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On Minkowski Sums of Simplices

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Abstract. We investigate the structure of the Minkowski sum of standard simplices in $\mathbb{R}^r$. In particular, we investigate the one-dimensional structure, the vertices, their degrees and the edges in the Minkowski sum polytope.

Keywords: polytope, Minkowski sum, zonotope

1. Introduction and Definitions

Let $[r] = \{1, 2, \ldots, r\}$. The *standard simplex* $\Delta_{[r]}$ of dimension $r-1$ is given by

$$\Delta_{[r]} = \{(x_1, \ldots, x_r) \in \mathbb{R}^r : x_i \geq 0 \text{ for all } i, x_1 + \cdots + x_r = 1\}.$$

Each subset $F \subseteq [r]$ yields a *face* Δ_F of $\Delta_{[r]}$ given by

$$\Delta_F = \{(x_1, \ldots, x_r) \in \Delta_{[r]} : x_i = 0 \text{ for } i \notin F\}.$$

Clearly, Δ_F itself is a simplex embedded in $\mathbb{R}^r$. If $\mathcal{F}$ is a family of subsets of $[r]$, then we can form the *Minkowski sum* of simplices

$$P_{\mathcal{F}} = \sum_{F \in \mathcal{F}} \Delta_F = \left\{ \sum_{F \in \mathcal{F}} x_F : x_F \in \Delta_F \text{ for each } F \in \mathcal{F} \right\}.$$

If $|F| = 2$ for all $F \in \mathcal{F}$, then the polytope $P_{\mathcal{F}}$ is called a *graphical zonotope*. The edge graphs of graphical zonotopes were studied by West et al. [5, 13], but several questions about them have gone unanswered. For example, it is not known if the set of integers that are the degrees of the vertices of a fixed graphical zonotope must be a set of consecutive integers. Minkowski sums of simplices have recently been studied by Feichtner and Sturmfels [4], and by Postnikov [11]. These later papers focus on the case when the collection $\mathcal{F}$ is a *building set*, i.e., $\mathcal{F}$ contains all singletons, and has the property that, for any $F_1, F_2 \in \mathcal{F}$, $F_1 \cap F_2 \neq \emptyset$ implies that $F_1 \cup F_2 \in \mathcal{F}$.

It turns out, see Proposition 5.5, that this property implies that the polytope $P_{\mathcal{F}}$ is simple. Applications of Minkowski sums of simplices appear in the paper of Morton et al. [10]. Minkowski sums of simplices have also appeared in the work of Conca [2] and of Herzog and Hibi [7], under the name transversal polymatroids.

In the remainder of this introductory section, we list some elementary properties of Minkowski sums of simplices, some of which have been noted in the papers [4] and [11]. We will denote by $\Delta_{\mathcal{F}}$ the simplicial complex with facets $\max(\mathcal{F})$.

Proposition 1.1. *If $\bigcup_{F\in\mathcal{F}} = [r]$ and the simplicial complex $\Delta_{\mathcal{F}}$ is connected, then the dimension of $P_{\mathcal{F}}$ is $r-1$.*

Proof. Every point $x \in P_{\mathcal{F}}$ satisfies $\sum_{i\in[r]} x_i = |\mathcal{F}|$. Suppose $c \in \mathbb{R}^r$ and there is a partition $[r] = I \cup J$ of $[r]$ into nonempty subsets, so that $c_i < c_j$ for all $i \in I$, $j \in J$. Because $\Delta_{\mathcal{F}}$ is connected, there are $i \in I$, $j \in J$, $G \in \mathcal{F}$ such that $\{e_i, e_j\} \subseteq G$. For each $F \in \mathcal{F}\backslash G$, pick an $x_F \in \Delta_F$. The points $z = \left(\sum_{F\in\mathcal{F}\backslash G} x_F\right) + e_i$ and $w = \left(\sum_{F\in\mathcal{F}\backslash G} x_F\right) + e_j$ are in $P_{\mathcal{F}}$ but $c^T z < c^T w$. Thus $\sum_{i\in[r]} x_i = |\mathcal{F}|$ is the only linear equation satisfied by all points of $P_{\mathcal{F}}$. ∎

In what follows, it will be useful to define $P_{\mathcal{F}}$ for $\mathcal{F} = \emptyset$ and $r > 0$ to be $0 \in \mathbb{R}^r$. The next proposition follows directly from the definition of $P_{\mathcal{F}}$.

Proposition 1.2. *Suppose that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ and there is a partition $[r] = I \cup J$ into subsets, so that $F \subseteq I$ for all $F \in \mathcal{F}_1$ and $F \subseteq J$ for all $F \in \mathcal{F}_2$. Then $P_{\mathcal{F}}$ is the Cartesian product $P_{\mathcal{F}_1} \times P_{\mathcal{F}_2}$.*

Corollary 1.3. *The dimension of the polytope $P_{\mathcal{F}}$ is given by $\dim(P_{\mathcal{F}}) = n - c$ where*

$$n = \left|\bigcup_{F\in\mathcal{F}} F\right| \in [r]$$

and c is the number of connected components of $\Delta_{\mathcal{F}}$.

From a more graph theoretic point of view we can also consider the following: Let $\Delta_1(\mathcal{F})$ be the one-dimensional skeleton of $\Delta_{\mathcal{F}}$.

Corollary 1.4. *The dimension of the polytope $P_{\mathcal{F}}$ is given by the number of edges in a spanning forest of $\Delta_1(\mathcal{F})$.*

A *face* of $P_{\mathcal{F}}$ is a subset of $P_{\mathcal{F}}$ on which a linear function is maximized. A vector $c = (c_1, \ldots, c_r) \in \mathbb{R}^r$ defines a partition $C = (C_1, C_2, \ldots, C_s)$ of $[r]$ into nonempty subsets, so that $c_{i_1} = c_{i_2}$ when i_1 and i_2 are in the same part of the partition, and $c_{i_1} < c_{i_2}$ whenever $i_1 \in C_{\ell_1}$, $i_2 \in C_{\ell_2}$, $\ell_1 < \ell_2$.

Proposition 1.5. *The face that maximizes $c^T x$ is the Minkowski sum of the simplices in the family*

$$\mathcal{F}^C := \{F \cap C_\ell \colon F \in \mathcal{F}, \ell = 1, \ldots, s, F \cap C_\ell \neq \emptyset, F \cap C_m = \emptyset \text{ for } m > \ell\}.$$

Proof. An often cited fact about Minkowski sums is that if the face on which $c^T x$ is maximized over P is G and the face on which $c^T x$ is maximized over Q is H, then the face on which $c^T x$ is maximized over the Minkowski sum $P + Q$ is $G + H$. The subset of Δ_F over which $c^T x$ is maximized is clearly $\operatorname{conv}(\{e_i \colon i \in F \cap C_\ell\})$, where ℓ is $\max\{j \colon C_j \cap F \neq \emptyset\}$. The proposition follows from this fact. ∎

By Corollary 1.3 and Proposition 1.5, the dimension of the face is determined by the number of connected components of the simplicial complex $\Delta_{\mathcal{F}C}$. If the face on which $c^T x$ is maximized is a facet, then $\Delta_{\mathcal{F}C}$ has one more connected component than $\Delta_{\mathcal{F}}$ and can be obtained from $\Delta_{\mathcal{F}}$ by splitting one of the components of $\Delta_{\mathcal{F}}$ in two. The coefficients of the vector c corresponding to C can be assumed to be 0 and 1. Therefore, all facets of $P_{\mathcal{F}}$ are of the form $\sum_{i\in D} x_i = t$ for some subset D of $[r]$ and integer t.

On the other hand, if the face maximizing $c^T x$ is an edge, then $\Delta_{\mathcal{F}C}$ has exactly one component of size two, say $\{i, j\}$, and otherwise all isolated elements. The corresponding face of $P_{\mathcal{F}}$ is an edge parallel to $e_i - e_j$. Vertices of $P_{\mathcal{F}}$ are points that maximize linear functions $c^T x$ in which all components of c are distinct. If $c_1 < c_2 < \cdots < c_r$ then component v_i of the vertex that maximizes $c^T x$ equals the number of sets F for which i is the largest element. From this we see as well that the vertices of $P_{\mathcal{F}}$ have integer coordinates (which, in itself is clear, since it is a Minkowski sum of lattice polytopes).

2. Minkowski Sum of a Fixed Number of Simplices

Suppose that $\mathcal{F}$ consists of k subsets $F_1, F_2, \ldots, F_k$ of $[r]$. We will for the most part write $\mathcal{F} = (F_1, F_2, \ldots, F_k)$ as an *ordered* k-tuple, since a lot will depend on the actual listing/order of the sets $F_1, \ldots, F_k$, although the combinatorics will not be effected by a different ordering of them. For each $i \in [r]$, define $N_{\mathcal{F}}(i) = \{j \in [k] \colon i \in F_j\}$. Let A be a subset of $[r]$ so that $N_{\mathcal{F}}(i_1) = N_{\mathcal{F}}(i_2)$ whenever i_1 and i_2 are in A. We would like to show how the combinatorial type of $P_{\mathcal{F}}$ can be inferred from that of $P_{\mathcal{F}'}$, where $\mathcal{F}'$ is obtained from $\mathcal{F}$ by replacing each appearance of A in a set $\mathcal{F}$ by the one-element set $m = \max(A)$. Afterward, we will restrict our attention to families in which all of the $N_{\mathcal{F}}(i)$ are distinct.

Proposition 2.1. *Suppose that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, and there is an $m \in [r]$ so that $F_i \cap F_j \subseteq \{m\}$ whenever $F_i \in \mathcal{F}_1$, $F_j \in \mathcal{F}_2$. Define $\mathcal{F}_1' = \{(F_i\backslash\{m\}) \cup \{r+1\} \colon m \in F_i \in \mathcal{F}_1\} \cup \{F_i \in \mathcal{F}_1 \colon m \notin F_i\}$. Then $P_{\mathcal{F}}$ has the same combinatorial type as the Cartesian product $P_{\mathcal{F}_1'} \times P_{\mathcal{F}_2}$.*

Proof. Let $[r+1] = A \cup B$ be a partition of $[r+1]$ for which $F_i \subseteq A$ for all $F_i \in \mathcal{F}_1'$ and $F_i \subseteq B$ for all $F_i \in \mathcal{F}_2$. The linear transformation $f \colon \mathbb{R}^{r+1} \to \mathbb{R}^r$ given by $f(x)_i = x_i$ if $i \neq m$, and $f(x)_m = x_m + x_{r+1}$ sends the affine space $\{x \in \mathbb{R}^{r+1} \colon \sum_{i\in A} x_i = |\mathcal{F}_1'|, \sum_{i\in B} x_i = |\mathcal{F}_2|\}$ onto $\{x \in \mathbb{R}^r \colon \sum_{i\in[r]} x_i = |\mathcal{F}|\}$. In particular, this means that $P_{\mathcal{F}}$ is an affine image of $P_{\mathcal{F}_1' \cup \mathcal{F}_2} = P_{\mathcal{F}_1'} \times P_{\mathcal{F}_2}$. ∎

Example 2.2. Let $\mathcal{F} = \{\{1,2\},\{2,3\}\}$, $\mathcal{F}_1 = \{\{1,2\}\}$, $\mathcal{F}_2 = \{\{2,3\}\}$. Then $\mathcal{F}_1' = \{\{1,4\}\}$ and $P_{\mathcal{F}_1' \cup \mathcal{F}_2} = P_{\mathcal{F}_1'} \times P_{\mathcal{F}_2}$ is the square conv$(\{(1,1,0,0), (1,0,1,0), (0,1,0,1), (0,0,1,1)\})$ which gets mapped to the rhombus $P_{\mathcal{F}} = \mathrm{conv}(\{(1,1,0,), (1,0,1), (0,2,0), (0,1,1)\})$. We will subsequently refer to this rhombus as $P(2)$.

Now let A be a subset of $[r]$ so that $N_{\mathcal{F}}(i_1) = N_{\mathcal{F}}(i_2)$ whenever i_1 and i_2 are in A. For each $F \in \mathcal{F}$ define $F' = (F\backslash A) \cup \{r+1\}$ if $A \subseteq F \in \mathcal{F}$ and $F' = F$ if $A \cap F = \emptyset$. Let $\mathcal{F}' = \{F' \colon F \in \mathcal{F}\}$. Consider the function $g_A \colon \mathbb{R}^{r+1} \to \mathbb{R}^r$ given by $g_A(x)_i = x_{r+1} x_i$ if $i \in A$, $g_A(x)_i = x_i$ if $i \in [r]\backslash A$.

Proposition 2.3. *The function g_A maps $P_{\{A\}} \times P_{\mathcal{F}'}$ onto $P_{\mathcal{F}}$. The restriction of g_A to $\{x \in P_{\{A\}} \times P_{\mathcal{F}'} : x_{r+1} \neq 0\}$ is one-to-one.*

Proof. Let $x \in P_{\{A\}} \times P_{\mathcal{F}'}$. Then there exist $x_A \in \Delta_A$ and $x_{F'} \in \Delta_{F'}$ for each $F' \in \mathcal{F}'$ so that $x = x_A + \sum_{F' \in \mathcal{F}'} x_{F'}$. Then $g_A(x) = \left(\sum_{F' \in \mathcal{F}'} (x_{F'})\right)_{r+1} x_A + \sum_{F' \in \mathcal{F}'} (x_{F'} - (x_{F'})_{r+1} e_{r+1}) = \sum_{F' \in \mathcal{F}'} \left[(x_{F'})_{r+1} x_A + x_{F'} - (x_{F'})_{r+1} e_{r+1}\right] = \sum_{F \in \mathcal{F}} x_F$, where each $x_F \in \Delta_F$. To show surjectivity, let $x = \sum_{F \in \mathcal{F}} x_F$, where each $x_F \in \Delta_F$. For each $F \in \mathcal{F}$, define $x_{F'} = x_F - \sum_{i \in A} (x_F)_i e_i + (\sum_{i \in A} (x_F)_i) e_{r+1}$. If $\sum_{i \in A} x_i > 0$, then let $(x_A)_i = (\sum_{i \in A} x_i)^{-1} x_i$ for all $i \in A$. If $\sum_{i \in A} x_i = 0$, let x_A be an arbitrary element of Δ_A. Then $x_F = g_A\left(x_A + \sum_{F' \in \mathcal{F}'} x_{F'}\right)$. If $g_A(x) = g_A(y)$ for $x, y \in \{x \in P_{\{A\}} \times P_{\mathcal{F}'} : x_{r+1} \neq 0\}$ then immediately $x_i = y_i$ for $i \notin A \cup \{r+1\}$. The requirement $\sum_{i \in A} x_i = \sum_{i \in A} y_i = 1$ implies $x_{r+1} = y_{r+1}$, and hence $x_i = y_i$ for $i \in A$. ∎

Let $c \in \mathbb{R}^{r+1}$ be a nonnegative vector. Let $C_A = \{i : c_i \geq c_j \text{ for all } j \in A\} \subseteq A$. Define a vector $c' \in \mathbb{R}^r$ by $c'_i = c_i$ if $i \notin A$, $c'_i = c_{r+1}$ if $i \in C_A$, and $c'_i = 0$ otherwise.

Proposition 2.4. *If Q is the face of $P_{\{A\}} \times P_{\mathcal{F}'}$ that maximizes $c^T x$, then $g_A(Q)$ is the face of $P_{\mathcal{F}}$ that maximizes $c'^T x$.*

Proof. Suppose that $x \in P_{\{A\}} \times P_{\mathcal{F}'}$. Then $c'^T g_A(x) = \sum_{i \in C_A} c_{r+1} x_{r+1} x_i + \sum_{i \notin A} c_i x_i \leq c_{r+1} x_{r+1} + \sum_{i \in [r] \setminus A} c_i x_i = c^T x - c_A$, with equality holding for $x \in Q$. ∎

Proposition 2.5. *If $|N_{\mathcal{F}}(i)| > 0$ for all $i \in A$, then the dimension of $P_{\{A\}} \times P_{\mathcal{F}'}$ equals the dimension of $P_{\mathcal{F}}$. If $|N_{\mathcal{F}}(i)| = 0$ for all $i \in A$, then the dimension of $P_{\mathcal{F}'}$ equals the dimension of $P_{\mathcal{F}}$.*

Proof. It is clear that in both cases, the simplicial complexes $\Delta_{\mathcal{F}}$ and $\Delta_{\mathcal{F}'}$ have the same number of components. ∎

Let $P_{\mathcal{F}''}$ be the face of $P_{\mathcal{F}}$ where $x_i = 0$ for all $i \in A$. Propositions 2.1 – 2.6 imply that the combinatorial type of $P_{\mathcal{F}}$ is that of $\Delta_A \times P_{\mathcal{F}'}$, except that (if $P_{\mathcal{F}''}$ is nonempty) the face $\Delta_A \times P_{\mathcal{F}''}$ is collapsed to a copy of $P_{\mathcal{F}''}$. In the case that $|A| = 2$, $P_{\mathcal{F}}$ is a *wedge* (see [8]) over $P_{\mathcal{F}'}$ with foot $P_{\mathcal{F}''}$. When $|A| > 2$, we can obtain $P_{\mathcal{F}}$ from $P_{\mathcal{F}'}$ by iterating the wedge construction, adding one element of A at a time.

Proposition 2.6. *For every vertex x of $P_{\mathcal{F}} \setminus P_{\mathcal{F}''}$ there is a unique $i \in A$ with $x_i > 0$. There are two kinds of edges of $P_{\mathcal{F}}$.*

(1) conv$(\{v, v + k(e_i - e_j)\})$, *where $i, j \in A$, k is a positive integer and v is a vertex of $P_{\mathcal{F}} \setminus P_{\mathcal{F}''}$.*
(2) conv$(\{v, v + k(e_i - e_j)\})$, *where k is a positive integer and v is a vertex of $P_{\mathcal{F}}$ for which there exists $(u, w) \in \Delta_A \times P_{\mathcal{F}'}$ so that $v = g_A(u, w)$ and* conv$(\{w, w + k(e_i - e_j)\})$ *is an edge of $P_{\mathcal{F}'}$.*

Proof. Every vertex x of $P_{\mathcal{F}}$ is the image under g_A of a vertex (u, w) of $\Delta_A \times P_{\mathcal{F}'}$. A vertex u of Δ_A has a unique nonzero coordinate. If $w_{r+1} > 0$ then $g_A(u, w)$ is in $P_{\mathcal{F}} \setminus P_{\mathcal{F}''}$. Every edge of $P_{\mathcal{F}}$ is the image under g_A of a pair (e, w) where e is an edge of Δ_A and w is a vertex of $P_{\mathcal{F}'}$, or a pair (u, f), where u is a vertex of Δ_A and f is an edge of $P_{\mathcal{F}'}$. Every edge of Δ_A is conv$(\{e_i, e_j\})$ for $i, j \in A$. ∎

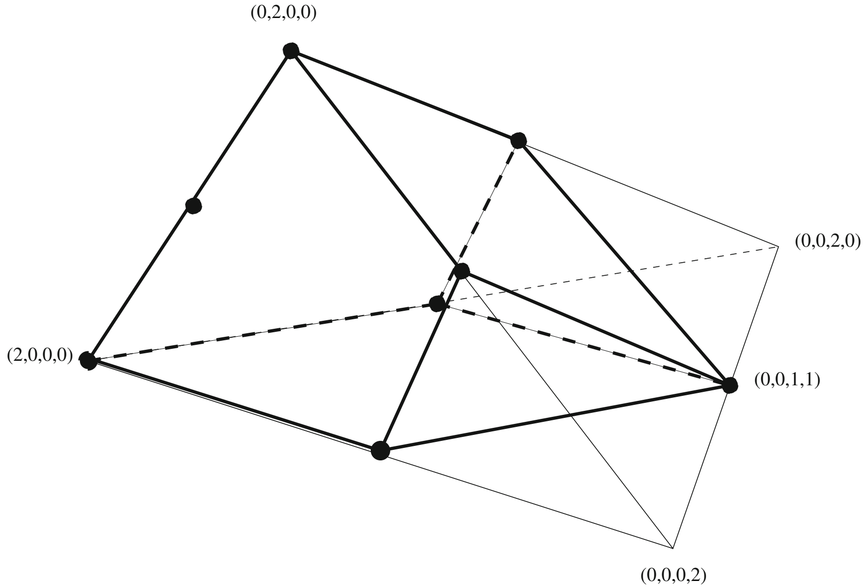

Figure 1: A sum of two triangles

Example 2.7. Consider the family $\mathcal{F} = (\{1,2,3\},\{1,2,4\})$ of subsets of $[4]$. Then $N_{\mathcal{F}}(i) = \{1,2\}$ for all i in $A = \{1,2\}$. The polytope $P_{\mathcal{F}}$ is drawn in Figure 1. The polytope $P_{\mathcal{F}'}$ is the rhombus that is the top face of the drawing. $P_{\mathcal{F}''}$ is the vertex $(0,0,1,1)$.

In applying Proposition 2.1, we consider first the case in which $\mathcal{F}$ consists of two sets, F and F'. In the special case where each of the sets $F \setminus F'$, $F \cap F'$, and $F' \setminus F$ has exactly one element, say 1, 2, and 3, respectively, then $F = \{1,2\}$ and $F' = \{2,3\}$, we have the rhombus $P(2)$ of Example 2.2.

We now argue that the generic Minkowski sum of two simplices roughly has the structure of such a rhombus if each of $F \setminus F'$, $F \cap F'$, and $F' \setminus F$ is nonempty.

By assigning the 1st, 2nd, and 3rd coordinate axis of $\mathbb{R}^3$ to these parts respectively, we can assign vertices of $P_{\mathcal{F}} = \Delta_F + \Delta_{F'}$ to the vertices of the rhombus of Example 2.2 the following way: A vertex $e_i + e_j$ of $P_{\mathcal{F}}$ is of type $(1,1,0)$ if $i \in F \setminus F'$ and $j \in F \cap F'$, of type $(0,2,0)$ if $i, j \in F \cap F'$, of type $(0,1,1)$ if $i \in F \cap F'$ and $j \in F' \setminus F$, and of type $(1,0,1)$ if $i \in F \setminus F'$ and $j \in F' \setminus F$. The following corollary describes the structure of a Minkowski sum of two standard simplices to be roughly that of the rhombus mentioned above.

Corollary 2.8. *If $F, F \subseteq [r]$ then the edges, or one-dimensional faces, of $P = \Delta_F + \Delta_{F'}$ are of the following types*:

(1) *Internal edges, where both the endpoints are of the same type $X \in \{(1,1,0), (0,2,0), (0,1,1), (1,0,1)\}$.*
(2) *Edges joining vertices of types X and Y, where X and Y are adjacent in $P(2)$.*

Proof. Each of the sets $F \setminus F'$, $F \cap F'$, and $F' \setminus F$ can play the role of the set A in Proposition 2.6. The two kinds of edges correspond to the two kinds of edges in the proposition. ∎

Theorem 2.9. *Let $F, F' \subseteq [r]$ and let u be a vertex of the polytope $P_{\mathcal{F}}$.*

(1) *If u is of type* $(1,1,0)$, $(0,2,0)$, *or* $(0,1,1)$, *then* $\deg(u) = |F \cup F'| - 1$.
(2) *If u is of type* $(1,0,1)$, *then* $\deg(u) = |F| + |F'| - 2$.

Proof. If u is of type $(0,2,0)$, say $u = 2e_i$, then u is adjacent to all $|F \cap F'| - 1$ other vertices of type $(0,2,0)$, and all type $(1,1,0)$ and $(0,1,1)$ vertices of the form $e_i + e_j$, where $j \in (F \setminus F') \cup (F' \setminus F)$. If u is of type $(1,1,0)$, say $u = e_i + e_j$, with $i \in F \setminus F'$ and $j \in F \cap F'$, then u is adjacent to two kinds of type $(1,1,0)$ vertices: $|F \cap F'| - 1$ vertices $e_i + e_k$ with $k \in (F \cap F') \setminus \{j\}$ and $|F \setminus F'| - 1$ vertices $e_k + e_j$ with $k \in F \setminus (F' \cap \{i\})$. Also, u is adjacent to $|F' \setminus F|$ type $(1,0,1)$ vertices $e_i + e_k$ with $k \in F' \setminus F$, and finally u is adjacent to the vertex $2e_j$. If u is of type $(1,0,1)$, say, $u = e_i + e_j$ with $i \in F \setminus F'$ and $j \in F' \setminus F$, then u is adjacent to $|(F \setminus F') \cup (F \setminus F')| - 2$ vertices of type $(1,0,1)$ obtained by replacing either e_i or e_j by an e_k for $k \in (F \setminus F') \cup (F \setminus F')$, and u is adjacent to $|F \cap F'|$ vertices of each type $(1,1,0)$ and $(0,1,1)$, obtained by replacing e_i or e_j by an e_k for $k \in F \cap F'$. ∎

Corollary 2.10. *Let $F, F' \subseteq [r]$ and $P = \Delta_F + \Delta_{F'}$.*

(1) *The number of vertices of P is* $|F| \cdot |F'| - |F \cap F'|(|F \cap F'| - 1)$.
(2) *The number of edges of P is given by*

$$\frac{1}{2}\Big[|F \setminus F'| \cdot |F' \setminus F|(|F| + |F'| - 2) \\ + |F \cap F'|(|F \cup F'| - 1)(|F \setminus F'| + |F' \setminus F| + 1)\Big].$$

Proof. The number of vertices of degree $|F| + |F'| - 2$ in P is $|F \setminus F'| \cdot |F' \setminus F|$. By Theorem 2.9 the remaining vertices of P all have degree $|F \cup F'| - 1$. The total number of edges is one half of the sum of the vertex degrees. ∎

Assuming that $F \cup F' = [r]$, then the maximum value of $|F| + |F'| - 2$ (provided $F \setminus F'$ and $F' \setminus F$ are nonempty) is $2r - 4$, which occurs when $F = [r-1]$ and $F' = [r] \setminus \{1\}$. Considering the distribution of the two possible degrees of $P = \Delta_F + \Delta_{F'}$, we have the following proposition:

Proposition 2.11. *Let $r \in \mathbb{N}$ be fixed. If $F, F' \subseteq [r]$ and $P = \Delta_F + \Delta_{F'}$ are of dimension $r - 1$, then the average degree $\overline{\deg}(P)$ satisfies*

$$r - 1 \le \overline{\deg}(P) < \frac{10}{9}(r-1).$$

Moreover, the lower bound is attained if and only if P is simple, that is if (i) $F \subseteq F'$, (ii) $F' \subseteq F$ *or* (iii) $|F \cap F'| = 1$. *Also,* $\overline{\deg}(P)/(r-1)$ *can become arbitrarily close to* $10/9$ *for large r.*

Proof. We introduce the variables x, y, and z by $x = |F \setminus F'|$, $y = |F' \setminus F|$, and $z = |F \cap F'|$. Here we have the boundary condition $x, y \geq 0$ and $x + y + z = r$, and since P is assumed to have dimension $r - 1$ we have $z \geq 1$ or $0 \leq x + y \leq r - 1$. By Corollary 2.10 we obtain that

$$\begin{aligned}
&\overline{\deg}(P) \\
&= 2\frac{|E(\Delta_1(\mathcal{F}))|}{|V(\Delta_1(\mathcal{F}))|} \\
&= \frac{|F \setminus F'| \cdot |F' \setminus F|(|F| + |F'| - 2) + |F \cap F'|(|F \cup F'| - 1)(|F \setminus F'| + |F' \setminus F| + 1)}{|F| \cdot |F'| - |F \cap F'|(|F \cap F'| - 1)} \\
&= \frac{xy(2r - 2 - x - y) + (r - 1)(r - x - y)(x + y + 1)}{(r - y)(r - x) - (r - x - y)(r - x - y - 1)}.
\end{aligned}$$

As a function of x and y we note that $\overline{\deg}(P) = \overline{\deg}(x, y)$ is symmetric, has the value of $r - 1$ on the boundary of the triangle bounded by $x = 0$, $y = 0$, and $x + y = r - 1$. By Theorem 2.9 the value $\overline{\deg}(x, y)$ is strictly larger than $r - 1$ inside the triangle. The maximum value $\overline{\deg}_{\max}(r)$ of $\overline{\deg}(x, y)$ occurs when $x = y = (r - 1)/3$, and we have $(10r - 13)/9 < \overline{\deg}_{\max}(r) < 10(r - 1)/9$, but $\overline{\deg}_{\max}(r) - (10r - 13)/9$ tends to zero when r tends to infinity. ∎

Remark 2.12. For any $\varepsilon > 0$ there is an r_0 such that for any $r \geq r_0$ we have

$$r - 1 \leq \overline{\deg}(P) < \frac{10r - 13}{9} + \varepsilon.$$

The *f-polynomial* $f_P(q)$ of a d-dimensional polytope P is $\sum_{i=0}^{d} f_i q^i$, where f_i is the number of i-dimensional faces of P. It is easy to see that $f_{P \times Q}(q) = f_P(q) f_Q(q)$. Postnikov [11] gave an elegant formula for $f_{P_\mathcal{F}}(q)$ in the case that $\mathcal{F}$ is a building set. If we assume that A, $\mathcal{F}'$, and $\mathcal{F}''$ are as in the discussion preceding Proposition 2.6, the f-polynomial can be decomposed as follows:

Proposition 2.13. $f_{P_\mathcal{F}}(q) = f_{\Delta_A}(q) f_{P_{\mathcal{F}'}}(q) - f_{\Delta_A}(q) f_{P_{\mathcal{F}''}}(q) + f_{P_{\mathcal{F}''}}(q)$.

In Example 2.7, $f_{P_\mathcal{F}}(q) = 7 + 11q + 6q^2 + q^3 = (2 + q)\left(4 + 4q + q^2\right) - (2 + q)(1) + 1$.

If $P_\mathcal{F}$ is the sum of two simplices Δ_F and $\Delta_{F'}$, then Proposition 2.1 shows that $P_\mathcal{F}$ has the same combinatorial type as $\Delta_F \times \Delta_{F'}$ when $|F \cap F'|$ is 0 or 1. This allows us to describe the f-polynomials of sums of two simplices quite easily, using the proposition with $A = F \cap F'$.

Corollary 2.14. *If $\mathcal{F} = \{F, F'\}$, where $F \cap F' = \{1, 2, \ldots, m\}$, then*

$$\begin{aligned}
f_{P_\mathcal{F}}(q) = {} & f_{\Delta_{F \cap F'}}(q) f_{\Delta_{(F \setminus F') \cup m} \times \Delta_{(F' \setminus F) \cup m}}(q) - f_{\Delta_{F \cap F'}}(q) f_{\Delta_{(F \setminus F')} \times \Delta_{(F' \setminus F)}}(q) \\
& + f_{\Delta_{(F \setminus F')} \times \Delta_{(F' \setminus F)}}(q).
\end{aligned}$$

We will now generalize the results that we obtained for the sum of two simplices to larger sums.

Definition 2.15. *For $k \in \mathbb{N}$, let $\mathcal{H}(k)$ be the family of k subsets of $\left[2^k - 1\right]$ so that for $i = 1, 2, \ldots, 2^k - 1$, $N_{\mathcal{H}(k)}(i)$ is the ith* (*in lexicographic order*) *nonempty subset of $[k]$. Then $P(k) := P_{\mathcal{H}(k)}$ is called the kth* master polytope.

Remark 2.16. There is no direct benefit to our choice of the lexicographic ordering on the subsets $[k]$ since *any* ordering of the subsets of $[k]$ will work just as well. Although Definition 2.15 of the master polytope does depend on the ordering of the subsets of $[k]$, any different ordering will clearly yield an equivalent polytope to the master polytope, obtained by a permutation of the coordinates. Hence, we will henceforth not distinguish between $P(k)$, as defined in Definition 2.15, and any other polytope obtained in the same way with a different ordering of the subsets of $[k]$.

Regarding the lexicographical ordering itself, it here denotes the order induced by the binary k-tuples corresponding to the subsets of $[k]$. For example, if $k = 2$ the lexicographic ordering of the nonempty subsets of $\{1,2\}$ is $\{1,2\} > \{1\} > \{2\}$, since the lexicographic order of the corresponding binary tuples is given by $(1,1) > (1,0) > (0,1)$. Hence, we have $\mathcal{H}(2) = (\{1,2\},\{1,3\})$ so that $N_{\mathcal{H}(2)}(1) = \{1,2\}$, $N_{\mathcal{H}(2)}(2) = \{1\}$, and $N_{\mathcal{H}(2)}(3) = \{2\}$.

Definition 2.17. *Let $\mathcal{F} = (F_1, \ldots, F_k)$ and u be a point in $P_{\mathcal{F}}$. Then $h_{\mathcal{F}}(u)$ is the point v in $P(k)$ for which, for $i = 1, 2, \ldots, 2^k - 1$, we set*

$$v_i = \begin{cases} \sum_{j:\, N_{\mathcal{F}}(j) = N_{\mathcal{H}(k)}(i)} u_j, & \text{if there is a } j \text{ with } N_{\mathcal{F}}(j) = N_{\mathcal{H}(k)}(i), \\ 0, & \text{otherwise}. \end{cases}$$

Theorem 2.18. *For $\mathcal{F} = (F_1, \ldots, F_k)$, the point $u \in P_{\mathcal{F}}$ is a vertex of $P_{\mathcal{F}}$ if and only if the following conditions are met*:

(1) *Each instance of $u_{i_\alpha} u_{i_\beta} > 0$, $N_{\mathcal{F}}(i_\alpha) = N_{\mathcal{F}}\left(i_\beta\right)$ implies that $i_\alpha = i_\beta$.*
(2) *$h_{\mathcal{F}}(u)$ is a vertex of the polytope $P(k)$.*

Proof. For a point u of $P_{\mathcal{F}}$, we first note that if $N_{\mathcal{F}}(i_\alpha) = N_{\mathcal{F}}\left(i_\beta\right)$ and $i_\alpha \neq i_\beta$, then u is a convex combination of v and w in $P_{\mathcal{F}}$ given by $v_{i_\alpha} = u_{i_\alpha} + u_{i_\alpha}$, $v_{i_\beta} = 0$, $v_i = u_i$ otherwise, $w_{i_\beta} = u_{i_\alpha} + u_{i_\beta}$, $w_{i_\alpha} = 0$, $w_i = u_i$ otherwise. Hence, the first condition is necessary for u to be a vertex of $P_{\mathcal{F}}$.

Let u be a point of $P_{\mathcal{F}}$ that satisfies the first condition. In this case the cardinality $|\{i \in [r] : u_i > 0\}|$ is at most $2^k - 1$. Also, if $u = u_{i_1} + \cdots + u_{i_m}$ where $m \in \left[2^k - 1\right]$ and each $u_{i_\ell} = a_{i_\ell} e_{i_\ell}$ where $a_{i_\ell} > 0$, then $h_{\mathcal{F}}(u)$ has the form $h_{\mathcal{F}}(u) = a_{i_1} e_{i'_1} + \cdots + a_{i_m} e_{i'_m}$, where i'_ℓ is the position in the lexicographic order of the subset $N_{\mathcal{F}}(i_\ell) \subseteq \left[2^k - 1\right]$.

If $c^T x$ is a linear function on $P_{\mathcal{F}}$ that is maximized at u, then we define the linear function c' by $c' := c_{i_1} x_{i_1} + \cdots + c_{i_m} x_{i_m}$. It is clear that c' is also maximized over $P_{\mathcal{F}}$ at u. This implies that the linear function $c_{i_1} x_{i'_1} + \cdots + c_{i_m} x_{i'_m}$ over $P(k)$ is maximized at $h_{\mathcal{F}}(u)$.

Assume that $h_{\mathcal{F}}(u)$ is a vertex of $P(k)$. Since $h_{\mathcal{F}}(u)$ is an extreme point of $P(k)$, there is a functional $c_{i_1} x_{i'_1} + \cdots + c_{i_m} x_{i'_m}$ on $P(k)$ that is maximized at $h_{\mathcal{F}}(u)$. In this case the corresponding functional $c_{i_1} x_{i_1} + \cdots + c_{i_m} x_{i_m}$ on $P_{\mathcal{F}}$ is maximized at u, showing that u is a vertex of $P_{\mathcal{F}}$. ∎

Let $X_1, \dots, X_h$ be the vertices of the polytope $P(k)$. Similar to the case when $k = 2$ in Corollary 2.8 we have the following theorem:

Theorem 2.19. *If $\mathcal{F} = (F_1, \dots, F_k)$, then the edges of $P_{\mathcal{F}}$ are of the following types*:

(1) *Internal edges, where both the endpoints are of type X_i for some $i \in \{1, \dots, m\}$.*

(2) *Edges joining vertices of types X_i and X_j, where X_i and X_j are adjacent in $P(k)$.*

Proof. We can partition $[r]$ into $\bigcup A_\ell$, where $A_\ell = \left\{ j \in [r] \colon N_{\mathcal{F}}(j) = N_{\mathcal{H}(k)}(\ell) \right\}$. Then $P_{\mathcal{F}}$ is the image of the Cartesian product $\prod_{A_\ell \subseteq [r]} \Delta_{A_\ell} \times P(k)$ under the composition of all of the maps g_{A_ℓ}, possibly followed by reordering the columns. An edge of the product corresponds to the product of an edge of one of the factors and vertices from the other factors, as in Proposition 2.6. ∎

Theorems 2.18 and 2.19 both reduce the structure of $P_{\mathcal{F}} \subseteq \mathbb{R}^r$ to considerations of the master polytope $P(k) \subseteq \mathbb{R}^{2^k - 1}$.

We conclude this section by investigating the polytope $P(3)$. Let

$$\mathcal{H} := (\{1,2,4,5\}, \{1,2,3,6\}, \{1,3,4,7\}).$$

Here we have that $N_{\mathcal{H}}(1) = \{1,2,3\}$, $N_{\mathcal{H}}(2) = \{1,2\}$, $N_{\mathcal{H}}(3) = \{2,3\}$, $N_{\mathcal{H}}(4) = \{1,3\}$, $N_{\mathcal{H}}(5) = \{1\}$, $N_{\mathcal{H}}(6) = \{2\}$, and $N_{\mathcal{H}}(7) = \{3\}$, so all of the nonempty subsets of $[3]$ are represented and hence $P(3) = P_{\mathcal{H}}$. (*Note*! Although $\mathcal{H}(3) = (\{1,2,3,4\}, \{1,2,5,6\}, \{1,3,5,7\})$, and $P(3) = P_{\mathcal{H}(3)}$ by Definition 2.15, the polytope $P_{\mathcal{H}}$ is equivalent to $P_{\mathcal{H}(3)}$ as remarked earlier.) The case of $k = |\mathcal{F}| = 3$ is the first interesting case for the mere reason that the polytope $P(3)$ does not have $2^{k(k-1)} = 64$ vertices, as was the case for $k = 2$, where the rhombus $P(2)$ has precisely $2^{k(k-1)} = 4$ vertices.

Example 2.20. The point $A = (0,1,1,1,0,0,0)$ in $P(3)$ is not a vertex, because $A = (B + C + D)/3$, where $B = (0,2,1,0,0,0,0)$, $C = (0,0,2,1,0,0,0)$, and $D = (0,1,0,2,0,0,0)$ and all the points B, C and D are points in the polytope $P(3)$.

Observation 2.21. The polytope $P(3)$ has 41 vertices in $\mathbb{R}^7$ given by the column vectors (without the last entry) in the following 7×10, 7×21, and 7×10 matrices. The last entry in each column is the degree of the vertex.

$$\begin{array}{cccccccccc}
3 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 \\
0 & 2 & 0 & 1 & 2 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6
\end{array}$$

$$
\begin{array}{ccccccccccccccccccccc}
2 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
6 & 6 & 6 & 6 & 8 & 6 & 8 & 6 & 6 & 6 & 8 & 6 & 6 & 8 & 6 & 6 & 6 & 6 & 6 & 8 & 8
\end{array}
$$

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
7 & 8 & 8 & 7 & 8 & 8 & 7 & 8 & 8 & 9
\end{array}
$$

These computations were verified using the computer program POLYMAKE [6]. Using POLYMAKE, we determined that the polytope $P(4)$ had vertices of all degrees in the set $\{14, 15, \dots, 28\}$ except for $\{16, 23, 26, 27\}$.

3. Function Representation of Integer Points of $P_{\mathcal{F}}$

The purpose of this section is to prove Theorem 3.5, a technical result that is useful for enumerating the vertices of $P_{\mathcal{F}}$. We have not found this specific result in the literature, but Proposition 3.4 is due to Edmonds [3] (see [7, Proposition 1.4]). In order to keep the presentation self-contained, we provide a detailed proof.

As in the previous section, we assume that $\mathcal{F} = (F_1, \dots, F_k)$, an ordered collection of k subsets of $[r]$. A function $f\colon [k] \to [r]$ that satisfies $f(i) \in F_i$ for each i will be called a *representation function* or a *rep-function* for short. For any function (and hence for a rep-function) $f\colon [k] \to [r]$ we define $u(f) := e_{f(1)} + \cdots + e_{f(k)}$. The following proposition is important and easily verified.

Proposition 3.1. *For functions $f, g\colon [k] \to [r]$, we have*

(1) $u(f) + u(g) = u(\min\{f, g\}) + u(\max\{f, g\})$.
(2) *If $f \neq g$ and $u(f) = u(g)$, then $u(f) \neq u(\min\{f, g\})$.*

If $u(f) = u(g)$, then we get by Proposition 3.1 that $u(f) = u(g) = (u(\min\{f, g\}) + u(\max\{f, g\}))/2$. Hence, if an integer point $u \in P_{\mathcal{F}}$ can be represented by two distinct functions f and g, then it is not a vertex of $P_{\mathcal{F}}$. The interesting part is the converse, which we will prove in the rest of this section. First we prove the following two lemmas:

Lemma 3.2. *If v is an integer point in $P_{\mathcal{F}}$ that is not a vertex of $P_{\mathcal{F}}$, and an edge of the inclusion-minimal face of $P_{\mathcal{F}}$ containing v is parallel to $e_{i_1} - e_{i_2}$, then $P_{\mathcal{F}}$ contains the points $v + e_{i_1} - e_{i_2}$ and $v - e_{i_1} + e_{i_2}$.*

Proof. If v is on a facet of $P_{\mathcal{F}}$ given by $\sum_{i\in T} x_i = t$ for some $T \subset [r]$ and integer t, then this equation is satisfied by all points in the inclusion-minimal face of $P_{\mathcal{F}}$ containing v, which means that i_1 and i_2 are either both in or both outside of T. Thus $v+e_{i_1}-e_{i_2}$ and $v-e_{i_1}+e_{i_2}$ will satisfy any linear equations that v satisfies. Furthermore, any inequality $\sum_{i\in T} x_i \le t$ that v satisfies strictly will also be satisfied by $v+e_{i_1}-e_{i_2}$ and $v-e_{i_1}+e_{i_2}$, because only one component is increased by 1 and one component is decreased by 1. ∎

Lemma 3.3. *If f and g are rep-functions and $u(g) = u(f)+te_{i_1}-te_{i_2}$ for $i_1 \neq i_2$ in $[r]$, then there exist rep-functions $f_1, f_2, \ldots f_{t-1}$ so that $u(f)+le_{i_1}-le_{i_2} = u(f_l)$ for $l = 1, 2, \ldots, t-1$.*

Proof. Define $G_{\mathcal{F}}$ to be the bipartite graph with vertex set $\{w_j\colon j \in [k]\} \cup \{v_i\colon i \in [r]\}$ and edges $\{(w_j, v_i)\}$ for all (i, j) with $i \in F_j$. For any rep-function h, let M_h be the set of edges (w_j, v_i) for which $h(j) = i$. For every $i \in [r] \setminus \{i_1, i_2\}$, the number of edges of M_g meeting v_i equals the number of edges of M_f meeting v_i. For every $j \in [k]$, w_j is met by exactly one edge from each of M_f and M_g. On the other hand, v_{i_1} is adjacent to t more edges of M_g than M_f, and v_{i_2} is adjacent to t more edges of M_f than M_g. Therefore, there exists a path P from v_{i_2} to v_{i_1} that alternates between edges of M_f and M_g. Let M^1 be the set of edges obtained from M_f by replacing the edges of M_f in the path by the edges of M_g in the path. Then, for $j = 1, 2, \ldots, k$, define $f_1(j) = i$, where (w_j, v_i) is an edge of M^1. Then $u(f_1) = u(f)+e_{i_1}-e_{i_2}$. We can continue this way to get $u(f_2), \ldots, u(f_{t-1})$. ∎

Proposition 3.4. *Every integer point v in $P_{\mathcal{F}}$ is $u(f)$ for some rep-function f.*

Proof. The proof is by induction on the dimension of the inclusion-minimal face of $P_{\mathcal{F}}$ containing v. From the first section, we know that the statement is true if v is a vertex. Suppose v is not a vertex. Suppose that there is an edge of the inclusion-minimal face of $P_{\mathcal{F}}$ containing v that is parallel to $e_{i_1}-e_{i_2}$. Then Lemma 3.2 allows us to build a segment parallel to $e_{i_1}-e_{i_2}$, containing v in its interior, and with endpoints on faces of $P_{\mathcal{F}}$ that are of lower dimension than the one containing v. By induction, the endpoints of the interval are $u(f)$ and $u(g)$ for some rep-functions f and g. Lemma 3.3 then gives us a rep-function for v. ∎

Theorem 3.5. *An integer point v in $P_{\mathcal{F}}$ is a vertex of $P_{\mathcal{F}}$ if and only if there is a unique rep-function f so that $u(f) = v$.*

Proof. Let v be an integer point in $P_{\mathcal{F}}$ that is not a vertex of $P_{\mathcal{F}}$. By Lemma 3.2 there are i_1 and i_2 in $[r]$ so that $P_{\mathcal{F}}$ contains the points $v-e_{i_1}+e_{i_2}$ and $v-e_{i_1}+e_{i_2}$. Let f and g be the rep-functions guaranteed by Proposition 3.4 for $v-e_{i_1}+e_{i_2}$ and $v-e_{i_1}+e_{i_2}$, respectively. Let $G_{\mathcal{F}}, M_f$, and M_g be as in the proof of Lemma 3.3. Then there are two edges of M_f adjacent to v_{i_2} that are not in M_g. Therefore, we can use these edges as initial edges in two different paths from v_{i_2} to v_{i_1} that alternate between edges of M_f and M_g. Swapping edges of M_f for edges of M_g along each of these alternating paths leads to two different rep-functions for v. ∎

The number of rep-functions for a given $\mathcal{F}$ is easy to count; it is $\Pi_{F\in\mathcal{F}}|F|$. By listing the rep-functions and the corresponding integer points $u(f)$, and striking out those $u(f)$ that appear more than once, one can list the vertices of $P_{\mathcal{F}}$. This was done by Sturmfels [1] for the polytopes $P(k)$ in the special cases of $k = 3, 4, 5$. He then conjectured that $P(3)$ had 41 vertices (consistent with Observation 2.21), $P(4)$ had 1015 vertices, and $P(5)$ had 59072 vertices.

4. Max-Degree As a Function of r and k

In this section we determine the function $d\colon \mathbb{N} \to \mathbb{N}$ given by

$$d(r) = \max_{\mathcal{F}} \{\deg_{\max}(P_{\mathcal{F}})\},$$

where the maximum is taken over all multi-subsets $(F_1, \ldots, F_k)$ of $\mathbb{P}([r])$, where $k \in \mathbb{N}$ can be any integer but r is fixed. Moreover, for each fixed $k \in \mathbb{N}$ we determine the function $d_k\colon \mathbb{N} \to \mathbb{N}$ defined by

$$d_k(r) = \max_{|\mathcal{F}| \le k} \{\deg_{\max}(P_{\mathcal{F}})\},$$

where the maximum here is taken over all multi-subsets $(F_1, \ldots, F_k)$ of $\mathbb{P}([r])$ where both k and r are fixed. Clearly $d(r) = \max_{k \in \mathbb{N}} \{d_k(r)\}$.

We start with the following lower bound for $d_k(r)$ and $d(r)$.

Lemma 4.1. *For $k, r \in \mathbb{N}$, we have $d_k(r) \ge k(r-k)$, and therefore, $d(r) \ge \lfloor r^2/4 \rfloor$.*

Proof. Let $k \in [r]$ and let for each $i \in [k]$, $F_i = \{i, k+1, k+2, \ldots, r\}$. Let $v = e_1 + e_2 + \cdots + e_k$. Let $1 \le i_2 \le k$ and $k+1 \le i_1 \le r$ and $c \in \mathbb{R}^r$ satisfy $c_i = 2$ if $i \in [k] \setminus \{i_2\}$, $c_{i_1} = c_{i_2} = 1$, and $c_i = 0$. Then $c^T x$ is maximized over $P_{\mathcal{F}}$ on the line segment from v to $v + (e_{i_1} - e_{i_2})$, so v and $v + (e_{i_1} - e_{i_2})$ are vertices of $P_{\mathcal{F}}$ and the line segment joining them is an edge. Therefore, $d_k(r) \ge k(r-k)$, so we have, in particular, that $d(r) \ge \lfloor r/2 \rfloor \lceil r/2 \rceil = \lfloor r^2/4 \rfloor$. ∎

Another polytope that has vertices of degree $\lfloor r^2/4 \rfloor$ is the graphical zonotope for the complete bipartite graph with $\lfloor r/2 \rfloor$ vertices on one side of the bipartition and $\lceil r/2 \rceil$ vertices on the other side. West [13] proved that the graphical zonotope for the complete bipartite graph has vertices of degree ℓ for all $r - 1 \le \ell \le \lfloor r^2/4 \rfloor$. On the other hand, every vertex of the polytope of Lemma 4.1 other than v has degree $r-1$.

For a fixed vertex u, each edge of P incident to u can be identified with a multiple of a difference $e_i - e_j$ of some pair of unit vectors, where $i, j \in [r]$ are distinct. Since the collection $\{\alpha(e_i - e_j) : \alpha \in \mathbb{N}\}$ is a set of parallel vectors, at most one multiple of $e_i - e_j$ can possibly correspond to an edge incident to u. From this alone we see that the maximum number of edges incident to u is at most $\binom{r}{2}$. However, more can be said:

For a vertex u of P, let $\vec{G}(u)$ be the directed graph with the vertex set $V\left(\vec{G}(u)\right) = [r]$ where a directed edge (i, j) is present if and only if $u + \alpha(e_i - e_j)$ is a neighbor of u in P for some $\alpha \in \mathbb{N}$.

Proposition 4.2. *For $r \in \mathbb{N}$ and $\mathcal{F} = (F_1, \ldots, F_k) \subseteq \mathbb{P}([r])$, the digraph $\vec{G}(u)$ is acyclic and its underlying graph $G(u)$ is simple and triangle-free.*

Proof. Assume there is a cycle $(i_1, i_2, \ldots, i_h)$ in $\vec{G}(u)$. Then $u, v_1, \ldots, v_h$ are all vertices of P, where $v_\ell = u + \alpha_\ell \left(e_{i_\ell} - e_{i_{\ell+1}}\right)$ $\left(\text{here we compute cyclically, so } e_{i_{h+1}} = e_{i_1}\right)$. This is however impossible since

$$\sum_{\ell=1}^{h} \frac{1}{\alpha_\ell} (v_\ell - u) = 0,$$

which means that there is no hyperplane containing u alone and having all the v_ℓ's strictly on one side of it. In particular, for $h = 2$, there are no directed 2-cycles and hence the underlying graph $G(u)$ is simple. Also for $h = 3$, there are no directed triangles in $\vec{G}(u)$ either.

Assume now that $G(u)$ has a triangle, which then does not correspond to a directed triangle in $\vec{G}(u)$, say, $v = u + \alpha(e_i - e_j)$, $v' = u + \beta(e_j - e_l)$, and $v'' = u + \gamma(e_i - e_l)$. In this case we have

$$v'' - u = \frac{\gamma}{\alpha}(v - u) + \frac{\gamma}{\beta}\left(v' - u\right),$$

which means that the vector $v'' - u$ is in the cone spanned by $v - u$ and $v' - u$. This contradicts the fact that uv'' is an edge of P. Hence, the underlying graph $G(u)$ of $\vec{G}(u)$ has no triangles. ∎

Theorem 4.3. *For $r \in \mathbb{N}$, we have $d(r) \leq \left\lfloor r^2/4 \right\rfloor$.*

Proof. The maximum degree of a vertex u of P is by Proposition 4.2 the maximum number of edges the simple triangle free graph $G(u)$ can have. By a theorem of Mantel [9] (a special case of Turán's Theorem [12]), the maximum number of edges of a simple triangle-free graph on r vertices is $\left\lfloor r^2/4 \right\rfloor$, hence the theorem holds. ∎

By Lemma 4.1 and Theorem 4.3 we have the following corollary:

Corollary 4.4. *For $r \in \mathbb{N}$, we have $d(r) = \left\lfloor r^2/4 \right\rfloor$.*

We now turn our attention to the computation of $d_k(r)$. Note that the Minkowski sum $P_{\mathcal{F}}$ provided in the proof of Lemma 4.1 that attains the overall maximum degree $d(r)$ has $k = |\mathcal{F}| = \lfloor r/2 \rfloor$. Therefore, when computing $d_k(r)$ we can assume $1 \leq k \leq r/2$.

First we need a variation of the theorem by Mantel [9].

Theorem 4.5. *Let $n \in \mathbb{N}$ and $1 \leq k \leq n/2$. If G is a triangle free simple graph on n vertices with a vertex cover of cardinality at most k, then $|E(G)| \leq k(n-k)$. Moreover, if $|E(G)| = k(n-k)$, then G is a complete bipartite graph with parts of cardinalities k and $n-k$.*

Proof. For $n \in \{1, 2\}$ the theorem is trivial. We proceed by induction and assume that G is a triangle free simple graph on $n > 2$ vertices with a vertex cover of cardinality at most k, and that $|E(G)|$ is the maximum number of edges for such graphs. Let $uv \in E(G)$ be an edge and since either u or v is in the vertex cover U of size k, we assume that $u \in U$. Since G is triangle-free, the set of neighbors $N(u)$ and $N(v)$ are disjoint. Let $G' = G - \{u, v\}$ be the simple graph obtained from G by removing the vertices u and v from G. By the disjointness of $N(u)$ and $N(v)$ we have $|E(G)| = |E(G')| + d(u) + d(v) - 1$.

Assume first that $v \in U$. In this case G' has a vertex cover of cardinality at most $k-2$, and by the induction hypothesis we have $|E(G)| = |E(G')| + d(u) + d(v) - 1 \leq (k-2)[(n-2)-(k-2)] + n - 1 < k(n-k)$.

Now assume that $v \notin U$. In this case G' has a vertex cover of cardinality at most $k-1$, and by the induction hypothesis we have $|E(G)| = |E(G')| + d(u) + d(v) - 1 \leq (k-1)[(n-2)-(k-1)] + n - 1 = k(n-k)$. Also by the induction hypothesis, $|E(G)| = k(n-k)$ can hold if and only if G' is a complete bipartite graph with parts of cardinalities $k-1$ and $n-k-1$, and $d(u)+d(v) = n$ $\big($i.e., $N(u) \cup N(v) = V(G)\big)$, which means that $|E(G)| = k(n-k)$ can hold if and only if $N(v) = U$ and $N(u) = V(G) \setminus U$, that is, G is a complete bipartite graph with parts of sizes k and $n-k$. This completes the proof. ∎

From Theorem 4.5 we obtain the following corollary:

Corollary 4.6. *For $r \in \mathbb{N}$ and $k \in \{1, \ldots, \lfloor r/2 \rfloor\}$, we have $d_k(r) = k(r-k)$.*

Proof. Consider a vertex u of $P_{\mathcal{F}}$. Then u can be represented uniquely as $u = e_{i_1} + \cdots + e_{i_k}$ with $i_j \in F_j$ for $j = 1, \ldots, k$ (note that some indices might coincide). As noted before, a neighbor v of u in P must have the form $v = u + \alpha(e_i - e_j)$ for some $\alpha \in \mathbb{N}$, and $i \in [r]$ and $j \in \{i_1, \ldots, i_k\}$. Since each directed edge $(i, j) \in V\left(\vec{G}(u)\right)$ has its head in $\{i_1, \ldots, i_k\}$, of cardinality at most k, the underlying graph $G(u)$ has a vertex cover of size at most k. Hence by Theorem 4.5 $G(u)$ has at most $k(r-k)$ edges.

In the proof of Lemma 4.1, an example of $P_{\mathcal{F}}$ with $|\mathcal{F}| \leq k$ and a vertex of degree $k(r-k)$ was given. This completes the argument. ∎

5. Simple Vertices

A *simple* vertex of a polytope is a vertex that is adjacent to exactly d other vertices of the polytope, where d is the dimension of the polytope. If $\mathcal{F}$ is a collection of distinct two-element sets, i.e., $P_{\mathcal{F}}$ is a graphical zonotope, then it is known from Shannon's theorem (see [14, p. 208]) that $P_{\mathcal{F}}$ has at least $2|\mathcal{F}|$ simple vertices. The family $\mathcal{F} = (\{1,2\}, \{1,3\}, \{1,2,3\})$, for which $P_{\mathcal{F}}$ is a pentagon, shows that this zonotopal theorem does not hold for more general Minkowski sums of simplices.

West [13] points out that simple vertices for graphical zonotopes can be obtained from depth first searches (DFS) on the graph. We will generalize this to set systems other than graphs and show that there are at least $d+1$ simple vertices, where d is the dimension of the polytope.

Let $J \subseteq [r]$. In what follows $\mathcal{F} \setminus J$ will denote the subcollection of $\mathcal{F}$ consisting of those sets whose intersection with J is empty.

Definition 5.1. *If $\mathcal{F}$ consists of a single set F, then for every $j \in F$, the vertex e_j of $P_{\mathcal{F}}$ will be called a DFS vertex with root j. In general, if $\mathcal{F}$ is connected, then a vertex v of $P_{\mathcal{F}}$ is called a DFS vertex with root j if*

(1) $v_j = |N_{\mathcal{F}}(j)| > 0$

(2) *If $\mathcal{F} \setminus \{j\}$ is nonempty and is the union of connected components $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_t$, then $v = v_j e_j + w^1 + w^2 + \cdots + w^t$, where, for all $k \in [t]$, w^k is a DFS vertex of $P_{\mathcal{F}_k}$ with root j_k so that $\{j_k, j\} \subseteq F$ for some $F \in \mathcal{F}$.*

Example 5.2. Let $\mathcal{F} = (\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,3,4,5\})$. The point $v = (0,1,0,1,2)$ is a DFS vertex with root 5, because $v = 2e_5 + w$, where w is a DFS vertex of $\mathcal{F} \setminus \{5\}$ with root 4. Note that $\mathcal{F} \setminus \{5\}$ is the set system of Example 2.20. On the other hand, $v' = (0,0,1,1,2)$ is not a DFS vertex. The root for v' would have to be 5. Then the root of $w' = v' - 2e_5$ would have to be 3 or 4, but $\{3,4\}$ is not contained in any set of $\mathcal{F} \setminus \{5\}$. This implies that w' cannot be decomposed further.

Note that conditions 1 and 2 of the definition of DFS vertex and the connectivity of $\mathcal{F}$ imply that the root is unique, since if a DFS vertex had two roots, say, i and j, then both $v_i = |N_{\mathcal{F}}(i)|$ and $v_j = |N_{\mathcal{F}}(j)|$, which is impossible. We state this formally:

Proposition 5.3. *The root of a DFS vertex is unique.*

For a DFS vertex v of a connected family $\mathcal{F}$, we define the directed graph $\vec{\Gamma}(v)$ with vertex set $[r]$ recursively as follows: If $\mathcal{F}$ consists of a single set F, and $v = e_j$ for some $j \in F$, then $\vec{\Gamma}(v)$ contains edges from j to all the other elements of F. Otherwise, if j is the root of v, $\vec{\Gamma}(v)$ contains edges from j to each of the roots of the DFS vertices of the connected components of $\mathcal{F} \setminus \{j\}$. The digraph $\vec{\Gamma}(v)$ also contains edges from j to every i for which $N_{\mathcal{F}}(i) \subseteq N_{\mathcal{F}}(j)$. After this definition has been applied recursively, we see that every vertex other than j is the head of exactly one directed edge of $\vec{\Gamma}(v)$ and that $\vec{\Gamma}(v)$ is a tree. If $|F| = 2$ for every $F \in \mathcal{F}$, then $\vec{\Gamma}(v)$ is a depth-first search tree, hence the name DFS vertex.

Proposition 5.4. *If v is a DFS vertex, then $\vec{\Gamma}(v)$ is $\vec{G}(v)$, the digraph of Proposition 4.2, with all directed edges reversed.*

Since $\vec{\Gamma}(v)$ is a tree with at most $r-1$ edges, Proposition 5.4 implies that a DFS vertex is a simple vertex.

Proof of Proposition 5.4. Suppose (k,l) is an edge in $\vec{\Gamma}(v)$. Let $a = (a_1, a_2, \ldots, a_r)$ be a permutation of $[r]$ that is an extension of the partial order defined by $\vec{\Gamma}(v)$, that is, if there is a directed path in $\vec{\Gamma}(v)$ from s to t, then $a_s > a_t$. Then it is clear that $a^T x$ is maximized over $P_{\mathcal{F}}$ at v. We can assume that the permutation a has been chosen so that $a_k = a_l + 1$. There is a subcollection $\mathcal{G}$ of $\mathcal{F}$ consisting of the sets that contain only elements of $[r]$ that can be reached from k by a directed path of $\vec{\Gamma}(v)$. Let $m = \left|N_{\mathcal{G}}(k) \cap N_{\mathcal{G}}(l)\right|$, and consider the point $w = v + m(e_l - e_k)$. Let a' be obtained from a by interchanging a_k and a_l. Then $a'^T x$ is maximized over $P_{\mathcal{F}}$ at w, so w is a vertex of $P_{\mathcal{F}}$. Furthermore, if we let $a'' = \frac{1}{2}(a + a')$, then the line segment from v to w is the subset of $P_{\mathcal{F}}$ on which $a''^T x$ is maximized over $P_{\mathcal{F}}$, so (l,k) is an edge of $\vec{G}(v)$.

To show that the reversed edges of $\vec{G}(v)$ are contained in $\vec{\Gamma}(v)$, suppose that (l,k) is an edge of $\vec{G}(v)$. Then $v_k > 0$, so k is one of the vertices that is a tail of an edge of $\vec{\Gamma}(v)$. If there is a directed path $(k = i_1, i_2, \ldots, i_t = l)$ with $t > 2$ in $\vec{\Gamma}(v)$, then the vector $e_l - e_k = (e_{i_2} - e_{i_1}) + (e_{i_3} - e_{i_2}) + \cdots + \left(e_{i_t} - e_{i_{t-1}}\right)$ is in the cone generated by $e_{i_2} - e_{i_1}, e_{i_3} - e_{i_2}, \ldots, e_{i_t} - e_{i_{t-1}}$. Because these latter vectors correspond to edges of

$P_{\mathcal{F}}$ leaving v, the vector $e_l - e_k$ is not parallel to an edge of $P_{\mathcal{F}}$ leaving v, so (l,k) is not an edge of $\vec{G}(v)$. If there is a directed path $(l = i_1, i_2, \ldots, i_t = k)$ in $\vec{\Gamma}(v)$, then the reversed path appears in $\vec{G}(v)$ which together with the edge (l,k) makes a directed cycle. If k and l are not contained in a directed path of $\vec{\Gamma}(v)$, let J be the set of elements of $[r]$ from which there are directed paths in $\vec{\Gamma}(v)$ to both k and l. Then k and l are in different components of $\mathcal{F} \setminus J$, so there is no edge in $P_{\mathcal{F}\setminus J}$ in the direction $e_l - e_k$. Therefore, the only way for (l,k) to be an edge of $\vec{G}(v)$ is for (k,l) to be an edge of $\vec{\Gamma}(v)$. ∎

Proposition 5.5. *If $\mathcal{F}$ is a building set, i.e., has the property that $F_1 \cap F_2 \neq \emptyset$ implies $F_1 \cup F_2 \in \mathcal{F}$ for all $F_1, F_2 \in \mathcal{F}$, then every vertex of $P_{\mathcal{F}}$ is a DFS vertex.*

Proof. Assume that $\mathcal{F}$ is connected. Suppose v is a vertex of $P_{\mathcal{F}}$ that maximizes $c^T x$. Let $c_m := \max\{c_i \colon i \in [r], |N_{\mathcal{F}}(i)| > 0\}$. Then $v_m = |N_{\mathcal{F}}(m)|$ and we can write $v = v_m e_m + w^1 + w^2 + \cdots + w^t$, where, for all $k \in [t]$, w^k is a vertex of $P_{\mathcal{F}_k}$. By induction, we can assume that each w_k is a DFS vertex of $P_{\mathcal{F}_k}$, because each $\mathcal{F}_k$ is a building set. The final condition, that $\{m_k, m\} \subseteq F$ for some $F \in \mathcal{F}$, for the root m_k of each DFS vertex w_k, follows from connectivity and the building set property of $\mathcal{F}$. ∎

Proposition 5.6. *If $\mathcal{F}$ is connected and $\bigcup_{F\in\mathcal{F}} = [r]$, then each $j \in [r]$ is a root of some DFS vertex of $P_{\mathcal{F}}$.*

Proof. By Definition 5.1 it is clear that we can for each $j \in [r]$ recursively obtain at least one DFS vertex v of $P_{\mathcal{F}}$ with root j. ∎

By Propositions 5.3, 5.4, and 5.6 we have the following corollary:

Corollary 5.7. *If $\mathcal{F}$ is connected, and $\bigcup_{F\in\mathcal{F}} = [r]$, then $P_{\mathcal{F}}$ has at least r simple vertices.*

By Corollary 1.3 we therefore have our main conclusion of this section.

Corollary 5.8. *If $P_{\mathcal{F}}$ has dimension d, then $P_{\mathcal{F}}$ has at least $d+1$ simple vertices.*

Proof. Suppose that $\mathcal{F}$ is the disconnected union of components $\mathcal{F}_1$ and $\mathcal{F}_2$, and that the dimension of $P_{\mathcal{F}_1}$ is d_1 and the dimension of $P_{\mathcal{F}_2}$ is d_2. If v_1 is a simple vertex of $P_{\mathcal{F}_1}$ and v_2 is a simple vertex of $P_{\mathcal{F}_2}$, then $v_1 + v_2$ is a simple vertex of $P_{\mathcal{F}}$. The dimension of $P_{\mathcal{F}}$ is $d_1 + d_2$ and $(d_1+1)(d_2+1) \geq d_1 + d_2 + 1$. ∎

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