a linear set of rank en contained in a $(2n-1)$-dimensional subspace T of $PG(n^2-1,q)$, such that $\Gamma\cap T^{\perp}=\emptyset$, it is possible to construct another linear set of T, say, Γ^*_T of rank en, as well. Moreover, under some extra hypothesis, Γ^*_T defines a geometric spread set (see Theorem 4.1). If $\mathbb{S}$ is the presemifield associated with Γ, the presemifield $\mathbb{S}^*_T$ associated with Γ^*_T will be called the *generalized translation dual* of $\mathbb{S}$ with respect to T, and the "classical" translation dual and the symplectic dual operations fall into this new procedure for $n=2$ and $n=3$, respectively.

2. Dual of Linear Sets

A $(t-1)$-spread of a projective space $PG(tk-1,q)$ is a partition of the points of $PG(tk-1,q)$ into $(t-1)$-dimensional subspaces.

Let $\Omega := PG(k-1,q^t) = PG(V,\mathbb{F}_{q^t})$. If we regard V as a vector space over $\mathbb{F}_q$ of rank kt, then $PG(V,\mathbb{F}_q)=PG(kt-1,q)$. Also, each point P of Ω defines a $(t-1)$-dimensional subspace X_P of $PG(kt-1,q)$ and $\mathcal{D}=\{X_P\colon P\in\Omega\}$ is a Desarguesian spread of $PG(kt-1,q)$ (see, e.g., [14]). Also, the incidence structure $\Pi_{k-1}(\mathcal{D}) := (\mathcal{D},\mathcal{L})$ whose points are the elements of $\mathcal{S}$ and whose lines are the $(2t-1)$-dimensional subspaces spanned by two elements of $\mathcal{S}$ is isomorphic to Ω. Such a structure is called *linear representation* of Ω over $\mathbb{F}_q$.

An m-dimensional $\mathbb{F}_q$-vector subspace U of V defines in $PG(kt-1,q)$ an $(m-1)$-dimensional projective subspace $M := PG(U,\mathbb{F}_q)$ and the linear set $L(U)$ of Ω can be seen as the set of points P of Ω such that $X_P\cap M\neq\emptyset$, i.e., $L(U)=\{P\in\Omega\colon X_P\cap M\neq\emptyset\}$.

Denote by $\perp$ the polarity of Ω defined by a non-singular bilinear form $<,>$ of V. If Tr is the trace of $\mathbb{F}_{q^t}$ over $\mathbb{F}_q$, the $\mathbb{F}_q$-bilinear form $Tr(<x,y>)$ of V (regarded as a vector space over $\mathbb{F}_q$) defines a polarity $\perp'$ of $PG(kt-1,q)$ and $M^{\perp'}=PG(U^{\perp'},\mathbb{F}_q)$ denotes the polar of M with respect to $\perp'$. Then

$$\Gamma^* := L\left(U^{\perp'}\right) = \left\{P\in PG\left(k-1,q^t\right) \,\middle|\, X_P\cap M^{\perp'}\neq\emptyset\right\} \tag{2.1}$$

is an $\mathbb{F}_q$-linear set of rank $kt-m$ of Ω. We call Γ^* the *dual* of Γ with respect to $\perp$ (see [21, §2]).

3. The Segre Variety $\mathcal{S}_{n,n}$ and Polarities

The incidence-geometric description of the Segre varieties is well known, see, e.g., [4] and [5, Section 25.5]. Here below, following the approach used in [18, §5], we describe the Segre variety in terms of linear maps.

Let $\mathbb{E} = End(\mathbb{F}_{q^n}, \mathbb{F}_q)$ be the vector space of all the endomorphisms of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. The Segre Variety $\mathcal{S}_{n,n}$ of the projective space $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_q) = PG(n^2-1, q)$ is the algebraic variety defined by the elements of $\mathbb{E}$ of rank 1 and the $(n-2)$-secant variety $\Omega(\mathcal{S}_{n,n})$ (the *secant variety of* $\mathcal{S}_{n,n}$, for short) of $\mathcal{S}_{n,n}$ is the hypersurface of $\mathbb{P}$ defined by the non-invertible elements of $\mathbb{E}$.

If $t_\alpha\colon x \in \mathbb{F}_{q^n} \mapsto \alpha x \in \mathbb{F}_{q^n}$, with $\alpha \in \mathbb{F}^*_{q^n}$ and $Tr\colon x \in \mathbb{F}_{q^n} \mapsto x + x^q + \cdots + x^{q^{n-1}} \in \mathbb{F}_q$, we have that

$$\mathcal{S}_{n,n} = \left\{ \langle t_\lambda \circ Tr \circ t_\mu \rangle : \lambda, \mu \in \mathbb{F}^*_{q^n} \right\},$$

where $\circ$ stands for composition of maps (see, e.g., [18, Prop. 5.1]).

Moreover, for any $\lambda \in \mathbb{F}^*_{q^n}$, the $(n-1)$-dimensional projective subspaces of $\mathbb{P}$

$$X(\lambda) = \left\{ \langle t_\alpha \circ Tr \circ t_\lambda \rangle : \alpha \in \mathbb{F}^*_{q^n} \right\} \quad \text{and} \quad X'(\lambda) = \left\{ \langle t_\lambda \circ Tr \circ t_\alpha \rangle : \alpha \in \mathbb{F}^*_{q^n} \right\}$$

are the maximal subspaces contained in $\mathcal{S}_{n,n}$ and $\mathcal{S}_{n,n} = \bigcup_{\lambda \in \mathbb{F}^*_{q^n}} X(\lambda) = \bigcup_{\lambda \in \mathbb{F}^*_{q^n}} X'(\lambda)$.

The sets $\mathcal{R}_1 = \left\{ X(\lambda) : \lambda \in \mathbb{F}^*_{q^n} \right\}$ and $\mathcal{R}_2 = \left\{ X'(\lambda) : \lambda \in \mathbb{F}^*_{q^n} \right\}$ are the *systems* of $\mathcal{S}_{n,n}$ and they satisfy the following properties: (a) the subspaces of $\mathcal{R}_1$ ($\mathcal{R}_2$, respectively) are mutually disjoint; (b) if $X(\mu)$ and $X'(\lambda)$ are $(n-1)$-dimensional subspaces belonging to different systems of $\mathcal{S}_{n,n}$, then $X(\mu) \cap X'(\lambda) = \{\langle t_\lambda \circ Tr \circ t_\mu \rangle\}$ is a point of $\mathbb{P}$; (c) each point of $\mathcal{S}_{n,n}$ belongs to a unique element of $\mathcal{R}_1$ and to a unique element of $\mathcal{R}_2$.

For each $\varphi \in \mathbb{E}$, where $\varphi(x) = \sum_{i=0}^{n-1} \beta_i x^{q^i}$, the *conjugate* $\overline{\varphi}$ of φ is defined by $\overline{\varphi}(x) = \sum_{i=0}^{n-1} \beta_i^{q^{n-i}} x^{q^{n-i}}$. Precisely, $\bar{\varphi}$ is the *adjoint* map of φ with respect to the non-degenerate $\mathbb{F}_q$-bilinear form of $\mathbb{F}_{q^n}$

$$\langle x, y \rangle = Tr(xy). \tag{3.1}$$

The map

$$T\colon \varphi \in \mathbb{E} \mapsto \overline{\varphi} \in \mathbb{E}$$

is an involutory $\mathbb{F}_q$-linear permutation of $\mathbb{E}$ and

$$X(\mu)^{\Phi_T} = X'(\mu), \qquad \text{for each } \mu \in \mathbb{F}^*_{q^n}, \tag{3.2}$$

where Φ_T denotes the collineation of $\mathbb{P}$ induced by T (for further details see [18, §5]).

If Γ is a geometric spread set in $\mathbb{P}$ defined by a presemifield $\mathbb{S}$, i.e., Γ is an $\mathbb{F}_s$-linear set of $\mathbb{P} = PG(n^2-1, q)$, $q = s^e$, of rank en disjoint from $\Omega(\mathcal{S}_{n,n})$, then Γ^{Φ_T} is an $\mathbb{F}_s$-linear set of $\mathbb{P}$ of rank en and, by (3.2), it is disjoint from $\Omega(\mathcal{S}_{n,n})$, as well. By [17, Lemma 2], it defines another presemifield $\mathbb{S}^t$, which is the so-called *transpose* of $\mathbb{S}$, introduced by Knuth in [8].

Now, we will prove some results which will be useful to prove the main result of the paper in the subsequent section.

Lemma 3.1. *Let* $\lambda_1,\dots,\lambda_s \in \mathbb{F}_{q^n}^*$, *then*

i) $dim\langle X'(\lambda_1),\dots,X'(\lambda_s)\rangle = sn-1$ *if and only if* $\lambda_1,\dots,\lambda_s$ *are* $\mathbb{F}_q$*-independent*;
ii) $dim\langle X(\lambda_1),\dots,X(\lambda_s)\rangle = sn-1$ *if and only if* $\lambda_1,\dots,\lambda_s$ *are* $\mathbb{F}_q$*-independent.*

In particular, if $X_1,\dots,X_r \in \mathcal{R}_i$, $i \in \{1,2\}$, *with* $dim\ \langle X_1,\dots,X_r\rangle = rn-1$, $r<n$, *and* $X_{r+1} \in \mathcal{R}_i$, *then either* $X_{r+1} \subset \langle X_1,\dots,X_r\rangle$ *or* $\langle X_1,\dots,X_r\rangle \cap X_{r+1} = \emptyset$.

Proof. We first prove i). Assume $dim\langle X'(\lambda_1),\dots,X'(\lambda_s)\rangle = sn-1$ and, by way of contradiction, suppose $\lambda_1,\dots,\lambda_s \in \mathbb{F}_{q^n}^*$ are $\mathbb{F}_q$-dependent. Hence, up to a rearrangement, there exist $\alpha_2,\dots,\alpha_s \in \mathbb{F}_q$, which are not all zero, such that $\lambda_1 = \alpha_2\lambda_2+\cdots+\alpha_s\lambda_s$. In such a case, it is easy to see that $X'(\lambda_1)\cap\langle X'(\lambda_2),\dots,X'(\lambda_s)\rangle \neq \emptyset$, a contradiction.

Conversely, let $\lambda_1,\dots,\lambda_s$ be $\mathbb{F}_q$-independent and, by way of contradiction, suppose that $dim\langle X'(\lambda_1),\dots,X'(\lambda_s)\rangle < sn-1$. Then, up to a rearrangement, we have $X'(\lambda_1)\cap\langle X'(\lambda_2),\dots,X'(\lambda_s)\rangle \neq \emptyset$, and hence there exist $\mu_1 \in \mathbb{F}_{q^n}^*$ and $\mu_2,\dots,\mu_s \in \mathbb{F}_{q^n}$ not all zero, such that

$$t_{\lambda_1}\circ Tr\circ t_{\mu_1} = \sum_{i=2}^{s}\left(t_{\lambda_i}\circ Tr\circ t_{\mu_i}\right). \tag{3.3}$$

It follows that

$$ker\left(t_{\lambda_1}\circ Tr\circ t_{\mu_1}\right) \supseteq \bigcap_{i=2}^{s} ker\left(t_{\lambda_i}\circ Tr\circ t_{\mu_i}\right).$$

If there is $a \in ker\left(t_{\lambda_1}\circ Tr\circ t_{\mu_1}\right)$ and $a \notin \bigcap_{i=2}^{s} ker\left(t_{\lambda_i}\circ Tr\circ t_{\mu_i}\right)$, then, from (3.3), it follows that $\lambda_2,\dots,\lambda_n$ are dependent over $\mathbb{F}_q$, a contradiction. Hence,

$$ker\left(t_{\lambda_1}\circ Tr\circ t_{\mu_1}\right) = \bigcap_{i=2}^{s} ker\left(t_{\lambda_i}\circ Tr\circ t_{\mu_i}\right),$$

which means

$$ker\left(t_{\lambda_1}\circ Tr\circ t_{\mu_1}\right) = ker\left(t_{\lambda_i}\circ Tr\circ \mu_i\right), \tag{3.4}$$

for each $i \in \{2,\dots,s\}$ such that $\mu_i \neq 0$. By Eq. (3.4), if $\mu_i \neq 0$ there exists $\beta_i \in \mathbb{F}_q$ such that $\mu_i = \beta_i\mu_1$. Substituting for μ_i in Eq. (3.3), we get $\lambda_1 = \sum_{i=2}^{s}\beta_i\lambda_i$, i.e., $\lambda_1,\dots,\lambda_s$ are $\mathbb{F}_q$-dependent, a contradiction.

Statement ii) follows from i) taking (3.2) into account. Finally, the last part of the statement follows from i) and ii) noting that if $\lambda_{r+1} = \sum_{i=1}^{r}\beta_i\lambda_i$, with $\beta_i \in \mathbb{F}_q$, then

$$t_{\lambda_{r+1}}\circ Tr\circ t_{\mu} = \sum_{i=1}^{r}\beta_i\left(t_{\lambda_i}\circ Tr\circ t_{\mu}\right),$$

for each $\mu \in \mathbb{F}_{q^n}$. ∎

Let $X_1,X_2,\dots,X_r$ be elements of $\mathcal{R}_i$, $i\in\{1,2\}$, and let $M = \langle X_1,X_2,\dots,X_r\rangle$. Since each map of X_i has rank 1, if $r \le n-1$, each element of M has rank at most $n-1$, i.e., $M \subset \Omega(\mathcal{S}_{n,n})$.

Let

$$\mathbb{T}_i := \left\{ M = \langle X_1, \ldots, X_{n-1} \rangle : X_j \in \mathcal{R}_i \text{ and } dim M = n^2 - n - 1 \right\}.$$

Since each element of $\mathbb{E} = End\left(\mathbb{F}_{q^n}, \mathbb{F}_q\right)$ of rank r is linear combination of r elements of $\mathbb{E}$ of rank 1, by applying Lemma 3.1, we have that $\Omega(\mathcal{S}_{n,n}) = \bigcup_{M \in \mathbb{T}_i} M$.

Denote by $\varphi_{a_0, a_1, \ldots, a_{n-1}}$ the element of $\mathbb{E}$ defined as

$$\varphi_{a_0, a_1, \ldots, a_{n-1}}(x) = a_0 x + a_1 x^q + \cdots + a_{n-1} x^{q^{n-1}}$$

and let $\perp$ be the polarity of $\mathbb{P}$ associated with the symmetric non-singular bilinear form

$$b\left(\varphi_{a_0, a_1, \ldots, a_{n-1}}, \varphi_{b_0, b_1, \ldots, b_{n-1}}\right) = Tr(a_0 b_0 + a_1 b_1 + \cdots + a_{n-1} b_{n-1}). \tag{3.5}$$

Lemma 3.2. *Let* $\perp$ *be the polarity of* $\mathbb{P}$ *defined by* (3.5)*. Then an element* X *belongs to* $\mathcal{R}_i$*,* $i \in \{1, 2\}$*, if and only if* $X^{\perp} \in \mathbb{T}_i$*.*

Proof. Taking Lemma 3.1 into account and by making direct computations it is possible to show that $|\mathbb{T}_i| = |\mathcal{R}_i| = \frac{q^n - 1}{q - 1}$ for each $i \in \{1, 2\}$. Hence, we may only prove the necessary condition. Consider the case $i = 1$ and let $X = X(\lambda) \in \mathcal{R}_1$, $\lambda \neq 0$. By (3.5) we have

$$b(t_\alpha \circ Tr \circ t_\lambda, t_\beta \circ Tr \circ t_\mu) = Tr(\alpha\beta Tr(\lambda\mu)),$$

hence if $Tr(\lambda\mu) = 0$ the element $X(\mu)$ of $\mathcal{R}_1$ belongs to $X(\lambda)^{\perp}$.

If $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\}$ is a basis of the $\mathbb{F}_q$-subspace $ker(Tr \circ t_\lambda)$ of $\mathbb{F}_{q^n}$ then, by Lemma 3.1,

$$M = \langle X(\mu_1), X(\mu_2), \ldots, X(\mu_{n-1}) \rangle$$

has dimension $n^2 - n - 1$ and it is contained in $X(\lambda)^{\perp}$. Then $X(\lambda)^{\perp} = M \in \mathbb{T}_1$.

In a similar way we can prove the case $i = 2$. ∎

Let $\Gamma = L(U)$ be an $\mathbb{F}_s$-linear set contained in a $(2n-1)$-dimensional subspace $T = PG(W, \mathbb{F}_q)$ of $\mathbb{P} = PG\left(n^2 - 1, q\right)$ of rank en ($q = s^e$) such that $\Gamma \cap T^{\perp} = \emptyset$, i.e., $U \cap W^{\perp} = \{\mathbf{0}\}$. The set

$$\Gamma_T^* := \Gamma^* \cap T,$$

where Γ^* is the dual of Γ with respect to $\perp$ as defined in Section 2, is an $\mathbb{F}_s$-linear set of rank en as well. Indeed, Γ_T^* is defined by the $\mathbb{F}_s$-subspace $U^{\perp'} \cap W$, where $\perp'$ is the polarity of $\mathbb{P}' = PG(\mathbb{E}, \mathbb{F}_s) = PG\left(n^2 e - 1, s\right)$ induced by $\perp$ as described in Section 2, and noting that $W^{\perp'} = W^{\perp}$, we get

$$\begin{aligned} rk_{\mathbb{F}_s} \Gamma_T^* &= dim_{\mathbb{F}_s}\left(U^{\perp'} \cap W\right) \\ &= n^2 e - dim_{\mathbb{F}_s}\left(U + W^{\perp'}\right) \\ &= n^2 e - dim_{\mathbb{F}_s}\left(U + W^{\perp}\right) \\ &= en. \end{aligned} \tag{3.6}$$

In the next theorem we will prove that the linear set Γ_T^* is a geometric spread set contained in T, providing suitable conditions on T and Γ. Indeed,

Theorem 3.3. *Let* Γ *be an* $\mathbb{F}_s$*-linear set of rank en of* $\mathbb{P} = PG(n^2-1,q)$, $q = s^e$, *contained in a* $(2n-1)$*-dimensional subspace* T *of* $\mathbb{P}$, *and let* $\perp$ *be the polarity of* $\mathbb{P}$ *defined by* (3.5). *If the following conditions are satisfied*:

(P_1) $\Gamma \cap T^{\perp} = \emptyset$;
(P_2) $T^{\perp} \cap \mathcal{S}_{n,n} = \emptyset$;
(P_3) $\langle X, T^{\perp}\rangle \cap \Gamma = \emptyset$, $\forall X \in \mathcal{R}_i$, *for a given* $i \in \{1,2\}$;

then, $\Gamma^*_T = \Gamma^* \cap T$ *is a geometric spread set.*

Proof. Suppose that (P_3) holds with $i = 1$, being the proof analogous in the other case. By the arguments stated above, (P_1) implies that $\Gamma^*_T = \Gamma^* \cap T$ is an $\mathbb{F}_s$-linear set of T of rank *en*. Since $\Omega(\mathcal{S}_{n,n}) = \bigcup_{M\in\mathbb{T}_1} M$, then Γ^*_T is a geometric spread set if and only if $\Gamma^*_T \cap M = \Gamma^* \cap T \cap M = \emptyset$ for each $M \in \mathbb{T}_1$. Let $\perp'$ be the polarity of $\mathbb{P}' = PG(\mathbb{E}, \mathbb{F}_s) = PG(n^2e-1, s)$ induced by $\perp$ as described at the end of Section 2. Note that if Y is a subspace of $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_q) = PG(n^2-1,q)$, then Y can be regarded as an $\mathbb{F}_s$-subspace of $\mathbb{P}'$ and it is easy to see that $Y^{\perp'} = Y^{\perp}$ (see [21, p. 4]). Now, regarding Γ^*, M, and T as $\mathbb{F}_s$-subspaces of $\mathbb{P}'$, we have that $\Gamma^* \cap T \cap M = \emptyset$ if and only if $\mathbb{P}' = \langle \Gamma, M^{\perp'}, T^{\perp'}\rangle = \langle \Gamma, M^{\perp}, T^{\perp}\rangle$. Since $M \in \mathbb{T}_1$, by Lemma 3.2, $M^{\perp} \in \mathcal{R}_1$, and hence, by (P_2), $M^{\perp} \cap T^{\perp} = \emptyset$. Then $\mathbb{P}' = \langle \Gamma, M^{\perp}, T^{\perp}\rangle$ if and only if $\langle M^{\perp}, T^{\perp}\rangle \cap \Gamma = \emptyset$ and, by (P_3) and by Lemma 3.2, the latter condition is satisfied for each $M \in \mathbb{T}_1$. ∎

Remark 3.4. We observe that by simply inspecting the hypotheses one can prove that the converse of Theorem 3.3 holds true, as well. Moreover, if the subspace T is non-singular with respect to the polarity $\perp$, i.e., $T \cap T^{\perp} = \emptyset$, then the geometric spread set Γ^*_T and the subspace T satisfy Conditions (P_1), (P_2), and (P_3). This means that, in such a case, the above procedure is involutory, i.e., it can be also applied to Γ^*_T yielding $\left(\Gamma^*_T\right)^*_T = \Gamma$.

4. Semifields and Polarities

Let $\mathbb{S}$ be a presemifield of dimension n over a subfield $\mathbb{F}_q$ of its left nucleus and let $\mathbb{F}_s$ $(q = s^e)$ be a subfield of $\mathbb{F}_q$ contained in the center of $\mathbb{S}$. If $\mathbb{S}$ is a proper semifield, then $n \geq 2$. Also, up to isotopy, we may assume that $\mathbb{S} = (\mathbb{F}_{q^n}, +, \star)$, where

$$x \star y = \varphi_y(x),$$

with $\varphi_y \in \mathbb{E} = End(\mathbb{F}_{q^n}, \mathbb{F}_q)$. The set

$$\mathcal{C}_{\mathbb{S}} = \left\{\varphi_y : x \in \mathbb{F}_{q^n} \mapsto x \star y \in \mathbb{F}_{q^n} \,|\, y \in \mathbb{F}_{q^n}\right\} \subset \mathbb{E}$$

is the *semifield spread set* associated with $\mathbb{S}$ (*spread set* for short): $\mathcal{C}_{\mathbb{S}}$ is an $\mathbb{F}_s$-subspace of $\mathbb{E}$ of rank *en* and each non-zero element of $\mathcal{C}_{\mathbb{S}}$ is invertible. Hence, we can associate with $\mathbb{S}$ the $\mathbb{F}_s$-linear set $\Gamma(\mathbb{S})$ of rank *en* of the projective space $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_q) = PG(n^2-1,q)$ defined by the non-zero elements of $\mathcal{C}_{\mathbb{S}}$. Such a linear set turns out to be disjoint from the variety $\Omega(\mathcal{S}_{n,n})$ of $\mathbb{P}$ defined by the non-invertible elements of $\mathbb{E}$ and hence it is a geometric spread set. Isotopic semifields produce in $\mathbb{P}$ geometric spread sets which are equivalent under the action of the subgroup $H(\mathcal{S}_{n,n})$ of $P\Gamma L(n^2,q)$ fixing the systems of $\mathcal{S}_{n,n}$, and conversely (see [15] and [10]). On the other hand, the transpose of $\mathbb{S}$ corresponds to a geometric spread set which is equivalent to $\Gamma(\mathbb{S})$ under a collineation of $\mathcal{G}\setminus H(\mathcal{S}_{n,n})$, where $\mathcal{G}=Aut(\mathcal{S}_{n,n})$ ([17, Lemma 2]).

We explicitly note that Theorem 3.3 relies on a linear set Γ of $\mathbb{P}$ and on a $(2n-1)$-dimensional subspace T of $\mathbb{P}$ containing Γ, satisfying some suitable assumptions. In what follows we will show with an example that when $n>2$ and Γ is a geometric spread set associated with a field, then applying Theorem 3.3 we can get geometric spread sets whose associated presemifields are not isotopic to any Knuth derivative of a field, this depending on a suitable choice of the subspace T satisfying the relevant hypotheses.

To this extent recall that a Generalized Twisted Field $\mathbb{G}$ with center of order q is a presemifield of type $\mathbb{G}=\mathbb{G}(q^n,c,t,h)=\left(\mathbb{F}_{q^n},+,\star\right)$ ($q=p^e$, p prime, $n>2$) with

$$x\star y = yx - cy^{q^t}x^{q^h}, \tag{4.1}$$

where $c\in\mathbb{F}^*_{q^n}$ such that $N(c)\neq 1$ and $N(\cdot)$ denotes the Norm function of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, and $1\leq h,t\leq n-1$, $h\neq t$, and $gcd(n,h,t)=1$.
With this in mind, let $\Gamma=\left\{\left\langle x\in\mathbb{F}_{q^n}\mapsto Ax\in\mathbb{F}_{q^n}\right\rangle : A\in\mathbb{F}^*_{q^n}\right\}$ be the geometric spread set in $\mathbb{P}=PG\left(\mathbb{E},\mathbb{F}_q\right)=PG\left(n^2-1,q\right)$, with $\mathbb{E}=End\left(\mathbb{F}_{q^n},\mathbb{F}_q\right)$ and $n>2$, associated with the finite field $\mathbb{F}_{q^n}$. The set Γ is called *Desarguesian* geometric spread set. If we choose

$$T=T(c,m,h,t)=\left\{\left\langle x\in\mathbb{F}_{q^n}\mapsto Ax+Bx^{q^m}-cB^{q^t}x^{q^{m+h}}\right\rangle\in\mathbb{F}_{q^n}: \right.$$
$$\left. A,B\in\mathbb{F}_{q^n},\,(A,B)\neq(0,0)\right\}, \tag{4.2}$$

where $1\leq m,h,t\leq n-1$, $gcd\,(n,h,t)=1$, $h\neq t$, $h+m\not\equiv 0\,(mod\,n)$ and such that $c\in\mathbb{F}^*_{q^n}$ with $N(c)\neq 1$, straightforward computations show that

$$T^\perp=\left\{\left\langle x\mapsto\sum_{i=0}^{n-1}a_ix^{q^i}\right\rangle : a_i\in\mathbb{F}_{q^n}\text{ (not all zero) with }a_0=0\right.$$
$$\left.\text{and }a_m=c^{q^{n-t}}a_{m+h}^{q^{n-t}}\right\},$$

and hence $\Gamma\cap T^\perp=T^\perp\cap\mathcal{S}_{n,n}=\emptyset$, i.e., Conditions (P_1) and (P_2) of Theorem 3.3 are satisfied. Also, for each $X(\lambda)\in\mathcal{R}_1$, $\lambda\in\mathbb{F}^*_{q^n}$, an element of $\left\langle X(\lambda),T^\perp\right\rangle$ is a map of shape

$$\psi(x)=\alpha\lambda x+\sum_{i=1,i\neq m}^{n-1}\left(\alpha\lambda^{q^i}+a_i\right)x^{q^i}+\left(\alpha\lambda^{q^m}+c^{q^{n-t}}a_{m+h}^{q^{n-t}}\right)x^{q^m},$$

for some $\alpha \in \mathbb{F}^*_{q^n}$, and it belongs to Γ if and only if

$$\begin{cases} \alpha\lambda = A, \\ \alpha\lambda^{q^m} + c^{q^{n-t}} a^{q^{n-t}}_{m+h} = 0, \\ \alpha\lambda^{q^{m+h}} + a_{m+h} = 0, \\ \alpha\lambda^{q^i} + a_i = 0, \qquad \forall i \neq 0, m, m+h. \end{cases}$$

Combining the second and the third equations of the previous system we have

$$c = \alpha^{q^t - 1} \lambda^{q^{m+t} - q^{m+h}},$$

a contradiction since $N(c) \neq 1$. Hence, T and Γ satisfy also Condition (P_3) of Theorem 3.3 and

$$\Gamma^*_T = \Gamma^* \cap T = \left\{ \left\langle x \in \mathbb{F}_{q^n} \mapsto Bx^{q^m} - cB^{q^t} x^{q^{m+h}} \in \mathbb{F}_{q^n} \right\rangle : B \in \mathbb{F}^*_{q^n} \right\}$$

is a geometric spread set. Also, it is clear that Γ^*_T defines a presemifield $\mathbb{S}^*_T$ with multiplication

$$x \circ y = yx^{q^m} - cy^{q^t} x^{q^{m+h}},$$

which is isotopic to a Generalized Twisted Field $\mathbb{G} = \mathbb{G}\big(q^n, c, t, h\big)$.

So, we have that starting from the Galois field $\mathbb{F} = \mathbb{F}_{q^n}$ of order q^n, $n > 2$, we may obtain, by means of the geometric procedure exposed in Theorem 3.3, every Generalized Twisted Field of order q^n and center $\mathbb{F}_q$. We point out that apart from the procedure recently described in [12], this does not occur in any other procedure known giving rise to a presemifield starting from a given one (see [6, Chapter 105], [11, Chapter 6, Section 6], and [2]).

4.1. The Generalized Translation Dual of a Semifield

We point out that Conditions (P_1), (P_2), and (P_3) of Theorem 3.3 strongly involve the linear set Γ. The following version of that theorem assures that the geometric procedure can be applied to any geometric spread set contained in T, i.e., to any $\mathbb{F}_s$-linear set Γ of $PG\big(n^2-1, q\big)$, $q = s^e$, of rank ne disjoint from the variety $\Omega(\mathcal{S}_{n,n})$. We stress the fact that in contrast to Theorem 3.3, the relevant conditions only concern with the subspace T of $\mathbb{P}$ and the variety $\mathcal{S}_{n,n}$. Precisely, we prove the following theorem.

Theorem 4.1. *Let T be a $(2n-1)$-dimensional subspace of $\mathbb{P} = PG\big(n^2-1, q\big)$, $q = s^e$, satisfying the following conditions*:

(T_1) $T \cap T^\perp = \emptyset$;
(T_2) $T^\perp \cap \mathcal{S}_{n,n} = \emptyset$;
(T_3) $\big\langle X, T^\perp \big\rangle \cap T \subset \Omega(\mathcal{S}_{n,n})$, $\forall X \in \mathcal{R}_i$, *for a given* $i \in \{1, 2\}$.

Then, for each $\mathbb{F}_s$-linear set Γ of rank ne contained in T and disjoint from the variety $\Omega(\mathcal{S}_{n,n})$ (i.e., a geometric spread set), the set $\Gamma^*_T = \Gamma^* \cap T$, where Γ^* is defined as in (2.1), turns out to be a geometric spread set as well.

Proof. Since $\Gamma \subset T$, Condition (T_1) trivially implies (P_1); moreover, if $\langle X, T^{\perp}\rangle \cap T$ is contained in $\Omega(\mathcal{S}_{n,n})$, then Γ is disjoint from $\langle X, T^{\perp}\rangle \cap T$ because Γ is a geometric spread set, and hence (T_3) implies (P_3). By Theorem 3.3, the assertion follows. ∎

Remark 4.2. Differently from what happens for Theorem 3.3, the converse of Theorem 4.1 does not hold true. We may show this in the following way. Let $\Gamma = \{\langle x \in \mathbb{F}_{q^n} \mapsto Ax \in \mathbb{F}_{q^n}\rangle : A \in \mathbb{F}^*_{q^n}\}$ be the Desarguesian geometric spread set in $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_q) = PG(n^2-1, q)$, with $\mathbb{E} = End(\mathbb{F}_{q^n}, \mathbb{F}_q)$ and $n > 2$, associated with the finite field $\mathbb{F}_{q^n}$. If we choose

$$T = \left\{\left\langle x \in \mathbb{F}_{q^n} \mapsto Ax + Bx^{q^h} \in \mathbb{F}_{q^n}\right\rangle : A, B \in \mathbb{F}_{q^n},\ (A,B) \neq (0,0)\right\},$$

with $1 \leq h \leq n-1$, then

$$T^{\perp} = \left\{\left\langle x \mapsto \sum_{i=0}^{n-1} a_i x^{q^i}\right\rangle : a_i \in \mathbb{F}_{q^n} \text{ (not all zero) with } a_0 = a_h = 0\right\}.$$

It can be easily seen that Condition (T_1) holds true; also T and Γ satisfy the conditions in Theorem 3.3. Hence,

$$\Gamma^*_T = \left\{\left\langle x \in \mathbb{F}_{q^n} \mapsto Bx^{q^h} \in \mathbb{F}_{q^n}\right\rangle : B \in \mathbb{F}^*_{q^n}\right\}$$

is a geometric spread set and it is clearly a Desarguesian geometric spread set, as well. Observe, on the other hand, that for each $X(\lambda) \in \mathcal{R}_1$ (or, equivalently, for each $X'(\lambda) \in \mathcal{R}_2$), with $\lambda \in \mathbb{F}^*_{q^n}$, any element of $\langle X(\lambda), T^{\perp}\rangle \cap T$ is a map of shape

$$\psi(x) = \alpha\lambda x + \alpha\lambda^{q^h} x^{q^h},$$

for some $\alpha \in \mathbb{F}^*_{q^n}$ and with $1 \leq h \leq n$. Hence, $\langle X(\lambda), T^{\perp}\rangle \cap T \subset \Omega(\mathcal{S}_{n,n})$ if and only if the map $\psi(x)$ is non-invertible and this happens if and only if the equation $y^{q^h-1} = -1$ admits its solution in $\mathbb{F}_{q^n}$, i.e., if and only if q is even or q is odd and the integer $\frac{n}{gcd(n,h)}$ is even. This means that Condition (T_3) is not always satisfied.

We want to stress the fact that the construction described in the previous theorems can be applied to any polarity of $\mathbb{P}$ interchanging a family $\mathcal{R}_i$ with $\mathbb{T}_j$, for some $i, j \in \{1,2\}$. However, as we will show in the next remark, up to the action of the automorphism group $\mathcal{G} = Aut(\mathcal{S}_{n,n})$ of the Segre variety $\mathcal{S}_{n,n}$, working only with the polarity $\perp$ is not restrictive.

Remark 4.3. Let τ be any polarity of $\mathbb{P}$ satisfying one of the following properties:

$(\star_{ii})$ $X \in \mathcal{R}_i \Rightarrow X^{\tau} \in \mathbb{T}_i$, for each $X \in \mathcal{R}_i$ and for a given $i \in \{1,2\}$;
$(\star_{ij})$ $X \in \mathcal{R}_i \Rightarrow X^{\tau} \in \mathbb{T}_j$, for each $X \in \mathcal{R}_i$ and for $\{i,j\} = \{1,2\}$.

Let Γ be an $\mathbb{F}_s$-linear set contained in a $(2n-1)$-dimensional subspace T of $\mathbb{P}$ fulfilling conditions of Theorem 3.3 or Theorem 4.1 with respect to the polarity τ, and denote by $\Gamma^{*'}_T$ the resulting geometric spread set. The collineation $\varphi := \tau \cdot \perp$, where

$\perp$ is the polarity of $\mathbb{P}$ defined by (3.5), either fixes or interchanges the two systems of $\mathcal{S}_{n,n}$ according to Property $(\star_{ii})$ or Property $(\star_{ij})$ is satisfied, respectively. If $\varphi' := \tau' \cdot \perp'$ is the collineation of $\mathbb{P}' = PG(\mathbb{E}, \mathbb{F}_s) = PG(n^2 e - 1, s)$ induced by φ, then $\left(\Gamma_T^{*'}\right)^{\varphi} = \left(\Gamma^{\tau'} \cap T\right)^{\varphi'} = \Gamma^{\perp'} \cap T^{\varphi'} = \Gamma^* \cap T^{\varphi} = \Gamma^*_{T^{\varphi}}$, i.e., the geometric spread set $\Gamma_T^{*'}$ is $\mathcal{G}$-equivalent to the geometric spread set $\Gamma^*_{T^{\varphi}}$. In particular, if $(\star_{ii})$ holds, then $\Gamma_T^{*'}$ and $\Gamma^*_{T^{\varphi}}$ are $H(\mathcal{S}_{n,n})$-equivalent, where $H(\mathcal{S}_{n,n})$ is the subgroup of $\mathcal{G}$ fixing the systems of $\mathcal{S}_{n,n}$. Finally, we observe that we can produce, by conjugation, a range of polarities satisfying $(\star_{ii})$; indeed, for each $\varphi \in H(\mathcal{S}_{n,n})$, it is easy to see that $\tau = \varphi^{-1} \perp \varphi$ satisfies this property.

Let $\Gamma = \Gamma(\mathbb{S})$ be the geometric spread set associated with a presemifield $\mathbb{S}$. In the light of Theorem 4.1, starting from Γ, it is possible to construct other geometric spread sets of type Γ_T^* depending on the choice of a subspace T containing Γ and satisfying Conditions (T_1), (T_2), and (T_3).

Now we prove that, up to the choice of the polarity, the procedure is well defined with respect to the action of $\mathcal{G}$. We need the following lemma.

Lemma 4.4. *Let Γ and Γ' be $\mathbb{F}_s$-linear sets of $\mathbb{P} = PG(n^2-1, q)$ $(q = s^e)$ of rank ne, $\mathcal{G}$-equivalent via $\varphi \in H(\mathcal{S}_{n,n})$, i.e., $\Gamma' = \Gamma^{\varphi}$ and let T be a $(2n-1)$-dimensional subspace of $\mathbb{P}$ satisfying (T_1), (T_2), and (T_3) of Theorem 4.1 relatively to Γ and with respect to the polarity $\perp$. Then the subspace $T' = T^{\varphi}$ satisfies Conditions (T_1), (T_2), and (T_3) as well relatively to Γ' and with respect to the polarity $\tau = \varphi^{-1} \perp \varphi$, and the geometric spread sets $\Gamma'^{*'}_{T'}$ and Γ_T^* are $\mathcal{G}$-equivalent.*

Proof. The proof of the first part easily follows from the assumptions and observing that $X^{\varphi} \in \mathcal{R}_i$ for each $X \in \mathcal{R}_i$ and $\Omega(\mathcal{S}_{n,n})^{\varphi} = \Omega(\mathcal{S}_{n,n})$. Also, it is clear that $\Gamma'^{*'}_{T'} = \left(\Gamma_T^*\right)^{\varphi}$. ∎

Now, we are able to prove the following result.

Theorem 4.5. *The geometric procedure described in Theorem 4.1 is well defined up to isotopism.*

Proof. If $\mathbb{S}'$ is a presemifield isotopic to $\mathbb{S}$ and $\mathbb{S}_T^*$ is the presemifield associated with Γ_T^*, by means of the procedure described in Theorem 4.1, then by Lemma 4.4 and Remark 4.3 there exist a $(2n-1)$-dimensional subspace T' and a polarity τ satisfying $(\star_{ii})$ such that $\mathbb{S}'^{*'}_{T'}$ is isotopic to $\mathbb{S}_T^*$. By Remark 4.3 the assertion follows. ∎

Hence, it makes sense to give the following definition.

Definition 4.6. *The presemifield $\mathbb{S}_T^*$ associated with Γ_T^* is called a translation dual of $\mathbb{S}$ or* the translation dual of $\mathbb{S}$ with respect to T. *Also, we will refer to the geometric procedure described in Theorem 4.1 as the generalized translation dual operation of a semifield.*

4.2. The Classical Translation Dual

If $n = 2$, then $\mathbb{P} = PG(\mathbb{E}, \mathbb{F}_q) = PG(3, q)$ and the whole space $\mathbb{P}$ satisfies Properties (T_1) and (T_2) of Theorem 4.1. Also, $\mathcal{S}_{2,2}$ is the hyperbolic quadric $\mathcal{Q} = Q^+(3, q)$ of $\mathbb{P}$ whose associated polar form is

$$\sigma(\varphi_{a,b}, \varphi_{a',b'}) = a^q a' + a a'^q - b^q b' - b'^q b,$$

where $\varphi_{a,b}(x) = ax + bx^q$. Finally, $\mathbb{T}_1 = \mathcal{R}_1$ and $\mathbb{T}_2 = \mathcal{R}_2$ are the reguli of $\mathcal{Q}$ and hence also (T_3) is trivially satisfied. On the other hand, the polarity τ induced by the quadric $\mathcal{Q}$ satisfies Properties $(\star_{11})$ and $(\star_{22})$ of Remark 4.3. These facts imply that, if Γ is any geometric spread set contained in $\mathbb{P}$, then the polarity $\perp$ defined by (3.5) and the polarity τ yield two geometric spread sets in $\mathbb{P}$, say Γ^* an $\Gamma^{*'}$, respectively. The linear set $\Gamma^{*'}$ defines the "classical" translation dual of a semifield [15]. However, by Remark 4.3, Γ^* an $\Gamma^{*'}$ are equivalent under the action of the subgroup $H(\mathcal{S}_{2,2})$ of $Aut(\mathcal{Q})$ fixing the reguli, i.e., the associated presemifields are isotopic. So, if $n = 2$, the construction described in Theorem 4.1 returns the classical translation dual operation.

When $n > 2$ and $\langle \Gamma \rangle_q$ has dimension less than $2n-1$, there are different $(2n-1)$-dimensional subspaces T of $\mathbb{P}$ containing Γ. In such a case, starting from a geometric spread set $\Gamma(\mathbb{S})$, it makes sense to search for suitable subspaces T of $\mathbb{P}$ satisfying conditions of Theorem 4.1. In this regard, by Remark 4.3, up to isotopisms and up to the transpose operation, working only with the polarity $\perp$ is not restrictive.

4.3. The Symplectic Dual

Let $n = 3$ and q **odd**. Let

$$T = \left\{ \left\langle x \in \mathbb{F}_{q^3} \mapsto ax + b^q x^q + b x^{q^2} \in \mathbb{F}_{q^3} \right\rangle \,\middle|\, a, b \in \mathbb{F}_{q^3},\ (a, b) \neq (0, 0) \right\}$$

be the 5-dimensional subspace of $\mathbb{P} = PG\big(\mathbb{E}, \mathbb{F}_q\big) = PG(8, q)$, where $\mathbb{E} = End\big(\mathbb{F}_{q^3}, \mathbb{F}_q\big)$, defined by the self-adjoint maps with respect to the non-degenerate bilinear form (3.1). It is easy to verify that

$$T^\perp = \left\{ \left\langle x \in \mathbb{F}_{q^3} \mapsto \beta^q x^q - \beta x^{q^2} \in \mathbb{F}_{q^3} \right\rangle \,\middle|\, \beta \in \mathbb{F}^*_{q^3} \right\},$$

and that $T^\perp \subset \Omega(\mathcal{S}_{3,3})$, $T^\perp \cap \mathcal{S}_{3,3} = \emptyset$, and $T \cap T^\perp = \emptyset$. Also, direct computations show that

$$\left\langle X(\lambda), T^\perp \right\rangle \cap T = \left\{ \langle t_\alpha \circ Tr \circ t_\lambda + t_\lambda \circ Tr \circ t_\alpha \rangle \colon \alpha \in \mathbb{F}^*_{q^3} \right\}$$

and such a subspace is contained in $\Omega(\mathcal{S}_{3,3})$, since $ker(t_\alpha \circ Tr \circ t_\lambda) \cap ker(t_\lambda \circ Tr \circ t_\alpha) \neq \{\mathbf{0}\}$ for each $\alpha \in \mathbb{F}^*_{q^3}$. So, by Theorem 4.1, if $\Gamma = \Gamma(\mathbb{S})$ is a geometric spread set contained in T, then Γ^*_T is a geometric spread set as well and the presemifield $\mathbb{S}^*_T$ arising from Γ^*_T is the symplectic dual of $\mathbb{S}$ as constructed in [17].

In [17, Theorem 4], the authors apply such a procedure to a Desarguesian geometric spread contained in T, proving that the symplectic dual of the Galois field $\mathbb{F}_{q^3}$

is isotopic to a Generalized Twisted Field. Also in [17] another geometric spread set Γ contained in T when $q = s^2$ is exhibited precisely, $\Gamma = L(U)$, where

$$U = \left\{ x \in \mathbb{F}_{s^6} \mapsto \alpha x + \beta^{s^2} \xi x^{s^2} + \beta \xi x^{s^4} \in \mathbb{F}_{s^6} \,\middle|\, \alpha, \beta \in \mathbb{F}_{s^3} \right\},$$

with $\xi \in \mathbb{F}_{s^2} \setminus \mathbb{F}_s$ such that $\xi^s = -\xi$. Moreover, it is proven that Γ is a Baer subgeometry of $\mathbb{P} = PG\big(5, s^2\big)$ isomorphic to $PG(5, s)$ [17, Theorem 1] and that the presemifield associated with Γ is isotopic to its symplectic dual $\big($i.e., the presemifield associated with $\Gamma^*_T\big)$ [17, Proposition 2].

This example shows how in some cases there is not much freedom in finding an appropriate subspace T to apply the procedure. In fact, since in the relevant case the geometric spread set Γ is a Baer subgeometry isomorphic to $PG(5, s)$, the subspace T is the unique 5-dimensional subspace of $\mathbb{P} = PG\big(8, s^2\big)$ containing it.

4.4. A Special Case

In this section we give an example of a subspace T satisfying the assumptions of Theorem 4.1. Indeed, suppose $n > 2$. Let

$$T = \Big\{ \Big\langle \varphi_{a_0, 0, \ldots, a_h, 0, \ldots, 0} \colon x \in \mathbb{F}_{q^n} \mapsto a_0 x + a_h x^{q^h} \in \mathbb{F}_{q^n} \Big\rangle :$$
$$a_0, a_h \in \mathbb{F}_{q^n}, (a_0, a_h) \neq (0, 0) \Big\}.$$

Then,

$$T^\perp = \left\{ \left\langle x \mapsto \sum_{i=0}^{n-1} a_i x^{q^i} \right\rangle \colon a_i \in \mathbb{F}_{q^n}, a_0 = a_h = 0, \right.$$
$$\left. (a_1, \ldots, a_{h-1}, a_{h+1}, \ldots, a_{n-1}) \neq (0, \ldots, 0) \right\},$$

and hence $T \cap T^\perp = T^\perp \cap \mathcal{S}_{n,n} = \emptyset$. Moreover, for each $X(\lambda) \in \mathcal{R}_1$ (or, equivalently, for each $X'(\lambda) \in \mathcal{R}_2$), with $\lambda \in \mathbb{F}^*_{q^n}$, we have that an element of $\langle X(\lambda), T^\perp \rangle \cap T$ is a map of shape

$$\psi(x) = \alpha \lambda x + \alpha \lambda^{q^h} x^{q^h},$$

for some $\alpha \in \mathbb{F}^*_{q^n}$, and it belongs to $\Omega(\mathcal{S}_{n,n})$ if and only if the map $\psi(x)$ is non-invertible and, as observed in Remark 4.2, this happens if and only if either q is even or q is odd and the integer $\frac{n}{gcd(n,h)}$ is even. It follows that, in this case, by applying Theorem 4.1, we can always construct the translation dual of any geometric spread set contained in T. Precisely, if $\Gamma = L(U)$ is a geometric spread set of rank ne ($q = s^e$) contained in T such that U does not have maps of the form $x \mapsto \lambda x^{q^h}$, for some element $\lambda \in \mathbb{F}^*_{q^n}$, then

$$\Gamma := \left\{ \left\langle x \in \mathbb{F}_{q^n} \mapsto yx + f(y) x^{q^h} \in \mathbb{F}_{q^n} \right\rangle \,\middle|\, y \in \mathbb{F}^*_{q^n} \right\},$$

where $f(y)$ is an $\mathbb{F}_s$-linear map of $\mathbb{F}_{q^n}$. Then

$$\Gamma_T^* := \left\{ \left\langle x \in \mathbb{F}_{q^n} \mapsto \bar{f}(z)x - zx^{q^h} \in \mathbb{F}_{q^n} \right\rangle \mid z \in \mathbb{F}_{q^n}^* \right\},$$

where $\bar{f}$ is the adjoint map of f as defined in (3.1). Hence, the procedure described in Theorem 4.1 can be applied to any geometric spread set defined by monomial or binomial functions, as the Desarguesian geometric spread set (see Remark 4.2) and the spread set associated with a Generalized Twisted Field.

5. Final Remarks

In [2] and [10] the authors presented a geometric construction giving rise to a presemifield $\mathbb{S}$ of order q^n whose center contains $\mathbb{F}_q$ starting from a Desarguesian n-spread $\mathcal{D}$ of a vector space $V = V(rn, q)$, and two subspaces of V satisfying suitable conditions. This procedure is called the *BEL-construction* in [2].

When $r = 2$, the roles of such subspaces can be switched yielding another presemifield which is denoted by $\mathbb{S}^{\mathbb{T}}$. In [9, Remark 3.1] it was stated that such a presemifield is, up to isotopisms, one of the six Knuth derivatives of the translation dual of $\mathbb{S}$. More precisely, in [16, Theorem 2.2], the authors prove that the classical translation dual $\mathbb{S}^{\perp}$ of $\mathbb{S}$ is isotopic to $\left(\mathbb{S}^{\mathbb{T}}\right)^{dtd}$, where d and t are the dual and the transpose operations of Knuth, respectively. So, it makes sense to ask whether or not the generalized translation dual and the switching operation are linked working with semifields having dimension greater than two over their left nucleus. Although, it must be observed that while the vector space underlying the switching operation is endowed with a Desarguesian spread, this is not expressly required in the geometric procedure behind the generalized translation dual presented here. For instance, with regard to the symplectic dual operation described in Section 4.3, it is possible to see that there is no Desarguesian spread of $PG(8, q)$, containing one of the two systems of the Segre variety $\mathcal{S}_{3,3}$, inducing a Desarguesian spread in the 5-dimensional subspace T. However, it would be interesting to study a possible link between the two procedures such as in the classical translation dual case.

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