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Integer Decomposition Property of Free Sums of Convex Polytopes

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Abstract. Let $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^e$ be integral convex polytopes of dimension d and e which contain the origin of $\mathbb{R}^d$ and $\mathbb{R}^e$, respectively. We say that an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ possesses the integer decomposition property if, for each $n \geq 1$ and for each $\gamma \in n\mathcal{P} \cap \mathbb{Z}^d$, there exist $\gamma^{(1)}, \ldots, \gamma^{(n)}$ belonging to $\mathcal{P} \cap \mathbb{Z}^d$ such that $\gamma = \gamma^{(1)} + \cdots + \gamma^{(n)}$. In the present paper, under some assumptions, the necessary and sufficient condition for the free sum of $\mathcal{P}$ and $\mathcal{Q}$ to possess the integer decomposition property will be presented.

Keywords: integral convex polytope, free sum, integer decomposition property

1. Introduction

A convex polytope is called *integral* if all of its vertices have integer coordinates. For an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$, let $\mathbb{Z}(\mathcal{P} \cap \mathbb{Z}^d)$ be the lattice generated by the integer points belonging to $\mathcal{P} \cap \mathbb{Z}^d$, i.e., $\mathbb{Z}(\mathcal{P} \cap \mathbb{Z}^d) = \{\sum_{v \in \mathcal{P} \cap \mathbb{Z}^d} z_v v \colon z_v \in \mathbb{Z}\}$.

Let $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^e$ be convex polytopes and suppose that $\mathbf{0}_d \in \mathcal{P}$ and $\mathbf{0}_e \in \mathcal{Q}$, where $\mathbf{0}_d \in \mathbb{R}^d$ denotes the origin of $\mathbb{R}^d$ and $\mathbf{0}_e \in \mathbb{R}^e$ denotes that of $\mathbb{R}^e$. We introduce the canonical injections $\mu \colon \mathbb{R}^d \to \mathbb{R}^{d+e}$ by setting $\mu(\alpha) = (\alpha, \mathbf{0}_e) \in \mathbb{R}^{d+e}$ with $\alpha \in \mathbb{R}^d$ and $\nu \colon \mathbb{R}^e \to \mathbb{R}^{d+e}$ by setting $\nu(\beta) = (\mathbf{0}_d, \beta) \in \mathbb{R}^{d+e}$ with $\beta \in \mathbb{R}^e$. In particular, $\mu(\mathbf{0}_d) = \nu(\mathbf{0}_e) = \mathbf{0}_{d+e}$, where $\mathbf{0}_{d+e}$ denotes the origin of $\mathbb{R}^{d+e}$. Then $\mu(\mathcal{P})$ and $\nu(\mathcal{Q})$ are convex polytopes of $\mathbb{R}^{d+e}$ with $\mu(\mathcal{P}) \cap \nu(\mathcal{Q}) = \mathbf{0}_{d+e} \in \mathbb{R}^{d+e}$. The *free sum* of $\mathcal{P}$ and $\mathcal{Q}$ is the convex hull of the set $\mu(\mathcal{P}) \cup \nu(\mathcal{Q})$ in $\mathbb{R}^{d+e}$. It is written as $\mathcal{P} \oplus \mathcal{Q}$. One has $\dim(\mathcal{P} \oplus \mathcal{Q}) = \dim \mathcal{P} + \dim \mathcal{Q}$.

For a convex polytope $\mathcal{P} \subset \mathbb{R}^d$ and for each integer $n \geq 1$, we write $n\mathcal{P}$ for the convex polytope $\{n\alpha \colon \alpha \in \mathcal{P}\} \subset \mathbb{R}^d$. We say that an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$

possesses the *integer decomposition property* (also known as *integrally closed*) if, for each $n \geq 1$ and for each $\gamma \in n\mathcal{P} \cap \mathbb{Z}^d$, there exist $\gamma^{(1)}, \ldots, \gamma^{(n)}$ belonging to $\mathcal{P} \cap \mathbb{Z}^d$ such that $\gamma = \gamma^{(1)} + \cdots + \gamma^{(n)}$.

Let $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^e$ be convex polytopes containing the origin (of $\mathbb{R}^d$ or $\mathbb{R}^e$). It is then easy to see that if the free sum of $\mathcal{P}$ and $\mathcal{Q}$ possesses the integer decomposition property, then each of $\mathcal{P}$ and $\mathcal{Q}$ possesses the integer decomposition property. On the other hand, the converse is not true in general. (See Example 1.3.) The purpose of the present paper is to show the following

Theorem 1.1. *Let $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^e$ be integral convex polytopes of dimension d and dimension e containing $\mathbf{0}_d$ and $\mathbf{0}_e$, respectively. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ satisfy $\mathbb{Z}(\mathcal{P} \cap \mathbb{Z}^d) = \mathbb{Z}^d$, $\mathbb{Z}(\mathcal{Q} \cap \mathbb{Z}^e) = \mathbb{Z}^e$ and*

$$(\mathcal{P} \oplus \mathcal{Q}) \cap \mathbb{Z}^{d+e} = \mu(\mathcal{P} \cap \mathbb{Z}^d) \cup \nu(\mathcal{Q} \cap \mathbb{Z}^e). \tag{1.1}$$

Then the free sum $\mathcal{P} \oplus \mathcal{Q}$ possesses the integer decomposition property if and only if the following two conditions are satisfied:

- *each of $\mathcal{P}$ and $\mathcal{Q}$ possesses the integer decomposition property;*
- *either $\mathcal{P}$ or $\mathcal{Q}$ satisfies that the defining hyperplane of each facet is of the form $\sum_{i=1}^{f} a_i z_i = b$, where each a_i is an integer, $b \in \{0, 1\}$, and $f \in \{d, e\}$.*

An integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is called a $(0, 1)$-*polytope* if each vertex of $\mathcal{P}$ belongs to $\{0, 1\}^d$. It then follows that Equation (1.1) is always satisfied if either $\mathcal{P}$ or $\mathcal{Q}$ is a $(0, 1)$-polytope. As an immediate corollary of Theorem 1.1, we also obtain the following.

Corollary 1.2. *Let $\mathcal{P} \subset \mathbb{R}^d$ be a $(0, 1)$-polytope of dimension d containing $\mathbf{0}_d$ and $\mathcal{Q} \subset \mathbb{R}^e$ an integral convex polytope of dimension e containing $\mathbf{0}_e$. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ satisfy $\mathbb{Z}(\mathcal{P} \cap \mathbb{Z}^d) = \mathbb{Z}^d$ and $\mathbb{Z}(\mathcal{Q} \cap \mathbb{Z}^e) = \mathbb{Z}^e$. Then the free sum $\mathcal{P} \oplus \mathcal{Q}$ possesses the integer decomposition property if and only if the following two conditions are satisfied*:

- *each of $\mathcal{P}$ and $\mathcal{Q}$ possesses the integer decomposition property;*
- *either $\mathcal{P}$ or $\mathcal{Q}$ satisfies that the defining hyperplane of each facet is of the form $\sum_{i=1}^{f} a_i z_i = b$, where each a_i is an integer, $b \in \{0, 1\}$, and $f \in \{d, e\}$.*

Example 1.3. Even if $\mathcal{P}$ and $\mathcal{Q}$ possess the integer decomposition property, the free sum $\mathcal{P} \oplus \mathcal{Q}$ may fail to possess the integer decomposition property. For example, let $\mathcal{P} \subset \mathbb{R}^3$ be the $(0, 1)$-polytope with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 0, 0)$. Then $\mathcal{P}$ possesses the integer decomposition property, but the free sum $\mathcal{P} \oplus \mathcal{P}$ fails to possess the integer decomposition property. In fact, $z_1 + z_2 + z_3 = 2$ is the defining hyperplane of a facet of $\mathcal{P}$.

The structure of the present paper is as follows. In Section 2, we will consider the condition for $\mathcal{P}$ and $\mathcal{Q}$ to satisfy Equation (1.1). In Section 3, a proof of Theorem 1.1 will be given.

2. When Does Equation (1.1) Hold?

Let $V(\mathcal{P})$ be the set of vertices of $\mathcal{P}$ and let $V(\mathcal{Q})$ be that of $\mathcal{Q}$. First, for $W \subset V(\mathcal{P}) \setminus \{\mathbf{0}_d\}$, let

$$\mathrm{int}(W) = (\mathrm{conv}(W \cup \{\mathbf{0}_d\}) \setminus \partial\, \mathrm{conv}(W \cup \{\mathbf{0}_d\})) \cap \mathbb{Z}^d,$$

where ∂ denotes the relative boundary. For $W \subset V(\mathcal{Q}) \setminus \{\mathbf{0}_e\}$, $\mathrm{int}(W)$ is also defined in the same way. Next, we define

$$\mathcal{W}(\mathcal{P}) = \{W \subset V(\mathcal{P}) \setminus \{\mathbf{0}_d\} \colon W \text{ is linearly independent and } \mathrm{int}(W) \neq \emptyset\}.$$

In a similar way, we also define $\mathcal{W}(\mathcal{Q})$. Note that the condition $\mathcal{W}(\mathcal{P}) = \emptyset$ is equivalent to the condition that $\mathcal{P}$ contains no integer point in the relative interior of any face containing the origin. Last, for any $W = \{w_1, \dots, w_m\} \in \mathcal{W}(\mathcal{P})$ (similarly, for any $W \in \mathcal{W}(\mathcal{Q})$), let

$$\min(W) = \min\left\{ \sum_{i=1}^{m} r_i \colon \sum_{i=1}^{m} r_i w_i \in \mathrm{int}(W) \right\}.$$

Then $0 < \min(W) < 1$.

Proposition 2.1. *Let $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^e$ be integral convex polytopes containing $\mathbf{0}_d$ and $\mathbf{0}_e$, respectively. Then the free sum $\mathcal{P} \oplus \mathcal{Q}$ satisfies Equation* (1.1) *if and only if*

- *$\mathcal{W}(\mathcal{P}) = \emptyset$ or $\mathcal{W}(\mathcal{Q}) = \emptyset$, or*
- *$\mathcal{W}(\mathcal{P}) \neq \emptyset$, $\mathcal{W}(\mathcal{Q}) \neq \emptyset$, and $\min(F) + \min(G) > 1$ for all $F \in \mathcal{W}(\mathcal{P})$ and $G \in \mathcal{W}(\mathcal{Q})$.*

Proof. **"Only if"** Assume that there exist $F \in \mathcal{W}(\mathcal{P})$ and $G \in \mathcal{W}(\mathcal{Q})$ such that $\min(F) + \min(G) \leq 1$. Then each of F and G is linearly independent. Let $F = \{v_1, \dots, v_n\}$ and let $G = \{w_1, \dots, w_m\}$. Then there are $0 < r_1, \dots, r_n < 1$, $0 < s_1, \dots, s_m < 1$ such that $\sum_{i=1}^n r_i v_i \in \mathrm{int}(F)$ and $\sum_{i=1}^m s_i w_i \in \mathrm{int}(G)$, where $0 < \sum_{i=1}^n r_i < 1$ and $0 < \sum_{i=1}^m s_i < 1$ with $\sum_{i=1}^n r_i + \sum_{i=1}^m s_i \leq 1$. Let us consider

$$\alpha = \sum_{i=1}^{n} r_i \mu(v_i) + \sum_{i=1}^{m} s_i \nu(w_i) \in \mathbb{R}^{d+e}.$$

Since $\sum_{i=1}^n r_i v_i \in \mathbb{Z}^d$, we have $\sum_{i=1}^n r_i \mu(v_i) \in \mathbb{Z}^{d+e}$. Similarly, $\sum_{i=1}^m s_i \nu(w_i) \in \mathbb{Z}^{d+e}$. Thus, $\alpha \in \mathbb{Z}^{d+e}$. Moreover, since $\sum_{i=1}^n r_i + \sum_{i=1}^m s_i \leq 1$, we have $\alpha \in \mathcal{P} \oplus \mathcal{Q}$. Hence, $\alpha \in (\mathcal{P} \oplus \mathcal{Q}) \cap \mathbb{Z}^{d+e}$. On the other hand, since $\sum_{i=1}^n r_i v_i \neq \mathbf{0}_d$ and $\sum_{i=1}^m s_i w_i \neq \mathbf{0}_e$, we see that $\alpha \notin \mu(\mathcal{P} \cap \mathbb{Z}^d) \cup \nu(\mathcal{Q} \cap \mathbb{Z}^e)$. These mean that Equation (1.1) is not satisfied.

"If" Assume that (1.1) is not satisfied. Since the inclusion $(\mathcal{P} \oplus \mathcal{Q}) \cap \mathbb{Z}^{d+e} \supset \mu(\mathcal{P} \cap \mathbb{Z}^d) \cup \nu(\mathcal{Q} \cap \mathbb{Z}^e)$ is always satisfied, we may assume that there is α belonging to $(\mathcal{P} \oplus \mathcal{Q}) \cap \mathbb{Z}^{d+e} \setminus \mu(\mathcal{P} \cap \mathbb{Z}^d) \cup \nu(\mathcal{Q} \cap \mathbb{Z}^e)$. Then α can be written as

$$\alpha = \sum_{i=1}^{n} r_i \mu(v_i) + \sum_{i=1}^{m} s_i \nu(w_i),$$

where $v_1, \ldots, v_n \in V(\mathcal{P}) \setminus \{\mathbf{0}_d\}$, $w_1, \ldots, w_m \in V(\mathcal{Q}) \setminus \{\mathbf{0}_e\}$, $0 \leq r_1, \ldots, r_n \leq 1$, $0 \leq s_1, \ldots, s_m \leq 1$ and $\sum_{i=1}^n r_i + \sum_{i=1}^m s_i \leq 1$. By Carathéodory's Theorem (cf. [5, Corollary 7.1i]), we can choose $\mu(v_1), \ldots, \mu(v_n)$, $\nu(w_1), \ldots, \nu(w_m)$ as linearly independent vectors of $\mathbb{R}^{d+e}$, that is, $v_1, \ldots, v_n$ are linearly independent in $\mathbb{R}^d$ and so are $w_1, \ldots, w_m$ in $\mathbb{R}^e$. Moreover, if $\sum_{i=1}^n r_i = 0$, then $\alpha \in \nu(\mathcal{Q} \cap \mathbb{Z}^e)$, a contradiction. Similarly, if $\sum_{i=1}^m s_i = 0$, then $\alpha \in \mu(\mathcal{P} \cap \mathbb{Z}^e)$, a contradiction. Thus, we also assume $\sum_{i=1}^n r_i > 0$ and $\sum_{i=1}^m s_i > 0$.

We consider $v = \sum_{i=1}^n r_i v_i \in \mathbb{Z}^d$. Since $\sum_{i=1}^n r_i > 0, \sum_{i=1}^m s_i > 0$, and $\sum_{i=1}^n r_i + \sum_{i=1}^m s_i \leq 1$, we have $0 < \sum_{i=1}^n r_i < 1$. Thus, $v \in \mathcal{P} \cap \mathbb{Z}^d$. Let $v_{i_1}, \ldots, v_{i_g}$ be all of v_i's such that $r_i > 0$ and let $F = \{v_{i_1}, \ldots, v_{i_g}\}$. Then F is also linearly independent and $v \in \mathrm{int}(F)$. Hence, $F \in \mathcal{W}(\mathcal{P})$. Similarly, let $w_{j_1}, \ldots, w_{j_h}$ be all of w_i's such that $s_i > 0$ and let $G = \{w_{j_1}, \ldots, w_{j_h}\}$. Then $G \in \mathcal{W}(\mathcal{Q})$. Now we see

$$\min(F) + \min(G) \leq \sum_{k=1}^{g} r_{i_k} + \sum_{k=1}^{h} s_{j_k} = \sum_{i=1}^{n} r_i + \sum_{i=1}^{m} s_i \leq 1,$$

as required. ∎

Example 2.2. (a) Let $\mathcal{P} \subset \mathbb{R}^d$ be a $(0, 1)$-polytope. Then we easily see that $\mathcal{W}(\mathcal{P}) = \emptyset$. Thus, if $\mathcal{P}$ or $\mathcal{Q}$ is a $(0, 1)$-polytope in Proposition 2.1, then Equation (1.1) always holds.

(b) Let $\mathcal{P} = \mathrm{conv}(\{(0,0), (1,0), (1,2)\}) \subset \mathbb{R}^2$ and let $\mathcal{Q} = \mathrm{conv}(\{0, 2\}) \subset \mathbb{R}^1$. Then $\mathcal{W}(\mathcal{Q}) \neq \emptyset$ but $\mathcal{W}(\mathcal{P}) = \emptyset$. Thus, Equation (1.1) holds.

(c) Let $\mathcal{P} = \mathcal{Q} = \mathrm{conv}(\{(0,0), (2,1), (1,2)\}) \subset \mathbb{R}^2$ and consider $W = \{(2,1), (1,2)\}$. Then we see that $\mathcal{W}(\mathcal{P}) = \{W\}$. On the other hand, we also have $\min(W) = 2/3$. Thus, Equation (1.1) holds.

3. Proof of Theorem 1.1

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d. A *configuration* arising from $\mathcal{P}$ is the finite set $\mathcal{A} = \{(\alpha, 1) \in \mathbb{Z}^{d+1} : \alpha \in \mathcal{P} \cap \mathbb{Z}^d\}$. We say that $\mathcal{A}$ is *normal* if

$$\mathbb{Z}_{\geq 0}\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{Q}_{\geq 0}\mathcal{A},$$

where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers and $\mathbb{Q}_{\geq 0}$ is the set of nonnegative rational numbers.

We recall the definition of the Ehrhart polynomial. Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d and, for each integer $n \geq 1$, write $i(\mathcal{P}, n)$ for the number of integer points belonging to $n\mathcal{P}$, i.e., $i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^d|$. It is known that $i(\mathcal{P}, n)$ is a polynomial in n of degree d with $i(\mathcal{P}, 0) = 1$. We call $i(\mathcal{P}, n)$ the *Ehrhart polynomial* of $\mathcal{P}$. See [4, Chapter IX] for details. We then define the integers $\delta_0, \delta_1, \delta_2, \ldots$ by the formula

$$(1-\lambda)^{d+1}\left[1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n)\lambda^n\right] = \sum_{n=0}^{\infty} \delta_n \lambda^n.$$

It then follows that $\delta_n = 0$ for $n > d$. The polynomial

$$\delta(\mathcal{P}) = \sum_{n=0}^{d} \delta_n \lambda^n$$

is called the *δ-polynomial* of $\mathcal{P}$.

Let $K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}, s]$ denote the Laurent polynomial ring in $d+1$ variables over a field K. If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathcal{P} \cap \mathbb{Z}^d$, then we write u_α for the Laurent monomial $t_1^{\alpha_1} \cdots t_d^{\alpha_d} \in K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$. The *toric ring* of $\mathcal{A}$ is the subring $K[\mathcal{A}]$ of $K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}, s]$ that is generated by those Laurent monomials $u_\alpha s$ with $\alpha \in \mathcal{P} \cap \mathbb{Z}^d$. Let $K[\{x_\alpha\}_{\alpha \in \mathcal{P} \cap \mathbb{Z}^d}]$ be the polynomial ring in $|\mathcal{P} \cap \mathbb{Z}^d|$ variables over K with each $\deg x_\alpha = 1$. We then define the surjective ring homomorphism $\pi\colon K[\{x_\alpha\}_{\alpha \in \mathcal{P} \cap \mathbb{Z}^d}] \to K[\mathcal{A}]$ by setting $\pi(x_\alpha) = u_\alpha s$ for each $\alpha \in \mathcal{P} \cap \mathbb{Z}^d$.

Finally, the Hilbert function of the toric ring $K[\mathcal{A}]$ of the configuration $\mathcal{A}$ arising from an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d is introduced. We write $(K[\mathcal{A}])_n$ for the subspace of $K[\mathcal{A}]$ spanned by those Laurent monomials of the form

$$\left(u_{\alpha^{(1)}} s\right) \left(u_{\alpha^{(2)}} s\right) \cdots \left(u_{\alpha^{(n)}} s\right),$$

with each $\alpha^{(i)} \in \mathcal{P} \cap \mathbb{Z}^d$. In particular, $(K[\mathcal{A}])_0 = K$ and $(K[\mathcal{A}])_1 = \sum_{\alpha \in \mathcal{P} \cap \mathbb{Z}^d} K u_\alpha s$. The *Hilbert function* of $K[\mathcal{A}]$ is the numerical function

$$H(K[\mathcal{A}], n) = \dim_K (K[\mathcal{A}])_n, \quad n = 0, 1, 2, \ldots.$$

Thus, in particular, $H(K[\mathcal{A}], 0) = 1$ and $H(K[\mathcal{A}], 1) = |\mathcal{P} \cap \mathbb{Z}^d|$. We then define the integers $h_0, h_1, h_2, \ldots$ by the formula

$$(1-\lambda)^{d+1} \left[\sum_{n=0}^{\infty} H(K[\mathcal{A}], n) \lambda^n \right] = \sum_{n=0}^{\infty} h_n \lambda^n.$$

A basic fact [1, Theorem 11.1] of Hilbert functions guarantees that $h_n = 0$ for $n \gg 0$. We say that the polynomial

$$h(K[\mathcal{A}]) = \sum_{n=0}^{\infty} h_n \lambda^n$$

is the *h-polynomial* of $K[\mathcal{A}]$.

Lemma 3.1. *Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d and $\mathcal{A} \subset \mathbb{Z}^{d+1}$ the configuration arising from $\mathcal{P}$. Suppose that $\mathcal{P}$ satisfies $\mathbb{Z}(\mathcal{P} \cap \mathbb{Z}^d) = \mathbb{Z}^d$. Then the following conditions are equivalent*:

(i) *$\mathcal{P}$ possesses the integer decomposition property;*
(ii) *$\mathcal{A}$ is normal;*
(iii) $\delta(\mathcal{P}) = h(K[\mathcal{A}])$.

Proof. It follows that $\mathcal{P}$ possesses the integer decomposition property if and only if, for $\alpha \in n\mathcal{P} \cap \mathbb{Z}^d$, one has $(\alpha, n) \in \mathbb{Z}_{\geq 0}\mathcal{A}$. Since $\mathbb{Z}(\mathcal{P} \cap \mathbb{Z}^d) = \mathbb{Z}^d$, i.e., $\mathbb{Z}\mathcal{A} = \mathbb{Z}^{d+1}$, it follows that $\mathcal{A}$ is normal if and only if $\mathbb{Z}_{\geq 0}\mathcal{A} = \mathbb{Z}^{d+1} \cap \mathbb{Q}_{\geq 0}\mathcal{A}$. Moreover, for $\alpha \in \mathbb{Q}^d$, one has $\alpha \in n\mathcal{P}$ if and only if $(\alpha, n) \in \mathbb{Q}_{\geq 0}\mathcal{A}$. Hence, (i) $\Leftrightarrow$ (ii) follows.

In general, one has $i(\mathcal{P}, n) \geq H(K[\mathcal{A}], n)$ for $n \in \mathbb{Z}_{\geq 0}$. Furthermore, it follows that $i(\mathcal{P}, n) = H(K[\mathcal{A}], n)$ for all $n \in \mathbb{Z}_{\geq 0}$ if and only if $\mathcal{P}$ possesses the integer decomposition property. Hence (i) $\Leftrightarrow$ (iii) follows. ∎

Lemma 3.2. *Let $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^e$ be integral convex polytopes of dimension d and e that contain the origin of $\mathbb{R}^d$ and $\mathbb{R}^e$, respectively. Let $\mathcal{A} \subset \mathbb{Z}^{d+1}$ and $\mathcal{B} \subset \mathbb{Z}^{e+1}$ be the configurations arising from $\mathcal{P}$ and $\mathcal{Q}$, respectively. Let $\mathcal{A} \oplus \mathcal{B} \subset \mathbb{Z}^{d+e+1}$ denote the configuration arising from the free sum $\mathcal{P} \oplus \mathcal{Q} \subset \mathbb{R}^{d+e}$. Suppose that*

$$(\mathcal{P} \oplus \mathcal{Q}) \cap \mathbb{Z}^{d+e} = \mu\left(\mathcal{P} \cap \mathbb{Z}^d\right) \cup \nu(\mathcal{Q} \cap \mathbb{Z}^e).$$

Then

$$h(K[\mathcal{A} \oplus \mathcal{B}]) = h(K[\mathcal{A}])h(K[\mathcal{B}]).$$

Furthermore, if $\mathcal{P} \oplus \mathcal{Q}$ possesses the integer decomposition property, then

$$\delta(\mathcal{P} \oplus \mathcal{Q}) = \delta(\mathcal{P})\delta(\mathcal{Q}).$$

Proof. Let $K[\mathcal{A}] \subset K\left[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}, s\right]$ and $K[\mathcal{B}] \subset K\left[t'_1, {t'}_1^{-1}, \ldots, t'_e, {t'}_e^{-1}, s'\right]$. Then $K[\mathcal{A} \oplus \mathcal{B}] = (K[\mathcal{A}] \otimes K[\mathcal{B}])/(s - s')$. Hence, $h(K[\mathcal{A} \oplus \mathcal{B}]) = h(K[\mathcal{A}] \otimes K[\mathcal{B}]) = h(K[\mathcal{A}])h(K[\mathcal{B}])$, as desired.

If, furthermore, $\mathcal{P} \oplus \mathcal{Q}$ possesses the integer decomposition property, then each of $\mathcal{P}$ and $\mathcal{Q}$ possesses the integer decomposition property. Lemma 3.1 then says that $\delta(\mathcal{P} \oplus \mathcal{Q}) = h(K[\mathcal{A} \oplus \mathcal{B}])$, $\delta(\mathcal{P}) = h(K[\mathcal{A}])$, and $\delta(\mathcal{Q}) = h(K[\mathcal{B}])$. Hence, $\delta(\mathcal{P} \oplus \mathcal{Q}) = \delta(\mathcal{P})\delta(\mathcal{Q})$, as required. ∎

We also recall the following theorem.

Theorem 3.3. ([2, Theorem 1.4]) *Let $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^e$ be integral convex polytopes containing the origin (of $\mathbb{R}^d$ or $\mathbb{R}^e$). Then the equality $\delta(\mathcal{P} \oplus \mathcal{Q}) = \delta(\mathcal{P})\delta(\mathcal{Q})$ holds if and only if either $\mathcal{P}$ or $\mathcal{Q}$ satisfies that the defining hyperplane of each facet is of the form $\sum_{i=1}^{f} a_i z_i = b$, where each a_i is an integer, $b \in \{0, 1\}$, and $f \in \{d, e\}$.*

We are now in the position to give a proof of Theorem 1.1.

Proof of Theorem 1.1. Assume that each of $\mathcal{P}$ and $\mathcal{Q}$ possesses the integer decomposition property and either $\mathcal{P}$ or $\mathcal{Q}$ satisfies the condition on its facets described in Theorem 1.1. It then follows from Theorem 3.3 that the condition on the facets is equivalent to satisfying that

$$\delta(\mathcal{P} \oplus \mathcal{Q}) = \delta(\mathcal{P})\delta(\mathcal{Q}). \tag{3.1}$$

Moreover, since each of $\mathcal{P}$ and $\mathcal{Q}$ possesses the integer decomposition property, we have the equalities $\delta(\mathcal{P}) = h(K[\mathcal{A}])$ and $\delta(\mathcal{Q}) = h(K[\mathcal{B}])$ by Lemma 3.1. In particular, one has

$$\delta(\mathcal{P})\delta(\mathcal{Q}) = h(K[\mathcal{A}])h(K[\mathcal{B}]). \tag{3.2}$$

Furthermore, since Equation (1.1) is satisfied, it follows from Lemma 3.2 that

$$h(K[\mathcal{A} \oplus \mathcal{B}]) = h(K[\mathcal{A}])h(K[\mathcal{B}]), \tag{3.3}$$

where $\mathcal{A} \oplus \mathcal{B} \subset \mathbb{Z}^{d+e+1}$ denotes the configuration arising from $\mathcal{P} \oplus \mathcal{Q} \subset \mathbb{R}^{d+e}$. Hence, by (3.1), (3.2), and (3.3), we obtain

$$\delta(\mathcal{P} \oplus \mathcal{Q}) = h(K[\mathcal{A} \oplus \mathcal{B}]).$$

Therefore, from Lemma 3.1, we conclude that $\mathcal{P} \oplus \mathcal{Q}$ possesses the integer decomposition property.

On the other hand, suppose that $\mathcal{P} \oplus \mathcal{Q}$ possesses the integer decomposition property. Then it is easy to see that each of $\mathcal{P}$ and $\mathcal{Q}$ possesses the integer decomposition property. Moreover, since $\mathcal{P} \oplus \mathcal{Q} \subset \mathbb{R}^{d+e}$ satisfies (1.1), the equality $\delta(\mathcal{P} \oplus \mathcal{Q}) = \delta(\mathcal{P})\delta(\mathcal{Q})$ holds by Lemma 3.2. Therefore, by Theorem 3.3, either $\mathcal{P}$ or $\mathcal{Q}$ satisfies the condition on its facets described in Theorem 1.1, as required. ∎

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