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A Hypergeometric Inequality

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Abstract. A sequence of coefficients that appeared in the evaluation of a rational integral has been shown to be unimodal. The original proof is based on a inequality for hypergeometric functions. A generalization is presented.

Keywords: hypergeometric function, unimodal polynomials, monotonicity

1. Introduction

A sequence of numbers $\{a_k \colon 0 \le k \le n\}$ is called *unimodal* if there is an index k^* such that $a_{k-1} \le a_k$ for $1 \le k \le k^*$ and $a_{k-1} \ge a_k$ for $k^* + 1 \le k \le n$. The prototypical example of unimodal sequences is $a_k = \binom{n}{k}$. A polynomial $P(x)$ is called unimodal if its sequence of coefficients is unimodal.

A simple criteria for unimodality of a polynomial was established in [4]:

Theorem 1.1. *If $P(x)$ is a polynomial with positive nondecreasing coefficients, then $P(x+1)$ is unimodal.*

The original motivation for this result was the question of unimodality of the polynomial

$$P_m(a) = \sum_{\ell=0}^{m} d_\ell(m) a^\ell \qquad (1.1)$$

with

$$d_\ell(m) = 2^{-2m} \sum_{k=\ell}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{\ell}. \qquad (1.2)$$

This example appeared in the evaluation of the formula

$$\int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}} = \frac{\pi}{2} \frac{P_m(a)}{[2(a+1)]^{m+1/2}} \qquad (1.3)$$

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given in [5]. A variety of proofs of (1.3) can be found in [3] and properties of the coefficients $\{d_\ell(m)\}$ have been reviewed in [26].

A property stronger than unimodality is that of *log-concavity*: a sequence of positive numbers $\{a_k : 0 \le k \le n\}$ is called *log-concave* if $a_k^2 \ge a_{k-1}a_{k+1}$ for $1 \le k \le n-1$. As before, a polynomial is called log-concave if its sequence of coefficients is log-concave. The log-concavity of $P_m(a)$ was first established by Kauers and Paule in [23] using computer algebra, in particular algorithms for automatically deriving recurrences for multiple sums and Cylindrical Algebraic Decomposition (CAD). A variety of classical techniques have been used in this problem by Chen et al. in [10–13].

The original proof of unimodality for $P_m(a)$ in [6] was based on the monotonicity of

$$T(m) = \sum_{r=2}^{m+1} \binom{2r}{r} \binom{m+1}{r} \frac{(r-1)}{2^r \binom{4m}{r}}. \tag{1.4}$$

This proof was revisited in [2] and the hypergeometric representation

$$T(m) = 1 - {}_1F_2 \left(\begin{matrix} \frac{1}{2}, -1-m \\ -4m \end{matrix} \middle| 2 \right) + \frac{m+1}{4m} {}_1F_2 \left(\begin{matrix} \frac{3}{2}, -m \\ 1-4m \end{matrix} \middle| 2 \right) \tag{1.5}$$

was used to give a new proof of the monotonicity of $T(m)$ and also to establish the value $\lim_{m \to \infty} T(m) = 2 - \sqrt{2}$.

The monotonicity of $T(m)$ yields a curious inequality, which is transformed via contiguous identities for hypergeometric functions to the form

$$
{}_2F_1 \left(\begin{matrix} \frac{3}{2}, -m-2 \\ -4m-4 \end{matrix} \middle| 2 \right) - {}_2F_1 \left(\begin{matrix} \frac{3}{2}, -m-1 \\ -4m \end{matrix} \middle| 2 \right)
$$
$$
> 3 \left[{}_2F_1 \left(\begin{matrix} \frac{1}{2}, -m-2 \\ -4m-4 \end{matrix} \middle| 2 \right) - {}_2F_1 \left(\begin{matrix} \frac{1}{2}, -m-1 \\ -4m \end{matrix} \middle| 2 \right) \right].
$$

The goal of this note is to prove the generalization of the previous inequality conjectured in [2]. The main result is stated next.

Theorem 1.2. *The inequality*

$$
{}_2F_1 \left(\begin{matrix} \frac{3}{2}, -m-2 \\ -4m-4 \end{matrix} \middle| 4x \right) - {}_2F_1 \left(\begin{matrix} \frac{3}{2}, -m-1 \\ -4m \end{matrix} \middle| 4x \right)
$$
$$
> 3 \left[{}_2F_1 \left(\begin{matrix} \frac{1}{2}, -m-2 \\ -4m-4 \end{matrix} \middle| 4x \right) - {}_2F_1 \left(\begin{matrix} \frac{1}{2}, -m-1 \\ -4m \end{matrix} \middle| 4x \right) \right]
$$

holds for $x \ge \frac{1}{2}$.

2. An Automatic Proof

This section provides an automatic proof of Theorem 1.2. The main result is first rewritten using

$$\ell_m(x) = {}_2F_1 \left(\begin{matrix} \frac{3}{2}, -m-2 \\ -4m-4 \end{matrix} \middle| 4x \right) \quad \text{and} \quad r_m(x) = 3\, {}_2F_1 \left(\begin{matrix} \frac{1}{2}, -m-2 \\ -4m-4 \end{matrix} \middle| 4x \right), \tag{2.1}$$

in the form

$$\ell_m(x) - \ell_{m-1}(x) > r_m(x) - r_{m-1}(x). \tag{2.2}$$

The expansion of the terms in the previous inequality becomes nicer if written as

$$L_m(x) := \ell_m(x) - r_m(x) > \ell_{m-1}(x) - r_{m-1}(x) = L_{m-1}(x).$$

Using any of the packages [14, 25] dealing with symbolic summation and holonomic closure properties that are available nowadays, it is easy to discover (and thus prove) a second order recurrence satisfied by $L_m(x)$. In theory, this would make this inequality accessible to the automatic inequality prover developed by Gerhold and Kauers [18, 20, 21]. These methods need as input only a defining recurrence and sufficiently many initial values. This method uses as an essential tool Cylindrical Algebraic Decomposition [8, 15, 16] (CAD), which was introduced to solve the problem of quantifier elimination over the field of real numbers. Given a quantified logical formula of polynomial equalities and inequalities, it computes an equivalent, quantifier-free formula. If the given formula does not contain any free variables, then the output is one of the logical constants *true* or *false*. The input class can be generalized to rational or algebraic functions, since these kinds of functions can be expressed in terms of polynomial equalities. For a recent overview on how and when to apply CAD see [22].

The run-time and memory requirements of CAD computations depend doubly exponential (in the worst case) on the number of variables and the degrees of the appearing polynomials. For the inequality discussed here, the calculations did not terminate within a reasonable amount of time. The recurrence is linear and only of order two, but the polynomial coefficients have degree 16 in m and degrees 9, 10 in x. This might explain the lack of termination in a reasonable time. Besides the computational complexity, this may also be because the method of Gerhold and Kauers is not an algorithm in the strict sense, since termination is not guaranteed for an arbitrary input [24, 28]. Despite this fact, this procedure has been applied successfully in proving different non-trivial inequalities, e.g., on orthogonal polynomials [1, 19, 27]. For the inequality at hand the approach does not seem to succeed and hence a different line of computer-assisted proof that also exploits CAD has been chosen.

The series expansion of $L_m(x)$ can be computed as follows:

$$L_m(x) = \sum_{k \geq 0} \left(\left(\tfrac{3}{2} \right)_k - 3 \left(\tfrac{1}{2} \right)_k \right) \frac{(-m-2)_k}{(-4m-4)_k k!} (4x)^k$$

$$= -2 + \sum_{k \geq 2} \frac{(-m-2)_k \left(\tfrac{1}{2} \right)_k}{(-4m-4)_k k!} (k-1) 2^{2k+1} x^k = -2 + \sum_{k \geq 2} c(m,k) x^k,$$

with coefficients

$$c(m,k) = \frac{(-m-2)_k \left(\tfrac{1}{2} \right)_k}{(-4m-4)_k k!} (k-1) 2^{2k+1}.$$

Here $(a)_k = a(a+1) \cdots (a+k-1)$ denotes the Pochhammer symbol (or rising factorial). Note that the sums above are finite (and thus the $L_m(x)$ are polynomials) because of the factor $(-m-2)_k$ in the numerator.

It is easy to see that for $k \geq 2$ and any $m \geq 2$ these coefficients are non-negative. The only negative contributions stem from the Pochhammer symbols in the numerator and denominator with the same number of factors. The coefficients $c(m, 1)$ vanish and the constant coefficients $c(m, 0) = -2$ for all m. Hence, the forward difference considered here may be written as

$$L_m(x) - L_{m-1}(x) = \sum_{k=2}^{m+1} \left(\frac{c(m, k)}{c(m-1, k)} - 1 \right) c(m-1, k)x^k + c(m, m+2)x^{m+2},$$

with the shift quotient $\rho(m, k) = c(m, k)/c(m-1, k)$ given by

$$\rho(m, k) = \frac{(m+2)(-k+4m+1)(-k+4m+2)(-k+4m+3)(-k+4m+4)}{8(m+1)(2m+1)(4m+1)(4m+3)(-k+m+2)}.$$

This shift quotient is a rational function, hence it is suitable as input for a CAD computation. Since $x \geq \frac{1}{2}$, and $c(m, m+2) > 0$ for $m \geq 0$, it would be sufficient to prove that $\rho(m, k) \geq 1$ for all m, k. This is not the case, but for $k, m \geq 7$ CAD quickly shows that the shift quotient is bounded from below by one:

In[1]:= **Resolve[ForAll[{m, k}, m ≥ 7&&7 ≤ k ≤ n+1, ρ[m, k] ≥ 1], {}, Reals]**
Out[1]= True

These simple observations gave rise to the idea to truncate $L_m(x) - L_{m-1}(x)$ at a fixed level that is provably positive and then to show that the tail of the expansion of $L_m(x) - L_{m-1}(x)$ is non-negative. All these steps can be carried out using CAD. The outline of the proof is as follows:

(1) Prove that $L_m(x) - L_{m-1}(x) > 0$ for $m \leq 6$.
(2) Prove that the truncated version of the inequality with k running from 2 to 6 is positive for all $m \geq 7$.
(3) Prove that the tails are non-negative.

The first cases for $m \leq 6$ are checked independently using CAD. Note that for $m = 1$ the inequality is strict only for $x > \frac{1}{2}$ $\left(L_1(\frac{1}{2}) = 0\right)$. In the problem considered here, the Mathematica implementation of CAD [29] has been used, but certainly any other computer algebra system capable of executing CAD [7,9,17] would be able to handle the given inequalities. In Mathematica, the commands can be executed as follows $\left(\text{where } L[m, x] = L_m(x)\right)$.

In[2]:= **Resolve[ForAll[x, x>$\frac{1}{2}$, L[1,x]−L[0,x]>0], {}, Reals]**
Out[2]= **True**

In[3]:= **Table[Resolve[ForAll[x, x≥$\frac{1}{2}$, L[m,x]−L[m−1,x]>0], {}, Reals], {m,2,6}]**
Out[3]= {**True, True, True, True, True**}

In the second step we consider the truncated power series that is a polynomial in m and x of fixed degree,

$$T_m(x) = -2 + \sum_{k=2}^{6} c(m, k)x^k$$

$$= -2 + \frac{3(m+2)x^2}{4m+3} + \frac{10m(m+2)x^3}{(2m+1)(4m+3)} + \frac{105(m-1)m(m+2)x^4}{2(2m+1)(4m+1)(4m+3)}$$

$$+ \frac{63(m-2)(m-1)(m+2)x^5}{(2m+1)(4m+1)(4m+3)} + \frac{1155(m-3)(m-2)(m-1)(m+2)x^6}{4(2m+1)(4m-1)(4m+1)(4m+3)}.$$

The following CAD computation quickly verifies that $T_{m+1}(x) - T_m(x) > 0$ holds for $m \geq 7$ and $x \geq \frac{1}{2}$.

In[4]:=**Resolve**[**ForAll**[$\{\mathbf{m}, \mathbf{x}\}, \mathbf{m} \geq 7$ && $\mathbf{x} \geq \frac{1}{2}, \mathbf{T}[\mathbf{m}, \mathbf{x}] - \mathbf{T}[\mathbf{m}-1, \mathbf{x}] > 0], \{\}, \mathbf{Reals}$]

Out[4]=**True**

The summary of the computations carried out so far is this: for $m \leq 7$, CAD shows that $L_m(x) > L_{m-1}(x)$. For $m \geq 7$ and $x \geq \frac{1}{2}$, we have

$$L_m(x) - L_{m-1}(x) = T_m(x) - T_{m-1}(x)$$

$$+ \sum_{k=7}^{m+1} \underbrace{(\rho(m,k)-1)c(m-1,k)x^k}_{\geq 0} + \underbrace{c(m+2, m+2)x^{m+2}}_{\geq 0}$$

$$\geq T_m(x) - T_{m-1}(x)$$

$$> 0,$$

where for the penultimate estimate we use that the shift quotient $\rho(m, k) \geq 1$ for $m, k \geq 7$ as shown above. This completes the proof.

3. A Different Approach

Using automated guessing it is possible to derive a second order ordinary differential equation satisfied by $y(x) = L_m(x)$ that (once found) is easy to check. This procedure is described here and is used to established the inequality in Theorem 1.2 for $x \geq 1$. This is not the optimal range, but the techniques employed here are simpler.

The equation obtained for $y(x) = L_m(x)$ is

$$x(3x-2)(4x-1)y''(x) - (12mx^2 - 20mx + 8m + 6x^2 - 11x + 8)y'(x) - 6(m+2)xy(x)$$

$$= 0.$$

The same technique is then used to find (and subsequently prove) the mixed difference-differential relation

$$d(m, x)L_{m+1}(x) = c_1(m, x)L_m'(x) + c_0(m, x)L_m(x), \tag{3.1}$$

with

$$d(m, x) = 2(m+2)(2m+3)(4m+5)(4m+7)\left(16m^2 - 104m + 425\right)$$

$$\times (1-4x)^2(3x-2)(6x+7),$$

$$c_1(m, x) = \left(16m^2 - 104m + 425\right)x(6x + 7)\left(16m^3\left(81x^4 - 165x^3 + 107x^2 - 25x + 2\right)\right.$$
$$+ 24m^2\left(270x^4 - 548x^3 + 368x^2 - 87x + 7\right)$$
$$+ m\left(10656x^4 - 21585x^3 + 15052x^2 - 3605x + 292\right)$$
$$+ 3\left(1920x^4 - 3889x^3 + 2824x^2 - 686x + 56\right)\Big),$$
$$c_0(m, x) = (m + 2)\left(16m^2 - 104m + 425\right)(6x + 7)\left(64m^3(1 - 4x)^2(3x - 2)\right.$$
$$+ 24m^2\left(27x^4 + 566x^3 - 671x^2 + 228x - 24\right)$$
$$+ m\left(1944x^4 + 19953x^3 - 23920x^2 + 8132x - 856\right)$$
$$+ 3\left(480x^4 + 3249x^3 - 3914x^2 + 1330x - 140\right)\Big).$$

Using CAD and these equations it is possible to show that $T_m(x) \geq 0$ for $m \geq 3$ and $x \geq 1$, and that $L_m(x) \geq 0$ for $m \leq 7$ and $x \geq 1$. Since

$$L_m(x) = T_m(x) + \sum_{k \geq 7} c(m, k)x^k$$

with non-negative coefficients $c(m, k)$, it follows that $L_m(x) \geq 0$ for all $x \geq 1$. The coefficients of $L'_m(x)$ are all non-negative (the constant coefficient -2 vanishes), and hence also $L'_m(x) \geq 0$ for all $m, k \geq 2$ and $x > 0$. The relation (3.1) then yields

$$L_{m+1}(x) - L_m(x) = \frac{c_1(m, x)}{d(m, x)}L'_m(x) + \frac{c_0(m, x) - d(m, x)}{d(m, x)}L_m(x).$$

A routine use of CAD shows the rational function coefficients on the right-hand side above are positive for $x \geq 1$. This completes the proof.

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