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# Partition-Theoretic Interpretations of Certain Modular Equations of Schröter, Russell, and Ramanujan

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**Abstract.** We show that certain modular equations studied by Schrööter, Russell, and Ramanujan yield elegant identities for colored partitions.

*Keywords*: partitions, colored partitions, theta-functions, modular equations

## 1. Introduction

In [6], Farkas and Kra established three theta constant identities and interpreted them in terms of partition identities. Perhaps the most elegant of their partition theorems is the following result [6, Theorem 3, p. 202].

**Theorem 1.1.** *Let $\mathcal{S}$ denote the set consisting of one copy of the positive integers and one additional copy of those positive integers that are multiples of* $7$*. Then for each positive integer* $k$*, the number of partitions of* $2k$ *into even elements of* $S$ *is equal to the number of partitions of* $2k+1$ *into odd elements of* $S$*.*

Subsequently, Hirschhorn [10] and Warnaar [21] gave different proofs of Theorem 1.1 by giving elegant derivations of the equivalent $q$-series identity that yields Theorem 1.1 immediately. It was observed by the referee of [10] that the identity proved by Farkas and Kra [6], Warnaar [21], and Hirschhorn [10] was, in fact, equivalent to a modular equation of degree 7 found in Ramanujan's notebooks [15, Chapter 19, Entry 19(i)] and [2, p. 314], but actually first proved by Guetzlaff [8] in 1834. Ramanujan discovered five modular equations of this type, which are also often called modular equations of Russell-type, after the English mathematician Russell, who also established these five modular equations, which are a particular subset of a somewhat larger, and generally more complicated, class of modular equations found by and named after him [16, 17].

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All five modular equations were previously and uniformly derived by Schröter and can be found in his paper [20]. We have carefully examined Russell's papers [16, 17], and we have searched a large segment of the literature on modular equations, and these five modular equations found independently by Schröter, Russell, and Ramanujan appear to be the only known modular equations of this simple type. The other modular equations found by either Russell or Ramanujan are more complicated and do not seem to have quite the elegant partition theoretic interpretations that these other five equations have. For a history of modular equations up to 1928, consult a paper by Hanna [9].

The purpose of this paper is to establish the five partition identities implied by the five aforementioned modular equations. In closing this introduction, we remark that Warnaar [21] considerably generalized the aforementioned $q$-series identity associated with the partition identity connected with 7's., and gave a combinatorial proof of his more general identity. (In fact, Warnaar's identity is equivalent to a theorem in Chapter 16 of Ramanujan's second notebook [15] and [2, p. 47]; see [1] for a verification of this claim.) Although indeed Warnaar's proof provides a combinatorial proof of the original theorem of Farkas and Kra as a special case, his argument does not provide a simple combinatorial proof that one hopes would exist for this elegant theorem. Thus, in the words of Hirschhorn [10], 7 still remains "mysterious". In a similar vein, the numbers 3, 5, 11, and 23, corresponding respectively to the partition identities arising from the modular equations of degrees 3, 5, 11, and 23 which follow, also appear to be manifestly "mysterious". It appears to be extremely difficult to construct simple bijective proofs of each of these five partition identities. Indeed, such proofs would be of enormous interest. Observe that adding 1 to each of the five numbers above yields the divisors of 24, except 1, 2, and 3, and so it might be worthwhile in seeking bijective proofs to simultaneously consider all five cases, which appear to be the only cases where primes other than two arise. In the work of Farkas and Kra [6] and Warnaar [21], a few further partition identities, slightly different in form but also related to modular equations, were derived. The author and Baruah [1] have established one partition identity corresponding to a modular equation of degree 15 that is precisely of the type discussed in this paper. They have also discovered some partition identities arising from other modular equations of degrees 3, 5, and 15. These latter theorems are similar but also of a somewhat different flavor than those discussed in this paper. We conjecture that the five partition identities examined in this paper are unique in that, in each case, the prime 3, 5, 7, 11, or 23 cannot be replaced by any other prime.

## 2. Modular Equations

We provide some definitions in preparation for defining a modular equation, as Ramanujan would have understood it.

The complete elliptic integral of the first kind associated with the *modulus* $k$, $0 < k < 1$, is defined by

$$K := K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

The complementary modulus $k'$ is defined by $k' := \sqrt{1-k^2}$; set $K' := K(k')$. If $q = \exp(-\pi K'/K)$, then one of the central theorems in the theory of elliptic functions asserts that [2, Entry 6, p. 101]

$$\varphi^2(q) = \frac{2}{\pi}K(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \tag{2.1}$$

where $\varphi$ denotes the classical theta function defined by

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2},$$

where ${}_2F_1\left(1/2, 1/2; 1; k^2\right)$ denotes an ordinary hypergeometric function, and where the last equality in (2.1) follows from expanding the integrand in a binomial series and integrating termwise. It is (2.1) upon which all of Ramanujan's modular equations ultimately rests.

Let $K$, $K'$, $L$, and $L'$ denote complete elliptic integrals of the first kind associated with the moduli $k, k', \ell$, and $\ell' := \sqrt{1-\ell^2}$, respectively, where $0 < k, \ell < 1$. Suppose that

$$n\frac{K'}{K} = \frac{L'}{L} \tag{2.2}$$

for some positive integer $n$. A relation between $k$ and $\ell$ induced by (2.2) is called a *modular equation of degree n*. In fact, modular equations are always algebraic equations. After Ramanujan, set $\alpha = k^2$ and $\beta = \ell^2$. In the sequel, we shall frequently say that $\beta$ has degree $n$ over $\alpha$.

After Ramanujan, define

$$\chi(q) := \left(-q; q^2\right)_\infty, \tag{2.3}$$

where, as usual,

$$(a;q)_\infty = (1-a)(1-aq)\left(1-aq^2\right)\cdots, \quad |q| < 1.$$

In his notebooks [15, Chapter 17, Entry 12(v)–(vii)], [2, p. 124], Ramanujan recorded the following evaluations of $\chi$ at different arguments in terms of $\alpha$ and $q$.

**Lemma 2.1.** *We have*

$$\chi(q) = \frac{2^{1/6}q^{1/24}}{\{\alpha(1-\alpha)\}^{1/24}},$$

$$\chi(-q) = \frac{2^{1/6}q^{1/24}(1-\alpha)^{1/12}}{\alpha^{1/24}},$$

$$\chi\left(-q^2\right) = \frac{2^{1/3}q^{1/12}(1-\alpha)^{1/24}}{\alpha^{1/12}}.$$

*Suppose that $\beta$ has degree n over $\alpha$. If we replace q by $q^n$ above, then the same evaluations hold with $\alpha$ replaced by $\beta$.*

We begin each of the following five sections by citing the requisite modular equation of degree $n$. We then use Lemma 2.1 to transcribe it into an equivalent $q$-series identity.

Finally, we give the partition-theoretic interpretation of the $q$-series identity. In our transcriptions, we make use of Euler's famous identity

$$\frac{1}{(q;q^2)_\infty} = (-q;q)_\infty, \tag{2.4}$$

i.e., the number of partitions of the positive integer $n$ into odd parts is identical to the number of partitions of $n$ into distinct parts.

## 3. Partitions With Multiples of 3

The following modular equation of degree 3 was first proved by Legendre [14] and only slightly later by Jacobi in his epic work [11, p. 68], [12, p. 124]. It can also be found in Cayley's book [3, p. 196]. This modular equation was later rediscovered by Ramanujan [15, Chapter 19, Entry 5(ii)], [2, p. 230] and by Schröter [20], and is also equivalent to one of the theorems of Farkas and Kra [6]. Another proof of this modular equation by Chowla [4], [5, pp. 70–73] has been neglected. Equation (3.4) and Theorem 3.4 below also appear in both the papers of Farkas and Kra [6, Theorem 2, p. 202] and Warnaar [21].

**Theorem 3.1.** *If* $\beta$ *has degree* 3 *over* $\alpha$*, then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{3.1}$$

We now translate (3.1) into a $q$-series identity by using Lemma 2.1. Multiply both sides of (3.1) by

$$\frac{2^{2/3}q^{1/3}}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}}$$

to deduce that

$$\begin{aligned} 4q&\frac{\alpha^{1/6}}{2^{2/3}q^{1/6}(1-\alpha)^{1/12}}\frac{\beta^{1/6}}{2^{2/3}q^{3/6}(1-\beta)^{1/12}} \\ &+\frac{2^{1/3}q^{1/12}(1-\alpha)^{1/6}}{\alpha^{1/12}}\frac{2^{1/3}q^{3/12}(1-\beta)^{1/6}}{\beta^{1/12}} \\ &=\frac{2^{1/3}q^{1/12}}{\{\alpha(1-\alpha)\}^{1/12}}\frac{2^{1/3}q^{3/12}}{\{\beta(1-\beta)\}^{1/12}}. \end{aligned} \tag{3.2}$$

Using Lemma 2.1, we can rewrite (3.2) in the form

$$\frac{4q}{\chi^2(-q^2)\chi^2(-q^6)} + \chi^2(-q)\chi^2\left(-q^3\right) = \chi^2(q)\chi^2\left(q^3\right). \tag{3.3}$$

Lastly, using the definition of $\chi$ in (2.3), rearranging, and employing Euler's theorem (2.4), we see that (3.3) can be written in the equivalent form

$$(-q;q^2)_\infty^2\left(-q^3;q^6\right)_\infty^2 - (q;q^2)_\infty^2\left(q^3;q^6\right)_\infty^2 = 4q\left(-q^2;q^2\right)_\infty^2\left(-q^6;q^6\right)_\infty^2. \tag{3.4}$$

We now easily see that (3.4) has the following interpretation in terms of partitions.

**Theorem 3.2.** *Let $\mathcal{S}$ denote the set of partitions into* 4 *distinct colors with two colors, say orange and blue, each appearing at most once, and the remaining two colors, say red and green, appearing at most once and only in multiples of* 3*. Let $A(N)$ denote the number of partitions of $2N+1$ into odd elements. Let $B(N)$ denote the number of partitions of $2N$ into even elements. Then, for $N \geq 1$,*

$$A(N) = 2B(N). \tag{3.5}$$

*Example 3.3.* Let $N = 2$. Then $B(2) = 3$ and $A(2) = 6$, and

$$\begin{aligned} 4_o &= 4_b = 2_o + 2_b, \\ 5_o &= 5_b = 3_o + 1_o + 1_b = 3_b + 1_o + 1_b \\ &= 3_r + 1_o + 1_b = 3_g + 1_o + 1_b. \end{aligned}$$

*Example 3.4.* Let $N = 3$. Then $B(3) = 8$, $A(3) = 16$, and

$$6_r = 6_g = 6_b = 6_o = 4_o + 2_o = 4_o + 2_b = 4_b + 2_o = 4_b + 2_b,$$

$$7_o = 7_b = 5_o + 1_o + 1_b = 5_b + 1_o + 1_b = 3_r + 3_g + 1_b,$$

11 further partitions of the type $3+3+1$.

## 4. Partitions With Multiples of 5

The following modular equation of degree 5 is originally due to Jacobi [11,12]. Schröter [20] rediscovered it, and Ramanujan recorded it as Entry 13(i) in Chapter 19 of his second notebook [15] and [2, p. 280]

**Theorem 4.1.** *If $\beta$ has degree* 5 *over $\alpha$, then*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{4.1}$$

Multiply both sides of (4.1) by

$$\frac{2^{4/3}q}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}}$$

to write (4.1) in the form

$$\begin{aligned} 16q^3 &\frac{\alpha^{1/3}}{2^{4/3}q^{1/3}(1-\alpha)^{1/6}} \frac{\beta^{1/3}}{2^{4/3}q^{5/3}(1-\beta)^{1/6}} \\ &+ \frac{2^{2/3}q^{1/6}(1-\alpha)^{1/3}}{\alpha^{1/6}} \frac{2^{2/3}q^{5/6}(1-\beta)^{1/3}}{\beta^{1/6}} + 8q \\ &= \frac{2^{2/3}q^{1/6}}{\{\alpha(1-\alpha)\}^{1/6}} \frac{2^{2/3}q^{5/6}}{\{\beta(1-\beta)\}^{1/6}}. \end{aligned} \tag{4.2}$$

Applying Lemma 2.1, we can rewrite (4.2) as

$$\frac{16q^3}{\chi^4(-q^2)\chi^4(-q^{10})}+\chi^4(-q)\chi^4\left(-q^5\right)+8q=\chi^4(q)\chi^4\left(q^5\right). \tag{4.3}$$

Using the definition of $\chi$ in (2.3), invoking Euler's identity (2.4), and rearranging, we deduce from (4.3) that

$$\begin{aligned}&(-q;q^2)_\infty^4\left(-q^5;q^{10}\right)_\infty^4-(q;q^2)_\infty^4\left(q^5;q^{10}\right)_\infty^4\\&=8q+16q^3\left(-q^2;q^2\right)_\infty^4\left(-q^{10};q^{10}\right)_\infty^4.\end{aligned} \tag{4.4}$$

It is now readily seen that (4.4) has the following partition-theoretic interpretation.

**Theorem 4.2.** *Let $\mathcal{S}$ denote the set of partitions into* 8 *distinct colors with four colors, say orange, blue, Orange, and Blue, each appearing at most once, and the remaining four colors, say red, green, Red, and Green, appearing at most once and only in multiples of* 5*. Let $A(N)$ denote the number of partitions of $2N+1$ into odd elements. Let $B(N)$ denote the number of partitions of $2N-2$ into even elements, with the convention that $B(1)=1$. Then, $A(0)=4$, and for $N\geq 1$,*

$$A(N)=8B(N). \tag{4.5}$$

*Example 4.3.* Let $N=2$. Then $B(2)=4$ and $A(2)=32$, and

$$\begin{aligned}&2_o=2_b=2_O=2_B,\\&5_o=5_b=5_O=5_B=5_r=5_g=5_R=5_G\\&\quad=3_o+1_o+1_b,\ \text{23 further partitions of the type 3+1+1.}\end{aligned}$$

*Example 4.4.* Let $N=3$. Then $B(3)=10$ and $A(3)=80$. In particular,

$$\begin{aligned}&4_o=4_b=4_O=4_B=2_o+2_b,\ 5\ \text{further representations of the form}\ 2+2,\\&7_o=7_b=7_O=7_B\\&\quad=5_o+1_o+1_b,\ 47\ \text{further representations of the form}\ 5+1+1\\&\quad=3_o+3_b+1_o,\ 23\ \text{further representations of the form}\ 3+3+1\\&\quad=3_o+1_o+1_b+1_O+1_B,\ 3\ \text{further representations of the form}\ 3+1+1+1+1.\end{aligned}$$

## 5. Partitions With Multiples of 7

As mentioned in the Introduction, the modular equation of degree 7 in the next theorem was discovered by Guetzlaff [8] in 1834. Schröter [18], in his doctoral dissertation in 1854, and Fiedler [7], in 1885, also proved Theorem 5.1. It can furthermore be found in Ramanujan's second notebook [15, Chapter 19, Entry 19(i)] and [2, p. 314]. Both (5.4) and Theorem 5.2 below can be found in the papers of Farkas and Kra [6] and Warnaar [21].

**Theorem 5.1.** *If* $\beta$ *has degree* 7 *over* $\alpha$*, then*

$$(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}=1. \tag{5.1}$$

Multiply both sides of (5.1) by

$$\frac{2^{1/3}q^{1/3}}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}}$$

to write (5.1) in the equivalent form

$$\begin{aligned}&2q\frac{\alpha^{1/12}}{2^{1/3}q^{1/12}(1-\alpha)^{1/24}}\frac{\beta^{1/12}}{2^{1/3}q^{7/12}(1-\beta)^{1/24}}\\&\quad+\frac{2^{1/6}q^{1/24}(1-\alpha)^{1/12}}{\alpha^{1/24}}\frac{2^{1/6}q^{7/24}(1-\beta)^{1/12}}{\beta^{1/24}}\\&\quad=\frac{2^{1/6}q^{1/24}}{\{\alpha(1-\alpha)\}^{1/24}}\frac{2^{1/6}q^{7/24}}{\{\beta(1-\beta)\}^{1/24}}.\end{aligned} \tag{5.2}$$

Applying Lemma 2.1 to (5.2), we see that

$$\frac{2q}{\chi(-q^2)\chi(-q^{14})}+\chi(-q)\chi\left(-q^7\right)=\chi(q)\chi\left(q^7\right). \tag{5.3}$$

Using the definition (2.3) and Euler's identity (2.4), we readily find that (5.3) can be recast in the form

$$\left(-q;q^2\right)_\infty\left(-q^7;q^{14}\right)_\infty-\left(q;q^2\right)_\infty\left(q^7;q^{14}\right)_\infty=2q\left(-q^2;q^2\right)_\infty\left(-q^{14};q^{14}\right)_\infty. \tag{5.4}$$

Interpreting (5.4) in terms of partitions, we deduce the following theorem.

**Theorem 5.2.** *Let* $\mathcal{S}$ *denote the set of partitions into* 2 *distinct colors with one color, say orange, appearing at most once, and the remaining color, say blue, appearing at most once and only in multiples of* 7*. Let* $A(N)$ *denote the number of partitions of* $2N+1$ *into odd elements. Let* $B(N)$ *denote the number of partitions of* $2N$ *into even elements. Then, for* $N\geq 0$*,*

$$A(N)=B(N). \tag{5.5}$$

A combinatorial proof of a considerable generalization of Theorem 5.2 has been given by Warnaar [21].

*Example 5.3.* Let $N=7$. Then $A(7)=B(7)=6$, and

$$14_o=14_b=12_o+2_o=10_o+4_o=8_o+6_o=8_o+4_o+2_o,$$

$$15_o=11_o+3_o+1_o=9_o+5_o+1_o=7_o+7_b+1_o=7_o+5_o+3_o=7_b+5_o+3_o.$$

*Example 5.4.* Let $N=8$. Then $A(8)=B(8)=7$, and

$$16_o=14_o+2_o=14_b+2_o=12_o+4_o=10_o+6_o=10_o+4_o+2_o=8_o+6_o+2_o,$$

$$\begin{aligned}17_o&=13_o+3_o+1_o=11_o+5_o+1_o=9_o+7_o+1_o\\&=9_o+7_b+1_o=9_o+5_o+3_o=7_o+7_b+3_o.\end{aligned}$$

## 6. Partitions With Multiples of 11

The modular equation in Theorem 6.1 is due to Schröter [19, 20], and was rediscovered by Ramanujan, who recorded it as Entry 7(i) in Chapter 20 of his second notebook [15] and [2, p. 363].

**Theorem 6.1.** *If* $\beta$ *has degree* 11 *over* $\alpha$*, then*

$$(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}+2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}=1. \tag{6.1}$$

Multiply both sides of (6.1) by

$$\frac{2^{2/3}q}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}}$$

to deduce that

$$\begin{aligned}4q^3&\frac{\alpha^{1/6}}{2^{2/3}q^{1/6}(1-\alpha)^{1/12}}\frac{\beta^{1/6}}{2^{2/3}q^{11/6}(1-\beta)^{1/12}}\\&+\frac{2^{1/3}q^{1/12}(1-\alpha)^{1/6}}{\alpha^{1/12}}\frac{2^{1/3}q^{11/12}(1-\beta)^{1/6}}{\beta^{1/12}}\\&+4q=\frac{2^{1/3}q^{1/12}}{\{\alpha(1-\alpha)\}^{1/12}}\frac{2^{1/3}q^{11/12}}{\{\beta(1-\beta)\}^{1/12}}.\end{aligned} \tag{6.2}$$

By Lemma 2.1, the identity (6.2) can be transformed into the equivalent identity

$$\frac{4q^3}{\chi^2(-q^2)\chi^2(-q^{22})}+\chi^2(-q)\chi^2\left(-q^{11}\right)+4q=\chi^2(q)\chi^2\left(q^{11}\right). \tag{6.3}$$

Invoking (2.3) and (2.4) in (6.3) and rearranging, we conclude that

$$\begin{aligned}&\left(-q;q^2\right)_\infty^2\left(-q^{11};q^{22}\right)_\infty^2-\left(q;q^2\right)_\infty^2\left(q^{11};q^{22}\right)_\infty^2\\&=4q+4q^3\left(-q^2;q^2\right)_\infty^2\left(-q^{22};q^{22}\right)_\infty^2.\end{aligned} \tag{6.4}$$

Transcribing (6.4) into a theorem about partitions, we deduce the following theorem.

**Theorem 6.2.** *Let* $\mathcal{S}$ *denote the set of partitions into* 4 *distinct colors with two colors, say orange and blue, each appearing at most once, and the remaining two colors, say red and green, each appearing at most once and only in multiples of* 11*. Let* $A(N)$ *denote the number of partitions of* $2N+1$ *into odd elements. Let* $B(N)$ *denote the number of partitions of* $2N-2$ *into even elements, with the convention that* $B(1)=1$*. Then,* $A(0)=2$*, and, for* $N\geq 1$*,*

$$A(N)=2B(N). \tag{6.5}$$

*Example 6.3.* Let $N = 5$. Then $B(5) = 9$ and $A(5) = 18$, and

$$\begin{aligned}
8_o &= 8_b = 6_o + 2_o = 6_o + 2_b = 6_b + 2_o = 6_b + 2_b = 4_o + 4_b \\
&= 4_o + 2_o + 2_b = 4_b + 2_o + 2_b, \\
11_o &= 11_b = 11_r = 11_g = 9_o + 1_o + 1_b = 9_b + 1_o + 1_b = 7_o + 3_o + 1_o = 7_o + 3_o + 1_b \\
&= 7_o + 3_b + 1_o = 7_o + 3_b + 1_b = 7_b + 3_o + 1_o = 7_b + 3_o + 1_b = 7_b + 3_b + 1_o \\
&= 7_b + 3_b + 1_b = 5_o + 5_b + 1_o = 5_o + 5_b + 1_b = 5_o + 3_o + 3_b = 5_b + 3_o + 3_b.
\end{aligned}$$

*Example 6.4.* Let $N = 6$. Then $B(6) = 14$ and $A(6) = 28$, and

$$\begin{aligned}
10_o &= 10_b = 8_o + 2_b = 8_o + 2_o = 8_b + 2_b = 8_b + 2_o \\
&= 6_b + 4_o = 6_b + 4_b = 6_o + 4_b = 6_o + 4_o = 4_b + 4_o + 2_b = 4_o + 4_b + 2_o \\
&= 6_b + 2_b + 2_o = 6_o + 2_b + 2_o, \\
13_o &= 13_b = 11_o + 1_o + 1_b = 11_b + 1_o + 1_b = 11_r + 1_o + 1_b = 11_g + 1_o + 1_b \\
&= 9_o + 3_o + 1_o, \text{ 7 further representations of the form } 9 + 3 + 1 \\
&= 7_o + 3_o + 3_b = 7_b + 3_o + 3_b \\
&= 7_o + 5_o + 1_o, \text{ 7 further representations of the form } 7 + 5 + 1 \\
&= 5_o + 5_b + 3_o = 5_o + 5_b + 3_b = 5_o + 3_o + 3_b + 1_o + 1_b = 5_b + 3_o + 3_b + 1_o + 1_b.
\end{aligned}$$

## 7. Partitions With Multiples of 23

The next theorem is also due to Schröter [19,20]. Ramanujan rediscovered this modular equation and recorded it as Entry 15(i) in Chapter 20 of his second notebook [15] and [2, p. 411]. According to Klein [13], Hurwitz also discovered this modular equation, but Klein does not provide a reference for Hurwitz's work.

**Theorem 7.1.** *If* $\beta$ *is of degree* 23 *over* $\alpha$*, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1. \tag{7.1}$$

We multiply both sides of (7.1) by

$$\frac{2^{1/3}q}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}}$$

and deduce that

$$\begin{aligned} 2q^3 &\frac{\alpha^{1/12}}{2^{1/3}q^{1/12}(1-\alpha)^{1/24}}\frac{\beta^{1/12}}{2^{1/3}q^{23/12}(1-\beta)^{1/24}} \\ &+\frac{2^{1/6}q^{1/24}(1-\alpha)^{1/12}}{\alpha^{1/24}}\frac{2^{1/6}q^{23/24}(1-\beta)^{1/12}}{\beta^{1/24}}+2q \\ &=\frac{2^{1/6}q^{1/24}}{\{\alpha(1-\alpha)\}^{1/24}}\frac{2^{1/6}q^{23/24}}{\{\beta(1-\beta)\}^{1/24}}. \end{aligned} \tag{7.2}$$

Using Lemma 2.1, we write (7.2) in the shape

$$\frac{2q^3}{\chi(-q^2)\chi(-q^{46})}+\chi(-q)\chi\left(-q^{23}\right)+2q=\chi(q)\chi\left(q^{23}\right). \tag{7.3}$$

Using (2.3) and (2.4), we transform (7.3) into $q$-products, and rearrange to deduce that

$$\begin{aligned} &\left(-q;q^2\right)_\infty\left(-q^{23};q^{46}\right)_\infty-\left(q;q^2\right)_\infty\left(q^{23};q^{46}\right)_\infty \\ &=2q+2q^3\left(-q^2;q^2\right)_\infty\left(-q^{46};q^{46}\right)_\infty. \end{aligned} \tag{7.4}$$

We now record the partition-theoretic interpretation of (7.4).

**Theorem 7.2.** *Let $\mathcal{S}$ denote the set of partitions into 2 distinct colors with one color, say orange, appearing at most once, and the remaining color, say blue, appearing at most once and only in multiples of 23. Let $A(N)$ denote the number of partitions of $2N+1$ into odd elements. Let $B(N)$ denote the number of partitions of $2N-2$ into even elements, with the convention that $B(1)=1$. Then, for $N\geq 1$,*

$$A(N)=B(N). \tag{7.5}$$

*Example 7.3.* Let $N=11$. Then $A(11)=B(11)=10$, and

$$\begin{aligned} 20_o&=18_o+2_o=16_o+4_o=14_o+6_o=14_o+4_o+2_o=12_o+8_o \\ &=12_o+6_o+2_o=10_o+8_o+2_o=10_o+6_o+4_o=8_o+6_o+4_o+2_o, \\ 23_o&=23_b=19_o+3_o+1_o=17_o+5_o+1_o=15_o+7_o+1_o=15_o+5_o+3_o \\ &=13_o+9_o+1_o=13_o+7_o+3_o=11_o+9_o+3_o=11_o+7_o+5_o. \end{aligned}$$

*Example 7.4.* Let $N=12$. Then $A(12)=B(12)=12$, and

$$\begin{aligned} 22_o&=20_o+2_o=18_o+4_o=16_o+6_o=16_o+4_o+2_o=14_o+8_o=14_o+6_o+2_o \\ &=12_o+10_o=12_o+8_o+2_o=12_o+6_o+4_o \\ &=10_o+8_o+4_o=10_o+6_o+4_o+2_o, \\ 25_o&=21_o+3_o+1_o=19_o+5_o+1_o=17_o+7_o+1_o=17_o+5_o+3_o=15_o+7_o+3_o \\ &=15_o+9_o+1_o=13_o+11_o+1_o=13_o+9_o+3_o=13_o+7_o+5_o \\ &=11_o+9_o+5_o=9_o+7_o+5_o+3_o+1_o. \end{aligned}$$

*Note added in proof.* In a recently submitted paper, Bijective proofs of partition identities arising from modular equations, Sun Kim has established bijective proofs of the Farkas-Kra identity, Warnaar's generalization, and further partition identities.

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## References

1. N.D. Baruah and B.C. Berndt, Partition identities and Ramanujan's modular equations, J. Combin. Theory Ser. A **114** (6) (2007) 1024–1045.
2. B.C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.
3. A. Cayley, An Elementary Treatise on Elliptic Functions, 2nd Ed., Dover Publications, New York, 1961.
4. S.D. Chowla, An elementary treatment of the modular equation of the third order, J. Indian Math. Soc. **17** (1927) 37–40.
5. S.D. Chowla, The Collected Papers of Sarvadaman Chowla, Vol. I, 1925–1935, Centre de Recherches Mathématiques, Montreal, 1999.
6. H.M. Farkas and I. Kra, Partitions and theta constant identities, In: The Mathematics of Leon Ehrenpreis, Contemp. Math. No. 251, American Mathematical Society, Providence, RI, (2000) pp. 197–203.
7. E. Fiedler, Ueber eine besondere Classe irrationaler Modulargleichungen der elliptischen Functionen, Vierteljahrsschr. Naturforsch. Gesell. (Zürich) **12** (1885) 129–229.
8. C. Guetzlaff, Aequatio modularis pro transformatione functionum ellipticarum septimi ordinis, J. Reine Angew. Math. **12** (1834) 173–177.
9. M. Hanna, The modular equations, Proc. London Math. Soc. (2) **28** (1928) 46–52.
10. M.D. Hirschhorn, The case of the mysterious sevens, Int. J. Number Theory **2** (2) (2006) 213–216.
11. C.G.J. Jacobi, Fundamenta Nova Theoriae Functionum Ellipticarum, Sumptibus Fratrum Bornträger, Regiomonti, 1829.
12. C.G.J. Jacobi, Gesammelte Werke, Erster Band, G. Reimer, Berlin, 1881.
13. F. Klein, Zur Theorie der elliptischen Modulfunctionen, Math. Ann. **17** (1) (1880) 62–70.
14. A.M. Legendre, Traité des Fonctions Elliptiques, Tome 1, Huzard-Courcier, Paris, 1825.
15. S. Ramanujan, Notebooks, Vols. 1–2, Tata Institute of Fundamental Research, Bombay, 1957.
16. R. Russell, On $\kappa\lambda - \kappa'\lambda'$ modular equations, Proc. London Math. Soc. **19** (1887) 90–111.
17. R. Russell, On modular equations, Proc. London Math. Soc. **21** (1890) 351–395.
18. H. Schröter, De aequationibus modularibus, Dissertatio Inauguralis, Albertina Litterarum Universitate, Regiomonti, 1854.
19. H. Schröter, Ueber Modulargleichungen der elliptischen Functionen, Auszug aus einem Schreiben an Herrn L. Kronecker, J. Reine Angew. Math. **58** (1861) 378–379.
20. H. Schröter, Beiträge zur Theorie der elliptischen Funktionen, Acta Math. **5** (1884) 205–208.
21. S.O. Warnaar, A generalization of the Farkas and Kra partition theorem for modulus 7, J. Combin. Theory Ser. A **110** (2005) 43–52.