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A Generalization of Euler Numbers to Finite Coxeter Groups

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Abstract. It is known that Euler numbers, defined as the Taylor coefficients of the tangent and secant functions, count alternating permutations in the symmetric group. Springer defined a generalization of these numbers for each finite Coxeter group by considering the largest descent class, and computed the value in each case of the classification. We consider here another generalization of Euler numbers for finite Coxeter groups, building on Stanley's result about the number of orbits of maximal chains of set partitions. We present a method to compute these integers and obtain the value in each case of the classification.

Keywords: Euler numbers, finite Coxeter groups, set partitions

1. Introduction

It is known since long ago [1] that the Euler numbers T_n, defined by

$$\sec(z) + \tan(z) = \sum_{n\geq 0} T_n \frac{z^n}{n!}, \tag{1.1}$$

count alternating permutations in the symmetric group $\mathfrak{S}_n$ $\big(\sigma$ is *alternating* if $\sigma(1) > \sigma(2) < \sigma(3) > \cdots\big)$. Since then, there has been a lot of interest in these numbers and permutations, as exposed in the recent survey of Stanley [12].

It can be shown that alternating permutations form the largest descent class in the symmetric group. Building on this, Springer [11] gave a characterization of the largest descent class of a finite Coxeter group, and computed its cardinality in each case of the classification. The analog of alternating permutations for other groups were studied by Arnol'd [4], who called these objects *snakes*. See also [8, Section 3] for an alternative proof of Springer's result. Another relevant reference is Saito's article [10], where a more general problem is considered.

In this article, we are interested in another construction that relates the number T_n with the symmetric group, and can also be generalized to finite Coxeter groups.

Namely, there is an action of $\mathfrak{S}_n$ on the maximal chains in the lattice of set partitions of size n, and Stanley [13] showed that the number of orbits is T_{n-1}. It is now well established that set partitions can be realized as an intersection lattice generated by reflecting hyperplanes, so that the construction can be generalized and gives an integer $K(W)$ for each finite Coxeter group W, with $K(A_n) = T_n$. (Note that this differs from Springer's construction, where the integer T_n is related with the group A_{n-1}.) We present a general method to compute $K(W)$ and apply it to obtain the value in each case of the classification. There is some similarity with a problem studied by Reading [9], which consists in the enumeration of maximal chains in the lattice of noncrossing partition (in both cases, there is a product formula for the reducible case and a recursion on maximal parabolic subgroup in the irreducible case).

2. Definitions

Let V be a Euclidean space, and W a finite subgroup of $GL(V)$ generated by orthogonal reflections. Let n be the rank of W, i.e., $n = \dim V$. We call *reflecting hyperplane* an hyperplane $H \subset V$ which is the fixed point set of some reflection in W. The following definition is now well established, see for example, [2, Chapter 4].

Definition 2.1. *The set partition lattice $\mathcal{P}(W)$ is the set of linear subspaces of V that are an intersection of reflecting hyperplanes. It is ordered by reverse inclusion, i.e., $\pi \leq \rho$ if $\rho \subset \pi$.*

Remark 2.2. We are mostly interested in the case where V is the standard geometric representation of a finite Coxeter group W. In this case, $\{0\} \in \mathcal{P}(W)$ and it is the maximal element. But in what follows, it will also be convenient to consider some reflection subgroup $U \subset W$. The definition is still valid and gives a subset $\mathcal{P}(U) \subset \mathcal{P}(W)$, and $\{0\} \notin \mathcal{P}(U)$ *a priori*.

In the case A_n of the classification, W is the symmetric group $\mathfrak{S}_{n+1}$ acting on $V = \left\{v \in \mathbb{R}^{n+1} \colon \sum v_i = 0\right\}$ by permuting coordinates. The reflecting hyperplanes are $H_{i,j} = \{v \in V \colon v_i = v_j\}$ where $i < j$. We recover the traditional definition of a set partition, for example, if $n = 6$, then

$$H_{1,7} \cap H_{2,4} \cap H_{4,5} = \{v \in V \colon v_1 = v_7, v_2 = v_4 = v_5\} \in \mathcal{P}(A_6)$$

corresponds to the set partition $17|245|3|6$.

Let $t \in W$ be a reflection, and $H = \mathrm{Fix}(t)$ be its fixed point set. Then $w(H) = \mathrm{Fix}\left(wtw^{-1}\right)$ for $w \in W$. So the natural action of W on linear subspaces of V gives an action of W on the reflecting hyperplanes, and on $\mathcal{P}(W)$. Since inclusion and rank are preserved, this extends to an action on the maximal chains in $\mathcal{P}(W)$.

Definition 2.3. *Let $\mathcal{M}(W)$ denote the set of maximal chains in $\mathcal{P}(W)$, i.e., sequences $C = (C_0, \dots, C_n) \in \mathcal{P}(W)^{n+1}$ where $C_0 < \cdots < C_n$ (this implies that C_i has rank i). We define an integer $K(W)$ as the number of orbits for the W-action on $\mathcal{M}(W)$, i.e., $K(W) = \#(\mathcal{M}(W)/W)$.*

An element of $\mathcal{M}(W)$ can be seen as a complete flag of V. Thus we can rephrase the definition: $K(W)$ is the number of W-orbits of complete flags in V where each element of the flag is a fixed point subspace of some $w \in W$.

Let us introduce some notations (see [5,6]). We recall that the complement in V of the reflecting hyperplanes is divided into connected regions called *chambers*, and W acts simply transitively on the chambers. Let $H_1, \dots, H_n$ be the reflecting hyperplanes that enclose one particular chamber R_0, the fundamental chamber. Then the corresponding orthogonal reflections $s_1, \dots, s_n$ form a set S of simple generators for W. According to this choice, there is a longest element w_0 (the unique group element that maximizes the length function). For any i, let $W_{(i)} \subset W$ be the (standard maximal parabolic) subgroup generated by the s_j with $j \neq i$. If $s \in S$, we also denote $W_{(s)} = W_{(i)}$ if $s = s_i$. An alternative description is that, if we define a line

$$L_i = \bigcap_{\substack{1 \le j \le n \\ j \neq i}} H_j, \tag{2.1}$$

then $w \in W_{(i)}$ if and only if $w(v) = v$ for any $v \in L_i$. The lines L_i are exactly those in $\mathcal{P}(W)$ that are incident to the fundamental chamber R_0.

For each line $L \in \mathcal{P}(W)$, we define two subgroups of W, respectively the stabilizer and the pointwise stabilizer:

$$\mathrm{Stab}(L) = \{ w \in W : w(L) = L \},$$
$$\mathrm{Stab}^*(L) = \{ w \in W : \forall x \in L,\, w(x) = x \}.$$

Note that $\mathrm{Stab}^*(L)$ is a subgroup of $\mathrm{Stab}(L)$ with index either 1 or 2. The group $\mathrm{Stab}^*(L)$ is generated by the reflections it contains and is itself a real reflection group, its reflecting hyperplanes being those of W containing L. So we can identify $\mathcal{P}(\mathrm{Stab}^*(L))$ with the interval $[V, L] \subset \mathcal{P}(W)$.

3. The General Method

We describe how the integer $K(W)$ can be computed inductively. To begin, in the reducible case we have:

Proposition 3.1. *Let W_1 and W_2 be two Coxeter groups of respective ranks m and n, then*

$$K(W_1 \times W_2) = \binom{m+n}{m} K(W_1) K(W_2).$$

Proof. First, note that there is natural identification $\mathcal{P}(W_1) \times \mathcal{P}(W_2) = \mathcal{P}(W_1 \times W_2)$. The idea is to shuffle an element of $\mathcal{M}(W_1)$ with one of $\mathcal{M}(W_2)$ and the details are as follows. Let $(x_0, \dots, x_m) \in \mathcal{M}(W_1)$ and $(y_0, \dots, y_n) \in \mathcal{M}(W_2)$. By elementary properties of the product order, we can form an element $C \in \mathcal{M}(W_1 \times W_2)$ by considering a sequence

$$C = \left((x_{i_0}, y_{j_0}), \dots, (x_{i_{m+n}}, y_{j_{m+n}}) \right),$$

where the indices are such that $i_0 = j_0 = 0$, $i_{m+n} = m$, $j_{m+n} = n$, and for $0 \le k < m+n$:

- either $i_{k+1} = i_k$ and $j_{k+1} = j_k + 1$,
- or $i_{k+1} = i_k + 1$ and $j_{k+1} = j_k$.

If I denotes the set of possible choices for the indices i_k and j_k, this defines a bijection

$$I \times \mathcal{M}(W_1) \times \mathcal{M}(W_2) \to \mathcal{M}(W_1 \times W_2).$$

Since the bijection commutes with the action of $W_1 \times W_2$ and $\#I = \binom{m+n}{m}$, the result is proved. ∎

We suppose now that W is irreducible. A natural approach to find $K(W)$ is to distinguish the maximal chains according to the coatom they contain (in terms of complete flags, we distinguish them according to the line they contain). Doing the same thing at the level of orbits will lead to Proposition 3.2 below.

Recall that we can identify $\mathcal{P}(\mathrm{Stab}^*(L))$ with $[V,L] \subset \mathcal{P}(W)$. There is also a natural way to see $\mathcal{M}(\mathrm{Stab}^*(L))$ as a subset of $\mathcal{M}(W)$, namely, $(C_0,\dots,C_{n-1}) \in \mathcal{M}(\mathrm{Stab}^*(L))$ is identified with $(C_0,\dots,C_{n-1},\{0\})$. Clearly, $[V,L]$ is stable by the action of $\mathrm{Stab}(L)$ and this extends to an action of $\mathrm{Stab}(L)$ on $\mathcal{M}(\mathrm{Stab}^*(L))$. With this at hand, we have:

Proposition 3.2. *Let $\mathcal{L} \subset \mathcal{P}(W)$ be a set of orbit representatives for the action of W on lines in $\mathcal{P}(W)$, then*:

$$K(W) = \sum_{L \in \mathcal{L}} \#\big(\mathcal{M}(\mathrm{Stab}^*(L))/\,\mathrm{Stab}(L)\big).$$

Proof. If $C = (C_0,\dots,C_n) \in \mathcal{M}(W)$, there is a unique $L \in \mathcal{L}$ such that the coatom C_{n-1} and L are in the same W-orbit. Moreover, L only depends on the W-orbit of C, so this defines a map $f\colon \mathcal{M}(W)/W \to \mathcal{L}$.

With the discussion above in mind, we identify $\mathcal{M}(\mathrm{Stab}^*(L))$ with the set of chains $C = (C_0,\dots,C_n) \in \mathcal{M}(W)$ satisfying $C_{n-1} = L$. Each element of $f^{-1}(L)$ is a W-orbit that can be represented by an element of $\mathcal{M}(\mathrm{Stab}^*(L))$, and two elements of $\mathcal{M}(\mathrm{Stab}^*(L))$ are in the same W-orbit if and only if they are in the same $\mathrm{Stab}(L)$-orbit. This permits to define a bijection between $f^{-1}(L)$ and $\mathcal{M}(\mathrm{Stab}^*(L))/\,\mathrm{Stab}(L)$. Now, we can write:

$$K(W) = \#(\mathcal{M}(W)/W) = \sum_{L \in \mathcal{L}} \#\left(f^{-1}(L)\right) = \sum_{L \in \mathcal{L}} \#\big(\mathcal{M}(\mathrm{Stab}^*(L))/\,\mathrm{Stab}(L)\big),$$

as announced. ∎

Now, let us describe how to find the set $\mathcal{L}$ of orbit representatives for the action of W on lines in $\mathcal{P}(W)$. We can use the lines L_i defined in Equation (2.1) from the previous section.

Proposition 3.3. *Each line $L \in \mathcal{P}(W)$ can be written $w(L_i)$ for some $w \in W$ and $1 \le i \le n$. If $w \in W$ and $i \neq j$, then $w(L_i) = L_j$ implies $w_0(L_i) = L_j$.*

Similar considerations appeared in the work of Armstrong, Reiner, and Rhoades [3], in the context of W-parking functions. Still, it is reasonable to include a short proof here.

Proof. Let us split the line L in two half-lines L^+ and L^-, and let R be a chamber incident to L^+. We also split L_i in two half-lines L_i^+ and L_i^-, where L_i^+ is the one incident to R_0. The group W acts simply transitively on the chambers, so there is $w \in W$ such that $w(R_0) = R$. Then $w^{-1}(L^+)$ is incident to R_0, so there is i such that $w^{-1}(L^+) = L_i^+$, and consequently, $L^+ = w\left(L_i^+\right)$ and $L = w(L_i)$.

Now, suppose we have $i \neq j$ and $w(L_i) = L_j$. We have either $w\left(L_i^+\right) = L_j^+$ or $w\left(L_i^+\right) = L_j^-$ (where L_j^+ and L_j^- are defined in the same way as with L_i). In the first case, R_0 and $w(R_0)$ are both incident to L_j^+. This implies $w\left(L_j^+\right) = L_j^+$ (note that $W_{(j)}$ acts simply transitively on the set of chambers incident to L_j^+), but this is a contradiction with $i \neq j$ and $w(L_i) = L_j$. So we have $w\left(L_i^+\right) = L_j^-$. Since L_j^- is incident to both $-R_0$ and $w(R_0)$, there is $u \in W_{(j)}$ such that $uw(R_0) = -R_0$, i.e., $uw = w_0$. Then, we have $w_0\left(L_i^+\right) = uw\left(L_i^+\right) = u\left(L_j^-\right) = L_j^-$. So $w_0(L_i) = L_j$. ∎

From the definition of L_i in Equation (2.1), $w_0(L_i) = L_j$ is equivalent to $w_0(H_i) = H_j$, which is also equivalent to $w_0 s_i w_0 = s_j$. Elementary properties of the longest element show that the map defined on the simple generators by $s \mapsto w_0 s w_0$ is an involutive automorphism of the Coxeter graph. One can also show that this automorphism is the identity if and only if the exponents of the group are all odd, see [5, Exercise 4.10]. So the set $\mathcal{L}$ can be obtained by taking $\{L_1, \dots, L_n\}$, quotiented by the action of w_0 which can be described in a precise way.

We have $\mathrm{Stab}^*(L_i) = W_{(i)}$, the standard maximal parabolic subgroup. To identify the group $\mathrm{Stab}(L_i)$, we have the following:

Proposition 3.4. *Either* $\mathrm{Stab}(L_i) = W_{(i)}$, *or* $\mathrm{Stab}(L_i) = < W_{(i)}, w_0 >$.

Proof. Suppose there is $w \in \mathrm{Stab}(L_i)$ with $w \notin W_{(i)}$, which means that $w\left(L_i^+\right) = L_i^-$. So $w(R_0)$ is incident to L_i^-. Since $W_{(i)}$ acts transitively on the chambers incident to L_i^-, there is $u \in W_{(i)}$ with $uw(R_0) = -R_0$, i.e., $uw = w_0$. It follows $w_0 \in \mathrm{Stab}(L_i)$ with $w_0 \notin W_{(i)}$. ∎

Since $W_{(i)}$ has rank $n-1$, by induction we can assume we already know the integer $K\left(W_{(i)}\right)$, which is useful in some situations.

Proposition 3.5. *With W, w_0, and $L_i \in \mathcal{L}$ as above, we have*:

- *If $w_0 s_i w_0 \neq s_i$, then*

$$\#\left(\mathcal{M}\left(W_{(i)}\right) / \mathrm{Stab}(L_i)\right) = K\left(W_{(i)}\right).$$

- *If $w_0 s_i w_0 = s_i$, and there is $u \in W_{(i)}$ such that $w_0 s_j w_0 = u s_j u$ for any $j \neq i$, then*

$$\#\left(\mathcal{M}\left(W_{(i)}\right) / \mathrm{Stab}(L_i)\right) = K\left(W_{(i)}\right).$$

- *If $w_0 s_i w_0 = s_i$, and the map $s \mapsto w_0 s w_0$ permutes nontrivially the connected components of the Coxeter graph of $W_{(i)}$, then*:

$$\#\left(\mathcal{M}\left(W_{(i)}\right) / \mathrm{Stab}(L_i)\right) = \frac{1}{2} K\left(W_{(i)}\right).$$

Proof. If $w_0 s_i w_0 \neq s_i$, then $w_0 \notin \operatorname{Stab}(L_i)$, hence $\operatorname{Stab}(L_i) = W_{(i)}$ using Proposition 3.4. This proves the first point.

Suppose $w_0 s_i w_0 = s_i$ and there exists u as above. It means that the action of u on $\mathcal{M}\big(W_{(i)}\big)$ is the same as the action of w_0. In either of the two cases given in Proposition 3.4, we find that the $\operatorname{Stab}(L_i)$-orbits are exactly the $W_{(i)}$-orbits. This proves the second point.

As for the third point, we suppose there are only two connected components in the Coxeter graph of $W_{(i)}$, the general case being similar. Let us write $W_{(i)} = W_1 \times W_2$. We have seen in the proof of Proposition 3.1 that the elements of $\mathcal{M}\big(W_{(i)}\big)$ are obtained by "shuffling" two elements of $\mathcal{M}(W_1)$ and $\mathcal{M}(W_2)$. So if $C = (C_0, \dots, C_{n-1}) \in \mathcal{M}\big(W_{(i)}\big)$, the element C_1 is a pair $(C_1', C_1'') \in \mathcal{P}(W_1) \times \mathcal{P}(W_2)$ where the respective ranks of C_1' and C_1'' are either 0 and 1, or 1 and 0. These two conditions are preserved by the action of $W_{(i)}$, and are reversed by the action of w_0. So the action of w_0 on $\mathcal{M}\big(W_{(i)}\big)/W_{(i)}$ has no fixed point and each orbit has cardinality 2. We can write:

$$\#\left(\mathcal{M}\left(W_{(i)}\right)/\operatorname{Stab}(L_i)\right) = \#\left(\left(\mathcal{M}\left(W_{(i)}\right)/W_{(i)}\right)/w_0\right),$$

and this proves the result. ∎

Let us summarize the situation. If w_0 is central in W, we can always apply the second case of Proposition 3.5, so that Proposition 3.2 gives

$$K(W) = \sum_{s \in S} K\left(W_{(s)}\right), \tag{3.1}$$

where each $W_{(s)}$ is a standard maximal parabolic subgroup of W. Furthermore, some of the terms are simplified using the product formula in Proposition 3.1. In particular, this equation can be directly obtained from the Coxeter graph.

When w_0 is not central, the map $s \mapsto w_0 s w_0$ is an involution on the set S of simple generators and we need to distinguish the two-element orbits and the fixed points. Indeed, we have:

$$K(W) = \sum_{\substack{\{s_i, s_j\} \subset S,\ s_i \neq s_j \\ w_0 s_i w_0 = s_j}} K\left(W_{(i)}\right) + \sum_{\substack{s_i \in S \\ w_0 s_i w_0 = s_i}} \#\left(\mathcal{M}\left(W_{(i)}\right)/\operatorname{Stab}(L_i)\right). \tag{3.2}$$

Some terms in the first sum (the second sum, respectively) can be further simplified using Proposition 3.1 (Proposition 3.5, respectively).

Note that Proposition 3.5 does not exhaust all the possibilities, so we do not have a general solution to find all the terms $\#\big(\mathcal{M}\big(W_{(i)}\big)/\operatorname{Stab}(L_i)\big)$ in the second sum of Equation (3.2). As we will see in the next section, the only case that cannot be treated directly will appear when $W = D_n$ with n odd.

4. The Case by Case Resolution

We follow the traditional notation for the classification of finite irreducible Coxeter groups, see [5]. We will denote $a_n = K(A_n)$, $b_n = K(B_n)$, $d_n = K(D_n)$. It will be convenient to take the conventions that $A_0 = B_0 = D_0$ (the trivial group with rank 0), $A_1 = B_1$, $D_2 = A_1 \times A_1$, and $D_3 = A_3$.

Proposition 4.1. (See [5, Exercise 4.10]) *In the groups $I_2(m)$ for m even, B_n, D_n for n even, G_2, H_3, H_4, E_7, and E_8, the longest element is central. In the other groups, i.e., $I_2(m)$ for m odd, A_n, D_n for n odd, and E_6, the map $s \mapsto w_0 s w_0$ is the unique nontrivial automorphism of the Coxeter graph.*

4.1. Case of A_n

We already know that $a_n = T_n$, but let us check how to prove it with our method. Here, w_0 is not central and $s \mapsto w_0 s w_0$ reverses the n vertices of the Coxeter graph. There is a fixed point only if n is odd, and it can be treated using the third case of Proposition 3.5. So Equation (3.2) gives, when $n \geq 2$:

$$a_n = \sum_{i=0}^{\lfloor n/2 \rfloor - 1} \binom{n-1}{i} a_i a_{n-1-i} + [n \bmod 2] \times \frac{1}{2} \binom{n-1}{(n-1)/2} a^2_{(n-1)/2}.$$

(Here, $[n \bmod 2]$ is considered as the natural number 0 or 1.) This can be rewritten as:

$$a_n = \frac{1}{2} \sum_{i=0}^{n-1} \binom{n-1}{i} a_i a_{n-1-i}. \tag{4.1}$$

Let us define

$$A(z) = \sum_{n \geq 0} a_n \frac{z^n}{n!}.$$

Multiplying Equation (4.1) by $\frac{z^{n-1}}{(n-1)!}$ and summing over $n \geq 2$ gives

$$A'(z) - 1 = \frac{1}{2} \left(A(z)^2 - 1 \right).$$

So $A(z)$ is the solution of the differential equation $A'(z) = \frac{1}{2}\left(A(z)^2 + 1\right)$ with the initial value $A(0) = 1$. It can be checked that $A(z) = \tan(z) + \sec(z)$ is the solution, so that $a_n = T_n$.

4.2. Case of B_n

In this group, the longest element is central. Equation (3.1) together with the product formula gives:

$$b_n = \sum_{i=0}^{n-1} \binom{n-1}{i} b_i a_{n-i-1}. \tag{4.2}$$

Now, let

$$B(z) = \sum_{n \geq 0} b_n \frac{z^n}{n!}.$$

Multiplying Equation (4.2) by $\frac{z^{n-1}}{(n-1)!}$ and summing over $n \geq 1$ gives

$$B'(z) = B(z) A(z).$$

So $B(z)$ is the solution of the differential equation $B'(z) = B(z)A(z)$ with initial value $B(0) = 1$. We can check that

$$B(z) = \frac{1}{1 - \sin(z)}$$

is a solution. This function also satisfies $B(z) = A'(z)$, so that

$$b_n = T_{n+1}.$$

A bijective proof of this will be given in [7].

4.3. Case of D_n

When n is even, the longest element of D_n is central and Equation (3.1) gives:

$$d_n = 2a_{n-1} + \sum_{2 \le i \le n-1} \binom{n-1}{i} d_i a_{n-1-i}. \tag{4.3}$$

In the case when n is odd, one cannot quite write the equation as immediately. The map $s \mapsto w_0 s w_0$ exchanges two vertices of the Coxeter graph, and this gives one term a_{n-1} coming from the first sum in Equation (3.2). As for the second sum, we are in the case where $s_i = w_0 s_i w_0$, and $W_{(i)} = D_i \times A_{n-1-i}$. If i is odd, we can apply the second case of Proposition 3.5 where u is chosen to be the longest element of the factor D_i. More care is needed when i is even, i.e., when we cannot directly apply Proposition 3.5. So we consider the set $\mathcal{M}(D_i \times A_{n-1-i})$, quotiented by $D_i \times A_{n-1-i}$, and further quotiented by the graph automorphism of the factor D_i $\big($the graph automorphism induces an action on $\mathcal{P}(D_i)\big)$. An argument similar to the one in Proposition 3.1 shows that the number of orbits can be factorized. Eventually, we obtain:

$$d_n = a_{n-1} + \sum_{\substack{2 \le i \le n-1 \\ i \text{ odd}}} \binom{n-1}{i} d_i a_{n-1-i} + \sum_{\substack{2 \le i \le n-1 \\ i \text{ even}}} \binom{n-1}{i} \bar{d}_i a_{n-1-i}, \tag{4.4}$$

where $\bar{d}_i$ is defined as follows: it is the number of orbits for the action on $\mathcal{M}(D_i)$ generated by D_i together with the graph automorphism (except that if $i = 4$, the graph automorphism is not unique but we only consider the one that exchanges two vertices). Note that for odd i, we can define $\bar{d}_i$ similarly but it is clear that $\bar{d}_i = d_i$. We need to compute $\bar{d}_n$ before solving the recursion for d_n.

Proposition 4.2. *We have $\bar{d}_0 = 1$ and for any $n \ge 1$,*

$$\bar{d}_n = a_{n-1} + \sum_{i=2}^{n-1} \binom{n-1}{i} \bar{d}_i a_{n-1-i}. \tag{4.5}$$

Proof. Although we cannot directly apply Propositions 3.2 and 3.1, the argument is completely similar, so we omit details. Let Γ denote the graph automorphism of D_n.

Suppose L_1 and L_2 are the two coatoms that are exchanged by Γ. Counting orbits of maximal chains having L_1 or L_2 as coatom, we obtain the first term a_{n-1}.

If $i \neq 1,2$, the number of orbits of maximal chains having L_i as coatom is the number of orbits in $\mathcal{M}\big(W_{(i)}\big)/<\mathrm{Stab}(L_i),\Gamma>$. This is also the number of orbits in $\mathcal{M}\big(W_{(i)}\big)/<W_{(i)},\Gamma>$, since either $\mathrm{Stab}(L_i)=W_{(i)}$ or $\mathrm{Stab}(L_i)=<W_{(i)},w_0>$ where w_0 has the same action as Γ. We have a decomposition $W_{(i)}=D_i\times A_{n-1-i}$ and the graph automorphism only acts on the factor D_i. So the argument of Proposition 3.1 shows that this number is $\bar{d}_i a_{n-1-i}$. ∎

Proposition 4.3. *If $n\geq 2$, we have $\bar{d}_n = 2a_{n+1}-(n+1)a_n$.*

Proof. The recursion in the previous proposition shows that the generating function $\bar{D}(z)=\sum_{n\geq 0}\bar{d}_n\frac{z^n}{n!}$ satisfies the differential equation

$$\bar{D}'(z)=(\bar{D}(z)-z)A(z),$$

with the initial condition $\bar{D}(0)=1$. This is solved by

$$\bar{D}(z)=\frac{2-\cos(z)-z\sin(z)}{1-\sin(z)}. \tag{4.6}$$

From this expression, we can get $\bar{D}(z)=(2-z)A'(z)+z-A(z)$, and it follows that $\bar{d}_n=2a_{n+1}-(n+1)a_n$ if $n\geq 2$. ∎

Proposition 4.4. *Let $n\geq 2$. We have $d_n-\bar{d}_n=a_n$ if n is even, and $d_n=\bar{d}_n$ otherwise.*

Proof. If $n\geq 2$, from (4.3), (4.4), and (4.5), we have:

$$d_n-\bar{d}_n=\chi[n\text{ even}]\times a_{n-1}+\sum_{\substack{2\leq i\leq n-1\\ n-i\text{ even}}}\binom{n-1}{i}\left(d_i-\bar{d}_i\right)a_{n-1-i}.$$

Here and in the sequel, χ means 1 or 0 depending on whether the condition within brackets is true or false. So the generating function

$$U(z)=1+\sum_{n\geq 2}\left(d_n-\bar{d}_n\right)\frac{z^n}{n!}$$

satisfies $U'(z)=U(z)\tan(z)$ and $U(0)=1$. This is solved by $U(z)=\sec(z)$ and the result follows. ∎

From the previous two propositions, we get that for $n\geq 2$,

$$d_n=\begin{cases}2T_{n+1}-nT_n, & \text{if } n \text{ is even},\\ 2T_{n+1}-(n+1)T_n, & \text{if } n \text{ is odd}.\end{cases}$$

From (4.6), we can separate the odd and even parts of $\bar{D}(z)$ (multiply the numerator and denominator by $1+\sin(z)$ and separate terms in the numerator). After some calculation, this leads to:

$$\sum_{n\geq 1}d_{2n}\frac{z^{2n}}{(2n)!}=\frac{\sin(z)(2\sin(z)-z)}{\cos(z)^2},$$

and

$$\sum_{n\geq 1} d_{2n+1} \frac{z^{2n+1}}{(2n+1)!} = \frac{\sin(z)(2-\cos(z)) - z}{\cos(z)^2}.$$

We can take the sum of these two equations to obtain $\sum_{n\geq 2} d_n \frac{z^n}{n!}$, but there seems to be no particular simplification. The first values of $\bar{d_n}$ for $n \geq 2$ are as follows:

$$1, 2, 7, 26, 117, 594, 3407, 21682, 151853, 1160026, 9600567\dots$$

And the first values of d_n for $n \geq 2$ are:

$$2, 2, 12, 26, 178, 594, 4792, 21682, 202374, 1160026, 12303332,\dots$$

4.4. Remaining Cases

For the dihedral group, we have:

$$K(I_2(m)) = \begin{cases} 1, & \text{if } m \text{ is odd,} \\ 2, & \text{if } m \text{ is even.} \end{cases}$$

Among the exceptional groups, E_6 is the only one where the longest element is not central. We apply Equation (3.2) and the calculation is the following:

$$\begin{aligned} K(E_6) &= K(D_5) + K(A_4 \times A_1) + \frac{1}{2} K(A_2 \times A_1 \times A_2) + K(A_5) \\ &= 26 + 25 + 15 + 16 \\ &= 82. \end{aligned}$$

The first two terms correspond to the terms where $s_i \neq s_j$ and $w_0 s_i w_0 = s_j$. The third term corresponds to a fixed point of the graph automorphism, the vertex of degree 3. It is treated using the second part of Proposition 3.5. The fourth term corresponds to the other fixed point of the graph automorphism, it is treated using the first part of Proposition 3.5.

For all the remaining groups, the longest element is central and we can apply Equation (3.1). This gives:

$$\begin{aligned} K(H_3) &= K(I_2(5)) + K(A_1 \times A_1) + K(A_2) = 4, \\ K(H_4) &= K(H_3) + K(I_2(5) \times A_1) + K(A_2 \times A_1) + K(A_3) = 12, \\ K(F_4) &= K(B_3) + K(A_2 \times A_1) + K(A_1 \times A_2) + K(B_3) = 16. \end{aligned}$$

Eventually, we have:

$$\begin{aligned} K(E_7) = K(E_6) &+ K(D_5 \times A_1) + K(A_4 \times A_2) \\ &+ K(A_3 \times A_1 \times A_2) + K(A_1 \times A_5) + K(D_6) + K(A_6) \end{aligned}$$

$$= 82 + 156 + 75 + 120 + 96 + 178 + 61$$
$$= 768,$$

and

$$\begin{aligned} K(E_8) &= K(E_7) + K(E_6 \times A_1) + K(D_5 \times A_2) + K(A_4 \times A_3) \\ &\quad + K(A_2 \times A_1 \times A_4) + K(A_6 \times A_1) + K(D_7) + K(A_7) \\ &= 768 + 574 + 546 + 350 + 525 + 427 + 594 + 272 \\ &= 4056. \end{aligned}$$

5. Final Remarks

Let us briefly mention some related results that will appear in [7]. Let c be a Coxeter element for W, and consider the set of *noncrossing partitions* $\mathcal{P}^{NC}(W,c)$ (see [2] for background on noncrossing partitions). This sets naturally embeds in $\mathcal{P}(W)$. It is not stable under the action of W, but we can consider how it is divided in equivalence classes (each class is the intersection of $\mathcal{P}^{NC}(W,c)$ with a W-orbit). There are $K(W)$ equivalence classes. We will show in [7] that we can in some sense compute the cardinality of each class, and that this leads to hook length formulas in type A and B.

Let us end this article which a more general question, which is not very precisely stated. Let G be a subgroup of $GL(\mathbb{R}^n)$ (there are probably some restrictions to consider, see below). We can define a set

$$\mathcal{P}(G) = \big\{\pi \subset \mathbb{R}^n : \exists g \in G, \pi = \mathrm{Fix}(g)\big\},$$

and let $\mathcal{M}(G)$ denote the set of maximal chains in $\mathcal{P}(G)$ with respect to inclusion. The group G acts on $\mathcal{P}(G)$ and $\mathcal{M}(G)$, and we can define $K(G) = \#(\mathcal{M}(G)/G)$. We have examined the case where G is a finite reflection group but we see that the definition is valid in a more general context. Note that a natural restriction on the group G is the requirement that $\mathcal{M}(G)$ is a set of complete flags. Suppose for example that G is the set of invertible upper-triangular matrices. Then $\mathcal{P}(G)$ is the set of all linear subspaces of $\mathbb{R}^n$, as can be seen using the LU decomposition. So $\mathcal{M}(G)$ is the complete flag variety $GL(\mathbb{R}^n)/G$. Using the Bruhat decomposition, we see that $K(G) = n!$. It might be of interest to examine the case of other groups.

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References

1. André, D.: Développement de sec x and tg x. C. R. Math. Acad. Sci. Paris 88, 965–979 (1879)
2. Armstrong, D.: Generalized noncrossing partitions and combinatorics of Coxeter groups. Mem. Amer. Math. Soc. 202(949) (2009)
3. Armstrong, D., Reiner, V., Rhoades, B.: Parking spaces. Adv. Math. 269, 647–706 (2015)

4. Arnol'd, V.I.: The calculus of snakes and the combinatorics of Bernoulli, Euler and Springer numbers for Coxeter groups. Russian Math. Surveys 47(1), 1–51 (1992)
5. Björner, A., Brenti, F.: Combinatorics of Coxeter Groups. Graduate Texts in Math., Vol. 231. Springer, New York (2005)
6. Humphreys, J.E.: Reflection Groups and Coxeter Groups. Cambridge University Press, Cambridge (1990)
7. Josuat-Vergès, M.: Refined enumeration of noncrossing chains and hook formulas. In preparation
8. Josuat-Vergès, M., Novelli, J.-C., Thibon, J.-Y.: The algebraic combinatorics of snakes. J. Combin. Theory Ser. A 119(8), 1613–1638 (2012)
9. Reading, N.: Chains in the noncrossing partition lattice. SIAM J. Discrete Math. 22(3), 875–886 (2008)
10. Saito, K.: Principal Γ-cone for a tree. Adv. Math. 212(2), 645–668 (2007)
11. Springer, T.A.: Remarks on a combinatorial problem. Nieuw Arch. Wisk. (3) 19, 30–36 (1971)
12. Stanley, R.P.: Some aspects of groups acting on finite posets. J. Combin. Theory Ser. A 32(2), 132–161 (1982)
13. Stanley, R.P.: A survey of alternating permutations. In: Brualdi, R.A., Hedayat, S., Kharaghani, H., Khosrovshahi, G.B., Shahriari, S. (eds.) Combinatorics and Graphs. Contemp. Math., Vol. 531, pp. 165–196. Amer. Math. Soc., Providence, RI (2010)