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# Recurrences for Eulerian Polynomials of Type B and Type D

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**Abstract.** We introduce new recurrences for the type B and type D Eulerian polynomials, and interpret them combinatorially. These recurrences are analogous to a well-known recurrence for the type A Eulerian polynomials. We also discuss their relationship to polynomials introduced by Savage and Visontai in connection to the real-rootedness of the corresponding Eulerian polynomials.

*Keywords*: Eulerian polynomials, descent number, type B Coxeter group, signed permutations, type D Coxeter group, even signed permutations, interlacing roots, compatible polynomials

## 1. Introduction

Let $S_n$ denote the group of permutations on the set $[n] := \{1, 2, \ldots, n\}$. Let $A_n(t)$ denote the type A Eulerian polynomial, that is,

$$A_n(t) = \sum_{\pi \in S_n} t^{\operatorname{des}(\pi)},$$

where the descent number of $\pi \in S_n$ is defined by

$$\operatorname{des}(\pi) = |\{i \in [n-1] \colon \pi(i) > \pi(i+1)\}|.$$

These polynomials were first introduced by Euler, although he did not define them via descents of permutations (see [9]). The following generating function identity, which is attributed to Euler, is an equivalent definition of these polynomials

$$\sum_{n \geq 0} A_n(t) \frac{x^n}{n!} = \frac{t-1}{t - \exp((t-1)x)}.$$

There are numerous recurrences for the type A Eulerian polynomials. For the purposes of this paper we call the reader's attention to the following recurrence which holds for $n \geq 1$, and is also attributed to Euler

$$A_n(t) = \sum_{k=0}^{n-1} A_k(t) \binom{n}{k} (t-1)^{n-k-1}. \tag{1.1}$$

The main goal of this paper is to find recurrences analogous to (1.1) for the type B and type D Eulerian polynomials, and interpret them combinatorially.

In Section 2 we establish notation for the group of signed permutations, denoted $B_n$. This is a Coxeter group of type B, and thus it has an analogous notion of descent. We let

$$B_n(t) = \sum_{\pi \in B_n} t^{\mathrm{des_B}(\pi)},$$

which are called the type B Eulerian polynomials. While there are several recurrences for these polynomials (see, for example, [3, 5, 6, 14]), we give a new recurrence, Equation (1.2) below, which we consider to be an analog of (1.1).

**Theorem 1.1.** *Let*

$$P_n(t) = \sum_{k=0}^{n-1} B_k(t) \binom{n}{k} (t-1)^{n-k-1}.$$

*For $n \geq 1$ we have*

$$B_n(t) = P_n(t) + t^n P_n\left(t^{-1}\right). \tag{1.2}$$

*Furthermore, if we define $B_n^+ = \{\pi \in B_n \colon \pi(n) > 0\}$, then for $n \geq 1$ we have*

$$P_n(t) = \sum_{\pi \in B_n^+} t^{\mathrm{des_B}(\pi)}.$$

*Remark 1.2.* Given a polynomial $f(t) = \sum_{k=0}^{n} c_k t^k$ of degree $n$, the reverse of $f$, denoted $\widetilde{f}$, is given by $\widetilde{f}(t) = t^n f(t^{-1}) = \sum_{k=0}^{n} c_{n-k} x^k$. Since $P_n(t)$ has degree $n-1$, Theorem 1.1 may be restated as

$$B_n(t) = P_n(t) + t\widetilde{P}_n(t).$$

The recurrence (1.2) appearing in Theorem 1.1 is new in the sense that it does not explicitly appear in the literature. However, we will show how this recurrence can be deduced from a well-known generating function identity. We will, in addition, provide a combinatorial proof of Theorem 1.1. We also note that the polynomials $\sum_{\pi \in B_n^+} t^{\mathrm{des_B}(\pi)}$ are a natural object, and have been studied in the literature (see, for example, [1]).

A recurrence similar to (1.2) also holds for the type D case. We begin Section 3 by establishing notation for the group of even signed permutations, denoted $D_n$. This is a Coxeter group of type D, and thus has an analogous notion of descent. We let

$$D_n(t) = \sum_{\pi \in D_n} t^{\mathrm{des_D}(\pi)},$$

which are called the type D Eulerian polynomials. Again, there are several recurrences for these polynomials (see, for example, [3–5, 14]). Here we give a new recurrence, Equation (1.3) below, which is analogous to (1.1) and (1.2).

**Theorem 1.3.** *Let*

$$Q_n(t) = \sum_{k=0}^{n-1} D_k(t) \binom{n}{k} (t-1)^{n-k-1}.$$

*For $n \geq 2$, we have*

$$D_n(t) = Q_n(t) + t^n Q_n\left(t^{-1}\right) = Q_n(t) + t\widetilde{Q}_n(t). \tag{1.3}$$

While there is a well-known generating function identity for the type D Eulerian polynomials, we see no easy way to deduce Theorem 1.3 from this identity. Instead we provide a combinatorial proof of Theorem 1.3.

It turns out that certain refinements of the polynomials $Q_n(t), t^n Q\left(t^{-1}\right), P_n(t)$, and $t^n P\left(t^{-1}\right)$ were introduced by Savage and Visontai in [13], where they proved Brenti's [3] conjecture that the type D Eulerian polynomials have only real roots. We discuss this connection in Section 4.

## 2. A Recurrence for the Eulerian Polynomials of Type B

As mentioned above, we let $B_n$ denote the group of signed permutations. An element $\pi \in B_n$ is a bijection on the integers $[-n, n]$ such that $\pi(-i) = -\pi(i)$, in particular $\pi(0) = 0$. We will often write elements in the "window" notation, i.e., $\pi = [\pi(1), \pi(2), \ldots, \pi(n)]$, and we will also use cycle notation. $B_n$ is a type B Coxeter group with generators $\tau_0, \tau_1, \ldots, \tau_{n-1}$ where $\tau_0 = [-1, 2, 3, \ldots, n] = (1, -1)$, and $\tau_j = (j, j+1)$ for $j = 1, \ldots, n-1$. Given $\pi \in B_n$, define

$$\mathrm{DES_B}(\pi) = \{i \in [0, n-1] \colon \pi(i) > \pi(i+1)\}$$

and

$$\mathrm{des_B}(\pi) = |\mathrm{DES_B}(\pi)|.$$

This definition coincides with the notion of Coxeter descent (i.e., $i \in \mathrm{DES_B}(\pi)$ if and only if the Coxeter length of $\pi\tau_i$ is less than the Coxeter length of $\pi$). The type B Eulerian polynomials, $B_n(t)$, satisfy the identity [14, Prop. 7.1 (b)]

$$\frac{B_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (2k+1)^n t^k,$$

from which one can deduce (see also [3, Theorem 3.4 (iv) with $q = 1$])

$$\sum_{n \geq 0} B_n(t) \frac{x^n}{n!} = \frac{(1-t)\exp(x(1-t))}{1 - t\exp(2x(1-t))}. \tag{2.1}$$

Next we show how Theorem 1.1 can be deduced from (2.1).

*First Proof of Theorem 1.1.* From (2.1) we have

$$\left(\sum_{n \geq 0} B_n(t) \frac{x^n}{n!}\right) \left(\frac{\exp(-x(1-t)) - t\exp(x(1-t))}{(1-t)}\right) = 1,$$

$$\left(\sum_{n\geq 0} B_n(t)\frac{x^n}{n!}\right)\left(\sum_{n\geq 0}\frac{x^n}{n!}(1-t)^{n-1}\left((-1)^n-t\right)\right)=1,$$

$$\sum_{n\geq 0}\frac{x^n}{n!}\sum_{k=0}^{n}\binom{n}{k}B_k(t)(1-t)^{n-k-1}\left((-1)^{n-k}-t\right)=1.$$

So for $n \geq 1$, we have

$$\sum_{k=0}^{n}\binom{n}{k}B_k(t)(1-t)^{n-k-1}\left((-1)^{n-k}-t\right)=0,$$

which implies

$$\begin{aligned} B_n(t) &= \sum_{k=0}^{n-1}\binom{n}{k}B_k(t)(1-t)^{n-k-1}\left(t+(-1)^{n-k-1}\right) \\ &= P_n(t)+t\sum_{k=0}^{n-1}\binom{n}{k}B_k(t)(1-t)^{n-k-1} \\ &= P_n(t)+t\sum_{k=0}^{n-1}\binom{n}{k}t^k B_k\left(t^{-1}\right)t^{n-k-1}\left(t^{-1}-1\right)^{n-k-1} \\ &= P_n(t)+t^n P_n\left(t^{-1}\right). \end{aligned}$$

The penultimate step uses the fact that $B_k(t)$ is symmetric of degree $k$ (see [3, Theorem 2.4]). ∎

Next we provide a combinatorial proof of Theorem 1.1. We begin with two lemmas, and then show they imply the theorem.

**Lemma 2.1.** *Define* $B_n^+ = \{\pi \in B_n \colon \pi(n) > 0\}$. *Then for* $n \geq 1$,

$$\sum_{\pi\in B_n^+} t^{\mathrm{des_B}(\pi)} = P_n(t) = \sum_{k=0}^{n-1}\binom{n}{k}B_k(t)(t-1)^{n-k-1}.$$

*Proof.* Following previous work in [11], we say that a signed permutation $\pi$ has a drop at position $i$ if $\pi(i) < i$. If $\pi$ has a drop at position $i$, then we say the drop size at position $i$ is $\min\{i-\pi(i), i\}$. The type B maximum drop of $\pi$, denoted $\mathrm{maxdrop_B}(\pi)$, is the maximum of all drop sizes occurring in $\pi$. In other words,

$$\mathrm{maxdrop_B}(\pi) := \max\left\{\max\{i-\pi(i)\colon \pi(i)>0\}, \max\{i\colon \pi(i)<0\}\right\}.$$

We also define

$$B_n^{(j)}(t) = \sum_{\substack{\pi\in B_n \\ \mathrm{maxdrop_B}(\pi)\leq j}} t^{\mathrm{des_B}(\pi)}.$$

The definition of type B maximum drop is motivated by the so-called type B bubble sort, in that $\mathrm{maxdrop_B}(\pi)$ is equal to the type B bubble sort complexity of $\pi$. For our purposes in this paper, we notice that $B_n^+ = \{\pi \in B_n \colon \mathrm{maxdrop_B}(\pi) \le n-1\}$. Thus

$$\sum_{\pi \in B_n^+} t^{\mathrm{des_B}(\pi)} = B_n^{(n-1)}(t).$$

It is proved (combinatorially) that for any nonnegative integers $m$ and $j$ we have [11, Theorem 1.6]

$$B_{m+j+1}^{(j)}(t) = \sum_{k=1}^{j+1} \binom{j+1}{k} B_{m+j+1-k}^{(j)}(t)(t-1)^{k-1}.$$

Setting $m = 0$ and replacing $j$ with $n-1$ we obtain

$$\begin{aligned} B_n^{(n-1)}(t) &= \sum_{k=1}^{n} \binom{n}{k} B_{n-k}^{(n-1)}(t)(t-1)^{k-1} \\ &= \sum_{k=1}^{n} \binom{n}{k} B_{n-k}(t)(t-1)^{k-1} \\ &= \sum_{k=0}^{n-1} \binom{n}{k} B_k(t)(t-1)^{n-k-1}, \end{aligned}$$

where the penultimate step follows from the fact that $B_k^{(j)}(t) = B_k(t)$ whenever $k \le j$. ∎

**Lemma 2.2.** *Define $B_n^- = \{\pi \in B_n \colon \pi(n) < 0\}$. Then for $n \ge 1$,*

$$\sum_{\pi \in B_n^-} t^{\mathrm{des_B}(\pi)} = t^n P_n\big(t^{-1}\big).$$

*Proof.* Define a bijection $\phi \colon B_n^+ \to B_n^-$ by $\phi(\pi(i)) = -\pi(i)$ for $i = 1, \ldots, n$. Then $\pi(1) < 0$ if and only if $\phi(\pi(1)) > 0$. And for $i = 1, 2, \ldots, n-1$ we have $\pi(i) > \pi(i+1)$ if and only if $\phi(\pi(i)) < \phi(\pi(i+1))$. Thus, $\mathrm{des_B}(\pi) + \mathrm{des_B}(\phi(\pi)) = n$. Then,

$$\sum_{\pi \in B_n^-} t^{\mathrm{des_B}(\pi)} = \sum_{\pi \in B_n^+} t^{\mathrm{des_B}(\phi(\pi))} = \sum_{\pi \in B_n^+} t^{n-\mathrm{des_B}(\pi)} = t^n \sum_{\pi \in B_n^+} t^{-\mathrm{des_B}(\pi)} = t^n P_n\big(t^{-1}\big). \quad ∎$$

*Second Proof of Theorem 1.1.* Using Lemmas 2.1 and 2.2, we have

$$B_n(t) = \sum_{\pi \in B_n^+} t^{\mathrm{des_B}(\pi)} + \sum_{\pi \in B_n^-} t^{\mathrm{des_B}(\pi)} = P_n(t) + t^n P_n\big(t^{-1}\big). \quad ∎$$

## 3. A Recurrence for the Eulerian Polynomials of Type D

Let $D_n$ denote the group of even signed permutations. We call $\pi$ an even signed permutation if $\pi$ is a signed permutation such that there are an even number of negative letters among $\pi(1), \pi(2), \ldots, \pi(n)$. $D_n$ is a Coxeter group of type D with generators

$\varepsilon_0, \tau_1, \ldots, \tau_{n-1}$ where $\varepsilon_0 = [-2,-1,3,4,5,\ldots,n] = (1,-2)$, and $\tau_j = (j, j+1)$ for $j = 1,\ldots,n-1$. Given $\pi \in D_n$, define

$$\mathrm{DES}(\pi) = \{i \in [n-1] \colon \pi(i) > \pi(i+1)\},$$

$$\mathrm{DES}_{\mathrm{D}}(\pi) = \begin{cases} \mathrm{DES}(\pi), & \text{if } \pi(1)+\pi(2) > 0, \\ \mathrm{DES}(\pi) \cup \{0\}, & \text{if } \pi(1)+\pi(2) < 0, \end{cases}$$

$$\mathrm{des}_{\mathrm{D}}(\pi) = |\mathrm{DES}_{\mathrm{D}}(\pi)|.$$

As in the type B case, this definition is motivated by the fact that it coincides with the notion of type D Coxeter descent. The type D Eulerian polynomials, $D_n(t)$, satisfy the following generating function formula due to Brenti [3]

$$\sum_{n\geq 0} D_n(t)\frac{x^n}{n!} = \frac{(1-t)\exp(x(1-t)) - xt(1-t)\exp(2x(1-t))}{1-t\exp(2x(1-t))}. \tag{3.1}$$

Although we can obtain a recurrence from (3.1) by an approach similar to that of the first proof of Theorem 1.1, doing so yields the following somewhat unpleasant formula

$$D_n(t) = \sum_{k=0}^{n-1} \binom{n}{k} D_k(t)$$
$$\times \sum_{j=0}^{n-k} \frac{(n-k)!}{j!} t^{n-j-k}(1-t)^{j-1}\left[t(n-j-k+1)^j - (n-j-k-1)^j\right].$$

Instead we will prove Theorem 1.3 in a manner similar to that of the second proof of Theorem 1.1. The first step is to obtain a type D analog of Lemma 2.1, which we accomplish by adapting the methods from [11], which in turn are extensions of methods of Chung, Claesson, Dukes, and Graham [8].

We begin by discussing standardization of permutations. Suppose we have a finite set $C = \{c_1, c_2, \ldots, c_n\} \subset \mathbb{N}$ with $c_1 < c_2 < \cdots < c_n$, and a permutation $\pi$ on $C$. The *standardization* of $\pi$, denoted $\mathrm{st}(\pi)$, is the permutation in $S_n$ obtained from $\pi$ by replacing $c_i$ with $i$. For example, $\mathrm{st}([4,5,2,9,7]) = [2,3,1,5,4]$. Given a signed permutation $\pi$, let $|\pi| = [|\pi(1)|, |\pi(2)|, \ldots, |\pi(n)|]$. We call $\pi$ a signed permutation on the set $C$, if $\pi$ is a word over $\mathbb{Z}$ and $|\pi|$ is permutation on $C$. We define the *signed standardization* of a signed permutation, denoted $\mathrm{sst}(\pi)$, by

$$\mathrm{sst}(\pi)(i) = \begin{cases} \mathrm{st}(|\pi|)(i), & \text{if } \pi(i) > 0, \\ -\mathrm{st}(|\pi|)(i), & \text{if } \pi(i) < 0. \end{cases}$$

For example, $\mathrm{sst}([-6,4,-2,9,7]) = [-3,2,-1,5,4]$. If the set $C$ is fixed, then the inverse of sst, denoted $\mathrm{sst}_C^{-1}$, is well defined. For example, if $C = \{2,4,6,7,9\}$ and $\pi = [-3,2,-1,5,4]$, then $\mathrm{sst}_C^{-1}(\pi) = [-6,4,-2,9,7]$. If we extend the definition of type D descent set for any word over $\mathbb{Z}$, then it is clear that both sst and $\mathrm{sst}_C^{-1}$

preserve the type D descent set of a word. For example, $\mathrm{DES_D}([-6,4,-2,9,7]) = \{0,2,4\} = \mathrm{DES_D}([-3,2,-1,5,4])$.

Given $S \subseteq [0,n-1]$ and $U \subseteq D_n$, define

$$U(S) = \{\pi \in U \colon \mathrm{DES_D}(\pi) \supseteq S\},$$

$$r_n(S) = \max\{i \colon [n-i,n-1] \subseteq S\},$$

with $r_n(S) = 0$ if $n-1 \notin S$. For our purposes $U$ will either be $D_n^+$, which we define by $D_n^+ = \{\pi \in D_n \colon \pi(n) > 0\}$, or be all of $D_n$. Let $\binom{[n]}{j}$ denote the set of $j$-element subsets of $[n]$. Given $\pi \in D_n^+(S)$, define a map $\psi$ by

$$\psi(\pi) = (\sigma, X),$$

where

$$\sigma = \mathrm{sst}([\pi(1),\pi(2),\ldots,\pi(n-i-1)])$$

and

$$X = \{\pi(n-i),\pi(n-i+1),\ldots,\pi(n)\},$$

where $i = r_n(S)$.

For example, consider $S = \{1,6,7\}$ and $\pi = [2,-3,5,1,-8,7,6,4] \in D_8^+(S)$. Then $r_8(S) = 2$ and $\psi(\pi) = (\sigma, X)$ where $\sigma = [2,-3,4,1,-5]$ and $X = \{4,6,7\}$.

**Lemma 3.1.** *Given any set $S \subseteq [0,n-1]$, the map*

$$\psi \colon D_n^+(S) \to D_{n-i-1}(S \cap [0,n-i-2]) \times \binom{[n]}{i+1}$$

*is a bijection, where $i = r_n(S)$. Consequently,*

$$|D_n^+(S)| = |D_{n-i-1}(S \cap [0,n-i-2])| \binom{n}{i+1}.$$

*Proof.* Given $\pi \in D_n^+(S)$, let $\psi(\pi) = (\sigma, X)$ as above. Since signed standardization preserves type D descent set, $\sigma \in D_{n-i-1}(S \cap [0,n-i-2])$. Since $[n-i,n-1] \subseteq \mathrm{DES_D}(\pi)$ and $\pi(n) > 0$, we have $X \in \binom{[n]}{i+1}$. Thus $\psi$ is well defined.

Next we describe the inverse map. Let $\sigma \in D_{n-i-1}(T)$ where $T = S \cap [0,n-i-2]$, and let $X = \{x_1,x_2,\ldots,x_{i+1}\} \in \binom{[n]}{i+1}$ where $x_1 < x_2 < \cdots < x_{i+1}$. Then set

$$\psi^{-1}(\sigma,X) = \mathrm{sst}^{-1}_{[n]\setminus X}(\sigma) * (x_{i+1},x_i,\ldots,x_1) \in D_n^+(T \cup [n-i,n-1]),$$

where $*$ denotes concatenation.

For example, let $n = 8$ and let $S = \{1,6,7\} \subseteq [0,7]$. Thus $i = 2$ and $T = \{1\}$. Consider $\sigma = [2,-3,4,1,-5] \in D_5(\{1\})$ and $X = \{4,6,7\}$. Then $[n] \setminus X = \{1,2,3,5,8\}$ and $\mathrm{sst}^{-1}_{[n]\setminus X}(\sigma) = [2,-3,5,1,-8]$, thus $\psi^{-1}(\sigma,X) = [2,-3,5,1,-8,7,6,4] \in D_8(\{1,6,7\})$.

Clearly, $\psi^{-1}(\psi(\pi)) = \pi$. Since $T \subseteq [0,n-i-2]$, we have $r_n(T \cup [n-i,n-1]) = i$. From this it follows that $\psi\big(\psi^{-1}(\sigma,X)\big) = (\sigma,X)$. ∎

**Lemma 3.2.** *Let* $D_n^+ = \{\pi \in D_n \colon \pi(n) > 0\}$. *Then for* $n \geq 2$

$$\sum_{\pi \in D_n^+} t^{\mathrm{des}_{\mathrm{D}}(\pi)} = Q_n(t) = \sum_{k=0}^{n-1} \binom{n}{k} D_k(t)(t-1)^{n-k-1}.$$

*Proof.* Given $U \subseteq D_n$ we have

$$\begin{aligned}
\sum_{\pi \in U} (t+1)^{\mathrm{des}_{\mathrm{D}}(\pi)} &= \sum_{\pi \in U} \sum_{j=0}^{\mathrm{des}_{\mathrm{D}}(\pi)} \binom{\mathrm{des}_{\mathrm{D}}(\pi)}{j} t^j \\
&= \sum_{\pi \in U} \sum_{S \subseteq \mathrm{DES}_{\mathrm{D}}(\pi)} t^{|S|} \\
&= \sum_{S \subseteq [0,n-1]} t^{|S|} \sum_{\pi \in U(S)} 1 \\
&= \sum_{S \subseteq [0,n-1]} t^{|S|} |U(S)|. \qquad (3.2)
\end{aligned}$$

Replacing $U$ with $D_n^+$ and using Lemma 3.1, we have

$$\begin{aligned}
\sum_{\pi \in D_n^+} (t+1)^{\mathrm{des}_{\mathrm{D}}(\pi)} &= \sum_{S \subseteq [0,n-1]} t^{|S|} |D_n^+(S)| \\
&= \sum_{S \subseteq [0,n-1]} t^{|S|} |D_{n-r_n(S)-1}(S \cap [0, n-r_n(S)-2])| \binom{n}{r_n(S)+1} \\
&= \sum_{i=0}^{n-1} \sum_{\substack{S \subseteq [0,n-1] \\ r_n(S)=i}} t^{|S|} |D_{n-i-1}(S \cap [0, n-i-2])| \binom{n}{i+1} \\
&= \sum_{i=0}^{n-1} \binom{n}{i+1} t^i \sum_{\substack{S \subseteq [0,n-1] \\ r_n(S)=i}} t^{|S|-i} |D_{n-i-1}(S \cap [0, n-i-2])|.
\end{aligned}$$

Recall that if $r_n(S) = i$, then $S \supseteq [n-i, n-1]$ and $n-i-1 \notin S$. Therefore, each such $S$ can be expressed as $S = T \cup [n-i, n-1]$ for some $T \subseteq [0, n-i-2]$. Thus,

$$\begin{aligned}
\sum_{\pi \in D_n^+} (t+1)^{\mathrm{des}_{\mathrm{D}}(\pi)} &= \sum_{i=0}^{n-1} \binom{n}{i+1} t^i \sum_{T \subseteq [0,n-i-2]} t^{|T|} |D_{n-i-1}(T)| \\
&= \sum_{i=0}^{n-1} \binom{n}{i+1} t^i \sum_{\pi \in D_{n-i-1}} (t+1)^{\mathrm{des}_{\mathrm{D}}(\pi)} \\
&= \sum_{i=0}^{n-1} \binom{n}{i+1} t^i D_{n-i-1}(t+1)
\end{aligned}$$

$$= \sum_{k=0}^{n-1} \binom{n}{k} t^{n-1-k} D_k(t+1),$$

where the second equality follows from (3.2), and the last equality is obtained by setting $i = n-1-k$. Finally, the lemma follows by replacing $t$ with $t-1$. ∎

Next we provide the analog of Lemma 2.2.

**Lemma 3.3.** *Define* $D_n^- = \{\pi \in D_n \colon \pi(n) < 0\}$. *Then for* $n \geq 2$,

$$\sum_{\pi \in D_n^-} t^{\mathrm{des_D}(\pi)} = t^n Q_n\big(t^{-1}\big).$$

*Proof.* We want a bijection $\varphi \colon D_n^+ \to D_n^-$. Note that we cannot flip all the signs as we did in the type B case, since we must ensure that $\varphi(\pi)$ has an even number of negative letters. So we define

$$\varphi(\pi) = \begin{cases} [-\pi(1), -\pi(2), \ldots, -\pi(n)], & \text{if } n \text{ is even,} \\ [\pi(1), -\pi(2), -\pi(3), \ldots, -\pi(n)], & \text{if } n \text{ is odd.} \end{cases}$$

We claim that $\mathrm{des_D}(\pi) + \mathrm{des_D}(\varphi(\pi)) = n$. The claim follows immediately if $n$ is even, just as in the type B case. So assume $n$ is odd. Clearly, $\pi(i) > \pi(i+1)$ if and only if $\varphi(\pi(i)) < \varphi(\pi(i+1))$ for $i = 2, 3, \ldots, n-1$. To complete the proof of this claim, we show that

$$|\mathrm{DES_D}(\pi) \cap [0,1]| + |\mathrm{DES_D}(\varphi(\pi)) \cap [0,1]| = 2.$$

Indeed, we have $\{0,1\} \subseteq \mathrm{DES_D}(\pi)$ if and only if $\pi(1) + \pi(2) < 0$ and $\pi(1) < \pi(2)$, which holds if and only if $-\pi(2) > \pi(1)$ and $-\pi(2) > -\pi(1)$. Thus,

$$0 \in \mathrm{DES_D}(\pi) \text{ and } 1 \in \mathrm{DES_D}(\pi) \iff -\pi(2) > |\pi(1)|. \tag{3.3}$$

A similar analysis yields

$$0 \notin \mathrm{DES_D}(\pi) \text{ and } 1 \notin \mathrm{DES_D}(\pi) \iff \pi(2) > |\pi(1)|, \tag{3.4}$$

$$0 \in \mathrm{DES_D}(\pi) \text{ and } 1 \notin \mathrm{DES_D}(\pi) \iff -\pi(1) > |\pi(2)|, \tag{3.5}$$

$$0 \notin \mathrm{DES_D}(\pi) \text{ and } 1 \in \mathrm{DES_D}(\pi) \iff \pi(1) > |\pi(2)|. \tag{3.6}$$

From (3.3)–(3.6) we deduce that $|\mathrm{DES_D}(\pi) \cap [0,1]| + |\mathrm{DES_D}(\varphi(\pi)) \cap [0,1]| = 2$, which verifies the claim that $\mathrm{des_D}(\pi) + \mathrm{des_D}(\varphi(\pi)) = n$.

Therefore, we have

$$\sum_{\pi \in D_n^-} t^{\mathrm{des_D}(\pi)} = \sum_{\pi \in D_n^+} t^{\mathrm{des_D}(\varphi(\pi))} = \sum_{\pi \in D_n^+} t^{n-\mathrm{des_D}(\pi)} = t^n \sum_{\pi \in D_n^+} t^{-\mathrm{des_D}(\pi)} = t^n Q_n\big(t^{-1}\big). \quad ∎$$

*Proof of Theorem 1.3.* Using Lemmas 3.2 and 3.3, we have (a copy of the proof of Theorem 1.1 replacing B with D and $P$ with $Q$),

$$D_n(t) = \sum_{\pi \in D_n^+} t^{\mathrm{des_D}(\pi)} + \sum_{\pi \in D_n^-} t^{\mathrm{des_D}(\pi)} = Q_n(t) + t^n Q_n\big(t^{-1}\big). \quad ∎$$

## 4. Real-Rootedness of Eulerian Polynomials

In examining the polynomials $P_n(t)$ and $Q_n(t)$ for small values of $n$, we observed that $P_n(t)$ and $t^n P_n\left(t^{-1}\right)$ have interlacing roots (see Definition 4.2 below), and that $Q_n(t)$ and $t^n Q_n\left(t^{-1}\right)$ also have interlacing roots. This led to the following conjecture.

*Conjecture 4.1.* (i) For $n \geq 1$, the polynomial $Q_n(t)$ has only real roots, and thus $t^n Q_n\left(t^{-1}\right)$ also has only real roots. Moreover $Q_n(t)$ and $t^n Q_n\left(t^{-1}\right)$ have interlacing roots.

(ii) For $n \geq 2$, the polynomial $P_n(t)$ has only real roots, and thus $t^n P_n\left(t^{-1}\right)$ also has only real roots. Moreover $P_n(t)$ and $t^n P_n\left(t^{-1}\right)$ have interlacing roots.

In this section, we confirm this conjecture by explaining its connection to polynomials studied by Savage and Visontai [13]. An alternate proof of this conjecture was given by Yang and Zhang [17] using the Hermite-Biehler theorem, and a result of Borcea and Brändén on Hurwitz stability.

First we begin with some background on the real-rootedness of Eulerian polynomials. A result first proved by Frobenius [10] is that the type A Eulerian polynomials have only real roots. Brenti [3] proved that the Eulerian polynomials of type B and of the exceptional finite Coxeter groups have only real roots. Brenti also conjectured that for every finite Coxeter group, the corresponding Eulerian polynomials have only real roots. This conjecture was settled by Savage and Visontai [13] who showed that the type D Eulerian polynomials do have only real roots. A $q$-analog of this result was proved by Yang and Zhang [16]. The proof in [13] includes an extension of techniques involving compatible polynomials. The notion of compatible polynomials was introduced by Chudnovsky and Seymour [7], and is related to the idea of interlacing roots, which we define next.

**Definition 4.2.** *Let $f$ be a polynomial with real roots $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\deg f}$, and let $g$ be a polynomial with real roots $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{\deg g}$. We say that $f$ interlaces $g$ if*

$$\cdots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1.$$

Note that in this case we must have $\deg f \leq \deg g \leq 1 + \deg f$. If $f$ interlaces $g$ or if $g$ interlaces $f$, then we also say that $f$ and $g$ have *interlacing roots*. The following remarkable theorem is due to Obreschkoff.

**Theorem 4.3.** ([12]) *Let $f, g \in \mathbb{R}[t]$ with* $\deg f \leq \deg g \leq 1 + \deg f$. *Then $f$ interlaces $g$ if and only if $c_1 f + c_2 g$ has only real roots for all $c_1, c_2 \in \mathbb{R}$.*

We turn our attention now to compatible polynomials. Call a set of polynomials $f_1, \ldots, f_k \in \mathbb{R}[t]$, *compatible* if for all nonnegative numbers $c_1, \ldots, c_k$ the polynomial $\sum_{i=1}^{k} c_i f_i(t)$ has only real roots. Call a set of polynomials $f_1, \ldots, f_k \in \mathbb{R}[t]$, *pairwise compatible* if for all $i, j \in [k]$ the polynomials $f_i$ and $f_j$ are compatible. For polynomials with positive leading coefficients, Chudnovsky and Seymour showed that these two notions are equivalent.

**Theorem 4.4.** ([7, 2.2]) *Let $f_1, \ldots, f_k \in \mathbb{R}[t]$ be a set of pairwise compatible polynomials with positive leading coefficients. Then $f_1, \ldots, f_k$ are compatible.*

Using the previous theorem, Savage and Visontai proved the following result, which is essential to their work in [13].

**Theorem 4.5.** ([13, Theorem 2.3]) *Let $f_1,\dots,f_k \in \mathbb{R}[t]$ be a sequence of polynomials with positive leading coefficients such that for all $1 \le i \le j \le k$ we have*

(a) *$f_i(t)$ and $f_j(t)$ are compatible, and*
(b) *$tf_i(t)$ and $f_j(t)$ are compatible.*

*Given a sequence of integers $0 \le \lambda_0 \le \lambda_1 \le \cdots \le \lambda_m \le k$, define*

$$g_p(t) = \sum_{r=0}^{\lambda_p - 1} t f_r(t) + \sum_{r=\lambda_p}^{k} f_p(t),$$

*for $1 \le p \le m$. Then for all $1 \le i \le j \le m$, we have*

(a') *$g_i(t)$ and $g_j(t)$ are compatible, and*
(b') *$tg_i(t)$ and $g_j(t)$ are compatible.*

The following lemma is useful in connection with the previous theorem (see [15, Lemma 3.4] and [13, Lemma 2.5]).

**Lemma 4.6.** *Let $f, g \in \mathbb{R}[t]$ be polynomials with nonnegative coefficients. Then, the following two statements are equivalent*:

(i) *$f(t)$ and $g(t)$ are compatible, and $tf(t)$ and $g(t)$ are compatible.*
(ii) *$f$ interlaces $g$.*

In their proof that the type D Eulerian polynomials have only real roots, Savage and Visontai used ascent sets of inversion sequences to construct a set of polynomials $T_{n,k}(t)$ for $0 \le k \le 2n-1$, and they used Theorem 4.5 to show that these polynomials are compatible. For $n \ge 2$, the involution on $B_n$ which changes the sign of the letter whose absolute value is one, is an involution which preserves type D descents (see [13, Lemma 3.9]). One can combine this fact along with [13, Theorem 3.12] to show that these polynomials may also be interpreted as follows (see [2] for a treatment of this topic that avoids inversion sequences).

$$T_{n,k}(t) = \begin{cases} 2 \displaystyle\sum_{\substack{\pi \in D_n \\ \pi(n) = n-k}} t^{\mathrm{des}_{\mathrm{D}}(\pi)}, & \text{for } 0 \le k \le n-1, \\ 2 \displaystyle\sum_{\substack{\pi \in D_n \\ \pi(n) = n-1-k}} t^{\mathrm{des}_{\mathrm{D}}(\pi)}, & \text{for } n \le k \le 2n-1. \end{cases}$$

In the proof of [13, Theorem 3.15], the authors establish that for all $0 \le i < j \le 2n-1$ the polynomials $T_{n,i}(t)$ and $T_{n,j}(t)$ are compatible, and the polynomials $tT_{n,i}(t)$ and $T_{n,j}(t)$ are also compatible, which implies the following.

**Theorem 4.7.** ([13]) *For $n \geq 4$, the set of polynomials*

$$T_{n,0}(t), T_{n,1}(t), \ldots, T_{n,2n-1}(t)$$

*are compatible, and the set of polynomials*

$$tT_{n,0}(t), tT_{n,1}(t), \ldots, tT_{n,n-1}(t), T_{n,n}(t), T_{n,n+1}(t), \ldots, T_{n,2n-1}(t)$$

*are also compatible.*

**Corollary 4.8.** *For $n \geq 2$, the polynomial $Q_n(t)$ interlaces $t^n Q_n\left(t^{-1}\right)$.*

*Proof.* For values of $n$ less than 4, this can be checked directly. For $n \geq 4$, by Theorem 4.7 we have that for all $c_1, c_2 \geq 0$ the polynomial

$$\frac{c_1}{2}\sum_{k=0}^{n-1} T_{n,k}(t) + \frac{c_2}{2}\sum_{k=n}^{2n-1} T_{n,k}(t) = c_1 Q_n(t) + c_2 t^n Q\left(t^{-1}\right)$$

has only real roots, and the polynomial

$$\frac{c_1}{2}\sum_{k=0}^{n-1} tT_{n,k}(t) + \frac{c_2}{2}\sum_{k=n}^{2n-1} T_{n,k}(t) = c_1 t Q_n(t) + c_2 t^n Q\left(t^{-1}\right)$$

has only real roots. By Lemma 4.6, $Q_n(t)$ interlaces $t^n Q_n\left(t^{-1}\right)$. ∎

In [13, Corollory 3.13] the authors remark that [13, Theorem 3.12] can be used to show that $B_n(t)$ has only real roots. Indeed, for $0 \leq k \leq 2n-1$ one can use [13, Theorem 3.12] to show that the following **s**-Eulerian polynomials can be interpreted as follows (again, see [2] for a treatment that avoids inversion sequences).

$$E_{n,k}^{(2,4,\ldots,2n)}(t) = \begin{cases} \displaystyle\sum_{\substack{\pi \in B_n \\ \pi(n)=n-k}} t^{\mathrm{des}_{\mathrm{B}}(\pi)}, & \text{for } 0 \leq k \leq n-1, \\ \displaystyle\sum_{\substack{\pi \in B_n \\ \pi(n)=n-1-k}} t^{\mathrm{des}_{\mathrm{B}}(\pi)}, & \text{for } n \leq k \leq 2n-1. \end{cases}$$

For simplicity of notation, we define $B_{n,k}(t) = E_{n,k}^{(2,4,\ldots,2n)}(t)$.

In their proof of [13, Theorem 1.1], the authors show that for all $0 \leq i < j < s_n$ the polynomials $E_{n,i}^{(\mathbf{s})}(t)$ and $E_{n,j}^{(\mathbf{s})}(t)$ are compatible, and the polynomials $tE_{n,i}^{(\mathbf{s})}(t)$ and $E_{n,j}^{(\mathbf{s})}(t)$ are also compatible. A special case is the following.

**Theorem 4.9.** ([13]) *For $n \geq 1$, the set of polynomials*

$$B_{n,0}(t), B_{n,1}(t), \ldots, B_{n,2n-1}(t)$$

*are compatible, and the set of polynomials*

$$tB_{n,0}(t), tB_{n,1}(t), \ldots, tB_{n,n-1}(t), B_{n,n}(t), B_{n,n+1}(t), \ldots, B_{n,2n-1}(t)$$

*are also compatible.*

In a manner analogous to Corollary 4.8, the previous theorem implies the following corollary.

**Corollary 4.10.** For $n \geq 1$ the polynomial $P_n(t)$ interlaces $t^n P_n\left(t^{-1}\right)$.

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