Annals of Combinatorics

On Triangulations with High Vertex Degree

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Abstract. We solve three enumerative problems concerning the families of planar maps. More precisely, we establish algebraic equations for the generating function of loopless triangulations in which all vertices have degree at least d, for a certain value d chosen in $\{3, 4, 5\}$.

The originality of the problem lies in the fact that degree restrictions are placed both on vertices and faces. Our proofs first follow Tutte's classical approach: We decompose maps by deleting the root-edge and translate the decomposition into an equation satisfied by the generating function of the maps under consideration. Then we proceed to solve the equation obtained using a recent technique that extends the so-called *quadratic method*.

Keywords: enumeration, asymptotic, triangulation, degree, catalytic variable

1. Introduction

The enumeration of planar maps (or *maps* for short) has received a lot of attention in the combinatorists community for nearly fifty years. This field of research, launched by Tutte, was originally motivated by the four-color conjecture. Tutte and his students considered a large number of map families corresponding to various constraints on face- or vertex-degrees. These seminal works, based on elementary decomposition techniques allied to a generating function approach, gave rise to many explicit results [20, 27–30]. Fifteen years later, some physicists became interested in the subject and developed their own tools [3, 11, 26] based on matrix integrals (see [34] for an introduction). Their techniques proved very powerful for map enumeration [7, 12]. More recently, a bijective approach based on *conjugacy classes of trees* emerged providing new insights on the subject [5, 6, 22, 25].

However, when one considers a map family defined by both face- and vertex-constraints, each of the above mentioned methods seems relatively ineffective and very few enumerative results are known. There are two major exceptions. First, and most importantly for this paper, the enumeration of loopless *triangulations* (faces have degree 3) in which all vertices have degree at least 3 and of 3-connected triangulations in which all vertices have degree at least 4 were performed by Gao and Wormald using

a compositional approach [17]. More recently, the enumeration of all *bipartite* maps (faces have an even degree) according to the degree distribution of the vertices was accomplished using conjugacy classes of trees [6, 24]. This result includes as a special case the enumeration of bipartite *cubic* maps (vertices have degree 3) performed by Tutte via a generating function approach [29, 31].

In this paper, we consider loopless triangulations in which all vertices have degree at least d, for a certain value d chosen in $\{3, 4, 5\}$. We establish algebraic equations for the generating function of each of these families. We also give the asymptotic behavior of the number of maps in each family. It is well-known that there is no triangulation in which all vertices have degree at least 6 (we shall prove this fact in Section 2). Hence, we have settled the problem of counting triangulations with 'high' vertex degree entirely.

As mentioned above, the loopless triangulations in which all vertices have degree at least 3 have already been enumerated by Gao and Wormald [17]. Our proof differs from theirs. Let us also mention that several families of triangulations defined by connectivity constraints have been enumerated, for instance: the general triangulations [16], the loopless triangulations (i.e., non-separable triangulations) [20, 22], the 3-connected triangulations (i.e., triangulations without multiple edges) [23, 27], the 4-connected triangulations [9] and the 5-connected triangulations [18]. Observe that the vertices of k-connected triangulations have degree at least k (except for the degenerated case of the triangle K_3). However, there is no equivalence between connectivity constraints and vertex-degree constraints. In the present paper, we shall focus on loopless triangulations but our approach can also be adapted to some other families of triangulations, in particular to general triangulations as well as 3-connected ones.

Our proofs first follow Tutte's classical approach, which consists in translating the decomposition obtained by deletion of the root-edge into a functional equation satisfied by the generating function. It is not clear at first sight why this approach should work here. As a matter of fact, finding a functional equation for triangulations with vertex degree at least 5 turns out to be rather complicated. But it eventually works if some of the constraints are relaxed at this stage of the solution. Our decomposition scheme requires to consider the set of near-triangulations and to take into account, beside the size of the map, the degree of its root-face. Consequently, in order to write a functional equation, we need to consider a *bivariate* generating function. We end up with an equation for the (bivariate) generating function in which the variable counting the degree of the root-face cannot be trivially eliminated. We then use a recent generalization of the *quadratic method* to get rid of the extra variable and compute an algebraic equation characterizing the *univariate* generating function (see [10] and [19, Section 2.9] for the quadratic method and [4] for its generalization).

This paper is organized as follows. In Section 2, we recall some definitions on planar maps and introduce the main notations. In Section 3, we recall the classic decomposition scheme due to Tutte (by deletion of the root-edge). We illustrate this scheme on the set of unconstrained non-separable near-triangulations. In Section 4, we apply the same decomposition scheme to the sets of near-triangulations in which any internal vertex has degree at least $3, 4, 5$. We obtain functional equation in which the variable x counting the degree of the root-face can not be trivially eliminated. In Section 5, we use techniques generalizing the quadratic method in order to get rid of the variable x.

We obtain algebraic equations for triangulations in which *any vertex not incident to the root-edge* has degree at least 3, 4, 5. In Section 6, we give algebraic equations for triangulations in which *any vertex* has degree at least 3, 4. Lastly, in Section 7 we study the asymptotic behavior of the number of maps in each family.

2. Preliminaries and Notations on Maps

We begin with some vocabulary on maps. A map is a proper embedding of a connected graph into the two-dimensional sphere, considered up to continuous deformations. A map is *rooted* if one of its edges is distinguished as the *root-edge* and attributed an orientation. Unless otherwise specified, all maps under consideration in this paper are rooted. The face at the right of the root-edge is called the *root-face* and the other faces are said to be *internal*. Similarly, the vertices incident to the root-face are said to be *external* and the others are said to be *internal*. Graphically, the root-face is usually represented as the infinite face when the map is projected on the plane (see Figure 1). The endpoints of the root-edge are distinguished as its *origin* and *end* according to the orientation of the root-edge. A map is a *triangulation* (*near-triangulation*, respectively) if all its faces (all its internal faces, respectively) have degree 3. For instance, the map of Figure 1 is a near-triangulation with root-face of degree 4. Lastly, a map is *non-separable* if it is loopless and 2-connected (the deletion of a vertex does not disconnect the map). For instance, the map in Figure 1 is non-separable. Observe that for a triangulation it is equivalent to be loopless or non-separable but this is not true for near-triangulations.
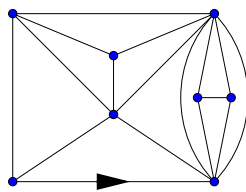

Figure 1: A non-separable near-triangulation.

In what follows, we enumerate 3 families of rooted non-separable triangulations. We recall some basic facts about these maps.

- By definition, a non-separable triangulation has no loop. Therefore, the faces of non-separable triangulations are always homeomorphic to a triangle: They have three distinct vertices and three distinct edges.
- Consider a triangulation with f faces, e edges, and v vertices. Given the incidence relation between edges and faces, we have $2e = 3f$. Hence, the number of edges of a triangulation is a multiple of 3. Moreover, given the Euler relation ($v - e + f = 2$), we see that a triangulation with $3n$ edges has $2n$ faces and $n + 2$ vertices.
- Observe that a non-separable map (not reduced to an edge) cannot have a vertex of degree one. Let us now prove, as promised, that *any triangulation has a vertex of degree less than* 6. Moreover, we prove that *this vertex can be chosen not to be incident to the root-edge*. Indeed, if all vertices not incident to the root-edge

have degree at least 6 the incidence relation between vertices and edges gives $2e \geq 6(v-2)+4$. This contradicts the fact that triangulations with $e = 3n$ edges have $v = n+2$ vertices. This property shows that, if one considers the sets of non-separable triangulations with vertex degree at least d, *the only interesting values of d are $d = 2$* (which corresponds to unconstrained non-separable triangulations) *and $d = 3, 4, 5$.*

Let $\mathbf{S}$ be the set of non-separable rooted *near-triangulations*. By convention, we exclude the map reduced to a vertex from $\mathbf{S}$. Thus, the smallest map in $\mathbf{S}$ is the map reduced to a straight edge (see Figure 2). This map is called the *link-map* and is denoted L. The vertices of other maps in $\mathbf{S}$ have degree at least 2. We consider three sub-families $\mathbf{T}, \mathbf{U}, \mathbf{V}$ of $\mathbf{S}$. The set $\mathbf{T}$ ($\mathbf{U}$ and $\mathbf{V}$, respectively) is the subset of non-separable near-triangulations in which any *internal* vertex has degree at least 3 (4 and 5, respectively). For each of the families $\mathbf{W} = \mathbf{S}, \mathbf{T}, \mathbf{U}, \mathbf{V}$, we consider the bivariate generating function $\mathbb{W}(x, z)$, where z counts the size (the number of edges) and x the degree of the root-face minus 2. That is to say, $\mathbb{W}(x) \equiv \mathbb{W}(x, z) = \sum_{n,d} a_{n,d} x^d z^n$ where $a_{n,d}$ is the number of maps in $\mathbf{W}$ with size n and root-face of degree $d + 2$. For instance, the link-map L, which is the smallest map in all our families, has contribution z to the generating function. Therefore, $\mathbb{W}(x) = z + o(z)$. Since the degree of the root-face is bounded by two times the number of edges, the generating function $\mathbb{W}(x, z)$ is a power series in the main variable z with polynomial coefficients in the secondary variable x. For each family $\mathbf{W} = \mathbf{S}, \mathbf{T}, \mathbf{U}, \mathbf{V}$, we will characterize the generating function $\mathbb{W}(x)$ as the unique power series solution of a functional equation (see Equation 3.1 and Propositions 4.1, 4.2, and 4.3).

Figure 2: The link-map L.

We also consider the set $\mathbf{F}$ of non-separable rooted *triangulations* and three of its subsets $\mathbf{G}, \mathbf{H}, \mathbf{K}$. The set $\mathbf{G}$ ($\mathbf{H}$ and $\mathbf{K}$, respectively) is the subset of non-separable triangulations in which any vertex *not incident to the root-edge* has degree at least 3 (4 and 5, respectively). As observed above, the number of edges of a triangulation is always a multiple of 3. To each of the families $\mathbf{L} = \mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{K}$, we associate the univariate generating function $\mathbb{L}(t) = \sum_n a_n t^n$ where a_n is the number of maps in $\mathbf{L}$ with $3n$ edges ($2n$ faces and $n+2$ vertices). For each family we will give an algebraic equation satisfied by $\mathbb{L}(t)$ (see Equation 3.3 and Theorems 5.1, 5.2, and 5.3).

There is a simple connection between the generating functions $\mathbb{F}(t)$ $\big(\mathbb{G}(t), \mathbb{H}(t),$ and $\mathbb{K}(t)$, respectively$\big)$ and $\mathbb{S}(x)$ $\big(\mathbb{T}(x), \mathbb{U}(x)$, and $\mathbb{V}(x)$, respectively$\big)$. Consider a non-separable near-triangulation distinct from L rooted on a digon (i.e., the root-face has degree 2). Deleting the external edge that is not the root-edge produces a non-separable triangulation (see Figure 3). This classical mapping (see e.g., [15, 32]) establishes a one-to-one correspondence between the set of triangulations $\mathbf{F}$ ($\mathbf{G}, \mathbf{H}$, and $\mathbf{K}$, respectively) and the set of near-triangulations in $\mathbf{S} - \{L\}$ $\big(\mathbf{T} - \{L\}, \mathbf{U} - \{L\},$ and $\mathbf{V} - \{L\},$ respectively$\big)$ rooted on a digon.

For $\mathbf{W} \in \{\mathbf{S}, \mathbf{T}, \mathbf{U}, \mathbf{V}\}$, the power series $\mathbb{W}(0) \equiv \mathbb{W}(0, z)$ is the generating function of near-triangulations in $\mathbf{W}$ rooted on a digon. Given that the link-map has contribution
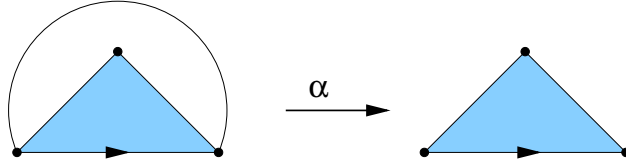
Figure 3: Near-triangulations rooted on a digon and triangulations.

z, we have

$$\mathbb{S}(0) = z + z\mathbb{F}\left(z^3\right), \ \mathbb{T}(0) = z + z\mathbb{G}\left(z^3\right),$$

$$\mathbb{U}(0) = z + z\mathbb{H}\left(z^3\right), \ \mathbb{V}(0) = z + z\mathbb{K}\left(z^3\right).$$

(2.1)

3. The Decomposition Scheme

In the following, we adopt Tutte's classical approach for enumerating maps. That is, we decompose maps by deleting their root-edge and translate this combinatorial decomposition into an equation satisfied by the corresponding generating function. In this section we illustrate this approach on unconstrained non-separable triangulations (this was first done in [20]). We give all the details on this simple case in order to prepare the reader to the more complicated cases of constrained non-separable triangulations treated in the next section.

We recall that **S** denotes the set of non-separable near-triangulations and $\mathbb{S}(x) = \mathbb{S}(x, z)$ the corresponding generation function. As observed before, the link-map L has contribution z to the generating function $\mathbb{S}(x)$. We decompose the other maps by deleting the root-edge. Let M be a non-separable triangulation distinct from L. Since M is non-separable, the root-edge of M is not an isthmus. Therefore, the face at the left of the root-edge is internal, hence has degree 3. Since M has no loop, the three vertices incident to this face are distinct. We denote by v the vertex not incident to the root-edge. When analyzing what can happen to M when deleting its root-edge, one is led to distinguish two cases (see Figure 4).
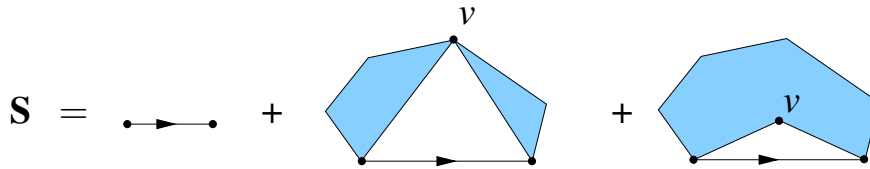

Figure 4: Decomposition of non-separable near-triangulations.

Either the vertex v is incident to the root-face, in which case the map obtained by deletion of the root-edge is separable (see Figure 5). Or v is not incident to the root-face and the map obtained by deletion of the root-edge is a non-separable near-triangulation

(see Figure 6). In the first case, the map obtained is in correspondence with an ordered pair of non-separable near-triangulations. This correspondence is bijective, that is, any ordered pair is the image of exactly one near-triangulation. In the second case, the degree of the root-face is increased by one. Hence the root-face of the near-triangulation obtained has degree at least 3. Here again, any near-triangulation in which the root-face has degree at least 3 is the image of exactly one near-triangulation.
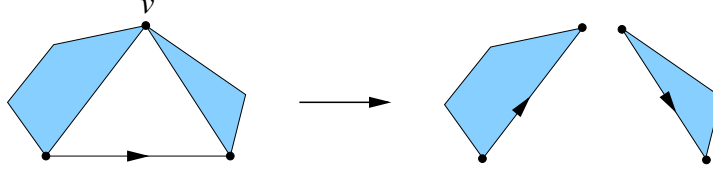
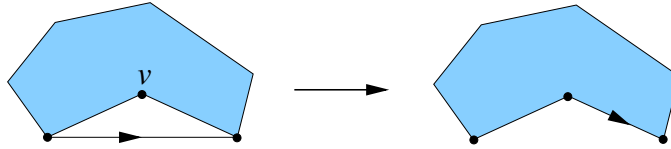Figure 5: Case 1. The vertex v is incident to the root-face.

Figure 6: Case 2. The vertex v is not incident to the root-face.

We want to translate this analysis into a functional equation. Observe that the degree of the root-face appears in this analysis. This is why we are *forced* to introduce the variable x counting this parameter in our generating function $\mathbb{S}(x, z)$. For this reason, following Zeilberger's terminology [33], the secondary variable x is said to be *catalytic*: We need it to write the functional equation, but we shall try to get rid of it later.

In our case, the decomposition easily translates into the following equation (details will be given in Section 4):

$$\mathbb{S}(x, z) = z + xz\mathbb{S}(x, z)^2 + \frac{z}{x}\left(\mathbb{S}(x, z) - \mathbb{S}(0, z)\right). \qquad (3.1)$$

The first summand of the right-hand side accounts for the link map, the second summand corresponds to the case in which the vertex v is incident to the root-face, and the third summand corresponds to the case in which v is not incident to the root-face.

It is an easy exercise to check that this equation defines the series $\mathbb{S}(x, z)$ uniquely as a power series in z with polynomial coefficients in x. By techniques presented in Section 5, we can derive from Equation 3.1 a polynomial equation satisfied by the series $\mathbb{S}(0, z)$ where the extra variable x does not appear anymore. This equation reads

$$\mathbb{S}(0, z) = z - 27z^4 + 36z^3\mathbb{S}(0, z) - 8z^2\mathbb{S}(0, z)^2 - 16z^4\mathbb{S}(0, z)^3. \qquad (3.2)$$

Given that $\mathbb{S}(0, z) = z + z\mathbb{F}(z^3)$, we deduce the algebraic equation

$$\mathbb{F}(t) = t(1 - 16t) - t(48t - 20)\mathbb{F}(t) - 8t(6t + 1)\mathbb{F}(t)^2 - 16t^2\mathbb{F}(t)^3, \qquad (3.3)$$

characterizing $\mathbb{F}(t)$ (the generating function of non-separable triangulations) uniquely as a power series in t. From this equation one can derive the asymptotic behavior of the coefficients of $\mathbb{F}(t)$, that is, the number of non-separable triangulations of a given size (see Section 7).

4. Functional Equations

In this section, we apply the decomposition scheme presented in Section 3 to the families $\mathbf{T}, \mathbf{U}, \mathbf{V}$ of non-separable near-triangulations in which all internal vertices have degree at least $3, 4, 5$. We obtain functional equations satisfied by the corresponding generating functions $\mathbb{T}(x), \mathbb{U}(x), \mathbb{V}(x)$.

Note that, when one deletes the root-edge of a map, the degree of its endpoints is lowered by one. Given the decomposition scheme, this remark explains why we are led to consider the near-triangulations where only *internal* vertices have a degree constraint. However, we need to control the degree of the origin of the root-edge since it may come from an internal vertex (see Figure 6). This leads to the following notations. Let $\mathbf{W}$ be one of the sets $\mathbf{S}, \mathbf{T}, \mathbf{U}, \mathbf{V}$. We define $\mathbf{W}_k$ as the set of maps in $\mathbf{W}$ such that the root-face has degree at least 3 and the origin of the root-edge has degree k. We also define $\mathbf{W}_\infty$ as the set of (separable) maps obtained by gluing the root-edge's end of a map in $\mathbf{W}$ with the root-edge's origin of a map in $\mathbf{W}$. The root-edge of the map obtained is chosen to be the root-edge of the second map. Generic elements of the sets $\mathbf{W}_k$ and $\mathbf{W}_\infty$ are shown in Figure 7. We also write $\mathbf{W}_{\geq k} \triangleq \mathbf{W}_\infty \cup \bigcup_{j \geq k} \mathbf{W}_j$. The notation $\mathbf{W}_{\geq k}$, which at first sight might seem awkward, allows to unify the two possible cases of our decomposition scheme (Figures 5 and 6). It shall simplify our arguments and equations (see for instance, Equations 4.1–4.4) which will prove a valuable property.

Figure 7: Generic elements of the sets $\mathbf{W}_k$ and $\mathbf{W}_\infty$.

The symbols $\mathbb{W}_k(x, z)$, $\mathbb{W}_\infty(x, z)$, and $\mathbb{W}_{\geq k}(x, z)$ denote the bivariate generating functions of the sets $\mathbf{W}_k$, $\mathbf{W}_\infty$, and $\mathbf{W}_{\geq k}$, respectively. In these series, as in $\mathbb{W}(x, z)$, the contribution of a map with n edges and root-face degree $d + 2$ is $x^d z^n$.

We are now ready to apply the decomposition scheme to the triangulations in $\mathbf{T}, \mathbf{U}, \mathbf{V}$. Consider a near-triangulation M distinct from L in $\mathbf{W} = \mathbf{S}, \mathbf{T}, \mathbf{U}, \mathbf{V}$. As observed before, the face at the left of the root-edge is an internal face incident to three distinct vertices. We denote by v the vertex not incident to the root-edge. If v is external, the deletion of the root-edge produces a map in $\mathbf{W}_\infty$ (see Figure 8). If v is internal and M is in $\mathbf{S}$ ($\mathbf{T}, \mathbf{U}$, and $\mathbf{V}$, respectively) then v has degree at least 2 (3, 4, and 5, respectively) and the map obtained by deleting the root-edge is in $\bigcup_{k \geq 2} \mathbf{S}_k$ $\left(\bigcup_{k \geq 3} \mathbf{T}_k, \bigcup_{k \geq 4} \mathbf{U}_k, \text{ and } \bigcup_{k \geq 5} \mathbf{V}_k, \text{ respectively} \right)$. Therefore, the deletion of the root-edge induces a mapping from $\mathbf{S} - \{L\}$ $\left(\mathbf{T} - \{L\}, \mathbf{U} - \{L\}, \text{ and } \mathbf{V} - \{L\}, \text{ respectively} \right)$ to $\mathbf{S}_{\geq 2}$ $\left(\mathbf{T}_{\geq 3}, \mathbf{U}_{\geq 4}, \text{ and } \mathbf{V}_{\geq 5}, \text{ respectively} \right)$.
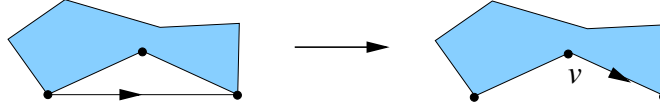
Figure 8: Mapping induced by deletion of the root-edge: The vertex v can be a separating point in which case the map is in $\mathbf{W}_\infty$.

This mapping is clearly bijective. Moreover, the map obtained after deleting the root-edge has size lowered by one and root-face degree increased by one. This analysis translates into the following equations:

$$\mathbb{S}(x) = z + \frac{z}{x}\mathbb{S}_{\geq 2}(x), \tag{4.1}$$

$$\mathbb{T}(x) = z + \frac{z}{x}\mathbb{T}_{\geq 3}(x), \tag{4.2}$$

$$\mathbb{U}(x) = z + \frac{z}{x}\mathbb{U}_{\geq 4}(x), \tag{4.3}$$

$$\mathbb{V}(x) = z + \frac{z}{x}\mathbb{V}_{\geq 5}(x). \tag{4.4}$$

In view of Equation (4.1), we will obtain a non-trivial equation for $\mathbb{S}(x)$ if we can express $\mathbb{S}_{\geq 2}(x)$ in terms of $\mathbb{S}(x)$. Similarly, we will obtain a non-trivial equation for $\mathbb{T}(x)$ if we can express $\mathbb{T}_{\geq 2}(x)$ and $\mathbb{T}_2(x)$ in terms of $\mathbb{T}(x)$. Similar statements hold for $\mathbb{U}(x)$ and $\mathbb{V}(x)$. Thus, our first task will be to evaluate $\mathbb{W}_{\geq 2}(x)$ for $\mathbb{W}$ in $\{\mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}\}$.

By definition, $\mathbf{W}_\infty$ is in bijection with $\mathbf{W}^2$, which translates into the functional equation

$$\mathbb{W}_\infty(x) = x^2\mathbb{W}(x)^2.$$

Observe that $\bigcup_{k\geq 2}\mathbf{W}_k$ is the set of maps in $\mathbf{W}$ for which the root-face has degree at least 3, that is, all maps except those rooted on a digon. Since $\mathbb{W}(0)$ is the generating function of maps in $\mathbf{W}$ rooted on a digon, we have

$$\sum_{k\geq 2}\mathbb{W}_k(x) = \mathbb{W}(x) - \mathbb{W}(0).$$

Given that $\mathbf{W}_{\geq 2} = \mathbf{W}_\infty \cup \bigcup_{k\geq 2}\mathbf{W}_k$, we obtain, for $\mathbb{W}$ in $\{\mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}\}$,

$$\mathbb{W}_{\geq 2}(x) = x^2\mathbb{W}(x)^2 + (\mathbb{W}(x) - \mathbb{W}(0)), \quad \text{for } \mathbb{W} \text{ in } \{\mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}\}. \tag{4.5}$$

Equations 4.1 and 4.5 already prove Equation 3.1 announced in Section 3:

$$\mathbb{S}(x) = z + xz\mathbb{S}(x)^2 + z\left(\frac{\mathbb{S}(x) - \mathbb{S}(0)}{x}\right).$$

In order to go further, we need to express $\mathbb{T}_2(x)$, $\mathbb{U}_2(x)$, $\mathbb{U}_3(x)$, $\mathbb{V}_2(x)$, $\mathbb{V}_3(x)$, and $\mathbb{V}_4(x)$ (see Equations 4.2–4.4). We begin with an expression of $\mathbb{W}_2(x)$ for $\mathbb{W}$ in

$\{\mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}\}$. Observe that for $\mathbf{W} = \{\mathbf{S}, \mathbf{T}, \mathbf{U}, \mathbf{V}\}$, the set $\mathbf{W}_2$ is in bijection with $\mathbf{W}$ by the mapping illustrated in Figure 9. Consequently we can write

$$\mathbb{W}_2(x) = xz^2\mathbb{W}(x), \quad \text{for } \mathbb{W} \text{ in } \{\mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}\}. \tag{4.6}$$
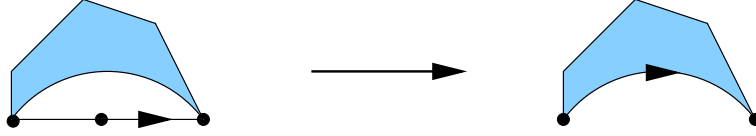

Figure 9: A bijection between $\mathbf{W}_2$ and $\mathbf{W}$.

This suffices to obtain an equation for the set $\mathbf{T}$:

$$\mathbb{T}(x) = z + \frac{z}{x}\mathbb{T}_{\geq 3}(x) \qquad \text{(by 4.2)}$$

$$= z + \frac{z}{x}\left(\mathbb{T}_{\geq 2}(x) - \mathbb{T}_2(x)\right)$$

$$= z + \frac{z}{x}\left(x^2\mathbb{T}(x)^2 + (\mathbb{T}(x) - \mathbb{T}(0)) - xz^2\mathbb{T}(x)\right). \qquad \text{(by 4.5 and 4.6)}$$

Proposition 4.1. *The generating function $\mathbb{T}(x)$ of non-separable near-triangulations in which all internal vertices have degree at least* 3 *satisfies*:

$$\mathbb{T}(x) = z + xz\mathbb{T}(x)^2 + z\left(\frac{\mathbb{T}(x) - \mathbb{T}(0)}{x}\right) - z^3\mathbb{T}(x). \tag{4.7}$$

In order to find an equation concerning the sets $\mathbf{U}$ and $\mathbf{V}$, we now need to express $\mathbb{U}_3(x)$ and $\mathbb{V}_3(x)$ in terms of $\mathbb{U}(x)$ and $\mathbb{V}(x)$, respectively. Let $\mathbf{W}$ be $\mathbf{U}$ or $\mathbf{V}$ and M be a map in $\mathbf{W}_3$. By definition, the root-face of M has degree at least 3 and its root-edge's origin u has degree 3. We denote by a and b the vertices preceding and following u on the root-face (see Figure 10). Since the map M is non-separable, the vertices a, b, and u are distinct. Let v be the third vertex adjacent to u. Since M cannot have loops, the vertex v is distinct from a, b, and u.

Suppose that M is in $\mathbf{U}_3$ ($\mathbf{V}_3$, respectively) and consider the operation of deleting u and the three adjacent edges. If the vertex v is internal it has degree $d \geq 4$ ($d \geq 5$, respectively) and the map obtained is in $\mathbf{U}_{d-1}$ ($\mathbf{V}_{d-1}$, respectively). If it is external, the map obtained is in $\mathbf{U}_\infty$ ($\mathbf{V}_\infty$, respectively). Thus, the map obtained is in $\mathbf{U}_{\geq 3}$ ($\mathbf{V}_{\geq 4}$, respectively). This correspondence is clearly bijective. It gives

$$\mathbb{U}_3(x) = z^3\mathbb{U}_{\geq 3}(x) = z^3(\mathbb{U}_{\geq 2}(x) - \mathbb{U}_2(x)), \tag{4.8}$$

$$\mathbb{V}_3(x) = z^3\mathbb{V}_{\geq 4}(x) = z^3(\mathbb{V}_{\geq 2}(x) - \mathbb{V}_2(x) - \mathbb{V}_3(x)). \tag{4.9}$$
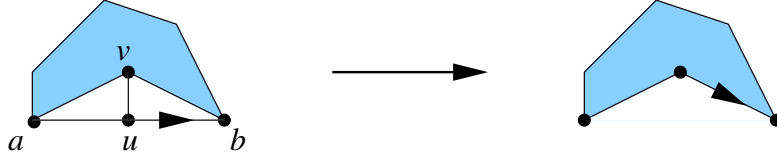
Figure 10: A bijection between $\mathbf{U}_3$ and $\mathbf{U}_{\geq 3}$ $\left(\mathbf{V}_3 \text{ and } \mathbf{V}_{\geq 4}, \text{ respectively}\right)$.

We are now ready to establish the functional equation concerning $\mathbf{U}$:

$$\mathbb{U}(x) = z + \frac{z}{x}\mathbb{U}_{\geq 4}(x) \qquad\qquad \text{(by 4.3)}$$

$$= z + \frac{z}{x}\left(\mathbb{U}_{\geq 2}(x) - \mathbb{U}_2(x) - \mathbb{U}_3(x)\right)$$

$$= z + \frac{z(1-z^3)}{x}\left(\mathbb{U}_{\geq 2}(x) - \mathbb{U}_2(x)\right) \qquad\qquad \text{(by 4.8)}$$

$$= z + \frac{z(1-z^3)}{x}\left(x^2\mathbb{U}(x)^2 + (\mathbb{U}(x) - \mathbb{U}(0)) - xz^2\mathbb{U}(x)\right). \qquad \text{(by 4.5 and 4.6)}$$

Proposition 4.2. *The generating function $\mathbb{U}(x)$ of non-separable near-triangulations in which all internal vertices have degree at least 4 satisfies*:

$$\mathbb{U}(x) = z + xz\left(1 - z^3\right)\mathbb{U}(x)^2 + z\left(1 - z^3\right)\left(\frac{\mathbb{U}(x) - \mathbb{U}(0)}{x}\right) - z^3\left(1 - z^3\right)\mathbb{U}(x). \quad (4.10)$$

We proceed to find an equation concerning the set $\mathbf{V}$. This will require significantly more work than the previous cases. We write

$$\mathbb{V}(x) = z + \frac{z}{x}\mathbb{V}_{\geq 4}(x) = z + \frac{z}{x}\left(\mathbb{V}_{\geq 2}(x) - \mathbb{V}_2(x) - \mathbb{V}_3(x) - \mathbb{V}_4(x)\right) \qquad (4.11)$$

and we want to express $\mathbb{V}_{\geq 2}(x)$, $\mathbb{V}_2(x)$, $\mathbb{V}_3(x)$, and $\mathbb{V}_4(x)$ in terms of $\mathbb{V}(x)$. We already have such expressions for $\mathbb{V}_{\geq 2}(x)$ and $\mathbb{V}_2(x)$ (by Equations 4.5 and 4.6). Moreover, Equation 4.9 can be rewritten as

$$\mathbb{V}_3(x) = \frac{z^3}{1+z^3}\left(\mathbb{V}_{\geq 2}(x) - \mathbb{V}_2(x)\right). \qquad (4.12)$$

It remains to express $\mathbb{V}_4(x)$ in terms of $\mathbb{V}(x)$. Unfortunately, this requires some efforts and some extra notations. We define $\mathbf{V}_{k,l}$ as the set of maps in $\mathbf{V}$ such that the root-face has degree at least 4, the root-edge's origin has degree k and the root-edge's end has degree l (see Figure 11). The set $\mathbf{V}_{k,\infty}$ is the set of maps obtained by gluing the root-edge's end of a map in $\mathbf{V}_k$ with the root-edge's origin of a map in $\mathbf{V}$. The root-edge of the new map obtained is the root-edge of the map in $\mathbf{V}_k$. The set $\mathbf{V}_{\infty,k}$ is the set of maps obtained by gluing the root-edge's end of a map in $\mathbf{V}$ with the root-edge's origin of map in $\mathbf{V}$ for which the root-face has degree at least 3 and the root-edge's end has
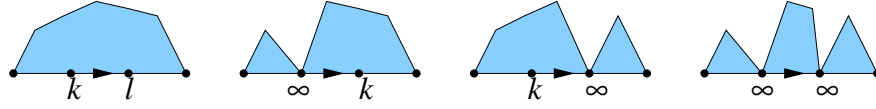
Figure 11: The sets $\mathbf{V}_{k,l}$, $\mathbf{V}_{\infty,k}$, $\mathbf{V}_{k,\infty}$, and $\mathbf{V}_{\infty,\infty}$.

degree k. The root-edge of the new map obtained is the root-edge of the second map. The set $\mathbf{V}_{\infty,\infty}$ is obtained by gluing 3 maps of $\mathbf{V}$ as indicated in Figure 11.

We also write $\mathbf{V}_{k,\geq l} \triangleq \bigcup_{i \geq l} \mathbf{V}_{k,i} \cup \mathbf{V}_{k,\infty}$ and

$$\mathbf{V}_{\geq k,\geq l} \triangleq \bigcup_{i \geq k, j \geq l} \mathbf{V}_{i,j} \cup \bigcup_{i \geq k} \mathbf{V}_{i,\infty} \cup \bigcup_{j \geq l} \mathbf{V}_{\infty,j} \cup \mathbf{V}_{\infty,\infty}.$$

As before, if $\mathbf{W}$ is any of these sets, the symbol $\mathbb{W}$ denotes the corresponding generating function, where the contribution of a map of size n and root-face degree $d+2$ is $x^d z^n$.

Moreover, we consider the subset $\mathbf{D}$ of $\mathbf{V}$ composed of maps for which the root-face is a digon. The set of maps in $\mathbf{D}$ for which the root-vertex has degree k will be denoted by $\mathbf{D}_k$. We write $\mathbf{D}_{\geq k} = \bigcup_{j \geq k} \mathbf{D}_j$. Lastly, if $\mathbf{E}$ is one of the set $\mathbf{D}$, $\mathbf{D}_k$, or $\mathbf{D}_{\geq k}$, the symbol $\mathbb{E}$ denotes the corresponding (univariate) generating function, where the contribution of a map of size n is z^n. As observed before, $\mathbb{D} = \mathbb{V}(0)$.

We can now embark on the decomposition of $\mathbf{V}_4$. We consider a map M in $\mathbf{V}_4$ with root-vertex v. By definition, v has degree 4. Let e_1, e_2, e_3, e_4 be the edges incident to v in counterclockwise order starting from the root-edge e_1. We denote by v_i ($i = 1, \ldots, 4$) the endpoint of e_i distinct from v. Since M is non-separable and its root-face has degree at least 3, the vertices v_1 and v_4 are distinct. Moreover, since M has no loop we have $v_1 \neq v_2$, $v_2 \neq v_3$, and $v_3 \neq v_4$. Therefore, only three configurations are possible: Either $v_1 = v_3$, the two other vertices being distinct, or symmetrically, $v_2 = v_4$, the other vertices being distinct, or v_1, v_2, v_3, v_4 are all distinct. The three cases are illustrated in Figure 12.
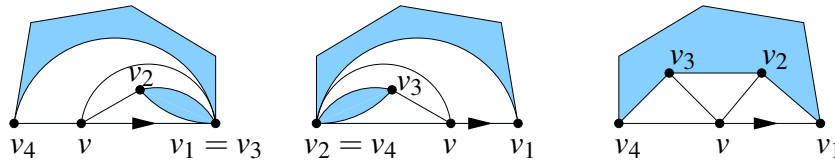

Figure 12: Three configurations for a map in $\mathbf{V}_4$.

In the case $v_1 = v_3$, the map can be decomposed into an ordered pair of maps in $\mathbf{V} \times \mathbf{D}_{\geq 4}$ (see Figure 13). This decomposition is clearly bijective. The symmetric case $v_2 = v_4$ admits a similar treatment. In the last case (v_1, v_2, v_3, v_4 are all distinct) the map obtained from M by deleting e_1, e_2, e_3, e_4 is in $\mathbf{V}_{\geq 4, \geq 4}$ (see Figure 14). Note that this case contains several subcases depending on whether v_2 and v_3 are separating points or not. But again the correspondence is clearly bijective.
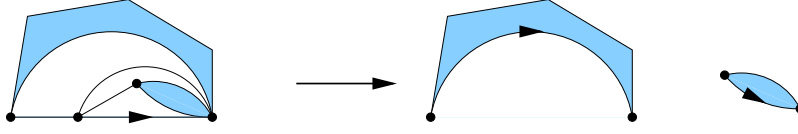
Figure 13: A bijection between maps of the first type in $\mathbf{V}_4$ and $\mathbf{V} \times \mathbf{D}_{\geq 4}$.
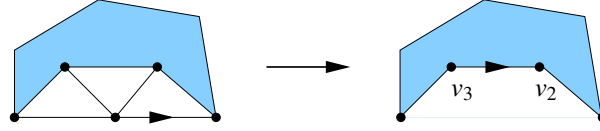
Figure 14: A bijection between maps of the third type in $\mathbf{V}_4$ and $\mathbf{V}_{\geq 4, \geq 4}$.

This correspondence gives

$$\mathbb{V}_4(x) = 2xz^4 \mathbb{V}(x)\mathbb{D}_{\geq 4} + \frac{z^4}{x}\mathbb{V}_{\geq 4, \geq 4}(x). \tag{4.13}$$

It remains to express the generating functions $\mathbb{D}_{\geq 4}$ and $\mathbb{V}_{\geq 4, \geq 4}(x)$ in terms of $\mathbb{V}(x)$. We start with $\mathbb{D}_{\geq 4}$.

We have $\mathbb{D}_{\geq 4} = \mathbb{D} - \mathbb{D}_1 - \mathbb{D}_2 - \mathbb{D}_3$. We know that $\mathbb{D} = \mathbb{V}(0)$. Moreover, the set $\mathbf{D}_1$ only contains the link-map and $\mathbf{D}_2$ is empty. Hence $\mathbb{D}_1 = z$ and $\mathbb{D}_2 = 0$. Lastly, the set $\mathbf{D}_3$ is in correspondence with $\mathbf{D}_{\geq 4}$ by the bijection represented in Figure 15. This gives $\mathbb{D}_3 = z^3 \mathbb{D}_{\geq 4}$.

Figure 15: A bijection between $\mathbf{D}_3$ and $\mathbf{D}_{\geq 4}$.

Putting these results together and solving for $\mathbb{D}_{\geq 4}$, we get

$$\mathbb{D}_{\geq 4} = \frac{\mathbb{V}(0) - z}{1 + z^3}. \tag{4.14}$$

We now want to express the generating function $\mathbb{V}_{\geq 4, \geq 4}(x)$. We first divide our problem as follows $\big($the equation uses the trivial bijections between the sets $\mathbf{V}_{\alpha, \beta}$ and $\mathbf{V}_{\beta, \alpha}\big)$:

$$\mathbb{V}_{\geq 4, \geq 4}(x) = \mathbb{V}_{\geq 2, \geq 2}(x) - \mathbb{V}_{2,2}(x) - 2\mathbb{V}_{2, \geq 3}(x) - \mathbb{V}_{3,3}(x) - 2\mathbb{V}_{3, \geq 4}(x). \tag{4.15}$$

We now treat separately the different summands in the right-hand-side of this equation.

- $\mathbf{V}_{\geq 2, \geq 2}$: It follows easily from the definitions that:

$$\mathbb{V}_{\geq 2, \geq 2}(x) = \sum_{k \geq 2, l \geq 2} \mathbb{V}_{k,l}(x) + 2 \sum_{k \geq 2} \mathbb{V}_{\infty, k}(x) + \mathbb{V}_{\infty, \infty}(x).$$

  - The set $\bigcup_{k \geq 2, l \geq 2} \mathbf{V}_{k,l}$ is the set of maps in $\mathbf{V}$ for which the root-face has degree at least 4. Thus,

$$\sum_{k \geq 2, l \geq 2} \mathbb{V}_{k,l}(x) = \mathbb{V}(x) - \mathbb{V}(0) - x[x]\mathbb{V}(x),$$

  where $[x]\mathbb{V}(x)$ is the coefficient of x in $\mathbb{V}(x)$.
  - By definition, the set $\bigcup_{k \geq 2} \mathbf{V}_{\infty, k}$ is in bijection with $\mathbf{V} \times \bigcup_{k \geq 2} \mathbf{V}_k$. Moreover, the set $\bigcup_{k \geq 2} \mathbf{V}_k$ is the set of maps in $\mathbf{V}$ for which the root-face has degree at least 3. This gives

$$\sum_{k \geq 2} \mathbb{V}_{\infty, k}(x) = x^2 \mathbb{V}(x) \left(\mathbb{V}(x) - \mathbb{V}(0) \right).$$

  - By definition, the set $\mathbf{V}_{\infty, \infty}$ is in bijection with $\mathbf{V}^3$, which gives

$$\mathbb{V}_{\infty, \infty}(x) = x^4 \mathbb{V}(x)^3.$$

Summing these contributions we get

$$\mathbb{V}_{\geq 2, \geq 2}(x) = \mathbb{V}(x) - \mathbb{V}(0) - x[x]\mathbb{V}(x) + 2x^2 \mathbb{V}(x) \left(\mathbb{V}(x) - \mathbb{V}(0) \right) + x^4 \mathbb{V}(x)^3. \quad (4.16)$$

- $\mathbf{V}_{2,2}$: The set $\mathbf{V}_{2,2}$ is empty (the face at the left of the root-edge would be of degree at least 4), hence

$$\mathbb{V}_{2,2}(x) = 0. \quad (4.17)$$

- $\mathbf{V}_{2, \geq 3}$: The set $\mathbf{V}_{2, \geq 3}$ is in bijection with $\mathbf{V}_{\geq 2}$ by the mapping illustrated in Figure 16. This gives $\mathbb{V}_{2, \geq 3}(x) = xz^2 \mathbb{V}_{\geq 2}(x)$. From this, using Equation 4.5, we obtain

$$\mathbb{V}_{2, \geq 3}(x) = xz^2 \left(\mathbb{V}(x) - \mathbb{V}(0) + x^2 \mathbb{V}(x)^2 \right). \quad (4.18)$$
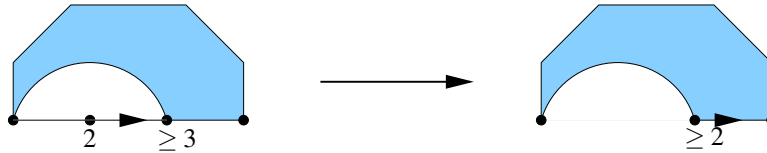

Figure 16: A bijection between $\mathbf{V}_{2, \geq 3}$ and $\mathbf{V}_{\geq 2}$.

- $\mathbf{V}_{3,3}$: We consider a map M in $\mathbf{V}_{3,3}$. We denote by v_1 the root-edge's origin, v_2 the root-edge's end, v_0 the vertex preceding v_1 on the root-face and v_3 the vertex following v_2 (see Figure 17). Since M is non-separable and its root-face has degree at least 4, the vertices v_i $(i = 1,\ldots,4)$ are all distinct. The third vertex v adjacent to v_1 is also the third vertex adjacent to v_2 (or the face at the left of the root-edge would not be a triangle). Since M has no loop, v is distinct from v_i, $i = 1,\ldots,4$. From these considerations, it is easily seen that the set $\mathbf{V}_{3,3}$ is in bijection with the set $\mathbf{V}_{\geq 3}$ by the mapping illustrated in Figure 17. (Note that this correspondence includes two subcases depending on v becoming a separating point or not.) We obtain

$$\mathbb{V}_{3,3}(x) = xz^5 \mathbb{V}_{\geq 3}(x) = xz^5 \left(\mathbb{V}_{\geq 2}(x) - \mathbb{V}_2(x) \right).$$

From this, using Equations 4.5 and 4.6, we get

$$\mathbb{V}_{3,3}(x) = xz^5 \left(\mathbb{V}(x) - \mathbb{V}(0) + x^2 \mathbb{V}(x)^2 - xz^2 \mathbb{V}(x) \right). \tag{4.19}$$
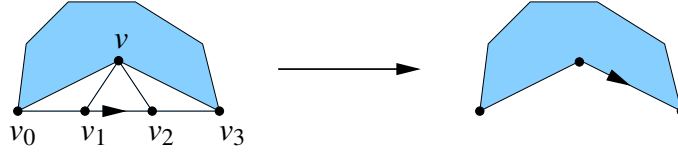

Figure 17: A bijection between $\mathbf{V}_{3,3}$ and $\mathbf{V}_{\geq 3}$.

- $\mathbf{V}_{3,\geq 4}$: Let M be a map in $\mathbf{V}_{3,\geq 4}$. We denote by v_1 the root-edge's origin, v_2 the root-edge's end, v_0 the vertex preceding v_1 on the root-face and v_3 the vertex following v_2 (see Figure 18). Since M is non-separable and its root-face has degree at least 4, the vertices v_i $(i = 1,\ldots,4)$ are all distinct. Let v be the third vertex adjacent to v_1. Two cases can occur: Either $v = v_3$ in which case the map decomposes into an ordered pair of maps in $\mathbf{V} \times \mathbf{D}_{\geq 3}$, or v is distinct from v_i $(i = 1,\ldots,4)$ in which case the map is in correspondence with a map in $\mathbf{V}_{\geq 4,\geq 3}$ (this includes two subcases depending on v becoming a separating point or not). In both cases the correspondence is clearly bijective. This gives

$$\mathbb{V}_{3,\geq 4}(x) = x^2 z^3 \mathbb{V}(x) \mathbb{D}_{\geq 3} + z^3 \mathbb{V}_{\geq 4,\geq 3}(x).$$

Given that $\mathbb{D}_{\geq 3} = \mathbb{D} - \mathbb{D}_1 - \mathbb{D}_2 = \mathbb{V}(0) - z$, we obtain

$$\mathbb{V}_{3,\geq 4}(x) = x^2 z^3 \mathbb{V}(x)(\mathbb{V}(0) - z) + z^3 \left(\mathbb{V}_{\geq 4,\geq 4}(x) + \mathbb{V}_{3,\geq 4}(x) \right),$$

and solving for $\mathbb{V}_{3,\geq 4}(x)$ we get

$$\mathbb{V}_{3,\geq 4}(x) = \frac{z^3}{1 - z^3} \left(x^2 \mathbb{V}(x)(\mathbb{V}(0) - z) + \mathbb{V}_{\geq 4,\geq 4}(x) \right). \tag{4.20}$$
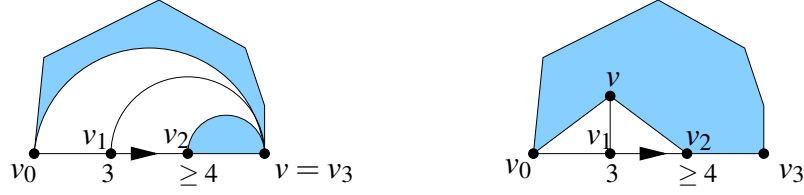
Figure 18: Two configurations for a map in $\mathbf{V}_{3,\geq 4}$.

We report Equations 4.16–4.20 in Equation 4.15 and solve for $\mathbb{V}_{\geq 4,\geq 4}$. We get

$$
\begin{aligned}
\mathbb{V}_{\geq 4,\geq 4}(x) =& \frac{1-z^3}{1+z^3}\Big(\mathbb{V}(x) - \mathbb{V}(0) - x[x]\mathbb{V}(x) + 2x^2\mathbb{V}(x)\,(\mathbb{V}(x) - \mathbb{V}(0)) \\
&+ x^4\mathbb{V}(x)^3 - xz^5\left(\mathbb{V}(x) - \mathbb{V}(0) + x^2\mathbb{V}(x)^2 - xz^2\mathbb{V}(x)\right) \\
&- 2xz^2\left(\mathbb{V}(x) - \mathbb{V}(0) + x^2\mathbb{V}(x)^2\right) - 2\frac{x^2 z^3}{1-z^3}\mathbb{V}(x)(\mathbb{V}(0) - z)\Big). \quad (4.21)
\end{aligned}
$$

Now, using Equations 4.5, 4.6, 4.12, 4.13, 4.14, and 4.21 we can replace $\mathbb{V}_{\geq 2}$, $\mathbb{V}_2$, $\mathbb{V}_3$, and $\mathbb{V}_4$ by their expressions in Equation 4.11. This establishes the following proposition.

Proposition 4.3. *The generating function* $\mathbb{V}(x) = \mathbb{V}(x,z)$ *of non-separable near-triangulations in which all internal vertices have degree at least* 5 *satisfies*:

$$
\begin{aligned}
\mathbb{V}(x) =& z + \frac{1}{1+z^3}\left(xz\mathbb{V}(x)^2 + z\frac{\mathbb{V}(x) - \mathbb{V}_0}{x} - z^3\mathbb{V}(x)\right) \\
&- \frac{z^5(1-z^3)}{1+z^3}\left(\frac{\mathbb{V}(x) - \mathbb{V}_0 - x\mathbb{V}_1}{x^2} - z^2\left(2+z^3\right)\frac{\mathbb{V}(x) - \mathbb{V}_0}{x} - 2\mathbb{V}(x)(\mathbb{V}_0 - z)\right) \\
&+ x^2\mathbb{V}(x)^3 - xz^2\left(2+z^3\right)\mathbb{V}(x)^2 + 2\mathbb{V}(x)\left(\mathbb{V}(x) - \mathbb{V}_0\right) + z^7\mathbb{V}(x)\Big), \quad (4.22)
\end{aligned}
$$

where $\mathbb{V}_0 = \mathbb{V}(0)$ *and* $\mathbb{V}_1 = [x]\mathbb{V}(x)$ *is the coefficient of* x *in* $\mathbb{V}(x)$.

5. Algebraic Equations for Triangulations with High Degree

In the previous section, we have exhibited functional equations concerning the families of *near-triangulations* **T**, **U**, **V**. By definition, the generating functions $\mathbb{T}(t)$, $\mathbb{U}(t)$, $\mathbb{V}(t)$ are power series in the main variable z with polynomial coefficients in the secondary variable x. We now solve these equations and establish algebraic equations for the families of *triangulations* **F**, **G**, **H** in which vertices not incident to the root-edge have degree at least 3, 4, 5, respectively. As observed in Section 2, the generating functions $\mathbb{F}(t)$, $\mathbb{G}(t)$, $\mathbb{H}(t)$ are closely related to the series $\mathbb{T}(0)$, $\mathbb{U}(0)$, $\mathbb{V}(0)$ (see Equation 2.1).

Let us look at Equations 4.7, 4.10, and 4.22 satisfied by the series $\mathbb{T}(x)$, $\mathbb{U}(x)$, and $\mathbb{V}(x)$, respectively. We begin with Equation 4.7. This equation is (after multiplication by x) a polynomial equation in the main unknown series $\mathbb{T}(x)$, the secondary unknown $\mathbb{T}(0)$ and the variables x, z. It is easily seen that this equation allows us to compute the coefficients of $\mathbb{T}(x)$ (hence those of $\mathbb{T}(0)$) iteratively. Moreover, we see by induction that the coefficients of this power series are polynomials in the secondary variable x. The same property holds for Equation 4.10 (4.22, respectively): It defines the series $\mathbb{U}(0)$ ($\mathbb{V}(0)$, respectively) uniquely as a power series in z with polynomial coefficients in x.

In some sense, Equations 4.7, 4.10, and 4.22 answer our enumeration problems. However, we want to *solve* these equations, that is, to derive from them some equations for the series $\mathbb{T}(0)$, $\mathbb{U}(0)$, and $\mathbb{V}(0)$. Certain techniques for performing such manipulations appear in the combinatorics literature. In the cases of Equations 4.7 and 4.10 which are quadratic in the main unknown series $\mathbb{T}(x)$ and $\mathbb{U}(x)$ we can routinely apply the so-called *quadratic method* [19, Section 2.9]. This method allows one to solve polynomial equations which are quadratic in the bivariate unknown series and have one unknown univariate series. This method also applies to Equation 3.2 concerning $\mathbb{S}(x)$ and allows to prove Equation 3.3. However, Equation 4.22 concerning $\mathbb{V}(x)$ is cubic in this series and involves two unknown univariate series $\big(\mathbb{V}(0)$ and $[x]\mathbb{V}(x)\big)$. Very recently, Bousquet-Mélou and Jehanne proposed a general method to solve polynomial equations of any degree in the bivariate unknown series and involving any number of unknown univariate series [4]. We present their formalism.

Let us begin with Equation 4.7 concerning $\mathbb{T}(0)$. We define the polynomial

$$P(T, T_0, X, Z) = XZ + X^2 Z T^2 + ZT - ZT_0 - XZ^3 T - XT.$$

Equation 4.7 can be written as

$$P(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0. \tag{5.1}$$

Let us consider the equation $P'_1(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0$, where P'_1 denotes the derivative of P with respect to its first variable. This equation can be written as

$$2x^2 z \mathbb{T}(x) + z - xz^3 - x = 0.$$

This equation is not satisfied for a generic x. However, considered as an equation in x, it is straightforward to see that it admits a unique power series solution $X(z)$.

Taking the derivative of Equation 5.1 with respect to x one obtains

$$\frac{\partial \mathbb{T}(x)}{\partial x} \cdot P'_1(\mathbb{T}(x), \mathbb{T}(0), x, z) + P'_3(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0,$$

where P'_3 denotes the derivative of P with respect to its third variable. Substituting the series $X(z)$ for x in that equation, we see that the series $X(z)$ is also a solution of the equation $P'_3(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0$. Hence, we have a system of three equations

$$P(\mathbb{T}(X(z)), \mathbb{T}(0), X(z), z) = 0,$$

$$P'_1(\mathbb{T}(X(z)), \mathbb{T}(0), X(z), z) = 0,$$

$$P'_3(\mathbb{T}(X(z)), \mathbb{T}(0), X(z), z) = 0,$$

for the three unknown series $\mathbb{T}(X(z))$, $\mathbb{T}(0)$, and $X(z)$. This polynomial system can be solved by elimination techniques using either resultant calculations or Gröbner bases. Performing these eliminations one obtains an algebraic equation for $\mathbb{T}(0)$:

$$\mathbb{T}(0) = z - 24z^4 + 3z^7 + z^{10} + \left(32z^3 + 30z^6 - 4z^9 - z^{12}\right)\mathbb{T}(0)$$

$$- 8z^2\left(1 + z^3\right)^2\mathbb{T}(0)^2 - 16z^4\mathbb{T}(0)^3.$$

Using the fact that $\mathbb{T}(0) = z + z\mathbb{G}(z^3)$ we get the following theorem.

Theorem 5.1. *Let **G** be the set of non-separable triangulations in which any vertex not incident to the root-edge has degree at least 3, and let $\mathbb{G}(t)$ be its generating function. The series $\mathbb{G}(t)$ is uniquely defined as a power series in t by the algebraic equation*:

$$16t^2\mathbb{G}(t)^3 + 8t\left(t^2 + 8t + 1\right)\mathbb{G}(t)^2$$

$$+ \left(t^4 + 20t^3 + 50t^2 - 16t + 1\right)\mathbb{G}(t) + t^2\left(t^2 + 11t - 1\right) = 0. \tag{5.2}$$

Similar manipulations lead to a cubic equation for the set **H**.

Theorem 5.2. *Let **H** be the set of non-separable triangulations in which any vertex not incident to the root-edge has degree at least 4, and let $\mathbb{H}(t)$ be its generating function. The series $\mathbb{H}(t)$ is uniquely defined as a power series in t by the algebraic equation*:

$$16t^2(t-1)^4\mathbb{H}(t)^3 + \left(t^8 + 12t^7 - 14t^6 - 84t^5\right.$$

$$+ 207t^4 - 192t^3 + 86t^2 - 16t + 1\right)\mathbb{H}(t)$$

$$+ 8t(t-1)^2\left(t^4 + 4t^3 - 13t^2 + 8t + 1\right)\mathbb{H}(t)^2$$

$$+ t^4(t-1)\left(t^3 + 5t^2 - 8t + 1\right) = 0. \tag{5.3}$$

For Equation 4.22 concerning $\mathbb{V}(0)$ the method is almost identical. We see that there is a polynomial $Q(V, V_0, V_1, x, z)$ such that Equation 4.22 can be written as $Q(\mathbb{V}(x), \mathbb{V}(0), [x]\mathbb{V}(x), x, z) = 0$. But we can show that there are exactly *two* series $X_1(z), X_2(z)$ such that $Q'_1(\mathbb{V}(X(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X(z), z) = 0$. Thus, we obtain a system of 6 equations

$$Q\left(\mathbb{V}(X_i(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X_i(z), z\right) = 0$$

$$Q'_1\left(\mathbb{V}(X_i(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X_i(z), z\right) = 0$$

$$Q'_3\left(\mathbb{V}(X_i(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X_i(z), z\right) = 0$$

for the 6 unknown series $\mathbb{V}(X_1(z))$, $\mathbb{V}(X_2(z))$, $X_1(z)$, $X_2(z)$, $\mathbb{V}(0)$, and $[x]\mathbb{V}(x)$. This system can be solved via elimination techniques though the calculations involved are heavy. We obtain the following theorem.

Theorem 5.3. *Let **K** be the set of non-separable triangulations in which any vertex not incident to the root-edge has degree at least* 5*, and let* $\mathbb{K}(t)$ *be its generating function. The series* $\mathbb{K}(t)$ *is uniquely defined as a power series in t by the algebraic equation*:

$$\sum_{i=0}^{6} P_i(t)\mathbb{K}(t)^i, \tag{5.4}$$

where the polynomials $P_i(t)$ $(i = 0, \ldots, 6)$ *are given in Section 9.1.*

6. Constraining the Vertices Incident to the Root-Edge

So far, we have established algebraic equations for the generating functions $\mathbb{G}(t)$, $\mathbb{H}(t)$, $\mathbb{K}(t)$ of triangulations in which *any vertex not incident to the root-edge* has degree at least $3, 4, 5$. The following theorems provide equations concerning the generating functions $\mathbb{G}^*(t)$, $\mathbb{H}^*(t)$ of triangulations in which *any vertex* has degree at least $3, 4$.

Theorem 6.1. *Let **G*** be the set of non-separable triangulations in which* any *vertex has degree at least* 3 *and let* $\mathbb{G}^*(t)$ *be its generating function. The series* $\mathbb{G}^*$ *is related to the series* $\mathbb{G}$ *of Theorem 5.1 by*

$$\mathbb{G}^*(t) = (1 - 2t)\mathbb{G}(t). \tag{6.1}$$

Theorem 6.2. *Let **H*** be the set of non-separable triangulations in which* any *vertex has degree at least* 4 *and let* $\mathbb{H}^*(t)$ *be its generating function. The series* $\mathbb{H}^*$ *is related to the series* $\mathbb{H}$ *of Theorem 5.2 by*

$$\mathbb{H}^*(t) = \frac{1 - 5t + 5t^2 - 3t^3}{1 - t}\mathbb{H}(t). \tag{6.2}$$

Let us make a few comments before proving these two theorems. First, observe that we can deduce from Theorems 5.1 and 6.1 (5.2 and 6.2, respectively) an algebraic equation for the generating function $\mathbb{G}^*$ ($\mathbb{H}^*$, respectively) of triangulations in which *any* vertex has degree at least 3 (4, respectively). The algebraic equation obtained for $\mathbb{G}^*$ coincides with the result of Gao and Wormald [17, Theorem 2]. From the algebraic equations we can routinely compute the first coefficients of our series:

$$\mathbb{G}^*(t) = t^2 + 3t^3 + 19t^4 + 128t^5 + 909t^6 + 6737t^7 + 51683t^8 + 407802t^9 + o\left(t^9\right),$$

$$\mathbb{H}^*(t) = t^4 + 3t^5 + 12t^6 + 59t^7 + 325t^8 + 1875t^9 + 11029t^{10} + 65607t^{11} + o\left(t^{11}\right).$$

Recall that the coefficient of t^n in the series $\mathbb{G}^*(t)$, $\mathbb{H}^*(t)$ is the number of triangulations with $3n$ edges ($2n$ triangles, $n + 2$ vertices) satisfying the required degree constraint. In the expansion of $\mathbb{G}^*(t)$, the smallest non-zero coefficient t^2 corresponds to the tetrahedron. In the expansion of $\mathbb{H}^*(t)$, the smallest non-zero coefficient t^4 corresponds to the octahedron (see Figure 19).

We were unable to find an equation that would permit to count non-separable triangulations in which *any vertex* has degree at least 5. However, we can use the algebraic equation 5.4 to compute the first coefficients of the series $\mathbb{K}(t)$:

$$\mathbb{K}(t) = t^{10} + 8t^{11} + 45t^{12} + 209t^{13} + 890t^{14} + 3600t^{15} + 14115t^{16} + 54306t^{17} + o\left(t^{18}\right).$$
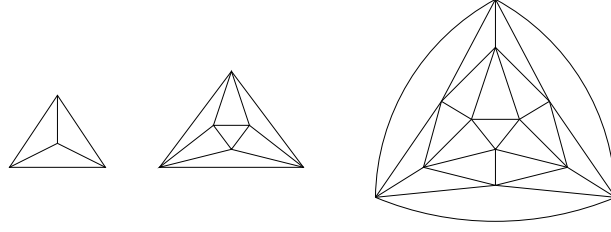
Figure 19: The platonic solids: tetrahedron, octahedron, icosahedron.

The first non-zero coefficient t^{10} corresponds to the icosahedron (see Figure 19).

In order to prove Theorems 6.1 and 6.2 we need some new notations. The set $\mathbf{G}_{i,j,k}$ $\left(\mathbf{H}_{i,j,k}, \text{ respectively}\right)$ is the set of triangulations such that the root-edge's origin has degree i, the root-edge's end has degree j, the third vertex of the root-face has degree k and all internal vertices have degree at least 3 (4, respectively). For $\mathbf{L} = \mathbf{G}, \mathbf{H}$ we define $\mathbf{L}_{\geq i,j,k} = \bigcup_{l \geq i} \mathbf{L}_{l,j,k}$ and with similar notation, $\mathbf{L}_{\geq i, \geq j, k}$ etc. If $\mathbf{L}$ is any of these sets, then $\mathbb{L}(t)$ is the corresponding generating function, where a map with $3n$ edges has contribution t^n.

Proof of Theorem 6.1. By definition, $\mathbf{G} = \mathbf{G}_{\geq 2, \geq 2, \geq 3}$ and $\mathbf{G}^* = \mathbf{G}_{\geq 3, \geq 3, \geq 3}$. Hence,

$$\mathbb{G}^*(t) = \mathbb{G}(t) - \mathbb{G}_{2,2,\geq 3}(t) - 2\mathbb{G}_{2,\geq 3,\geq 3}(t). \tag{6.3}$$

- The set $\mathbf{G}_{2,2,\geq 3}$ is empty, hence $\mathbb{G}_{2,2,\geq 3}(t) = 0$.
- The set $\mathbf{G}_{2,\geq 3,\geq 3}$ is in bijection with $\mathbf{G}_{\geq 1,\geq 1,\geq 3} = \mathbf{G}$ by the mapping represented in Figure 20. This gives $\mathbb{G}_{2,\geq 3,\geq 3}(t) = t\mathbb{G}(t)$.

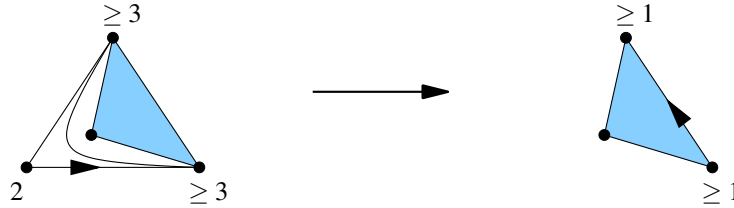Plugging these results in 6.3 proves the theorem.   ∎

Figure 20: A bijection between $\mathbf{G}_{2,\geq 3,\geq 3}$ and $\mathbf{G}$ $\left(\mathbf{H}_{2,\geq 3,\geq 3} \text{ and } \mathbf{H}, \text{ respectively}\right)$.

Proof of Theorem 6.2. By definition, $\mathbf{H}^* = \mathbf{H}_{\geq 4, \geq 4, \geq 4}$. Hence,

$$\mathbb{H}^*(t) = \mathbb{H}_{\geq 3, \geq 3, \geq 4}(t) - \mathbb{H}_{3,3,\geq 4}(t) - 2\mathbb{H}_{3,\geq 4,\geq 4}(t). \tag{6.4}$$

Recall that $\mathbf{H} = \mathbf{H}_{\geq 1, \geq 1, \geq 4} = \mathbf{H}_{\geq 2, \geq 2, \geq 4}$.

- Clearly, $\mathbb{H}_{\geq 3, \geq 3, \geq 4}(t) = \mathbf{H}_{\geq 2, \geq 2, \geq 4}(t) - \mathbf{H}_{2,2,\geq 4}(t) - 2\mathbf{H}_{2, \geq 3, \geq 3}(t)$.

- The set $\mathbb{H}_{2,2,\geq 4}(t)$ is empty, hence $\mathbf{H}_{2,2,\geq 4}(t) = 0$.
- The set $\mathbf{H}_{2,\geq 3,\geq 3}$ is in bijection with $\mathbf{H}_{\geq 1,\geq 1,\geq 4} = \mathbf{H}$ by the mapping represented in Figure 20, hence $\mathbf{H}_{2,\geq 3,\geq 3}(t) = t\mathbb{H}(t)$. This gives

$$\mathbb{H}_{\geq 3,\geq 3,\geq 4}(t) = (1 - 2t)\mathbb{H}(t). \tag{6.5}$$

- The set $\mathbf{H}_{3,3,\geq 4}$ is in bijection with $\mathbf{H}_{\geq 1,\geq 1,\geq 4} = \mathbf{H}$ by the mapping represented in Figure 21. This gives

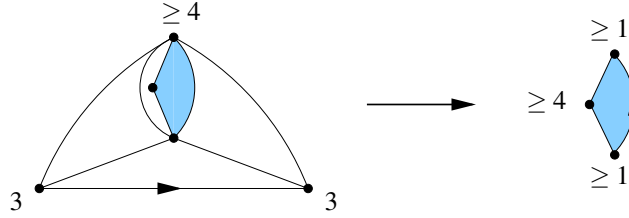$$\mathbb{H}_{3,3,\geq 4}(t) = t^2 \mathbb{H}(t). \tag{6.6}$$

Figure 21: A bijection between $\mathbf{H}_{3,3,\geq 4}$ and $\mathbf{H}$.

- For any integer k greater than 2, the set $\mathbf{H}_{\geq k,\geq k,3}$ is in bijection with the set $\mathbf{H}_{\geq k-1,\geq k-1,\geq 3}$ by the mapping represented in Figure 22. This gives

$$\mathbb{H}_{\geq k,\geq k,3}(t) = t\mathbb{H}_{\geq k-1,\geq k-1,\geq 3}(t), \quad \text{for all } k \geq 2. \tag{6.7}$$
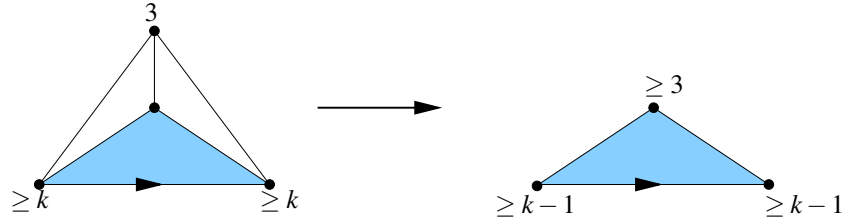
Figure 22: A bijection between $\mathbf{H}_{\geq k,\geq k,3}$ and $\mathbf{H}_{\geq k-1,\geq k-1,\geq 3}$.

Using Equation 6.7 for $k = 4$ and then for $k = 3$ (and trivial symmetry properties), we get

$$\mathbb{H}_{3,\geq 4,\geq 4}(t) = \mathbb{H}_{\geq 4,\geq 4,3}(t) = t\mathbb{H}_{\geq 3,\geq 3,\geq 3}(t)$$
$$= t\mathbb{H}_{\geq 3,\geq 3,\geq 4}(t) + t\mathbb{H}_{\geq 3,\geq 3,3}(t)$$
$$= t\mathbb{H}_{\geq 3,\geq 3,\geq 4}(t) + t^2\mathbb{H}_{\geq 2,\geq 2,\geq 3}(t).$$

- By Equation 6.5, we have $\mathbb{H}_{\geq 3,\geq 3,\geq 4}(t) = (1 - 2t)\mathbb{H}(t)$.

   – Using Equation 6.7 for $k = 2$ gives

$$\mathbb{H}_{\geq 2, \geq 2, \geq 3}(t) = \mathbb{H}_{\geq 2, \geq 2, \geq 4}(t) + \mathbb{H}_{\geq 2, \geq 2, 3}(t) = \mathbb{H}(t) + t\mathbb{H}_{\geq 1, \geq 1, \geq 3}(t).$$

   Given that $\mathbf{H}_{\geq 1, \geq 1, \geq 3} = \mathbf{H}_{\geq 2, \geq 2, \geq 3}$, we get $\mathbb{H}_{\geq 2, \geq 2, \geq 3}(t) = \frac{1}{1-t}\mathbb{H}(t)$.
   Thus, we obtain

$$\mathbb{H}_{3, \geq 4, \geq 4}(t) = \frac{t\left(1 - 2t + 2t^2\right)}{1 - t}\mathbb{H}(t). \tag{6.8}$$

Plugging Equations 6.5, 6.6, and 6.8 in Equation 6.4 proves the theorem. ∎

7. Asymptotics

In Section 5, we established algebraic equations for the generating functions $\mathbb{L} = \mathbb{F}$, $\mathbb{G}$, $\mathbb{H}$, $\mathbb{K}$ of non-separable triangulations in which any vertex not incident to the root-edge has degree at least $d = 2, 3, 4, 5$ (Equations 3.3, 5.2, 5.3, and 5.4). We will now derive the asymptotic form of the number $l_n = f_n, g_n, h_n, k_n$ of maps with $3n$ edges in each family by analyzing the singularities of the generating function $\mathbb{L} = \mathbb{F}, \mathbb{G}, \mathbb{H}, \mathbb{K}$ (l_n is the coefficient of t^n in $\mathbb{L}$). The principle of this method is a general correspondence between the expansion of a generating function at its dominant singularities and the asymptotic form of its coefficients [13, 14].

Lemma 7.1. *Each of the generating functions $\mathbb{L} = \mathbb{F}, \mathbb{G}, \mathbb{H}, \mathbb{K}$ has a unique dominant singularity $\rho_L > 0$ and a singular expansion with singular exponent $\frac{3}{2}$ at ρ_L, in the sense that*

$$\mathbb{L}(t) = \alpha_L + \beta_L\left(1 - \frac{t}{\rho_L}\right) + \gamma_L\left(1 - \frac{t}{\rho_L}\right)^{3/2} + O\left(\left(1 - \frac{t}{\rho_L}\right)^2\right), \tag{7.1}$$

with $\gamma_L \neq 0$. The dominant singularities of the series $\mathbb{F}$ and $\mathbb{G}$ are respectively $\rho_F = \frac{2}{27}$ and $\rho_G = \frac{3\sqrt{3}-5}{2}$. The dominant singularities ρ_H and ρ_K of the series $\mathbb{H}$ and $\mathbb{K}$ are defined by algebraic equations given in Section 9.2.

Proof. (sketch) The (systematic) method we follow is described in [14, Chapter VII.4]). Calculations were performed using the Maple package *gfun* [21].

    Let us denote generically by ρ_L the radius of convergence of the series $\mathbb{L}$ and by $Q(\mathbb{L}, t)$ the algebraic equation satisfied by $\mathbb{L}$ (Equations 3.3, 5.2, 5.3, and 5.4). It is known that the singular points of the series $\mathbb{L}$ are among the roots of the polynomial $R(t) = D(t)\Delta(t)$ where $D(t)$ is the dominant coefficient of $Q(y, t)$ and $\Delta(t)$ is the discriminant of $Q(y, t)$ considered as a polynomial in y. Moreover, since the series $\mathbb{L}$ has non-negative coefficients, we know (by Pringsheim's Theorem) that the point $t = \rho_L$ is singular. In our cases, the smallest positive root of $R(t)$ is found to be indeed a singular point of the series $\mathbb{L}$. (This requires to solve some connection problems that we do not detail.) Moreover, no other root of $R(t)$ has the same modulus. This proves that the series $\mathbb{L}$ has a unique dominant singularity.

    The second step is to expand the series $\mathbb{L}$ near its singularity ρ_L. This calculation can be performed using *Newton's polygon method* (see [14, Chapter VII.4]) which is implemented in the *algeqtoseries* Maple command [21]. ∎

From Lemma 7.1, we can deduce the asymptotic form of the number $l_n = f_n, g_n, h_n,$ k_n of non-separable triangulations of size n in each family.

Theorem 7.2. *The number $l_n = f_n, g_n, h_n, k_n$ of non-separable triangulations of size n ($3n$ edges) in which any vertex not incident to the root-edge has degree at least $d = 2, 3, 4, 5$ has asymptotic form*

$$l_n \sim \lambda_L n^{-5/2} \left(\frac{1}{\rho_L} \right)^n.$$

The growth constants ρ_F, ρ_G, ρ_H, and ρ_K are given in Lemma 7.1. Numerically,

$$\frac{1}{\rho_F} = 13.5, \quad \frac{1}{\rho_G} \approx 10.20, \quad \frac{1}{\rho_H} \approx 7.03, \quad \frac{1}{\rho_K} \approx 4.06.$$

Remark 7.3. The subexponential factor $n^{-5/2}$ is typical of planar maps families $\big($see for instance [2] where 15 classical families of maps are listed all displaying this subexponential factor $n^{-5/2}\big)$.

Remark 7.4. Using Theorems 6.1 and 6.2, it is easily seen that the series $\mathbb{L}^* = \mathbb{G}^*, \mathbb{H}^*$ has dominant singularity $\rho_L = \rho_G, \rho_H$ with singular exponent $\frac{3}{2}$ at ρ_L:

$$\mathbb{L}(t) = \alpha_L^* + \beta_L^* \left(1 - \frac{t}{\rho_L} \right) + \gamma_L^* \left(1 - \frac{t}{\rho_L} \right)^{3/2} + O\left(\left(1 - \frac{t}{\rho_L} \right)^2 \right).$$

Therefore, we obtain the asymptotic form

$$l_n^* \sim \lambda_L^* n^{-5/2} \left(\frac{1}{\rho_L} \right)^n,$$

for the number $l_n^* = g_n^*, h_n^*$ of non-separable triangulations of size n with vertex degree at least $d = 3, 4$. Hence, the numbers l_n^* and l_n are equivalent up to a (known) constant multiplicative factor $\frac{\lambda_L^*}{\lambda_L}$:

$$\frac{\lambda_G^*}{\lambda_G} = \frac{\gamma_G^*}{\gamma_G} = 1 - 2\rho_G = 6 - 3\sqrt{3},$$

$$\frac{\lambda_H^*}{\lambda_H} = \frac{\gamma_H^*}{\gamma_H} = \frac{1 - 5\rho_H + 5\rho_H^2 - 3\rho_H^3}{1 - \rho_H}.$$

We do not have such precise information about the asymptotic form of the number k_n^* of non-separable triangulations of size n ($3n$ edges) with vertex degree at least 5. However, we do know that $k_n^* = \Theta(k_n) = \Theta\left(n^{-5/2}\rho_K^{-n}\right)$. Indeed, we clearly have $k_n^* \leq k_n$ and, in addition, $k_n^* \geq k_{n-9} \sim \rho_K^9 k_n$. The latter inequality is proved by observing that the operation of replacing the root-face of a triangulation by an icosahedron is an injection from the set of triangulations of size n in which any vertex *not incident to the root-edge* has degree 5 to the set of triangulations of size $n+9$ in which *any* vertex has degree at least 5.

8. Concluding Remarks

We have established algebraic equations for the generating functions of loopless triangulations (i.e., non-separable triangulations) in which *any vertex not incident to the root-edge* has degree at least $d = 3, 4, 5$. We have also established algebraic equations for loopless triangulations in which *any vertex* has degree at least $d = 3, 4$. However, have not found a similar result for $d = 5$. The algebraic equations we have obtained can be converted into differential equations (using for instance the *algeqtodiffeq* Maple command available in the *gfun* package [21]) from which one can compute the coefficients of the series in a linear number of operations. Moreover, the asymptotic form of their coefficients can also be found routinely from the algebraic equations.

The approach we have adopted is based on a classic decomposition scheme allied with a generating function approach. Alternatively, it is possible to obtain some of our results by a compositional approach. This is precisely the method followed by Gao and Wormald to obtain the algebraic equation concerning loopless triangulations in which any vertex has degree at least 3 ([17]). This substitution approach can also be extended to obtain the algebraic equation concerning loopless triangulations in which any vertex has degree at least 4. However, we do not see how to apply this method to loopless triangulations in which vertices not incident to the root-edge have degree at least 5.

Recently, Poulalhon and Schaeffer gave a bijective proof based on the *conjugacy classes of tree* for the number of loopless triangulations ([22]). However, it is dubious that this approach should apply for the families $\mathbb{H}$, $\mathbb{K}$ of loopless triangulations in which vertices have degree at least $d = 4, 5$. Indeed, for a large number of families of maps **L**, the generating function $\mathbb{L}(t)$ is *Lagrangean*, that is, there exists a series $\mathbb{X}(t)$ and two rational functions Ψ, Φ satisfying

$$\mathbb{L}(t) = \Psi(\mathbb{X}(t)) \quad \text{and} \quad \mathbb{X}(t) = t\Phi(\mathbb{X}(t))$$

(see for instance, [2] where 15 classical families are listed together with a Lagrangean parametrization). Often, a parametrization can be found such that the series $\mathbb{X}(t)$ looks like the generating function of a family of trees (i.e., $\Phi(x)$ is a series with non-negative coefficients) suggesting that a bijective approach exists based on the enumeration of certain trees [6–8]. However, it is known that an algebraic series is Lagrangean if and only if the genus of the algebraic equation is 0 ([1, Chapter 15]). In our case, the algebraic equations defining the series $\mathbb{F}$, $\mathbb{G}$, $\mathbb{H}$, and $\mathbb{K}$ have respective genus 0, 0, 2, and 25. (The genus can be computed using the Maple command *genus*.) Thus, whereas the series $\mathbb{F}$, $\mathbb{G}$ are Lagrangean (with a parametrization given in Section 9.3), the series $\mathbb{H}$, $\mathbb{K}$ are not.

Lastly, we claim some generality to our approach. Here, we have focused on loopless triangulations, but it is possible to practice the same kind of manipulations for *general* triangulations and for 3-connected ones. The method should also apply to some other families of maps, like quadrangulations. Thus, a whole new class of map families is expected to have algebraic generating functions.

9. Appendix

9.1. Coefficients of the Algebraic Equation 5.4

The coefficients $P_i(t)(i = 0, \ldots, 6)$ in the algebraic Equation 5.4 are:

$$
\begin{aligned}
P_0(t) =\ & t^{10}\big(-1 + 82552t^{11} - 163081t^{12} + 277796t^{13} - 308156t^{14} - 443851t^{16} \\
& + t^{34} + 13t + 32t^{31} + 454t^5 - 2434t^6 - 5762t^8 + 4373t^7 \\
& - 53961t^{10} + 23037t^9 + 354387t^{15} + 163964t^{20} - 28454t^{21} \\
& - 38408t^{22} + 36713t^{23} - 11737t^{24} + t^{33} + 2t^{32} - 278t^{25} \\
& + 242t^{28} - 1678t^{27} + 2714t^{26} + 36t^{29} - 64t^{30} - 70t^2 + 180t^3 \\
& - 195t^4 - 273662t^{19} + 122688t^{18} + 262614t^{17}\big),
\end{aligned}
$$

$$
\begin{aligned}
P_1(t) =\ & \big(1 - 594873t^{11} + 1078572t^{12} - 1457943t^{13} + 1921912t^{14} \\
& + 1327736t^{16} + 1462t^{38} - 3168t^{37} - 611t^{39} + 25956t^{35} \\
& - 56515t^{34} - 3826t^{36} - 21t - 467567t^{31} - 4545t^5 + 3916t^6 \\
& + 60304t^8 - 13364t^7 + 275068t^{10} - 142715t^9 - 2t^{42} + 9t^{43} \\
& + t^{44} - 2338117t^{15} - 4673450t^{20} + 5167054t^{21} - 1145738t^{22} \\
& - 2425736t^{23} + 2298353t^{24} + 66635t^{33} + 90827t^{32} + 559893t^{25} \\
& - 874518t^{28} + 2995671t^{27} - 3225500t^{26} - 526335t^{29} + 763474t^{30} \\
& + 68t^{41} + 75t^{40} + 193t^2 - 988t^3 + 2913t^4 + 1719643t^{19} \\
& - 945302t^{18} + 541155t^{17}\big),
\end{aligned}
$$

$$
\begin{aligned}
P_2(t) =\ & t\big(8 + 2011979t^{11} - 1422607t^{12} + 2174211t^{13} - 4910332t^{14} \\
& - 9095603t^{16} - 814t^{38} + 688t^{37} + 306t^{39} - 16997t^{35} \\
& + 43703t^{34} + 1292t^{36} - 4t + 370239t^{31} - 3000t^5 + 20421t^6 \\
& - 268574t^8 + 72382t^7 - 1309172t^{10} + 527412t^9 + 8t^{42} \\
& + 5383141t^{15} + 31153077t^{20} - 16211612t^{21} - 2143067t^{22} \\
& + 7886923t^{23} - 2902691t^{24} - 50536t^{33} - 26161t^{32} \\
& - 4609909t^{25} + 156674t^{28} - 3199107t^{27} + 6488106t^{26} \\
& + 970079t^{29} - 902321t^{30} + 12t^{41} + 4t^{40} - 556t^2 + 3851t^3
\end{aligned}
$$

$$- 8840t^4 - 18494688t^{19} - 9439987t^{18} + 17752182t^{17}),$$

$$
\begin{aligned}
P_3(t) =& t^2 \big(16 + 1278321t^{11} - 2978655t^{12} + 1697247t^{13} + 5975715t^{14} \\
& + 54631824t^{16} + 166t^{38} - 90t^{37} - 32t^{39} + 3984t^{35} - 13104t^{34} \\
& - 868t^{36} - 192t - 105251t^{31} + 17247t^5 - 36981t^6 + 521925t^8 \\
& - 74982t^7 + 835782t^{10} - 1142394t^9 - 29427957t^{15} - 39935486t^{20} \\
& + 7773505t^{21} + 6824437t^{22} - 5541795t^{23} - 1619262t^{24} + 18648t^{33} \\
& + 4941t^{32} + 5146785t^{25} + 349680t^{28} + 880004t^{27} - 3411645t^{26} \\
& - 600239t^{29} + 358687t^{30} + 16t^{40} + 1046t^2 - 2554t^3 - 397t^4 \\
& + 60017232t^{19} - 26467945t^{18} - 34977363t^{17} \big),
\end{aligned}
$$

$$
\begin{aligned}
P_4(t) =& 9t^5(t-1)^2 \big(8 + 722739t^{11} - 1888278t^{12} + 1483343t^{13} + 679876t^{14} \\
& + 1099122t^{16} - 84t - 20t^{31} + 9250t^5 - 17908t^6 + 144652t^8 - 22565t^7 \\
& + 87721t^{10} - 234335t^9 - 1820089t^{15} - 5409t^{20} - 64607t^{21} + 41918t^{22} \\
& - 12628t^{23} - 1362t^{24} + 8t^{32} + 6200t^{25} - 189t^{28} + 1127t^{27} - 3809t^{26} \\
& - 103t^{29} + 84t^{30} + 368t^2 - 583t^3 - 2069t^4 \\
& + 110521t^{19} - 69119t^{18} - 243772t^{17} \big),
\end{aligned}
$$

$$
\begin{aligned}
P_5(t) =& 81t^8(t-1)^4 \big(1 + 25926t^{11} - 14080t^{12} + 2973t^{13} \\
& - 369t^{14} + 348t^{16} - 9t + 2118t^5 - 2936t^6 + 23913t^8 - 4134t^7 \\
& - 6330t^{10} - 25946t^9 - 970t^{15} + 12t^{20} - 22t^{21} + 12t^{22} - 3t^{23} \\
& + t^{24} + 30t^2 - 15t^3 - 747t^4 + 42t^{19} - 219t^{18} + 405t^{17} \big),
\end{aligned}
$$

$$P_6(t) = 59049t^{15}(t+1)(t-1)^9.$$

9.2. Algebraic Equations for the Dominant Singularity of the Series $\mathbb{H}(t)$ and $\mathbb{K}(t)$

The dominant singularity ρ_H (ρ_K, respectively) of the generating function $\mathbb{H}(t)$ ($\mathbb{K}(t)$, respectively) is the smallest positive root of the polynomial $r_H(t)$ ($r_K(t)$, respectively) where $r_H(t) = 2 - 17t + 22t^2 - 10t^3 + 2t^4$, and

$$
\begin{aligned}
r_K(t) =& 256 - 5504t + 51744t^2 - 265664t^3 + 755040t^4 \\
& - 1069751t^5 + 1411392t^6 - 9094370t^7 + 30208920t^8
\end{aligned}
$$

$$- 14854607t^9 - 106655904t^{10} + 169679596t^{11}$$

$$+ 1693392t^{12} + 58535932t^{13} - 263701752t^{14} - 751005332t^{15}$$

$$+ 2215033200t^{16} - 2276240390t^{17} + 2301677920t^{18}$$

$$- 1558097344t^{19} - 2448410184t^{20} + 6223947236t^{21}$$

$$- 7440131352t^{22} + 6100648148t^{23} + 1602052848t^{24}$$

$$- 9604816702t^{25} + 6144202392t^{26} + 996698032t^{27}$$

$$+ 551560496t^{28} - 3299013583t^{29} - 728097928t^{30}$$

$$+ 4881643814t^{31} - 3845803168t^{32} + 494467523t^{33}$$

$$+ 1677669800t^{34} - 1787552140t^{35} + 825330824t^{36}$$

$$+ 1529759t^{37} - 340280968t^{38} + 301075034t^{39}$$

$$- 121555768t^{40} - 1710967t^{41} + 37850432t^{42}$$

$$- 27659392t^{43} + 9430688t^{44} - 152352t^{45}$$

$$- 1901664t^{46} + 1245152t^{47} - 400416t^{48}$$

$$+ 47744t^{49} + 30720t^{50} - 22528t^{51}$$

$$+ 7680t^{52} - 1792t^{53} + 256t^{54}.$$

9.3. Lagrangean Parametrization for the Series $\mathbb{F}(t)$, $\mathbb{G}(t)$, and $\mathbb{G}^*(t)$

The series $\mathbb{F}(t)$ has the following Lagrangean parametrization:

$$\mathbb{F}(t) = \frac{\mathbb{X}(1 + \mathbb{X})}{2},$$

where

$$\mathbb{X} \equiv \mathbb{X}(t) = 2t(1 + \mathbb{X}(t))^3.$$

The series $\mathbb{G}(t)$ and $\mathbb{G}^*(t)$ have the following Lagrangean parametrization:

$$\mathbb{G}(t) = 2t\mathbb{Y}(1 + \mathbb{Y})\left(1 - \mathbb{Y} - \mathbb{Y}^2\right),$$

$$\mathbb{G}^*(t) = 4t^2(1 + \mathbb{Y})\left(1 - \mathbb{Y} - \mathbb{Y}^2\right)\left(1 + 3\mathbb{Y} + 6\mathbb{Y}^2 + 2\mathbb{Y}^3\right),$$

where

$$\mathbb{Y} \equiv \mathbb{Y}(t) = 2t(1 + \mathbb{Y}(t))\left(1 + 4\mathbb{Y}(t) + 2\mathbb{Y}(t)^2\right).$$

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