Ann. Comb. 13 (2010) 425–452
DOI 10.1007/s00026-009-0037-6
Published online January 12, 2010

Annals of Combinatorics

Orientations, Lattice Polytopes, and Group Arrangements I: Chromatic and Tension Polynomials of Graphs*

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Received January 15, 2007

AMS Subject Classification: 05A99, 05B35, 05C15, 06A07, 52B20, 52B40

Abstract. This is the first one of a series of papers on association of orientations, lattice polytopes, and group arrangements to graphs. The purpose is to interpret the integral and modular tension polynomials of graphs at zero and negative integers. The whole exposition is put under the framework of subgroup arrangements and the application of Ehrhart polynomials. Such a viewpoint leads to the following main results of the paper: (i) the reciprocity law for integral tension polynomials; (ii) the reciprocity law for modular tension polynomials; and (iii) a new interpretation for the value of the Tutte polynomial $T(G; x, y)$ of a graph G at $(1, 0)$ as the number of cut-equivalence classes of acyclic orientations on G.

Keywords: subgroup arrangements, hyperplane arrangements, orientations, acyclic orientations, oriented cuts, cut-equivalence relation, coloring polytopes, tension polytopes, chromatic polynomial, tension polynomial, characteristic polynomial, Tutte polynomial, reciprocity law

1. Introduction

The chromatic polynomial $\chi(G, q)$ of a graph G, which was introduced by Birkhoff [3] as the number of proper colorings of G with q colors, is one of the most mysterious but fruitful polynomials in graph theory. It is the root of many other important polynomials such as the characteristic polynomial for graded posets and the Tutte polynomial for matroids; see [5, 7, 22, 25]. Stimulated by the Four Color Conjecture (FCC) of the time and the work of Rota et al. on the foundations of combinatorics, a flood of research on such polynomials has continued for about three decades in connection with graphs, matroids, posets, and simplicial complexes, and the wave is still in its trend in connection with convexity, toric geometry, Ehrhart polynomials, and topological invariants.

* The research was supported by the RGC Competitive Earmarked Research Grants 600703 and 600506.

The key stimulation behind the research of characteristic polynomials and their analogs is the geometric idea, which dates back to Veblen's equivalent formulation of FCC [27] as solving a system of linear inequalities over the field $GF(4)$. Such a geometric idea is implicit in the work of Crapo and Rota [13], Greene and Zaslavsky [16], Kung [19], Zaslavsky [28], and many others. More recent activities include the work of Athanasiadis [1], Björner and Ekedahl [4], Chen [8], Ehrenborg and Readdy [14], Reiner [21], and Sagan [23]. To continue this geometric trend, we propose to study systematically the chromatic polynomials, flow polynomials, characteristic polynomials, and Tutte polynomials in a series of papers by associating to graphs with orientations, lattice polytopes, and abelian subgroup arrangements, and then applying the theory of Ehrhart polynomials. The present paper is the first one of a series of papers on such investigations, concentrating on chromatic and tension polynomials. The main task is to generalize Stanley's interpretation [24] of chromatic polynomials at negative integers to integral and modular tension polynomials introduced by Kochol [18]. Our approach is to directly introduce group arrangements, lattice polytopes, and equivalence relations on orientations of graphs in a rigorous systematical way. We follow the books [5, 6, 15, 20, 25, 29] for various notions on graphs, polytopes, and hyperplane arrangements. For Ehrhart polynomials we refer to the papers [9, 10]. The whole exposition is self-contained.

Let $G = (V, E)$ be a graph with possible loops and multiple edges; we write $V = V(G)$ and $E = E(G)$. An *orientation* of G is an assignment that each edge of G is assigned an arrow. If an edge is incident with two vertices, there are two ways to assign an arrow; if it is incident with one vertex, there is only one way to assign an arrow. A graph with an orientation is called a *digraph*. A *circuit* is a connected graph having degree 2 at every vertex; a *direction* of a circuit is an orientation such that there is an arrow in and an arrow out at every vertex; a circuit with a direction is called a *directed circuit*. An orientation is said to be *acyclic* if it has no directed circuit. We denote by $O(G)$ the set of all orientations of G, and by $O_{\mathrm{AC}}(G)$ the set of all acyclic orientations. For positive integers q, let $[q] = \{1, 2, \ldots, q\}$; we define the counting function

$$\bar{\chi}(G, q) := \#\left\{(\varepsilon, f) \,|\, \varepsilon \in O_{\mathrm{AC}}(G), f \in [q]^V, f(u) \geq f(v) \,\text{for}\, u \stackrel{\varepsilon}{\to} v\right\}.$$

For each acyclic orientation $\varepsilon \in O_{\mathrm{AC}}(G)$, let $\bar{\Delta}^{+}_{\mathrm{CL}}(G, \varepsilon)$ denote the 0-1 polytope in the Euclidean space $\mathbb{R}^V$ whose vertices are vectors $f\colon V(G) \to \{0, 1\}$ such that

$$f(u) \geq f(v), \text{ for all } u \stackrel{\varepsilon}{\to} v.$$

We call $\bar{\Delta}^{+}_{\mathrm{CL}}(G, \varepsilon)$ the *coloring polytope* and its interior $\Delta^{+}_{\mathrm{CL}}(G, \varepsilon)$ the *open coloring polytope* of the digraph (G, ε). Let

$$\chi(G, \varepsilon; t) := L(\Delta^{+}_{\mathrm{CL}}(G, \varepsilon), t + 1)$$

denote the Ehrhart polynomial of $\Delta^{+}_{\mathrm{CL}}(G, \varepsilon)$ in variable $t + 1$, and let

$$\bar{\chi}(G, \varepsilon; t) := L(\bar{\Delta}^{+}_{\mathrm{CL}}(G, \varepsilon), t - 1)$$

denote the Ehrhart polynomial of $\bar{\Delta}^{+}_{\mathrm{CL}}(G, \varepsilon)$ in variable $t - 1$. The following theorem is a reformulation of Stanley's interpretation for the values of chromatic polynomials at negative integers by geometric ideas; see [24].

Theorem 1.1. (Stanely) *Let G be a loopless graph with possible multiple edges. Then the chromatic polynomials $\chi(G,t)$ and $\bar{\chi}(G,t)$ can be written as*

$$\chi(G,t) = \sum_{\varepsilon \in O_{\mathrm{AC}}(G)} \chi(G,\varepsilon;t), \tag{1.1}$$

$$\bar{\chi}(G,t) = \sum_{\varepsilon \in O_{\mathrm{AC}}(G)} \bar{\chi}(G,\varepsilon;t). \tag{1.2}$$

Moreover, for each orientation $\varepsilon \in O_{\mathrm{AC}}(G)$, we have the reciprocity law:

$$\chi(G,\varepsilon;-t) = (-1)^{|V|}\bar{\chi}(G,\varepsilon;t), \tag{1.3}$$

$$\chi(G,-t) = (-1)^{|V|}\bar{\chi}(G,t). \tag{1.4}$$

In particular,

$$\bar{\chi}(G,\varepsilon;1) = (-1)^{|V|}\chi(G,\varepsilon;-1) = 1,$$

and $|\chi(G,-1)|$ counts the number of acyclic orientations of G.

Let (H_i,ε_i) be disubgraphs of G, $i = 1,2$. We introduce a *coupling* of ε_1 and ε_2, defined as a function $[\varepsilon_1,\varepsilon_2] : E \to \mathbb{Z}$ by

$$[\varepsilon_1,\varepsilon_2](x) = \begin{cases} 1, & \text{if } \varepsilon_1(x) = \varepsilon_2(x), x \in E(H_1) \cap E(H_2), \\ -1, & \text{if } \varepsilon_1(x) \neq \varepsilon_2(x), x \in E(H_1) \cap E(H_2), \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

Let A be an abelian group. A *tension* of (G,ε) with values in A is a function $f : E(G) \to A$ such that for any directed circuit (C,ε_C),

$$\sum_{x \in C} [\varepsilon,\varepsilon_C](x) f(x) = 0. \tag{1.6}$$

The *tension group* $T(G,\varepsilon;A)$ of (G,ε) with the coefficient group A is the abelian group of all tensions of (G,ε) with values in A. Tensions with values in the field $\mathbb{R}$ of real numbers (the ring $\mathbb{Z}$ of integers) are called *real* (*integral*) *tensions*. Let q be a positive integer. A real tension f is called a *q-tension* if $|f(e)| < q$ for all $e \in E(G)$. We denote by $\tau(G,q)$ the number of nowhere-zero tensions of (G,ε) with values in A whose order is q, and by $\tau_{\mathbb{Z}}(G,q)$ the number of nowhere-zero integral q-tensions. It turns out that both functions $\tau(G,q)$ and $\tau_{\mathbb{Z}}(G,q)$ are independent of the chosen orientation ε and the group structure of A. Moreover, $\tau(G,q)$ and $\tau_{\mathbb{Z}}(G,q)$ are polynomial functions of positive integers q, having degree $r(G) := |V| - k(G)$, where $k(G)$ is the number of connected components of G. We call $\tau(G,q)$ and $\tau_{\mathbb{Z}}(G,q)$ the *modular tension polynomial* and the *integral tension polynomial* of G, respectively.

To interpret the values of the integral tension polynomial $\tau_{\mathbb{Z}}(G,t)$ at negative integers, let us introduce the counting function

$$\bar{\tau}_{\mathbb{Z}}(G,q) := \#\{(\varepsilon,f) \mid \varepsilon \in O_{\mathrm{AC}}(G), f \in T(G,\varepsilon;\mathbb{Z}), 0 \le f \le q\}.$$

For each $\varepsilon \in O_{\mathrm{AC}}(G)$, let $\bar{\Delta}^{+}_{\mathrm{TN}}(G,\varepsilon)$ be the 0-1 polytope in the tension vector space $T(G,\varepsilon;\mathbb{R})$ whose vertices are vectors $f \in T(G,\varepsilon;\mathbb{R})$ such that $f(x) = 0$ or 1. We call

$\bar{\Delta}^{+}_{\mathrm{TN}}(G,\varepsilon)$ the *tension polytope* and its interior $\Delta^{+}_{\mathrm{TN}}(G,\varepsilon)$ the *open tension polytope* of digraph (G,ε). Let

$$\tau_{\mathbb{Z}}(G,\varepsilon;t) := L\left(\Delta^{+}_{\mathrm{TN}}(G,\varepsilon),t\right) \text{ and } \bar{\tau}_{\mathbb{Z}}(G,\varepsilon;t) := L\left(\bar{\Delta}^{+}_{\mathrm{TN}}(G,\varepsilon),t\right)$$

denote the Ehrhart polynomials of $\Delta^{+}_{\mathrm{TN}}(G,\varepsilon)$ and $\bar{\Delta}^{+}_{\mathrm{TN}}(G,\varepsilon)$, respectively. Our first main result is the following theorem.

Theorem 1.2. *Let G be a loopless graph with possible multiple edges. Then* $\tau_{\mathbb{Z}}(G,q)$ *and* $\bar{\tau}_{\mathbb{Z}}(G,q)$ *are polynomial functions of degree* $r(G)$ *in positive integers q, and can be written as*

$$\tau_{\mathbb{Z}}(G,t) = \sum_{\varepsilon\in O_{\mathrm{AC}}(G)} \tau_{\mathbb{Z}}(G,\varepsilon;t), \tag{1.7}$$

$$\bar{\tau}_{\mathbb{Z}}(G,t) = \sum_{\varepsilon\in O_{\mathrm{AC}}(G)} \bar{\tau}_{\mathbb{Z}}(G,\varepsilon;t). \tag{1.8}$$

Moreover, for each orientation $\varepsilon \in O_{\mathrm{AC}}(G)$*, we have the* reciprocity law:

$$\tau_{\mathbb{Z}}(G,\varepsilon;-t) = (-1)^{r(G)}\bar{\tau}_{\mathbb{Z}}(G,\varepsilon;t), \tag{1.9}$$

$$\tau_{\mathbb{Z}}(G,-t) = (-1)^{r(G)}\bar{\tau}_{\mathbb{Z}}(G,t). \tag{1.10}$$

In particular,

$$\bar{\tau}_{\mathbb{Z}}(G,\varepsilon;0) = (-1)^{r(G)}\tau_{\mathbb{Z}}(G,\varepsilon;0) = 1;$$

and $|\tau_{\mathbb{Z}}(G,0)|$ *counts the number of acyclic orientations of G.*

To interpret the values of the modular tension polynomial $\tau(G,t)$ at negative integers, we define an equivalence relation $\sim$ on $O(G)$. Recall that a *cut* of G is a nonempty subset $[S,T] \subseteq E(G)$, where $\{S,T\}$ is a partition of V and $[S,T]$ is the set of all edges between S and T. Let (G,ε) be a digraph. If $[S,T]$ is a cut, we denote by $(S,T)_\varepsilon$ the set of all edges having arrows from S to T, and by $(T,S)_\varepsilon$ the set of all edges having arrows from T to S. A cut is called a *bond* if it does not contain properly any cut. A cut $[S,T]$ of (G,ε) is said to be *directed* if either $[S,T] = (S,T)_\varepsilon$ or $[S,T] = (T,S)_\varepsilon$, i.e., all edges of $[S,T]$ have arrows from S to T or all have arrows from T to S. A minimal directed cut is called a *directed bond*. A cut of (G,ε) is said to be *oriented* if it can be decomposed into a disjoint union of directed bonds. Two orientations $\varepsilon,\varrho \in O(G)$ are said to be *cut-equivalent*, denoted $\varepsilon\sim\varrho$, if the edge set

$$E(\varepsilon\neq\varrho) := \{x\in E \mid \varepsilon(x)\neq\varrho(x)\}$$

is empty or is an oriented cut with respect to either orientation ε or ϱ. It turns out that $\sim$ is indeed an equivalence relation on $O(G)$. Moreover, if $\varepsilon\sim\varrho$ and ε is acyclic, so is ϱ. Thus $\sim$ induces an equivalence relation on the set $O_{\mathrm{AC}}(G)$ of acyclic orientations of G. Let $[O_{\mathrm{AC}}(G)]$ denote the set of cut-equivalence classes of acyclic orientations of G. We introduce the counting function

$$\bar{\tau}(G,q) := \#\left\{([\varepsilon],f) \mid [\varepsilon]\in[O_{\mathrm{AC}}(G)],\, f\in T(G,[\varepsilon];\mathbb{Z}),\, 0\le f\le q\right\},$$

where $T(G,[\varepsilon];\mathbb{Z}) = T(G,\varepsilon;\mathbb{Z})$ and $\varepsilon\in[\varepsilon]$ is any fixed representative. Our second main result is the following theorem.

Theorem 1.3. *Let $G = (V, E)$ be a loopless graph with possible multiple edges. Then $\tau(G, q)$ and $\bar{\tau}(G, q)$ are polynomial functions of degree $r(G)$ in positive integers q, and can be written as*

$$\tau(G, t) = \sum_{\varepsilon \in [\mathcal{O}_{\mathrm{AC}}(G)]} \tau_{\mathbb{Z}}(G, \varepsilon; t), \tag{1.11}$$

$$\bar{\tau}(G, t) = \sum_{\varepsilon \in [\mathcal{O}_{\mathrm{AC}}(G)]} \bar{\tau}_{\mathbb{Z}}(G, \varepsilon; t), \tag{1.12}$$

and satisfy the reciprocity law

$$\tau(G, -t) = (-1)^{r(G)} \bar{\tau}(G, t). \tag{1.13}$$

In particular, $|\tau(G, 0)|$ counts the number of cut-equivalence classes of acyclic orientations of G.

After searching literature on tension polynomials, we found that formulas equivalent to (1.7) and (1.11) were observed by Kochol [18] in different forms with incomplete proof. It is interesting that certain bounds were obtained in [18] for the modular and integral tension polynomials $\tau(G, q)$ and $\tau_{\mathbb{Z}}(G, q)$. However, the interpretations for the values of the polynomials at zero and negative integers are not considered there.

Corollary 1.4. *Let $T(G; x, y)$ be the Tutte polynomial of a graph G. If G is loopless, then*

$$T(G; t, 0) = \bar{\tau}(G, t-1) = (-1)^{r(G)} \tau(G, 1-t). \tag{1.14}$$

In particular,

$$T(G; 1, 0) = (-1)^{r(G)} \tau(G, 0) = \bar{\tau}(G, 0)$$

counts the number of cut-equivalence classes of acyclic orientations of G.

At the end, we remark that the field $GF(4)$ in Veblen's equivalent formulation [27] for FCC can be replaced by any abelian group of 4 elements, i.e., every planar graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$ can be properly colored with 4 colors if and only if the system of linear inequalities

$$x_i - x_j \neq 0, \quad (i, j) \in E$$

has a solution over an abelian group of 4 elements. For an arbitrary graph G, the chromatic number of G is the minimal order of an abelian group such that the above system of linear inequalities has a solution. The fact that the field can be replaced by any abelian group is that graphs are torsion free or their torsion is 1.

2. Valuations on Abelian Groups

Let Ω be a finitely generated abelian group, not necessarily finite and free. By a *flat* of Ω we mean a coset of a subgroup of Ω. Let $\Gamma \subseteq \Omega$ be a subgroup. Let $\mathrm{Tor}(\Gamma)$ denote the torsion subgroup of Γ. The *size* of Γ is defined as

$$|\Gamma| := |\mathrm{Tor}(\Gamma)|\, t^{\mathrm{rank}\,(\Gamma)}.$$

If Γ is finite, then $|\Gamma|$ is the number of elements of Γ. For each flat F in Ω, the *rank*, the *torsion subgroup*, and the *size* of F are defined as the rank, the torsion subgroup, and the size of Γ, respectively. For any subset $X \subseteq \Omega$ and $a \in \Omega$, we define the translate

$$a+X := \{a+x \,|\, x \in X\}.$$

Let $\mathscr{L}(\Omega)$ denote the set of all flats in Ω, called the *semilattice* of flats in Ω. An *affine subgroup arrangement* (or just *arrangement*) of Ω is a collection $\mathcal{A}$ of finitely many flats of Ω. Let $\mathcal{A}$ be an arrangement of Ω. Let $\mathscr{L}(\mathcal{A})$ be the set of all non-empty intersections of flats in $\mathcal{A}$, including $\Omega = \bigcap_{A\in\emptyset} A$, called the *semilattice* of $\mathcal{A}$. Notice that $\mathscr{L}(\mathcal{A})$ is closed under intersection. We introduce the *characteristic polynomial* of $\mathcal{A}$ as

$$\chi(\mathcal{A},t) = \sum_{X\in\mathscr{L}(\mathcal{A})} \frac{\mu(X,\Omega)|\mathrm{Tor}(\Omega)|}{|\mathrm{Tor}(\Omega/\langle X\rangle)|} t^{\operatorname{rank}\langle X\rangle},$$

where μ is the Möbius function of the poset $\mathscr{L}(\mathcal{A})$, whose partial order is the set inclusion, and $\langle X\rangle = \{x-y \,|\, x,y \in X\}$. Let $\mathscr{B}(\Omega)$ denote the Boolean algebra generated by $\mathscr{L}(\Omega)$, i.e., every element of $\mathscr{B}(\Omega)$ is obtained from flats of Ω by taking unions, intersections, and complements finitely many times.

A *valuation* on Ω with values in an abelian group A is a map $\nu\colon \mathscr{B}(\Omega) \to A$ such that

$$\nu(\emptyset) = 0, \tag{2.1}$$

$$\nu(X\cup Y) = \nu(X) + \nu(Y) - \nu(X\cap Y), \tag{2.2}$$

for $X, Y \in \mathscr{B}(\Omega)$. A valuation ν on Ω is said to be *translate-invariant* if for any $a \in \Omega$ and $X \in \mathscr{B}(\Omega)$,

$$\nu(a+X) = \nu(X); \tag{2.3}$$

and ν is said to satisfy *productivity* if

$$\nu(A+B) = \nu(A)\nu(B) \tag{2.4}$$

for subgroups $A, B \subset \Omega$ such that $A+B$ is a direct sum, and $A+B$ is a direct summand of Ω.

Let $X \subseteq \Omega$ be a subset. The *indicator function* $1_X\colon \Omega \to \mathbb{Z}$ of X is defined by

$$1_X(x) = \begin{cases} 1, & \text{if } x \in X, \\ 0, & \text{if } x \notin X. \end{cases}$$

Let $\mathscr{F}(\Omega)$ denote the abelian group of all functions $f\colon \Omega \to \mathbb{Z}$ such that $f(\Omega)$ is a finite set and $f^{-1}(n) \in \mathscr{B}(\Omega)$ for $n \in \mathbb{Z}$. Then each such function can be written as a linear combination

$$f = \sum a_i 1_{F_i},$$

where F_i are flats of Ω and $a_i \in \mathbb{Z}$. The *integral* of f with respect to a valuation ν is defined as

$$\nu(f) = \int_\Omega f \,\mathrm{d}\nu(x) = \sum a_i \nu(F_i) = \sum_{n\in\mathbb{Z}} n\nu\left(f^{-1}(n)\right).$$

By Groemer's Extension Theorem [17], a set function $\nu\colon \mathscr{L}(\Omega)\to A$ can be extended to a valuation $\nu\colon \mathscr{B}(\Omega)\to A$ if and only if the *inclusion-exclusion formula*

$$\nu(L)=\sum_{k=1}^{n}(-1)^{k-1}\sum_{i_1<\cdots<i_k}\nu(L_{i_1}\cap\cdots\cap L_{i_k}) \tag{2.5}$$

is satisfied, provided that $L, L_1,\dots,L_n\in\mathscr{L}(\Omega)$ and $L=L_1\cup\cdots\cup L_n$.

Theorem 2.1. *For any finitely generated abelian group Ω, there is a unique translate-invariant valuation $\lambda\colon \mathscr{B}(\Omega)\to\mathbb{Q}[t]$ such that the productivity is satisfied and*

$$\lambda(\Omega)=|\mathrm{Tor}(\Omega)|\,t^{\mathrm{rank}\,(\Omega)}=|\Omega|.$$

Moreover, for any finite subset $S\subseteq\Omega$ and subgroup $\Gamma\subseteq\Omega$, we have

$$\lambda(S)=|S|,\quad \lambda(\Gamma)=\frac{|\Omega|}{|\Omega/\Gamma|}.$$

Proof. For a subgroup $\Gamma\subseteq\Omega$, let Z_Γ be a free subgroup of Γ such that $\Gamma=\mathrm{Tor}(\Gamma)+Z_\Gamma$ is a direct sum. Let Z be a free subgroup of Ω such that $Z_\Gamma\subseteq Z$ and $\Omega=T+Z$ is a direct sum, where $T=\mathrm{Tor}(\Omega)$. Since $\Omega=\bigsqcup_{a\in T}(a+Z)$, by additivity and productivity we have $\lambda(\Omega)=|T|\lambda(Z)=\lambda(T)\lambda(Z)$. Thus $\lambda(T)=|T|$ and $\lambda(Z)=t^{\mathrm{rank}\,(Z)}$. Since λ is translate-invariant, it follows that $\lambda(\{0\})=1$. So $\lambda(S)=|S|$ for any finite subset $S\subseteq\Omega$.

Let $a_1,\dots,a_k$ be a collection of generators of Z. Then

$$\mathbb{Z}a_1+\mathbb{Z}a_2=\mathbb{Z}a_1+\mathbb{Z}(a_1+a_2)=\mathbb{Z}(a_1+a_2)+\mathbb{Z}a_2$$

and the sums are direct. Again by the productivity we have

$$\lambda(\mathbb{Z}a_1+\mathbb{Z}a_2)=\lambda(\mathbb{Z}a_1)\lambda(\mathbb{Z}(a_1+a_2))=\lambda(\mathbb{Z}(a_1+a_2))\lambda(\mathbb{Z}a_2).$$

Thus $\lambda(\mathbb{Z}a_1)=\lambda(\mathbb{Z}a_2)$. Since $\lambda(Z)=t^{\mathrm{rank}\,(Z)}$, it follows that $\lambda(\mathbb{Z}a_1)=t$. So $\lambda(H)=t^{\mathrm{rank}\,(H)}$ for any direct summand H of Z.

Let $\tilde{Z}_\Gamma=\{z\in Z\,|\,mz\in Z_\Gamma \text{ for some } m\}$. Then $\tilde{Z}_\Gamma$ is a direct summand of Ω. Since $\tilde{Z}_\Gamma=\bigsqcup_{[a]\in\tilde{Z}_\Gamma/Z_\Gamma}(a+Z_\Gamma)$, we have $\lambda(\tilde{Z}_\Gamma)=|\tilde{Z}_\Gamma/Z_\Gamma|\lambda(Z_\Gamma)=t^{\mathrm{rank}\,(\Gamma)}$. Hence

$$\lambda(\Gamma)=\lambda(\mathrm{Tor}(\Gamma))\lambda(Z_\Gamma)=\frac{|\mathrm{Tor}(\Gamma)|}{\left|\tilde{Z}_\Gamma/Z_\Gamma\right|}t^{\mathrm{rank}\,(\Gamma)}.$$

Notice that

$$|\Omega|=|\mathrm{Tor}(\Gamma)|\,|\mathrm{Tor}(\Omega)/\mathrm{Tor}(\Gamma)|\,t^{\mathrm{rank}\,(\Omega)},$$

$$|\Omega/\Gamma|=|\mathrm{Tor}(\Omega)/\mathrm{Tor}(\Gamma)|\,\left|\tilde{Z}_\Gamma/Z_\Gamma\right|\,t^{\mathrm{rank}\,(\Omega)-\mathrm{rank}\,(\Gamma)}.$$

It follows that $\lambda(\Gamma)|=|\Omega)|/|\Omega/\Gamma|$.

Next we show that the inclusion-exclusion formula (2.5) is satisfied. We proceed by induction on n, the number of flats in (2.5). For $n=1$, it is obviously true. Assume it is true for the case $n-1$, and consider the case n. Let $L, L_1, \dots, L_n \in \mathscr{L}(\Omega)$ be such that $L = L_1 \cup \dots \cup L_n$. Without loss of generality, we may assume that L contains the zero element 0, i.e., L is a subgroup. Notice that one of the flats $L_1, \dots, L_n$ must have the same rank as L, say, $\operatorname{rank}(L_n) = \operatorname{rank}(L)$. Fix an element $a \in L_n$; we set

$$\Gamma := L_n - a = \{x - a \mid x \in L_n\}.$$

Then Γ is a subgroup, and L_n is a coset of Γ. Clearly, Γ is a subgroup of L with the same rank. Thus the quotient group L/Γ, consisting of all cosets of Γ in L, is a finite set. Notice that all cosets $F \in L/\Gamma$, except L_n, are contained in $L_1 \cup \dots \cup L_{n-1}$. Hence, $F = (L_1 \cap F) \cup \dots \cup (L_{n-1} \cap F)$. By induction, for $F \in L/\Gamma$ such that $F \neq L_n$, we have

$$\lambda(F) = \sum_{I \subseteq [n-1], I \neq \emptyset} (-1)^{|I|-1} \lambda\left(\bigcap_{i \in I} L_i \cap F \right).$$

On the other hand, it is routine to check that for any flat L' of L,

$$\lambda(L') = \sum_{F \in L/\Gamma} \lambda(L' \cap F).$$

In particular, for the flats $L_I = \bigcap_{i \in I} L_i$ of L with non-empty $I \subseteq [n]$, we have

$$\lambda(L_I) = \sum_{F \in L/\Gamma} \lambda(L_I \cap F).$$

Now the right-hand side of (2.5) can be written as

$$\begin{aligned}
\text{RHS} &= \sum_{I \subseteq [n], I \neq \emptyset} (-1)^{|I|-1} \sum_{F \in L/\Gamma} \lambda(L_I \cap F) \\
&= \sum_{F \in L/\Gamma} \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|-1} \lambda(L_I \cap F) \\
&= \left\{ \sum_{\substack{F \in L/\Gamma \\ F = L_n}} + \sum_{\substack{F \in L/\Gamma \\ F \neq L_n}} \right\} \left\{ \sum_{\substack{I \subseteq [n] \\ n \in I}} + \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset, n \notin I}} \right\}.
\end{aligned}$$

Clearly, the RHS is decomposed into the following four sums:

$$\begin{aligned}
\sum_{\substack{F \in L/\Gamma \\ F = L_n}} \sum_{\substack{I \subseteq [n] \\ n \in I}} &= \sum_{I \subseteq [n-1]} (-1)^{|I|} \lambda(L_I \cap L_n) \\
&= \lambda(L_n) + \sum_{I \subseteq [n-1], I \neq \emptyset} (-1)^{|I|} \lambda(L_I \cap L_n);
\end{aligned}$$

$$\sum_{\substack{F \in L/\Gamma \\ F \neq L_n}} \sum_{\substack{I \subseteq [n] \\ n \in I}} = 0 \quad (\text{since } L_I \cap F = \emptyset);$$

$$\sum_{\substack{F\in L/\Gamma\\ F=L_n}}\sum_{\substack{I\subseteq[n]\\ I\neq\emptyset,n\notin I}} = \sum_{I\subseteq[n-1],I\neq\emptyset}(-1)^{|I|-1}\lambda(L_I\cap L_n);$$

$$\sum_{\substack{F\in L/\Gamma\\ F\neq L_n}}\sum_{\substack{I\subseteq[n]\\ I\neq\emptyset,n\notin I}} = \sum_{\substack{F\in L/\Gamma\\ F\neq L_n}}\sum_{\substack{I\subseteq[n-1]\\ I\neq\emptyset}}(-1)^{|I|-1}\lambda(L_I\cap F)$$

$$= \sum_{\substack{F\in L/\Gamma\\ F\neq L_n}}\lambda(F)=\lambda(L)-\lambda(L_n).$$

It follows that the right-hand side of (2.5) equals $\lambda(L)$, as desired. The proof of uniqueness is trivial. ∎

Theorem 2.2. *Let $\mathcal{A}$ be an affine subgroup arrangement of a finitely generated abelian group Ω. Then*

$$\chi(\mathcal{A},t)=\lambda\left(\Omega-\bigcup_{F\in\mathcal{A}}F\right). \tag{2.6}$$

Proof. For each flat $Y\in\mathscr{L}(\mathcal{A})$, let $Y^0=Y-\bigcup_{X<Y}X$. Then $\{X^0\,|\,X\in\mathscr{L}(\mathcal{A})\}$ is a collection of disjoint subsets. Since $Y=\bigsqcup_{X\leq Y}X^0$ for all $Y\in\mathscr{L}(\mathcal{A})$, we have

$$1_Y=\sum_{X\leq Y}1_{X^0},\quad Y\in\mathscr{L}(\mathcal{A}).$$

Applying the Möbius inversion, we obtain

$$1_{Y^0}=\sum_{X\leq Y}\mu(X,Y)\,1_X,\quad Y\in\mathscr{L}(\mathcal{A}).$$

In particular, for $Y=\Omega$, we have $\Omega^0=\Omega-\bigcup_{X\in\mathcal{A}}X$, and

$$1_{\Omega^0}=\sum_{X\in\mathscr{L}(\mathcal{A})}\mu(X,\Omega)\,1_X.$$

Applying the valuation λ to both sides, we see that

$$\begin{aligned}\lambda\left(\Omega^0\right)&=\sum_{X\in\mathscr{L}(\mathcal{A})}\mu(X,\Omega)\lambda(X)\\ &=\sum_{X\in\mathscr{L}(\mathcal{A})}\mu(X,\Omega)\cdot\frac{|\Omega|}{|\Omega/\langle X\rangle|}\\ &=\sum_{X\in\mathscr{L}(\mathcal{A})}\frac{\mu(X,\Omega)|\mathrm{Tor}(\Omega)|}{|\mathrm{Tor}(\Omega/\langle X\rangle)|}t^{\operatorname{rank}\langle X\rangle}\\ &=\chi(\mathcal{A},t).\end{aligned}$$

∎

Remark 2.3. When Ω is a finite-dimensional vector space, the unique translate-invariant valuation λ was obtained by Ehrenborg and Readdy [14]; see also Chen [11] for the generalization to infinite-dimensional vector spaces.

3. Chromatic and Modular Tension Polynomials

Let $G = (V, E)$ be a loopless graph with possible multiple edges. Let A be an abelian group. A *coloring* of G with the color set A is a function $f\colon V \to A$; f is said to be *proper* if $f(u) \neq f(v)$ for any adjacent vertices u, v. We denote by $K(G, A)$ the set of all colorings of G, and by $K_{\mathrm{nz}}(G, A)$ the set of all proper colorings. If $|A| = q$ is finite, it is well-known that the counting function

$$\chi(G, q) := |K_{\mathrm{nz}}(G, A)| \tag{3.1}$$

is a polynomial function of q, depending only on the order of A, not on the group structure; $\chi(G, t)$ is called the *chromatic polynomial* of G.

Let ε be an orientation of G. We denote by $T(G, \varepsilon; A)$ the abelian group of all tensions of the digraph (G, ε) with values in A, called the *tension group* of (G, ε), and by $T_{\mathrm{nz}}(G, \varepsilon; A)$ the set of all nowhere-zero tensions. If A is finite, we shall see that $|T(G, \varepsilon; A)|$ and $|T_{\mathrm{nz}}(G, \varepsilon; A)|$ depend only on the order of A, but not on the abelian group structure. So, for $|A| = q$, we define the counting function

$$\tau(G, q) := |T_{\mathrm{nz}}(G, \varepsilon; A)|\,.$$

We shall see that $\tau(G, q)$ is a polynomial function of positive integers $q = |A|$, and is independent of the chosen orientation ε and the abelian group structure of A.

Note that a coloring of G may be viewed as a potential function on G. There is a natural *difference operator* $\delta\colon A^V \to A^E$ defined by

$$(\delta f)(x) = f(u) - f(v), \tag{3.2}$$

where $x = uv$ is an edge with the orientation $u \overset{\varepsilon}{\to} v$. When a graph is viewed as a 1-dimensional simplicial complex, the difference operator δ is known as *coboundary operator*, and the tension group $T(G, \varepsilon; A)$ is known as *cohomology group*. The following Proposition 3.1 and its Corollary 3.2 state the relation between colorings and tensions; see [2, 6].

Proposition 3.1. (a) $\operatorname{Im}\delta = T(G, \varepsilon; A)$.
(b) $\delta\colon K(G, A) \to T(G, \varepsilon; A)$ *is a group homomorphism with* $\operatorname{Ker}\delta \simeq A^{k(G)}$, *where* $k(G)$ *is the number of connected components of* G.
(c) *The restriction* $\delta\colon K_{\mathrm{nz}}(G, A) \to T_{\mathrm{nz}}(G, \varepsilon; A)$ *is well defined.*

Proof. (a) Let $\tilde{f} \in T(G, \varepsilon; A)$. We construct a coloring $f \in K(G, A)$ as follows. Let G_0 be a connected component of G. Fix a vertex $v_0 \in G_0$ and an element $a \in A$. For each $v \in V(G_0)$, let $P = v_0 v_1 \cdots v_n$ be a shortest path from v_0 to $v = v_n$, and let ϱ be an orientation of P such that $v_{i-1} \overset{\varrho}{\to} v_i$. We define $f(v_0) = a$, and

$$f(v) = a - \sum_{i=1}^{n} [\varepsilon, \varrho]\,(v_{i-1}, v_i)\, \tilde{f}\,(v_{i-1}, v_i)\,.$$

It is routine to check that f is well defined. Moreover, f is proper if and only if $\tilde{f}$ is nowhere-zero. Since a is arbitrarily given for each component, we see that $\operatorname{Ker}(d) \simeq A^{k(G)}$.

(b) and (c) are trivial by the construction of f from $\tilde{f}$. ∎

Corollary 3.2. *The chromatic polynomial* $\chi(G,t)$ *and the modular tension polynomial* $\tau(G,t)$ *are related by*

$$\chi(G,t) = t^{k(G)}\tau(G,t), \tag{3.3}$$

where $k(G)$ *is the number of connected components of* G*.*

Proof. It follows from (b) and (c) of Proposition 3.1. ∎

Let F be a maximal forest of G. For each edge $e \in F$, let B_e be the unique bond, whose edge set consists of the edge e and the edges x of $F^c (:= E - F)$ such that the circuit of $F \cup x$ contains the edge e. Let ε_e be a direction of B_e such that $\varepsilon_e(e) = \varepsilon(e)$. It is easy to see that $[\varepsilon, \varepsilon_e]$ is a tension of (G, ε). The following lemma may be found in the book of Berge [2].

Lemma 3.3. *Let* F *be a spanning forest of* G*. Then the linear system* (1.6)*, whose equations are indexed by all circuits, is equivalent to the linear system*

$$f = \sum_{e\in F} f(e)\,[\varepsilon, \varepsilon_e], \tag{3.4}$$

with equations indexed by edges $x \in F^c$*. In other words,* (3.4) *solves the linear system* (1.6) *with* $|F|$ *free variables.*

Proof. For any $f \in A^E$, the function $f_e := f(e)[\varepsilon, \varepsilon_e]$ is a tension of (G, ε). Set $f' = f - \sum_{e\in F} f_e$. Then f' is a tension of (G, ε) if and only if f is a tension. Since $\varepsilon_e(e) = \varepsilon(e)$, we have $f_e(e) = f(e)$. For any $x \in F$ but $x \neq e$, we have $f_e(x) = 0$ since $x \notin B_e$. This means that $f' \equiv 0$ on F.

For any $y \in F^c$, let C_y be the unique circuit contained in $F \cup y$. Let ε_y be a direction of C_y. If f' is a tension, then $\sum_{x\in C_y} [\varepsilon, \varepsilon_y](x) f'(x) = 0$ by definition. Since $f'(x) = 0$ for any $x \in C_y$ but y, we have $f'(y) = 0$. This means that $f' \equiv 0$ on $E(G)$. Thus $f = \sum_{e\in F} f_e$. ∎

Corollary 3.4. *The tension group* $T(G, \varepsilon; A)$ *is isomorphic to the A-free abelian group* $A^{r(G)}$*, where* $r(G) = |V| - k(G)$*. Moreover, for any subset* $X \subseteq E(G)$*, the abelian group*

$$T_X(G, \varepsilon; A) := \{f \in T(G, \varepsilon; A) \mid f(x) = 0 \text{ for } x \in X\}$$

is isomorphic to $A^{r(G)-r\langle X\rangle}$*, where* $\langle X\rangle = (V, X)$*. In particular, if* $|A| = q$ *is finite, then*

$$|T(G, \varepsilon; A)| = q^{r(G)}, \quad |T_X(G, \varepsilon; A)| = q^{r(G)-r\langle X\rangle}. \tag{3.5}$$

Proof. Let F_X be a spanning forest of X in the sense that each component of F_X is a spanning tree of a component of $\langle X\rangle$. Then F_X can be extended to a forest F of G. By Lemma 3.3, we see that $T_X(G, \varepsilon; A)$ is of rank $|F| - |F_X| = r(G) - r\langle X\rangle$. ∎

To see that $|T_{\text{nz}}(G, \varepsilon; A)|$ is independent of ε and A, let ϱ be another orientation of G. We introduce an involution $P_{\varrho,\varepsilon} \colon A^E \to A^E$, defined by

$$(P_{\varrho,\varepsilon} f)(x) = [\varrho, \varepsilon](x) f(x), \quad f \in A^E. \tag{3.6}$$

Obviously, $P_{\varepsilon,\varepsilon}$ is the identity map. For orientations ε, ϱ, and ς,

$$P_{\varsigma,\varrho} P_{\varrho,\varepsilon} = P_{\varsigma,\varepsilon}. \tag{3.7}$$

Lemma 3.5. *The involution $P_{\varrho,\varepsilon}$ is a group isomorphism. Moreover,*

$$P_{\varrho,\varepsilon}(T(G,\varepsilon;A)) = T(G,\varrho;A),$$

$$P_{\varrho,\varepsilon}(T_{\rm nz}(G,\varepsilon;A)) = T_{\rm nz}(G,\varrho;A).$$

Proof. It is obvious that $P_{\varrho,\varepsilon}$ is a group isomorphism. Let $f \in T(G,\varepsilon;A)$. For any directed circuit (C,ε_C), we have

$$\begin{aligned}\sum_{x\in C}[\varrho,\varepsilon_C](x)\,(P_{\varrho,\varepsilon}f)(x) &= \sum_{x\in C}[\varrho,\varepsilon_C](x)[\varrho,\varepsilon](x)f(x)\\ &= \sum_{x\in C}[\varepsilon,\varepsilon_C](x)f(x) = 0.\end{aligned}$$

Thus $P_{\varrho,\varepsilon}f \in T(G,\varrho;A)$. Similarly, for $g \in T(G,\varrho;A)$, we have $P_{\varepsilon,\varrho}g \in T(G,\varepsilon;A)$. Since $P_{\varrho,\varepsilon}$ is an involution, the first identity follows immediately. The second identity follows from the fact that for $x \in E$, $(P_{\varrho,\varepsilon}f)(x) \neq 0$ if and only if $f(x) \neq 0$. ∎

The *coloring arrangement* of G with the color set A is the subgroup arrangement $\mathcal{A}_{\rm CL}(G,A)$ of the abelian group $K := K(G,A)$, consisting of the subgroups

$$K_e = K_e(G,\varepsilon;A) := \left\{f \in K(G,A) \,\middle|\, f(u) = f(v)\right\}, \quad e = uv \in E. \tag{3.8}$$

The set of proper colorings with values in A is the complement $K(G,A) - \bigcup_{e\in E} K_e$. The *tension arrangement* of the digraph (G,ε) with the abelian group A is the subgroup arrangement $\mathcal{A}_{\rm TN}(G,\varepsilon;A)$ of the abelian group $T := T(G,\varepsilon;A)$, consisting of the subgroups

$$T_e = T_e(G,\varepsilon;A) := \left\{f \in T(G,\varepsilon;A) \,\middle|\, f(e) = 0\right\}, \quad e \in E. \tag{3.9}$$

The set of nowhere-zero tensions of (G,ε) with values in A is the complement $T(G,\varepsilon;A) - \bigcup_{e\in E} T_e$.

Theorem 3.6. *Let A be an abelian group, either $|A| = q$ is finite or $A \in \{\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{C}\}$. Then the chromatic polynomial $\chi(G,t)$ is given by*

$$\chi(G,q) = \lambda\left(K(G,A) - \bigcup_{e\in E} K_e\right)\Bigg|_{t=q} = \chi\left(\mathcal{A}_{\rm CL}(G,A),q\right), \tag{3.10}$$

and the tension polynomial $\tau(G,t)$ is given by

$$\tau(G,q) = \lambda\left(T(G,\varepsilon;A) - \bigcup_{e\in E} T_e\right)\Bigg|_{t=q} = \chi\left(\mathcal{A}_{\rm TN}(G,\varepsilon;A),q\right). \tag{3.11}$$

Proof. For finite $|A| = q$ or $A = \mathbb{Z}$, (3.10) and (3.11) are consequences of Theorem 2.2 by applying the valuation λ to the subgroup arrangements $\mathcal{A}_{\rm CL}(G,A)$ and $\mathcal{A}_{\rm TN}(G,\varepsilon;A)$. For $A = \mathbb{F} \in \{\mathbb{Q},\mathbb{R},\mathbb{C}\}$ to be an infinite field, (3.10) and (3.11) follow from the valuation λ on the hyperplane arrangements $\mathcal{A}_{\rm CL}(G,\mathbb{F})$ and $\mathcal{A}_{\rm TN}(G,\varepsilon;\mathbb{F})$, such that $\lambda(W) = t^{\dim(W)}$ for vector spaces W over $\mathbb{F}$; see [11, 14]. ∎

If the abelian group A is a finite field, the counting formula (3.10) was observed by Athanasiadis [1] and Björner and Ekedahl [4].

4. Integral Chromatic Polynomials

In this section we consider colorings of a graph $G = (V, E)$ with values in the field $\mathbb{R}$ of real numbers. The set of proper colorings is the complement of the hyperplane arrangement $\mathcal{A}_{\rm CL}(G, \mathbb{R})$. Let q be a positive integer, and let $(0, q) = \{a \in \mathbb{R} \mid 0 < a < q\}$, $[0, q] = \{a \in \mathbb{R} \mid 0 \le a \le q\}$. Then the set of proper colorings with the color set $\{1, 2, \ldots, q-1\}$ is the set of lattice points of the dilatation $q\Delta^+_{\rm CL}(G)$ (dilated by q) of the non-convex polyhedron

$$\Delta^+_{\rm CL}(G) := \big\{ f \in \mathbb{R}^V \mid f(u) \ne f(v) \text{ for } e = uv \in E, 0 < f < 1 \big\}. \tag{4.1}$$

For each $e = uv \in E$, the non-equality $f(u) \ne f(v)$ can be split into two inequalities:

$$f(u) > f(v) \quad \text{and} \quad f(u) < f(v),$$

which can be interpreted as two orientations on the edge $e = uv$. Thus, for each orientation ε of G, we have a system of linear inequalities, indexed by elements of $E(G)$. The system of such linear inequalities has a common solution $f \in \mathbb{R}^V$ if and only if the orientation ε is acyclic.

Recall that $O_{\rm AC}(G)$ is the set of all acyclic orientations of G. For each $\varepsilon \in O_{\rm AC}(G)$, we define a 0-1 open polytope

$$\Delta^+_{\rm CL}(G, \varepsilon) := \Big\{ f \in \mathbb{R}^V \mid f(u) > f(v) \text{ for } u \overset{\varepsilon}{\to} v, 0 < f < 1 \Big\}. \tag{4.2}$$

Such polytopes are in one-to-one correspondence with the connected components of the complement $\mathbb{R}^V - \bigcup \mathcal{A}_{\rm CL}(G, \mathbb{R})$. We call $\Delta^+_{\rm CL}(G, \varepsilon)$ the *chromatic open polytope* and its closure $\bar{\Delta}^+_{\rm CL}(G, \varepsilon)$ the *chromatic polytope* of the digraph (G, ε). Let $\chi(G, \varepsilon; q)$ denote the number of colorings of the digraph (G, ε) with color set $[q] := \{1, 2, \ldots, q\}$, i.e.,

$$\chi(G, \varepsilon; q) := \# \Big\{ f \colon V(G) \to [q] \,\Big|\, f(u) > f(v) \text{ for } u \overset{\varepsilon}{\to} v \Big\}. \tag{4.3}$$

Each coloring f counted in $\chi(G, \varepsilon; q)$ can be viewed as a lattice point satisfying $0 < f < q+1$ and $f(u) > f(v)$ for $u \overset{\varepsilon}{\to} v$. Then $\chi(G, \varepsilon; q)$ counts the number of lattice points of the open lattice polytope $(q+1)\Delta^+_{\rm CL}(G, \varepsilon)$, i.e.,

$$\chi(G, \varepsilon; q) = \big| (q+1)\Delta^+_{\rm CL}(G, \varepsilon) \cap \mathbb{Z}^V \big|. \tag{4.4}$$

We call $\chi(G, \varepsilon; q)$ the *chromatic polynomial of G with respect to* ε. Let

$$\bar{\chi}(G, \varepsilon; q) := \# \Big\{ f \colon V(G) \to [q] \,\Big|\, f(u) \ge f(v) \text{ for } u \overset{\varepsilon}{\to} v \Big\}. \tag{4.5}$$

Notice that the color set $\{1, 2 \ldots, q\}$ can be replaced by $\{0, 1, \ldots, q-1\} = [0, q-1] \cap \mathbb{Z}$, and the counting function $\bar{\chi}(G, \varepsilon; q)$ remains unchanged. Similarly, a coloring f counted in $\bar{\chi}(G, \varepsilon; q)$ with color set $\{0, 1, \ldots, q-1\}$ can be viewed as a lattice point satisfying $0 \le f \le q-1$ and $f(u) \ge f(v)$ for $u \overset{\varepsilon}{\to} v$. Thus $\bar{\chi}(G, \varepsilon; q)$ counts the number of lattice points of the lattice polytope $(q-1)\bar{\Delta}^+_{\rm CL}(G, \varepsilon)$, i.e.,

$$\bar{\chi}(G, \varepsilon; q) = \big| (q-1)\bar{\Delta}^+_{\rm CL}(G, \varepsilon) \cap \mathbb{Z}^V \big|. \tag{4.6}$$

Now we have seen that $\chi(G,\varepsilon;q)$ is the Ehrhart polynomial $L(\Delta^+_{\rm CL}(G,\varepsilon),q+1)$ of the lattice open polytope $\Delta^+_{\rm CL}(G,\varepsilon)$ at $q+1$, and $\bar{\chi}(G,\varepsilon;q)$ is the Ehrhart polynomial $L(\bar{\Delta}^+_{\rm CL}(G,\varepsilon),q-1)$ of the closed polytope $\bar{\Delta}^+_{\rm CL}(G,\varepsilon)$ at $q-1$.

Proof of Theorem 1.1. Let $\varepsilon \in O_{\rm AC}(G)$. The polytope $\bar{\Delta}^+_{\rm CL}(G,\varepsilon)$ is the convex hull of the lattice points $f \in \mathbb{Z}^V$ such that $f(w)=0$ or 1 for all $w \in V$ and $f(u) \geq f(v)$ for $u \xrightarrow{\varepsilon} v$. The reciprocity law is a straightforward consequence of the reciprocity law of Ehrhart polynomials. In fact, the reciprocity law (1.3) follows from (4.4) and (4.6) as

$$\begin{aligned}\chi(G,\varepsilon;-t) &= L(\Delta^+_{\rm CL}(G,\varepsilon),-t+1)\\ &= L(\Delta^+_{\rm CL}(G,\varepsilon),-(t-1))\\ &= (-1)^{|V|}L(\bar{\Delta}^+_{\rm CL}(G,\varepsilon),t-1)\\ &= (-1)^{|V|}\bar{\chi}(G,\varepsilon;t).\end{aligned}$$

In particular, $\bar{\chi}(G,\varepsilon;1) = L(\bar{\Delta}^+_{\rm CL}(G,\varepsilon),0) = 1$ and

$$\chi(G,\varepsilon;-1) = L(\Delta^+_{\rm CL}(G,\varepsilon),0) = (-1)^{\dim \Delta^+_{\rm CL}(G,\varepsilon)} = (-1)^{|V|}.$$

Let $t=q$ be a positive integer. Then (1.4) follows from the disjoint union

$$(q+1)\Delta^+_{\rm CL}(G) = \bigsqcup_{\varepsilon \in O_{\rm AC}(G)} (q+1)\Delta^+_{\rm CL}(G,\varepsilon),$$

and (1.2) follows from the definition of $\bar{\chi}(G,q)$. The reciprocity law (1.4) is a straightforward consequence of (1.1)–(1.3). The interpretation for $(-1)^{|V|}\chi(G,-1)$ follows from (1.1) as

$$\chi(G,-1) = \sum_{\varepsilon \in O_{\rm AC}(G)} \chi(G,\varepsilon;-1) = \sum_{\varepsilon \in O_{\rm AC}(G)} (-1)^{|V|} = (-1)^{|V|}\,|O_{\rm AC}(G)|. \qquad \blacksquare$$

Remark 4.1. One may also define integral q-colorings f of a graph G as functions $f\colon V(G) \to \mathbb{Z}$ such that $|f(v)| < q$ for all $v \in V(G)$. Then the counting function $\chi_{\mathbb{Z}}(G,q)$, defined as the number of proper integral q-colorings of G, is a polynomial function of positive integers q of degree $|V(G)|$. However, this polynomial $\chi_{\mathbb{Z}}(G,q)$ is essentially the same as $\chi(G,q)$, up to a change of variable, i.e.,

$$\chi_{\mathbb{Z}}(G,q) = \chi(G,2q-1).$$

5. Integral Tension Polynomials

In this section we consider the tensions of digraph (G,ε) with values in $\mathbb{Z}$ and $\mathbb{R}$. Given a positive integer q; let $T(G,\varepsilon;q)$ be the set of all real q-tensions of (G,ε). We denote by $T_{\mathbb{Z}}(G,\varepsilon;q)$ the set of all integral q-tensions of (G,ε), and by $T_{{\rm nz}\mathbb{Z}}(G,\varepsilon;q)$ the set of all nowhere-zero integral q-tensions, i.e.,

$$T_{\mathbb{Z}}(G,\varepsilon;q) := \left\{ f \in T(G,\varepsilon;\mathbb{Z}) \,\middle|\, |f(x)| < q \text{ for } x \in E \right\}, \tag{5.1}$$

$$T_{{\rm nz}\mathbb{Z}}(G,\varepsilon;q) := \{ f \in T_{\mathbb{Z}}(G,\varepsilon;q) \mid f(x) \neq 0 \text{ for } x \in E \}. \tag{5.2}$$

Clearly, $T_{\mathrm{nz}\mathbb{Z}}(G,\varepsilon;q)$ is the set of lattice points of the dilatation $q\Delta_{\mathrm{TN}}(G,\varepsilon)$ (dilated by q) of the non-convex polyhedron

$$\Delta_{\mathrm{TN}}(G,\varepsilon) := \left\{ f \in T(G,\varepsilon;\mathbb{R}) \;\middle|\; 0 < |f(x)| < 1 \text{ for } x \in E \right\}.$$

It follows that

$$\tau_{\mathbb{Z}}(G,q) := |T_{\mathrm{nz}\mathbb{Z}}(G,\varepsilon;q)| = L(\Delta_{\mathrm{TN}}(G,\varepsilon),q) \tag{5.3}$$

is an Ehrhart polynomial function of degree $\dim \Delta_{\mathrm{TN}}(G,\varepsilon)$ in the positive integer variable q. We shall see that $|T_{\mathrm{nz}\mathbb{Z}}(G,\varepsilon;q)|$ is independent of the chosen orientation ε; and call $\tau_{\mathbb{Z}}(G,q)$ the *integral tension polynomial* of G.

Lemma 5.1. *For orientations ε, $\varrho \in O(G)$ and the involution $P_{\varrho,\varepsilon}$, we have*

$$P_{\varrho,\varepsilon}(\Delta_{\mathrm{TN}}(G,\varepsilon)) = \Delta_{\mathrm{TN}}(G,\varrho),$$
$$P_{\varrho,\varepsilon}(T_{\mathrm{nz}\mathbb{Z}}(G,\varepsilon;q)) = T_{\mathrm{nz}\mathbb{Z}}(G,\varrho;q).$$

Proof. It follows trivially from Lemma 3.5. ∎

For nowhere-zero real tensions $f \in T_{\mathrm{nz}}(G,\varepsilon;\mathbb{R})$, the non-equality $f(x) \neq 0$ can be split into two inequalities:

$$f(x) > 0 \quad \text{and} \quad f(x) < 0,$$

which can be interpreted as two orientations on the edge x; one is the same as $\varepsilon(x)$ and the other is opposite to $\varepsilon(x)$. Let ϱ be an orientation of G. We define a convex cone

$$T(G,\varepsilon;\varrho) := \{ f \in T(G,\varepsilon;\mathbb{R}) \mid [\varrho,\varepsilon](x)f(x) > 0 \text{ for } x \in E \}$$

of the tension vector space $T(G,\varepsilon;\mathbb{R})$. It is clear that the complement

$$T(G,\varepsilon;\mathbb{R}) - \bigcup_{e \in E(G)} T_e$$

is a disjoint union of open convex cones $T(G,\varepsilon;\varrho)$, where $\varrho \in O(G)$, some of them may be empty. Each such cone $T(G,\varepsilon;\varrho)$ is isomorphic to the open convex cone

$$T^{+}(G,\varrho) := \{ f \in T(G,\varrho;\mathbb{R}) \mid f(x) > 0 \text{ for } x \in E \}.$$

We introduce the open polytope

$$\Delta_{\mathrm{TN}}(G,\varepsilon;\varrho) := \{ f \in T(G,\varepsilon;\varrho) \mid |f(x)| < 1 \text{ for } x \in E \}$$

and the 0-1 open polytope

$$\Delta^{+}_{\mathrm{TN}}(G,\varepsilon) := \left\{ f \in T^{+}(G,\varepsilon) \mid 0 < f(x) < 1 \text{ for } x \in E \right\}.$$

We call $\Delta^{+}_{\mathrm{TN}}(G,\varepsilon)$ the *tension open polytope* and its closure $\bar{\Delta}^{+}_{\mathrm{TN}}(G,\varepsilon)$ the *tension polytope* of the digraph (G,ε). Let $\tau_{\mathbb{Z}}(G,\varepsilon;q)$ denote the number of nowhere-zero tensions with values in $\{0,1,\ldots,q-1\}$. In other words, $\tau_{\mathbb{Z}}(G,\varepsilon;q)$ is the number of positive integral q-tensions of (G, ε). Clearly, $\tau_{\mathbb{Z}}(G, \varepsilon; q)$ counts the number of lattice points of $q\Delta^{+}_{\mathrm{TN}}(G, \varepsilon)$, i.e.,

$$\tau_{\mathbb{Z}}(G, \varepsilon; q) := \left|q\Delta^{+}_{\mathrm{TN}}(G, \varepsilon) \cap \mathbb{Z}^{E}\right| = L(\Delta^{+}_{\mathrm{TN}}(G, \varepsilon), q). \tag{5.4}$$

We call $\tau_{\mathbb{Z}}(G, \varepsilon; q)$ the *positive tension polynomial with respect to* ε. Similarly, let $\bar{\tau}_{\mathbb{Z}}(G, \varepsilon; q)$ denote the number of integral tensions of (G, ε) with values in $\{0, 1, \ldots, q\}$. In other words, $\bar{\tau}_{\mathbb{Z}}(G, \varepsilon; q)$ is the number of non-negative integral $(q+1)$-tensions of (G, ε). Then $\bar{\tau}_{\mathbb{Z}}(G, \varepsilon; q)$ counts the number of lattice points of $q\bar{\Delta}^{+}_{\mathrm{TN}}(G, \varepsilon)$, i.e.,

$$\bar{\tau}_{\mathbb{Z}}(G, \varepsilon; q) := \left|q\bar{\Delta}^{+}_{\mathrm{TN}}(G, \varepsilon) \cap \mathbb{Z}^{E}\right| = L(\bar{\Delta}^{+}_{\mathrm{TN}}(G, \varepsilon), q). \tag{5.5}$$

We call $\bar{\tau}_{\mathbb{Z}}(G, \varepsilon; q)$ the *non-negative tension polynomial of G with respect to* ε.

Lemma 5.2. *For orientations* ε, $\varrho \in O(G)$ *and the involution* $P_{\varrho,\varepsilon}$,

$$P_{\varrho,\varepsilon}\left(\Delta_{\mathrm{TN}}(G, \varepsilon; \varrho)\right) = \Delta^{+}_{\mathrm{TN}}(G, \varrho), \tag{5.6}$$

$$\Delta_{\mathrm{TN}}(G, \varepsilon) = \bigsqcup_{\varrho \in O(G)} \Delta_{\mathrm{TN}}(G, \varepsilon; \varrho). \tag{5.7}$$

Proof. Let $f \in \Delta_{\mathrm{TN}}(G, \varepsilon; \varrho)$. Since $P_{\varrho,\varepsilon} f \in T(G, \varrho; \mathbb{R})$ and $P_{\varrho,\varepsilon} f(x) = [\varrho, \varepsilon](x) f(x) > 0$ for all $x \in E$, we have $P_{\varrho,\varepsilon} f \in \Delta^{+}_{\mathrm{TN}}(G, \varrho)$. Conversely, for any $g \in \Delta^{+}_{\mathrm{TN}}(G, \varrho)$, we obviously have $P_{\varepsilon,\varrho} g \in \Delta_{\mathrm{TN}}(G, \varepsilon; \varrho)$. Hence (5.6) is valid.

The right-hand side of (5.7) is obviously contained in the left-hand side. Let $f \in \Delta_{\mathrm{TN}}(G, \varepsilon)$, and let ϱ be an orientation of G such that $\varrho(x) = \varepsilon(x)$ if $f(x) > 0$, and $\varrho(x) \neq \varepsilon(x)$ if $f(x) < 0$. Then $f \in \Delta_{\mathrm{TN}}(G, \varepsilon; \varrho)$. Thus the left-hand side of (5.7) is contained in its right-hand side. ∎

Lemma 5.2 shows that for orientations ε, ϱ and positive integers q,

$$\tau(G, \varrho; q) = \left|q\Delta^{+}_{\mathrm{TN}}(G, \varrho) \cap \mathbb{Z}^{E}\right| = \left|q\Delta_{\mathrm{TN}}(G, \varepsilon; \varrho) \cap \mathbb{Z}^{E}\right|.$$

Notice that $\tau_{\mathbb{Z}}(G, \varrho; q) \equiv 0$ if and only if $\Delta^{+}_{\mathrm{TN}}(G, \varrho) = \emptyset$. The following lemma characterizes ϱ such that $\Delta^{+}_{\mathrm{TN}}(G, \varrho)$ is non-empty.

Lemma 5.3. *Let* $\varepsilon \in O(G)$. *Then the digraph* (G, ε) *has a positive real tension if and only if* (G, ε) *has no directed circuit. In other words,* $\Delta^{+}_{\mathrm{TN}}(G, \varepsilon)$ *is non-empty if and only if* ε *is acyclic.*

Proof. Trivial. ∎

Theorem 1.2 gives an interpretation for the integral tension polynomial $\tau_{\mathbb{Z}}(G, t)$ at zero and negative integers. This interpretation is similar to that of Stanley's result on the chromatic polynomial at negative integers; see [24].

Proof of Theorem 1.2. It is clear that $\Delta^{+}_{\mathrm{TN}}(G, \varepsilon)$ is a 0-1 open polytope. Since $\tau_{\mathbb{Z}}(G, \varepsilon; q)$ and $\bar{\tau}_{\mathbb{Z}}(G, \varepsilon; q)$ are Ehrhart polynomials by (5.4) and (5.5), we automatically have the reciprocity law (1.9), $\tau_{\mathbb{Z}}(G, \varepsilon; 0) = (-1)^{r(G)}$, and $\bar{\tau}_{\mathbb{Z}}(G, \varepsilon; 0) = 1$ from the standard properties of Ehrhart polynomials. The degree follows from Corollary 3.4.

By identity (5.7) and Lemma 5.3, we have the disjoint union

$$\Delta_{\mathrm{TN}}(G,\varepsilon)=\bigsqcup_{\varrho\in O_{\mathrm{AC}}(G)} P_{\varepsilon,\varrho}(\Delta^{+}_{\mathrm{TN}}(G,\varrho)).$$

Now the identity (1.7) follows immediately. The formula (1.8) follows from the definition of $\bar{\tau}_{\mathbb{Z}}(G,q)$. The reciprocity law (1.10) follows from the reciprocity law of Ehrhart polynomials. ∎

6. Interpretation of Modular Tension Polynomial

In this section we shall give a decomposition for the modular tension polynomial similar to that of Theorem 1.2. Let $G=(V,E)$ be a graph with possible loops and multiple edges. Recall that $O_{\mathrm{AC}}(G)$ is the set of all orientations of G without directed circuit. Given a positive integer q; we define $\mathrm{Mod}_q\colon \mathbb{R}^E\to(\mathbb{R}/q\mathbb{Z})^E$ by

$$(\mathrm{Mod}_q f)(x)=f(x) \bmod q,\quad f\in\mathbb{R}^E. \tag{6.1}$$

Then $\mathrm{Mod}_q(\mathbb{Z}^E)=(\mathbb{Z}/q\mathbb{Z})^E$ is a subgroup of the torus $(\mathbb{R}/q\mathbb{Z})^E$, and $\mathrm{Mod}_q(T(G,\varepsilon;\mathbb{Z}))$ is a subgroup of $\mathrm{Mod}_q(\mathbb{Z}^E)$. For $\varepsilon,\varrho\in O(G)$, we define an involution $Q_{\varrho,\varepsilon}\colon [0,q]^E\to[0,q]^E$ by

$$(Q_{\varrho,\varepsilon}f)(x)=\begin{cases} f(x), & \text{if } \varrho(x)=\varepsilon(x),\\ q-f(x), & \text{if } \varrho(x)\neq\varepsilon(x),\end{cases}\qquad f\in[0,q]^E. \tag{6.2}$$

For $\varepsilon,\varrho,\rho\in O(G)$ and $f\in[0,q]^E$, it is straightforward to check that

$$\left(Q_{\rho,\varrho}P_{\varrho,\varepsilon}f\right)(x)=\begin{cases} f(x), & \text{if } \rho(x)=\varrho(x)=\varepsilon(x),\\ -f(x), & \text{if } \rho(x)=\varrho(x)\neq\varepsilon(x),\\ q-f(x), & \text{if } \rho(x)\neq\varrho(x)=\varepsilon(x),\\ q+f(x), & \text{if } \rho(x)\neq\varrho(x)\neq\varepsilon(x),\end{cases}$$

and

$$\left(P_{\varepsilon,\rho}Q_{\rho,\varrho}P_{\varrho,\varepsilon}f\right)(x)=\begin{cases} f(x), & \text{if } \rho(x)=\varrho(x)=\varepsilon(x),\\ f(x), & \text{if } \rho(x)=\varrho(x)\neq\varepsilon(x),\\ f(x)-q, & \text{if } \rho(x)\neq\varrho(x)=\varepsilon(x),\\ f(x)+q, & \text{if } \rho(x)\neq\varrho(x)\neq\varepsilon(x).\end{cases} \tag{6.3}$$

Lemma 6.1. *The restrictions*

$$\mathrm{Mod}_q\colon T_{\mathbb{Z}}(G,\varepsilon;q)\to T(G,\varepsilon;\mathbb{Z}/q\mathbb{Z}) \ \text{ and}$$

$$\mathrm{Mod}_q\colon T_{\mathrm{nz}\mathbb{Z}}(G,\varepsilon;q)\to T_{\mathrm{nz}}(G,\varepsilon;\mathbb{Z}/q\mathbb{Z})$$

are surjective.

Proof. Let $\tilde{f} \in T(G, \varepsilon; \mathbb{Z}/q\mathbb{Z})$. We construct an integral q-tension $f \in T_{\mathbb{Z}}(G, \varepsilon; q)$ as follows. We first identify $\mathbb{Z}/q\mathbb{Z}$ as the set $\{0, 1, \ldots, q-1\}$. Let G_0 be a connected component of G; and let v_0 be a fixed vertex of G_0. For each vertex v of G_0, let $P = v_0 v_1 \cdots v_n$ be a shortest path from v_0 to $v = v_n$ and $P(v_0, v_i) = v_0 v_1 \cdots v_i$, $1 \le i \le n$. Let ϱ be an orientation of P such that $v_{i-1} \xrightarrow{\varrho} v_i$, $1 \le i \le n$. Define

$$f(v_0, v_1) = \tilde{f}(v_0, v_1) + \frac{q}{2}\left([\varepsilon, \varrho](v_0, v_1) - 1\right),$$

$$f(v_{i-1}, v_i) = \begin{cases} \tilde{f}(v_{i-1}, v_i) + q[\varepsilon, \varrho](v_{i-1}, v_i), & \text{if } -q < s_i < 0, \\ \tilde{f}(v_{i-1}, v_i), & \text{if } \ 0 \le s_i < q, \\ \tilde{f}(v_{i-1}, v_i) - q[\varepsilon, \varrho](v_{i-1}, v_i), & \text{if } \ q \le s_i < 2q, \end{cases}$$

where

$$s_i = [\varepsilon, \varrho](v_{i-1}, v_i)\tilde{f}(v_{i-1}, v_i) + \sum_{x \in P(v_0, v_{i-1})} [\varepsilon, \varrho](x) f(x).$$

Let $Q(v_0, w_n)$ be another shortest path from v_0 to $v = w_n$. It is routine by the construction of f to verify that

$$0 \le a_i := \sum_{x \in P(v_0, v_i)} [\varepsilon, \varrho](x) f(x) < q,$$

$$0 \le b_i := \sum_{x \in Q(v_0, w_i)} [\varepsilon, \varrho](x) f(x) < q,$$

and $a_i \equiv b_i \bmod q$ for $0 \le i \le n$. Let $g \colon V \to [0, q) \cap \mathbb{Z}$ be defined by

$$g(v_0) := 0 = a_0 \quad \text{and} \quad g(v) := g(v_n) = a_n.$$

Then g is a well-defined coloring of G with values in $\{0, 1, \ldots, q-1\}$. It follows that the function f is a well-defined tension of (G, ε). Since $[\varepsilon, \varrho](v_{i-1}, v_i) f(v_{i-1}, v_i) = a_i - a_{i-1}$ for $1 \le i \le n$, we see that $|f| < q$. So $f \in T_{\mathbb{Z}}(G, \varepsilon; q)$. Finally, it is clear that if $\tilde{f}$ is nowhere-zero, then f is nowhere-zero.

A non-constructive proof can be followed from Corollary 3.4 and a result of Tutte [26] on regular abelian groups. ∎

Recall that an oriented cut in a digraph (G, ε) is a disjoint union of directed bonds. However, an oriented cut may be decomposed into a disjoint union of bonds that are not necessarily directed. For instance, the edge set $\{a, b, c, d\}$ of the digraph in Figure 1 is an oriented cut, for it can be written as disjoint union of two directed bonds $\{a, b\}$ and $\{c, d\}$. However, the same cut is the union of the bonds $\{a, d\}$ and $\{b, c\}$, which are not directed.

Proposition 6.2. *Let $U \subseteq E(G)$ be a non-empty subset of a digraph (G, ε). Then U is an oriented cut of (G, ε) if and only if for any directed circuit (C, ε_C),*

$$\sum_{x \in U \cap C} [\varepsilon, \varepsilon_C](x) = 0. \tag{6.4}$$

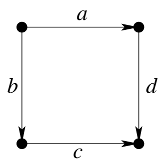

Figure 1: An oriented cut may be decomposed into a disjoint union of undirected bonds.

Proof. "⇒": Let U be decomposed into a disjoint union of directed bonds B. Then for any directed circuit $(C, \varepsilon_{\rm C})$ of G, the intersection $B\cap C$ has even number of edges, half of them agreeing with the orientation $\varepsilon_{\rm C}$. It is obvious that

$$\sum_{x\in B\cap C} [\varepsilon, \varepsilon_{\rm C}]\,(x) = 0.$$

Hence (6.4) is valid by adding up these sums for all directed bonds B in the decomposition of U.

"⇐": It is clear that U does not contain loops. Choose an edge $e \in U$ whose ending vertex is v. Without loss of generality, we may assume that the graph G is connected. Let S be a subset of V, consisting of vertices u such that there exists a path P from u to v and

$$\sum_{x\in U\cap P} [\varepsilon, \varepsilon_{\rm P}]\,(x) > 0,$$

where $\varepsilon_{\rm P}$ is the direction of the path P from u to v. Similarly, let T be a subset of V, consisting of vertices w such that there exists a path Q from w to v and

$$\sum_{y\in U\cap Q} [\varepsilon, \varepsilon_{\rm Q}]\,(x) \leq 0,$$

where $\varepsilon_{\rm Q}$ is the direction of the path Q from w to v. It is clear that S and T are well defined, and $S\cap T = \emptyset$, $S\cup T = V$; so $[S,T]$ is a cut. We first claim that $[S,T]$ is contained in U. In fact, if there is an edge $e' = u'v' \in [S,T]$ such that $e' \notin U$, then $u' \in S$, $v' \in T$. Let P' be a path from u' to v and $\sum_{x\in U\cap P'} [\varepsilon, \varepsilon_{\rm P'}]\,(x) > 0$. Then $Q' = (v', e')P'$ is a path from v' to v and $\sum_{x\in U\cap Q'} [\varepsilon, \varepsilon_{\rm Q'}]\,(x) > 0$. So $v' \in S$, a contradiction.

Next we claim that $[S,T]$ is a directed cut with all arrows from S to T. Suppose that there is an edge e' with initial vertex $v' \in T$ and ending vertex $u' \in S$. Let P' be a path from u' to v such that $\sum_{x\in U\cap P'} [\varepsilon, \varepsilon_{\rm P'}]\,(x) > 0$. Then $Q' = (v', e')P'$ is a path from v' to v and

$$\sum_{x\in U\cap Q'} [\varepsilon, \varepsilon_{\rm Q'}]\,(x) = 1 + \sum_{x\in U\cap P'} [\varepsilon, \varepsilon_{\rm P'}]\,(x) > 0.$$

So $v' \in S$; this is a contradiction.

Now the edge set $[S,T]$ is a directed cut. Let $U' = U - [S,T]$. Then by induction, U' can be decomposed into a disjoint union of directed bonds. So is U. ∎

Corollary 6.3. *Let G be a graph with possible loops and multiple edges, and let $U \subseteq E(G)$ be a non-empty subset. Then U is a cut if and only if $|U\cap C|$ is even for any circuit C.*

Proof. The necessity is obvious. To prove the sufficiency, we still proceed by induction on $|U|$. For $|U| = 1$, to have $|U \cap C|$ to be even for a circuit C, the intersection $U \cap C$ must be empty. So U is a bridge of G. Of course it is a cut. Let $|U| \geq 2$. Choose an edge $e \in U$ and all circuits $C_1, \ldots, C_n$ that contain e. For each C_i, choose an edge $e_i \in U \cap (C_i - \{e\})$ since $|U \cap C_i|$ is even. It is clear that $B = e \cup \{e_i \mid 1 \leq i \leq n\}$ is a bond of G. Now for any circuit C, we have $|B \cap C| = even$ and $|U \cap C| = even$. It follows that $|(U - B) \cap C| = even$. If $U - B = \emptyset$, then $U = B$ is a cut. If $U - B \neq \emptyset$, then by induction $U - B$ is a cut. Hence $U = (U - B) \cup B$ is a cut. ∎

Remark 6.4. Proposition 6.2 is equivalent to the statement that the tension space and the flow space are orthogonal complement in $\mathbb{R}^E$ (see [5, 6, 15]); the directed circuit (C, ε_C) can be replaced by directed Eulerian subgraphs. Corollary 6.3 is well known; the cut C can be replaced by Eulerian subgraph. The inclusion of Proposition 6.2 and Corollary 6.3 here is for easy comparison with similar results on signed graphs and matroids by the same approach in subsequent work.

Lemma 6.5. (a) *The relation $\sim$ is an equivalence relation on $O(G)$.*
(b) *Let ε, $\varrho \in O(G)$ be cut-equivalent. If ε is acyclic, so is ϱ.*
(c) *Let ε, $\varrho \in O(G)$ be cut-equivalent. Then $Q_{\varrho,\varepsilon}\colon q\bar{\Delta}^+_{\mathrm{TN}}(G, \varepsilon) \to q\bar{\Delta}^+_{\mathrm{TN}}(G, \varrho)$ is a bijection, sending lattice points to lattice points. In particular,*

$$Q_{\varrho,\varepsilon}\left(q\Delta^+_{\mathrm{TN}}(G, \varepsilon)\right) = q\Delta^+_{\mathrm{TN}}(G, \varrho),$$

$$\tau_{\mathbb{Z}}(G, \varepsilon; q) = \tau_{\mathbb{Z}}(G, \varrho; q),$$

$$\bar{\tau}_{\mathbb{Z}}(G, \varepsilon; q) = \bar{\tau}_{\mathbb{Z}}(G, \varrho; q).$$

Proof. (a) The reflexivity and symmetry property are obvious. Let $\varepsilon_i \in O(G)$ ($i = 1, 2, 3$) be such that $\varepsilon_1 \sim \varepsilon_2$ and $\varepsilon_2 \sim \varepsilon_3$. Let $\{P_1, T_1\}$ and $\{P_2, T_2\}$ be partitions of the vertex set V such that

$$[P_1, T_1] = E(\varepsilon_1 \neq \varepsilon_2), \quad [P_2, T_2] = E(\varepsilon_2 \neq \varepsilon_3).$$

If $(P_1 \cap P_2) \cup (T_1 \cap T_2) = \emptyset$, we have $P_1 = T_2$, $P_2 = T_1$; if $(P_1 \cap T_2) \cup (P_2 \cap T_1) = \emptyset$, we have $P_1 = P_2$, $T_1 = T_2$. In either case the two cuts are the same. Hence $\varepsilon_1 = \varepsilon_3$. If it is neither of the two cases, let $S = (P_1 \cap P_2) \cup (T_1 \cap T_2)$, $T = (P_1 \cap T_2) \cup (P_2 \cap T_1)$. Then $[S, T]$ is a cut and can be written as a disjoint union of sets

$$[P_1 \cap P_2, P_1 \cap T_2],\ [P_1 \cap P_2, P_2 \cap T_1],\ [T_1 \cap T_2, P_1 \cap T_2],\ [T_1 \cap T_2, P_2 \cap T_1].$$

It is routine to verify that $[S, T] = E(\varepsilon_1 \neq \varepsilon_3)$.

Next we show that the cut $[S, T]$ is an oriented cut. Let (C, ε_C) be a directed circuit. Then

$$\sum_{x \in C \cap E(\varepsilon_1 \neq \varepsilon_3)} [\varepsilon_1, \varepsilon_C](x) = \left\{ \sum_{x \in C \cap E(\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3)} + \sum_{x \in C \cap E(\varepsilon_1 \neq \varepsilon_2 = \varepsilon_3)} \right\} [\varepsilon_1, \varepsilon_C](x)$$

$$= \left\{ \sum_{x \in C \cap E(\varepsilon_2 \neq \varepsilon_3)} - \sum_{x \in C \cap E(\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3)} \right\} [\varepsilon_2, \varepsilon_C](x)$$

$$-\left\{\sum_{x\in C\cap E(\varepsilon_1\neq\varepsilon_2)} - \sum_{x\in C\cap E(\varepsilon_1\neq\varepsilon_2\neq\varepsilon_3)}\right\}[\varepsilon_2,\varepsilon_C](x)$$

$$=\left\{\sum_{x\in C\cap E(\varepsilon_2\neq\varepsilon_3)} - \sum_{x\in C\cap E(\varepsilon_1\neq\varepsilon_2)}\right\}[\varepsilon_2,\varepsilon_C](x)$$

$$=0.$$

The last equality follows from Proposition 6.2.

(b) Suppose ϱ is not acyclic, i.e., there is a directed circuit (C,ϱ_C) of the digraph (G,ϱ). Since (G,ε) has no directed circuit, the circuit C must intersect $E(\varepsilon\neq\varrho)$. Thus any bond of $E(\varepsilon\neq\varrho)$ that intersects (C,ϱ_C) must have opposite directed edges. This is a contrary to that $E(\varepsilon\neq\varrho)$ is an oriented cut.

(c) Let C be a circuit of G with an orientation ε_C. For any $f\in q\bar{\Delta}^+_{\mathrm{TN}}(G,\varepsilon)$, we have

$$\sum_{x\in C}[\varepsilon,\varepsilon_C](x)f(x)=0.$$

Notice that ε and ϱ have the same direction at edges of $C\cap E(\varepsilon=\varrho)$ and have the opposite directions at edges of $C\cap E(\varepsilon\neq\varrho)$. Then

$$\sum_{x\in C}[\varrho,\varepsilon_C](x)\,(Q_{\varrho,\varepsilon}f)(x)=\sum_{x\in C\cap E(\varepsilon=\varrho)}[\varepsilon,\varepsilon_C](x)f(x)$$

$$-\sum_{x\in C\cap E(\varepsilon\neq\varrho)}[\varepsilon,\varepsilon_C](x)\,(q-f(x))$$

$$=\sum_{x\in C}[\varepsilon,\varepsilon_C](x)f(x)-q\sum_{x\in C\cap E(\varepsilon\neq\varrho)}[\varepsilon,\varepsilon_C](x)$$

$$=0.$$

This means that $Q_{\varrho,\varepsilon}f$ is a non-negative q-tension of (G,ϱ). The other identities can be verified trivially. ∎

Part (b) of Lemma 6.5 shows that the cut-equivalence relation $\sim$ can be viewed as an equivalence relation on the set $\mathcal{O}_{\mathrm{AC}}(G)$ of acyclic orientations. We denote by $[\mathcal{O}_{\mathrm{AC}}(G)]$ the quotient set $\mathcal{O}_{\mathrm{AC}}(G)/\!\sim$ of cut-equivalence classes. For $\varepsilon\in\mathcal{O}_{\mathrm{AC}}(G)$, we denote by $[\varepsilon]$ the equivalence class in $\mathcal{O}_{\mathrm{AC}}(G)/\!\sim$, and define

$$\tau_{\mathbb{Z}}(G,[\varepsilon];q):=\tau_{\mathbb{Z}}(G,\varepsilon;q),\quad \bar{\tau}_{\mathbb{Z}}(G,[\varepsilon];q):=\bar{\tau}_{\mathbb{Z}}(G,\varepsilon;q). \tag{6.5}$$

Let $\bar{\tau}(G,q)$ denote the number of pairs $(f,[\varepsilon])$, where $[\varepsilon]$ is a cut-equivalence class of a fixed representative ε in $[\mathcal{O}_{\mathrm{AC}}(G)]$ and f is an integral $(q+1)$-tension of (G,ε), i.e.,

$$\bar{\tau}(G,q):=\#\big\{(f,\varepsilon)\,|\,\varepsilon\in[\mathcal{O}_{\mathrm{AC}}(G)],\ f\text{ is an integral tension of }(G,\varepsilon)$$

$$\text{such that }0\le f(x)\le q\text{ for }x\in E\big\}. \tag{6.6}$$

Let f be a real-valued tension of (G, ε). We define an orientation ε_f for $x \in E$ as

$$\varepsilon_f(x) = \begin{cases} \varepsilon(x), & \text{if } f(x) > 0, \\ -\varepsilon(x), & \text{if } f(x) \le 0. \end{cases} \tag{6.7}$$

Let ϱ be an orientation of G. We define a 0-1 function $f_\varrho \colon E \to \{0, 1\}$ for $x \in E$ as

$$f_\varrho(x) = \begin{cases} 1, & \text{if } \varrho(x) = \varepsilon(x), \\ 0, & \text{if } \varrho(x) \ne \varepsilon(x). \end{cases} \tag{6.8}$$

Lemma 6.6. *Let $f_i \in T(G, \varepsilon; q)$, and let $\varepsilon_i = \varepsilon_{f_i}$ be orientations defined by* (6.7). *If $f_1(x) \equiv f_2(x) \bmod q$ for all $x \in E(G)$, then ε_1 and ε_2 are cut-equivalent.*

Proof. Since $f_1(x) \equiv f_2(x) \bmod q$ for all $x \in E(G)$, we have

$$f_1(x) = f_2(x) + k(x)q, \quad x \in E,$$

where $k(x) \in \{-1, 0\}$ if $f_2(x) > 0$, $k(x) \in \{0, 1\}$ if $f_2(x) < 0$, and $k(x) = 0$ if $f_2(x) = 0$. More precisely,

$$f_1(x) = \begin{cases} f_2(x), & \text{if } \varepsilon_1(x) = \varepsilon_2(x), \\ f_2(x) - q, & \text{if } \varepsilon_1(x) \ne \varepsilon_2(x) = \varepsilon(x), \\ f_2(x) + q, & \text{if } \varepsilon_1(x) \ne \varepsilon_2(x) \ne \varepsilon(x). \end{cases}$$

Let $g_i = P_{\varepsilon_i, \varepsilon} f_i$. Then g_i are non-negative q-tensions of (G, ε_i). Moreover,

$$g_1(x) = \begin{cases} g_2(x), & \text{if } \varepsilon_1(x) = \varepsilon_2(x), \\ q - g_2(x), & \text{if } \varepsilon_1(x) \ne \varepsilon_2(x). \end{cases} \tag{6.9}$$

Let C be a circuit of G with a direction ε_C. Since g_i are positive tensions of (G, ε_i), then

$$\sum_{x \in C} [\varepsilon_1, \varepsilon_C](x) g_1(x) = \sum_{x \in C} [\varepsilon_2, \varepsilon_C](x) g_2(x) = 0.$$

Applying (6.9) and cancelling the common terms on both sides, we have

$$\begin{aligned} \sum_{x \in C \cap E(\varepsilon_1 \ne \varepsilon_2)} [\varepsilon_1, \varepsilon_C](x) g_1(x) &= \sum_{x \in C \cap E(\varepsilon_1 \ne \varepsilon_2)} [\varepsilon_2, \varepsilon_C](x) g_2(x) \\ &= -\sum_{x \in C \cap E(\varepsilon_1 \ne \varepsilon_2)} [\varepsilon_1, \varepsilon_C](x) \left(q - g_1(x)\right). \end{aligned}$$

It follows that

$$\sum_{x \in C \cap E(\varepsilon_1 \ne \varepsilon_2)} [\varepsilon_1, \varepsilon_C](x) = 0.$$

By Proposition 6.2, $E(\varepsilon_1 \ne \varepsilon_2)$ is an oriented cut. Hence $\varepsilon_1 \sim \varepsilon_2$. ∎

Lemma 6.7. *Let $\varrho, \rho, \sigma \in O_{AC}(G)$ be such that $\rho \sim \sigma \sim \varrho$, and $f \in q\Delta_{TN}(G, \varepsilon; \varrho)$. Then*

(a) $\varepsilon_f = \varrho$.
(b) $P_{\varepsilon,\rho} Q_{\rho,\varrho} P_{\varrho,\varepsilon} (q\Delta_{\text{TN}}(G, \varepsilon; \varrho)) = q\Delta_{\text{TN}}(G, \varepsilon; \rho)$.
(c) $P_{\varepsilon,\rho} Q_{\rho,\varrho} P_{\varrho,\varepsilon} f = P_{\varepsilon,\sigma} Q_{\sigma,\varrho} P_{\varrho,\varepsilon} f$ *if and only if* $\rho = \sigma$.
(d) $T(G, \varepsilon; q) \cap \text{Mod}_q^{-1}(\text{Mod}_q f) = \{P_{\varepsilon,\alpha} Q_{\alpha,\varrho} P_{\varrho,\varepsilon} f \mid \alpha \sim \varrho\}$.

Proof. (a) Notice that $\varepsilon_f = \frac{f}{|f|}\varepsilon$ by (6.7), and $[\varrho, \varepsilon](x) f(x) > 0$ for all $x \in E(G)$. It then follows that $\varepsilon_f = \varrho$.

(b) Recall Lemma 5.2 and Lemma 6.5(c); we have

$$P_{\varrho,\varepsilon} (q\Delta_{\text{TN}}(G, \varepsilon; \varrho)) = q\Delta^+_{\text{TN}}(G, \varrho), \quad Q_{\rho,\varrho} \left(q\Delta^+_{\text{TN}}(G, \varrho)\right) = q\Delta^+_{\text{TN}}(G, \rho),$$

and $P_{\varepsilon,\rho} \left(q\Delta^+_{\text{TN}}(G, \rho)\right) = q\Delta^+_{\text{TN}}(G, \varepsilon; \rho)$. The required identity follows immediately.

(c) Let $g = P_{\varepsilon,\rho} Q_{\rho,\varrho} P_{\varrho,\varepsilon} f$ and $h = P_{\varepsilon,\sigma} Q_{\sigma,\varrho} P_{\varrho,\varepsilon} f$. Then $g \in q\Delta_{\text{TN}}(G, \varepsilon; \rho)$ and $h \in q\Delta_{\text{TN}}(G, \varepsilon; \sigma)$ by (b). Thus $\rho = \varepsilon_g = \varepsilon_h = \sigma$ by (a).

(d) Let $\alpha \in O(G)$ be such that $\alpha \sim \varrho$. Then by (6.3),

$$(P_{\varepsilon,\alpha} Q_{\alpha,\varrho} P_{\varrho,\varepsilon} f)(x) = f(x) + k(x)q, \quad \text{where } x \in E(G),\ k(x) \in \mathbb{Z}.$$

Clearly, $(P_{\varepsilon,\alpha} Q_{\alpha,\varrho} P_{\varrho,\varepsilon} f)(x) \equiv f(x) \bmod q$. Thus $P_{\varepsilon,\alpha} Q_{\alpha,\varrho} P_{\varrho,\varepsilon} f \in \text{Mod}_q^{-1}(\text{Mod}_q f)$.

Let $g \in T(G, \varepsilon; q)$ be such that $\text{Mod}_q g = \text{Mod}_q f$. Then

$$g(x) = f(x) + h(x)q, \quad \text{where} \quad x \in E(G),\ h(x) \in \{-1, 0, 1\}. \tag{6.10}$$

Since f is nowhere-zero, by (a) and (6.8) we have $\varrho = \varepsilon_f = \frac{f}{|f|}\varepsilon$. Notice that

(1) $\varepsilon_g(x) = \varepsilon_f(x) = \varepsilon(x) \Leftrightarrow f(x) > 0, g(x) > 0$;
(2) $\varepsilon_g(x) = \varepsilon_f(x) \neq \varepsilon(x) \Leftrightarrow f(x) < 0, g(x) \leq 0$;
(3) $\varepsilon_g(x) \neq \varepsilon_f(x) = \varepsilon(x) \Leftrightarrow f(x) > 0, g(x) \leq 0$;
(4) $\varepsilon_g(x) \neq \varepsilon_f(x) \neq \varepsilon(x) \Leftrightarrow f(x) < 0, g(x) > 0$.

The Equation (6.10) implies that the function g must have the form

$$g(x) = \begin{cases} f(x), & \text{if } \varepsilon_g(x) = \varepsilon_f(x) = \varepsilon(x), \\ f(x), & \text{if } \varepsilon_g(x) = \varepsilon_f(x) \neq \varepsilon(x), \\ f(x) - q, & \text{if } \varepsilon_g(x) \neq \varepsilon_f(x) = \varepsilon(x), \\ f(x) + q, & \text{if } \varepsilon_g(x) \neq \varepsilon_f(x) \neq \varepsilon(x). \end{cases}$$

On the other hand, recall the formula (6.3); we have

$$\left(P_{\varepsilon,\varepsilon_g} Q_{\varepsilon_g,\varepsilon_f} P_{\varepsilon_f,\varepsilon} f\right)(x) = \begin{cases} f(x), & \text{if } \varepsilon_g(x) = \varepsilon_f(x) = \varepsilon(x), \\ f(x), & \text{if } \varepsilon_g(x) = \varepsilon_f(x) \neq \varepsilon(x), \\ f(x) - q, & \text{if } \varepsilon_g(x) \neq \varepsilon_f(x) = \varepsilon(x), \\ f(x) + q, & \text{if } \varepsilon_g(x) \neq \varepsilon_f(x) \neq \varepsilon(x). \end{cases}$$

Therefore, $g = P_{\varepsilon,\varepsilon_g} Q_{\varepsilon_g,\varepsilon_f} P_{\varepsilon_f,\varepsilon} f = P_{\varepsilon,\varepsilon_g} Q_{\varepsilon_g,\varrho} P_{\varrho,\varepsilon} f$. ∎

Proposition 6.8. *Fix any orientation* $\varepsilon \in O(G)$*; the number of orientations of G that are cut-equivalent to* ε *is the number of* 0-1 *tensions of the digraph* (G, ε)*, i.e.,*

$$\bar{\tau}_{\mathbb{Z}}(G, \varepsilon; 1) = \left|\bar{\Delta}^{+}_{\mathrm{TN}}(G, \varepsilon) \cap \mathbb{Z}^{E}\right|.$$

Proof. Let ϱ be an orientation which is cut-equivalent to ε. We identity ϱ as a 0-1 function $I_\varrho = f_{-\varrho}$ given by (6.8). We claim that I_ϱ is a 0-1 tension of (G, ε). In fact, for any circuit C with an orientation ε_C,

$$\sum_{x \in C} [\varepsilon, \varepsilon_C](x) I_\varrho(x) = \sum_{x \in C \cap E(\varrho \neq \varepsilon)} [\varepsilon, \varepsilon_C](x) = 0.$$

The last equality follows from Proposition 6.2. So $I_\varrho \in \bar{\Delta}^{+}_{\mathrm{TN}}(G, \varepsilon) \cap \mathbb{Z}^{E}$.

Conversely, for any 0-1 tension f of (G, ε), we identify f as an orientation $-\varepsilon_f$ defined by (6.7). We claim that $-\varepsilon_f \sim \varepsilon$. Since $E(-\varepsilon_f \neq \varepsilon) = \{x \in E \mid f(x) = 1\}$, then for any circuit C with an orientation ε_C,

$$\sum_{x \in C \cap E(-\varepsilon_f \neq \varepsilon)} [\varepsilon, \varepsilon_C](x) = \sum_{x \in C} [\varepsilon, \varepsilon_C](x) f(x) = 0.$$

It follows from Theorem 6.2 that $E(-\varepsilon_f \neq \varepsilon)$ is an oriented cut. Hence $-\varepsilon_f \sim \varepsilon$.

Now it is easy to verify $f = f_{\varepsilon_f}$ and $\varrho = -\varepsilon_{f_{-\varrho}} = \varepsilon_{f_\varrho}$. So $\varrho \mapsto f_{-\varrho}$ is a one-to-one correspondence from the set $[\varepsilon]$ of orientations cut-equivalent to ε to the set $\bar{\Delta}^{+}_{\mathrm{TN}}(G, \varepsilon) \cap \mathbb{Z}^{E}$ of 0-1 tensions of (G, ε). ∎

Proof of Theorem 1.3. We have seen that $\tau(G, t)$ is a polynomial of degree $r(G)$. Fix an orientation $\varrho \in O_{\mathrm{AC}}(G)$. For any $f \in \Delta_{\mathrm{TN}}(G, \varepsilon; \varrho)$, by Lemma 6.7 and Proposition 6.8, we have

$$\begin{aligned}
\left|T(G, \varepsilon; q) \cap \mathrm{Mod}_q^{-1}(\mathrm{Mod}_q f)\right| &= \#\left\{P_{\varepsilon,\rho} Q_{\rho,\varrho} P_{\varrho,\varepsilon} f \mid \rho \sim \varrho\right\} \\
&= \#\{\rho \in O_{\mathrm{AC}}(G) \mid \rho \sim \varrho\} \\
&= \bar{\tau}_{\mathbb{Z}}(G, \varrho; 1) \qquad (6.11)
\end{aligned}$$

We claim that

$$\bigsqcup_{\rho \sim \varrho} q\Delta_{\mathrm{TN}}(G, \varepsilon; \rho) = T(G, \varepsilon; q) \cap \mathrm{Mod}_q^{-1} \mathrm{Mod}_q \left(\bigsqcup_{\rho \sim \varrho} q\Delta_{\mathrm{TN}}(G, \varepsilon; \rho) \right). \qquad (6.12)$$

Obviously, the left-hand side of (6.12) is contained in its right side. For any $g \in \bigcup_{\rho \sim \varrho} q\Delta_{\mathrm{TN}}(G, \varepsilon; \rho)$ and any $h \in T(G, \varepsilon; q) \cap \mathrm{Mod}_q^{-1}(\mathrm{Mod}_q g)$, by Lemma 6.7, there exists a $\sigma \in O(G)$ such that $\sigma \sim \rho$ and $h = P_{\varepsilon,\sigma} Q_{\sigma,\rho} P_{\rho,\varepsilon} g \in q\Delta_{\mathrm{TN}}(G, \varepsilon; \sigma)$. As $\rho \sim \varrho$ and $\sim$ is an equivalence relation, we have $\sigma \sim \varrho$. This means that h belongs to the left-hand side of (6.12). Moreover, applying (6.11) and Lemma 6.5(c), we see that

$$\left|T(G, \varepsilon; q) \cap \mathrm{Mod}_q^{-1}(\mathrm{Mod}_q g)\right| = \bar{\tau}_{\mathbb{Z}}(G, \rho; 1) = \bar{\tau}_{\mathbb{Z}}(G, \varrho; 1).$$

Apply (5.6) of Lemma 5.2 and Lemma 6.5(c) to each set $q\Delta_{\mathrm{TN}}(G, \varepsilon; \rho)$ in the left-hand side of (6.12); we see that

$$|q\Delta_{\mathrm{TN}}(G, \varepsilon; \rho)| = \left|q\Delta^{+}_{\mathrm{TN}}(G, \rho)\right| = \tau_{\mathbb{Z}}(G, \rho; q) = \tau_{\mathbb{Z}}(G, \varrho; q).$$

Notice that the left-hand side of (6.12) is a disjoint union of $\bar{\tau}_{\mathbb{Z}}(G, \varrho; 1)$ sets. Thus

$$\tau_{\mathbb{Z}}(G, \varrho; q)\,\bar{\tau}_{\mathbb{Z}}(G, \varrho; 1) = \left|\mathrm{Mod}_q\left(\bigsqcup_{\rho\sim\varrho} q\Delta_{\mathrm{TN}}(G, \varepsilon; \rho)\right)\right| \bar{\tau}_{\mathbb{Z}}(G, \varrho; 1).$$

Since $\bar{\tau}_{\mathbb{Z}}(G, \varrho; 1) \geq 1$, it follows that

$$\tau_{\mathbb{Z}}(G, \varrho; q) = \left|\mathrm{Mod}_q\left(\bigsqcup_{\rho\sim\varrho} q\Delta_{\mathrm{TN}}(G, \varepsilon; \rho)\right)\right|.$$

Now applying (5.7) of Lemma 5.2 and Lemma 5.3, we have

$$\begin{aligned} T_{\mathrm{nz}\mathbb{Z}}(G, \varepsilon; q) &= q\Delta_{\mathrm{TN}}(G, \varepsilon) \cap \mathbb{Z}^E \\ &= \bigsqcup_{\varrho\in O_{\mathrm{AC}}(G)} q\Delta_{\mathrm{TN}}(G, \varepsilon; \varrho) \cap \mathbb{Z}^E \\ &= \bigsqcup_{\varrho\in[O_{\mathrm{AC}}(G)]} \bigsqcup_{\rho\sim\varrho} q\Delta_{\mathrm{TN}}(G, \varepsilon; \rho) \cap \mathbb{Z}^E. \end{aligned}$$

Since (6.12) and Mod_q is surjective (Lemma 6.1), we obtain the following disjoint decomposition

$$\begin{aligned} T_{\mathrm{nz}}(G, \varepsilon; \mathbb{Z}/q\mathbb{Z}) &= \mathrm{Mod}_q\left(T_{\mathrm{nz}\mathbb{Z}}(G, \varepsilon; q)\right) \\ &= \bigsqcup_{\varrho\in[O_{\mathrm{AC}}(G)]} \mathrm{Mod}_q\left(\bigsqcup_{\rho\sim\varrho} q\Delta_{\mathrm{TN}}(G, \varepsilon; \rho) \cap \mathbb{Z}^E\right). \end{aligned} \tag{6.13}$$

Counting both sides of (6.13), the identity (1.11) is obtained as

$$\tau(G, q) = \sum_{\varrho\in[O_{\mathrm{AC}}(G)]} \tau_{\mathbb{Z}}(G, \varrho; q).$$

The identity (1.12) follows from its definition. The reciprocity law follows from the reciprocity law of the Ehrhart polynomials $\tau_{\mathbb{Z}}(G, \varepsilon; q)$ and $\bar{\tau}_{\mathbb{Z}}(G, \varepsilon; q)$. Since $\bar{\tau}_{\mathbb{Z}}(G, \varepsilon; 0) = 1$ for all $\varepsilon \in [O_{\mathrm{AC}}(G)]$, we see that

$$\tau(G, 0) = (-1)^{r(G)}\bar{\tau}(G, 0) = (-1)^{r(G)}\left|[O_{\mathrm{AC}}(G)]\right|. \qquad \blacksquare$$

7. Connection with the Tutte Polynomial

The Tutte polynomial (see [5]) of a graph $G = (V, E)$ is a polynomial in two variables

$$T(G; x, y) = \sum_{A\subseteq E(G)} (x-1)^{r\langle E\rangle - r\langle A\rangle}(y-1)^{n\langle A\rangle}, \tag{7.1}$$

where $\langle A\rangle = (V, A)$, $r(G) = |V| - k(G)$, $k(G)$ is the number of connected components of G, $r\langle A\rangle = |V| - k(\langle A\rangle)$, $k(\langle A\rangle)$ is the number of connected components of $\langle A\rangle$, and $n\langle A\rangle = |A| - r\langle A\rangle$. The polynomial $T(G;x,y)$ satisfies the Deletion-Contraction Relation

$$T(G;x,y) = \begin{cases} xT(G/e;x,y), & \text{if } e \text{ is a bridge,} \\ yT(G-e;x,y), & \text{if } e \text{ is a loop,} \\ T(G-e;x,y) + T(G/e;x,y), & \text{otherwise.} \end{cases}$$

It is well known that the chromatic polynomial $\chi(G,t)$ is related to $T(G;x,y)$ by

$$\chi(G,t) = (-1)^{r(G)} t^{k(G)} T(G; 1-t, 0). \tag{7.2}$$

Since $\chi(G,t) = t^{k(G)}\tau(G,t)$, it follows that

$$\tau(G,t) = (-1)^{r(G)} T(G; 1-t, 0).$$

Thus

$$\bar{\tau}(G,t) = (-1)^{r(G)}\tau(G,-t) = T(G; t+1, 0).$$

We conclude the information as the following proposition.

Proposition 7.1. *Let $G = (V,E)$ be a graph with possible loops and multiple edges. Then*

$$T(G;t,0) = \bar{\tau}(G,t-1) = (-1)^{r(G)}\tau(G,1-t). \tag{7.3}$$

In particular, $T(G;1,0) = (-1)^{r(G)}\tau(G,0) = \bar{\tau}(G,0)$ counts the number of cut-equiv -alence classes of acyclic orientations of G.

Example 7.2. Let G be the labelled graph in Figure 2 with 4 vertices and 5 edges. Its

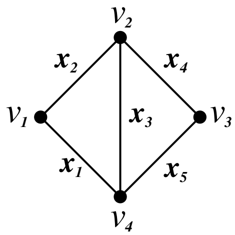

Figure 2: A labelled double triangle graph.

chromatic polynomial and modular tension polynomial are given by

$$\chi(G,t) = t(t-1)(t-2)^2, \quad \tau(G,t) = (t-1)(t-2)^2.$$

Take the directions $v_1v_2v_4v_1$ and $v_2v_3v_4v_2$ for the left and right circuits in Figure 2; we see that the number of nowhere-zero integral q-tensions of G is the number of integral solutions of the system of linear equalities and inequalities

$$x_1 + x_2 + x_3 = 0, \quad -x_3 + x_4 + x_5 = 0, \quad -q < x_i < q, \quad x_i \neq 0.$$

Counting the number of integral solutions of the system gives the integral tension polynomial

$$\tau_{\mathbb{Z}}(G,q) = 6(q-1)^2(2q-3) + \frac{(q-1)q(2q-1)}{3}.$$

Then $|\chi(G,-1)| = |\tau(G,-1)| = |\tau_{\mathbb{Z}}(G,0)| = 18$ is the number of acyclic orientations of G, and $|\tau(G,0)| = 4$ is the number of cut-equivalence classes of acyclic orientations; see Figure 3 below.

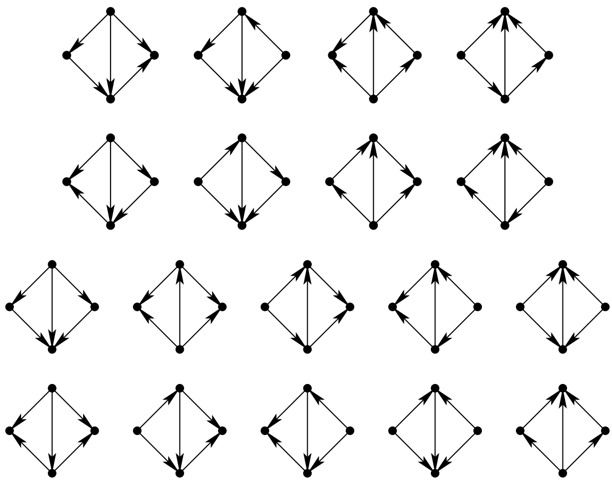

Figure 3: Cut-equivalence classes of acyclic orientations.

There are total of 32 orientations. The other 14 cyclic orientations constitute 10 cut-equivalence classes, each of 6 equivalence classes contains exactly one cyclic orientation, and each of the other 4 equivalence classes contains exactly two cyclic orientations.

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