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Permutation Diagrams, Fixed Points and Kazhdan-Lusztig *R*-Polynomials

Federico Incitti

Department of Mathematics, Royal Institute of Technology, Lindstedts väg 25, 10044 Stockholm, Sweden
incitti@math.kth.se

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Abstract. In this paper, we give an algorithm for computing the Kazhdan-Lusztig R-polynomials in the symmetric group. The algorithm is described in terms of permutation diagrams. In particular we focus on how the computation of the polynomial is affected by certain fixed points. As a consequence of our methods, we obtain explicit formulas for the R-polynomials associated with some general classes of intervals, generalizing results of Brenti and Pagliacci.

Keywords: permutation diagram, fixed point, Bruhat order, Kazhdan-Lusztig polynomial, R-polynomial

1. Introduction

In [10], Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials with integer coefficients, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials. They are related to the algebraic geometry and topology of Schubert varieties, and also play a crucial role in representation theory. In order to prove the existence of these polynomials, Kazhdan and Lusztig used another family of polynomials which arises from the multiplicative structure of the Hecke algebra associated with W. These are known as the R-polynomials. In [4], Dyer related the computation of the R-polynomials to the enumeration of the paths in the Bruhat graph of W which are increasing with respect to certain orderings.

In this paper, we give an algorithm for generating all the increasing paths between two permutations in the Bruhat graph of the symmetric group. By Dyer's result, this allows to compute the R-polynomial associated with the two permutations. In particular, we obtain a recursive formula, which is closely related to the one given in [3], but with some differences that will be pointed out. The algorithm is given in terms of a pictorial way of describing the Bruhat order in the symmetric group, namely the diagram of a pair of permutations, which was already introduced in [9]. In particular, we analyse the effects of certain fixed points on the computation of the polynomial. As a consequence of our methods, we obtain closed expressions of the R-polynomials for some general classes of intervals, generalizing results that appeared in [3] and [11].

Some basic definitions and notation are collected in Section 2. In Section 3, we describe the diagram of a pair of permutations, showing how the poset structure of the corresponding interval is related to it. Section 4 is devoted to the description of the main algorithm, which we call the *stair method*. In Section 5, we derive some consequences, in particular on how the computation of the R-polynomial is affected by certain fixed points. Finally, in Section 6, we obtain explicit formulas for the R-polynomials associated with some general classes of intervals.

2. Preliminaries

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{Z}$ be the set of integers. For $n \in \mathbf{N}$, let $[n] = \{1, 2, \ldots, n\}$ and for $n, m \in \mathbf{N}$, with $n \le m$, let $[n, m] = \{n, n+1, \ldots, m\}$. We refer to [12] for general poset theory. Given a poset P, we denote by $\lhd$ the covering relation. The *Hasse diagram* of P is the directed graph having P as vertex set such that there is an edge from x to y if and only if $x \lhd y$. Given $x, y \in P$ with $x < y$, we set $[x, y] = \{z \in P\colon x \le z \le y\}$, and call it an *interval* of P. An *atom* (respectively, *coatom*) of $[x, y]$ is an element $z \in [x, y]$ such that $x \lhd z$ (respectively, $z \lhd y$).

We refer to [7] for basic notions about Coxeter groups. Given a Coxeter group W with the set of generators S, the set of *reflections* is

$$T = \{wsw^{-1}\colon w \in W,\ s \in S\}.$$

Given $x \in W$, the *length* of x, denoted by $\ell(x)$, is the minimal k such that x can be written as a product of k generators. The *Bruhat graph* of W, denoted by $BG(W)$ (or simply BG) is the directed graph having W as vertex set such that there is a directed edge $x \to y$ if and only if $y = xt$, with $t \in T$, and $\ell(x) < \ell(y)$. For convenience, we label the edge by the reflection t:

$$x \xrightarrow{t} y.$$

Finally, the *Bruhat order* of W is the partial order induced by BG; in other words, given $x, y \in W$, $x \le y$ in the Bruhat order if and only if there is a path from x to y in BG:

$$x = x_0 \to x_1 \to \cdots \to x_k = y.$$

It is known that W, partially ordered by the Bruhat order, is a graded poset with rank function given by the length. Given $x, y \in W$ with $x < y$, we set $\ell(x, y) = \ell(y) - \ell(x)$, and call it the *length* of the pair (x, y). Note that $\ell(x, y)$ is the distance between x and y in the Hasse diagram of $[x, y]$.

Kazhdan-Lusztig polynomials and R-polynomials of W can be defined in several equivalent ways. Following [1], we start with defining the R-polynomials and then we use them to define the Kazhdan-Lusztig polynomials.

Theorem 2.1. *There exists a unique family of polynomials* $\{R_{x,y}(q)\}_{x,y \in W} \subseteq \mathbf{Z}[q]$ *satisfying the following conditions*:

(1) $R_{x,y}(q) = 0$, *if* $x \not\leq y$;
(2) $R_{x,y}(q) = 1$, *if* $x = y$;
(3) *if* $x < y$ *and* $s \in S$ *is such that* $ys \lhd y$, *then*

$$R_{x,y}(q) = \begin{cases} R_{xs,ys}(q), & \text{if } xs \lhd x, \\ qR_{xs,ys}(q) + (q-1)R_{x,ys}(q), & \text{if } xs \rhd x. \end{cases}$$

The existence of such a family is a consequence of the invertibility of certain basis elements of the Hecke algebra $\mathcal{H}$ of W and is proved in [7, §§7.4 and 7.5]. The polynomials whose existence and uniqueness are guaranteed by Theorem 2.1 are called the *R-polynomials* of W. Theorem 2.1 can be used to compute the polynomials $\{R_{x,y}(q)\}_{x,y \in W}$, by induction on $\ell(y)$.

Theorem 2.2. *There exists a unique family of polynomials* $\{P_{x,y}(q)\}_{x,y \in W} \subseteq \mathbf{Z}[q]$ *satisfying the following conditions*:

(1) $P_{x,y}(q) = 0$, *if* $x \not\leq y$;
(2) $P_{x,y}(q) = 1$, *if* $x = y$;
(3) *if* $x < y$, *then* $\deg(P_{x,y}(q)) < \ell(x,y)/2$ *and*

$$q^{\ell(x,y)} P_{x,y}\left(q^{-1}\right) - P_{x,y}(q) = \sum_{x < z \leq y} R_{x,z}(q) P_{z,y}(q).$$

A proof appears in [7, §§7.9, 7.10 and 7.11]. The polynomials whose existence and uniqueness are guaranteed by Theorem 2.2 are called the *Kazhdan-Lusztig polynomials* of W. By Theorem 2.2, knowing the R-polynomials is equivalent to knowing the Kazhdan-Lusztig polynomials. In fact condition (3) can be recursively used to compute one family from the other, by induction on $\ell(x,y)$.

In order to give a combinatorial interpretation of the R-polynomials, another family of polynomials, known as the *$\widetilde{R}$-polynomials*, has been introduced. The following is [1, Proposition 5.3.1].

Proposition 2.3. *Let* $x, y \in W$ *with* $x < y$. *Then there is a unique polynomial* $\widetilde{R}_{x,y}(q) \in \mathbf{Z}_{\geq 0}[q]$ *such that*

$$R_{x,y}(q) = q^{\ell(x,y)/2} \widetilde{R}_{x,y}\left(q^{1/2} - q^{-1/2}\right).$$

The advantage of the $\widetilde{R}$-polynomials over the R-polynomials is that they have nonnegative integer coefficients. In fact there is a nice combinatorial interpretation for them. We start by giving some notation.

Given $x, y \in W$ with $x < y$, we denote by $Paths(x,y)$ the set of paths in BG from x to y. The *length* of $\Delta = (x_0, x_1, \ldots, x_k) \in Paths(x,y)$, denoted by $|\Delta|$, is the number k of its directed edges. Now let $\prec$ be a fixed *reflection ordering* on the set T of reflections (see, e.g., [1, § 5.2] for the definition of a reflection ordering and for a proof of existence). A path $\Delta = (x_0, x_1, \ldots, x_k) \in Paths(x,y)$, with

$$x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} x_k,$$

is said to be *increasing* with respect to the order $\prec$ if

$$t_1 \prec t_2 \prec \cdots \prec t_k.$$

We denote by $Paths^{\prec}(x, y)$ the set of paths in $Paths(x, y)$ which are increasing with respect to $\prec$. The following result is due to Dyer [4].

Theorem 2.4. *Let W be a Coxeter group and $x, y \in W$ with $x < y$. Set $\ell = \ell(x, y)$. Then*

$$\widetilde{R}_{x,y}(q) = \sum_{k=1}^{\ell} c_k q^k, \tag{2.1}$$

where

$$c_k = \left|\left\{\Delta \in Paths^{\prec}(x, y) \colon |\Delta| = k\right\}\right|,$$

for every $k \in [\ell]$.

A few remarks on equation (2.1) follow. Every path from x to y in BG necessarily has a length of the same parity as ℓ, so $c_k = 0$ if $k \not\equiv \ell \pmod 2$. Furthermore, by the EL-shellability of the Bruhat order (see, e.g., [2]), in BG there is exactly one increasing path from x to y of length ℓ, thus $c_\ell = 1$.

Finally, let us introduce the notion of absolute length of a pair. We recall that the *absolute length* of $x \in W$, denoted by $a\ell(x)$, is defined as the minimum k such that x can be written as a product of k reflections. By [5, Theorem 1.2], $a\ell(x)$ is the (oriented) distance between e and x in BG. By [5, Theorem 2.3], the coefficient of q^k in $\widetilde{R}_{x,y}(q)$ is non-zero if and only if there is a path in BG of length k. These facts suggest the following definition.

Definition 2.5. *Let $x, y \in W$ with $x < y$. The* absolute length *of (x, y), denoted by $a\ell(x, y)$, is the (oriented) distance between x and y in BG.*

By the results in [5], this notion generalizes that of absolute length of an element. In fact, given $x \in W$, we have

$$a\ell(x) = a\ell(e, x).$$

On the other hand, by [3, Proposition 6.1], there exists $m \equiv \ell(x, y) \pmod 2$ such that the coefficient of q^k in $\widetilde{R}_{x,y}(q)$ is non-zero if and only if $k \in [m, \ell(x, y)]$. It turns out that this m is exactly $a\ell(x, y)$. Putting all together, we have the following.

Corollary 2.6. *Let W be a Coxeter group and $x, y \in W$ with $x < y$. Set $\ell = \ell(x, y)$ and $a\ell = a\ell(x, y)$. Then*

$$\widetilde{R}_{x,y}(q) = q^{\ell} + c_{\ell-2}\, q^{\ell-2} + \cdots + c_{a\ell+2}\, q^{a\ell+2} + c_{a\ell}\, q^{a\ell},$$

where

$$c_k = \left|\left\{\Delta \in Paths^{\prec}(x, y) \colon |\Delta| = k\right\}\right| \geq 1,$$

for every $k \in [a\ell, \ell - 2]$ with $k \equiv \ell \pmod 2$.

3. Diagram of a Pair of Permutations

In this section, we describe a way of "drawing" the Bruhat order in the symmetric group, which was already introduced in [9]. We present it here using a slightly different notation, which is more suitable for our purposes.

We denote by S_n the *symmetric group*, that is the set of all bijections on $[n]$, and call its elements *permutations*. To denote a permutation $x \in S_n$ we often use the one-line notation: $x = x_1 x_2 \cdots x_n$ means that $x(i) = x_i$ for every $i \in [n]$. The *diagram* of a permutation $x \in S_n$ is the subset of $\mathbf{N}^2$ defined by

$$Diag(x) = \{(i, x(i)) \colon i \in [n]\}.$$

The symmetric group S_n is known to be a Coxeter group, with generators given by the simple transpositions $(i, i+1)$, for $i \in [n-1]$. The length of a permutation $x \in S_n$ is given by $\ell(x) = inv(x)$, where

$$inv(x) = \left|\left\{(i, j) \in [n]^2 \colon i < j, x(i) > x(j)\right\}\right|$$

is the number of *inversions* of x.

There is a nice combinatorial characterization of the Bruhat order relation in the symmetric group. In order to give that, we introduce the following notation: for $x \in S_n$ and $(h, k) \in [n]^2$, we set

$$x[h, k] = \left|\{i \in [n] \colon i \leq h, x(i) \geq k\}\right|, \tag{3.1}$$

that is, $x[h, k]$ is the number of points of the diagram of x lying in the upper-left quarter plane with origin at (h, k). Given $x, y \in S_n$ and $(h, k) \in [n]^2$, we set

$$(x, y)[h, k] = y(h, k) - x(h, k). \tag{3.2}$$

The characterization is the following (see e.g., [1, Theorem 2.1.5]).

Theorem 3.1. *Let $x, y \in S_n$. Then*

$$x \leq y \Leftrightarrow (x, y)[h, k] \geq 0, \textit{ for every } (h, k) \in [n]^2.$$

It is useful to extend the notation introduced in (3.1) and (3.2) to every $(h, k) \in \mathbf{R}^2$. We call the mapping $(h, k) \mapsto (x, y)[h, k]$, which associates with every $(h, k) \in \mathbf{R}^2$ the integer $(x, y)[h, k]$, the *multiplicity mapping* of the pair (x, y). Theorem 3.1 can be reformulated as follows:

$$x \leq y \Leftrightarrow (x, y)[h, k] \geq 0, \textit{ for every } (h, k) \in \mathbf{R}^2.$$

Definition 3.2. *Let $x, y \in S_n$. The* diagram *of the pair (x, y) is the collection of*:

(1) *the diagram of x*;
(2) *the diagram of y*;
(3) *the multiplicity mapping $(h, k) \mapsto (x, y)[h, k]$.*

We pictorially represent the diagram of a pair (x, y) with the following convention: the diagram of x (respectively, y) is denoted by black dots (respectively, empty circles) and, if $x < y$, then the mapping $(h, k) \mapsto (x, y)[h, k]$ is represented by colouring the preimages of different positive integers with different levels of grey, with the rule that with a lower integer corresponds a lighter grey. An example is shown in Figure 1, where $(x, y) = (315472986, 782496315)$.

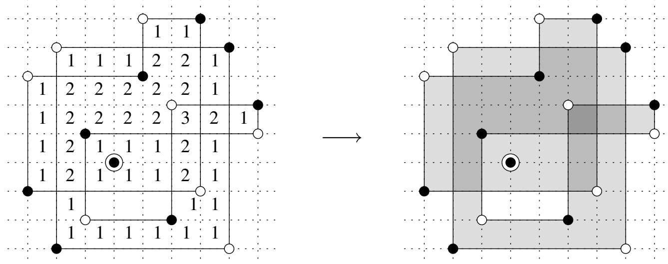


Figure 1: Diagram of a pair of permutations.

A nice way of obtaining the multiplicity mapping of (x, y), starting from the diagrams of x and y, is the following. Consider, for every $h \in [n]$, the vertical segment having $(h, x(h))$ and $(h, y(h))$ as extremes, and say that it is *positive* or *negative* depending on whether $x(h) < y(h)$ or $x(h) > y(h)$. Similarly, for every $k \in [n]$, the horizontal segment having $(x^{-1}(k), k)$ and $(y^{-1}(k), k)$ as extremes is said to be *positive* if $x^{-1}(k) < y^{-1}(k)$ and *negative* if $x^{-1}(k) > y^{-1}(k)$. Now, for every $(h, k) \in (\mathbf{R}\setminus\mathbf{N})^2$, let $left^+(h, k)$ and $left^-(h, k)$ be, respectively, the number of positive and negative vertical segments at the left of (h, k), namely those intersecting the segment having $(0, k)$ and (h, k) as extremes. Define in a similar way $right^{\pm}(h, k)$, $down^{\pm}(h, k)$ and $up^{\pm}(h, k)$. Then

$$\begin{aligned}(x, y)[h, k] &= left^+(h, k) - left^-(h, k)\\ &= right^-(h, k) - right^+(h, k)\\ &= down^+(h, k) - down^-(h, k)\\ &= up^-(h, k) - up^+(h, k).\end{aligned}$$

Note that the diagram of a pair (x, y) contains information about the cycle structure of $x^{-1}y$. In fact there is a trivial bijection between the disjoint cycles in which $x^{-1}y$ is decomposed, and the closed paths in the diagram of (x, y) made of vertical and horizontal segments. For the example shown in Figure 1, we have

$$x^{-1}y = (1, 5, 7)(2, 8)(3, 9, 6)$$

and in the diagram there are exactly three closed paths, whose vertical edges correspond to the indices $(1, 5, 7)$, $(2, 8)$ and $(3, 9, 6)$.

It is useful to give the following definitions.

Definition 3.3. *Let $x, y \in S_n$ with $x < y$. The* support *of the pair* (x, y) *is*

$$\Omega(x, y) = \left\{(h, k) \in \mathbf{R}^2 : (x, y)[h, k] > 0\right\},$$

and the support index set *of* (x, y) *is*

$$I_\Omega(x, y) = \left\{i \in [n] : (i, x(i)) \in \overline{\Omega(x, y)}\right\},$$

where $\overline{\Omega(x, y)}$ *denotes the* (*topological*) *closure of the set* $\Omega(x, y)$.

We also set

$$Fix(x, y) = Fix(x^{-1}y) = \{i \in [n] : x(i) = y(i)\}.$$

Note that $[n]\backslash I_\Omega(x, y) \subseteq Fix(x, y)$. Finally, set

$$Fix_\Omega(x, y) = Fix(x, y) \cap I_\Omega(x, y),$$

$fix(x, y) = |Fix(x, y)|$ and $fix_\Omega(x, y) = |Fix_\Omega(x, y)|$.

A combinatorial characterization of the covering relation in the Bruhat order of the symmetric group is given in terms of free rises. We recall that, given $x \in S_n$, a *free rise* of x is a pair (i, j), with $i < j$ and $x(i) < x(j)$, such that there is no $k \in \mathbf{N}$, with $i < k < j$ and $x(i) < x(k) < x(j)$. It is known that, given $x, y \in S_n$, then $x \lhd y$ if and only if $y = x(i, j)$, where (i, j) is a free rise of x.

Given $a, b, c, d \in [n]$ with $a < b$ and $c < d$, we set

$$Rect(a, b, c, d) = \left\{(h, k) \in \mathbf{R}^2 : a \leq h < b, c < k \leq d\right\}.$$

Now, let $x, y \in S_n$ with $x < y$. In order to describe the atoms of the interval $[x, y]$, we say that a free rise (i, j) of x is *good* with respect to y if

$$Rect(i, j, x(i), x(j)) \subseteq \Omega(x, y).$$

The following gives a characterization of the atoms of an interval.

Proposition 3.4. *Let $x, y \in S_n$ with $x < y$. Then z is an atom of $[x, y]$ if and only if $z = x(i, j)$, where (i, j) is a free rise of x good with respect to y.*

Proof. Let (i, j) be a free rise of x, and let $z = x(i, j)$. We know that $x \lhd z$. Thus z is an atom of $[x, y]$ if and only if $z \leq y$. As it can be easily checked, for every $(h, k) \in \mathbf{R}^2$, we have

$$(z, y)[h, k] = \begin{cases} (x, y)[h, k] - 1, & \text{if } (h, k) \in Rect(i, j, x(i), x(j)), \\ (x, y)[h, k], & \text{otherwise.} \end{cases}$$

Then, by Theorem 3.1, $z \leq y$ if and only if $(x, y)[h, k] \geq 1$ for every $(h, k) \in Rect\,(i, j, x(i), x(j))$, that is, if and only if $Rect(i, j, x(i), x(j)) \subseteq \Omega(x, y)$. ∎

Symmetrically, we can define a *free inversion* of y, as a pair (i, j), with $i < j$ and $y(i) > y(j)$, such that there is no $k \in \mathbf{N}$ with $i < k < j$ and $y(i) > y(k) > y(j)$. Note that $x \lhd y$ if and only if $x = y(i, j)$, where (i, j) is a free inversion of y.

Given $x, y \in S_n$ with $x < y$, we say that a free inversion (i, j) of y is *good* with respect to x if

$$Rect(i, j, y(j), y(i)) \subseteq \Omega(x, y).$$

The next result gives a characterization of the coatoms of an interval, and its proof is dual to that of Proposition 3.4.

Proposition 3.5. *Let $x, y \in S_n$ with $x < y$. Then w is a coatom of $[x, y]$ if and only if $w = y(i, j)$, where (i, j) is a free inversion of y good with respect to x.*

Going back to the example shown in Figure 1, the free rises of x good with respect to y, and the free inversions of y good with respect to x are illustrated in Figure 2. Note that among the free rises of x, exactly 3 are not good with respect to y, namely $(2, 4)$, $(4, 9)$ and $(6, 9)$. And among the free inversions of y, only $(1, 3)$ is not good with respect to x. We can conclude that

$$a(x, y) = c(x, y) = 11.$$

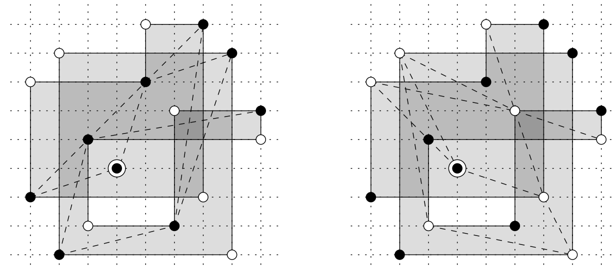


Figure 2: Good free rises and good free inversions.

By Proposition 3.4, if z is an atom of $[x, y]$, then $z = x(i, j)$ with (i, j) being a free rise of x, good with respect to y. Note that in this case the diagram of (z, y) is obtained from that of (x, y) by "removing" the rectangle $Rect(i, j, x(i), x(j))$. By Proposition 3.5, a similar consideration holds for coatoms. In Figure 3 two examples are shown. This means that the structure of the interval $[x, y]$, as an abstract poset, reflects this process of "unmounting" the diagram of (x, y), by throwing away rectangles step by step. This way of viewing the intervals in the symmetric group is crucial for the considerations that follow.

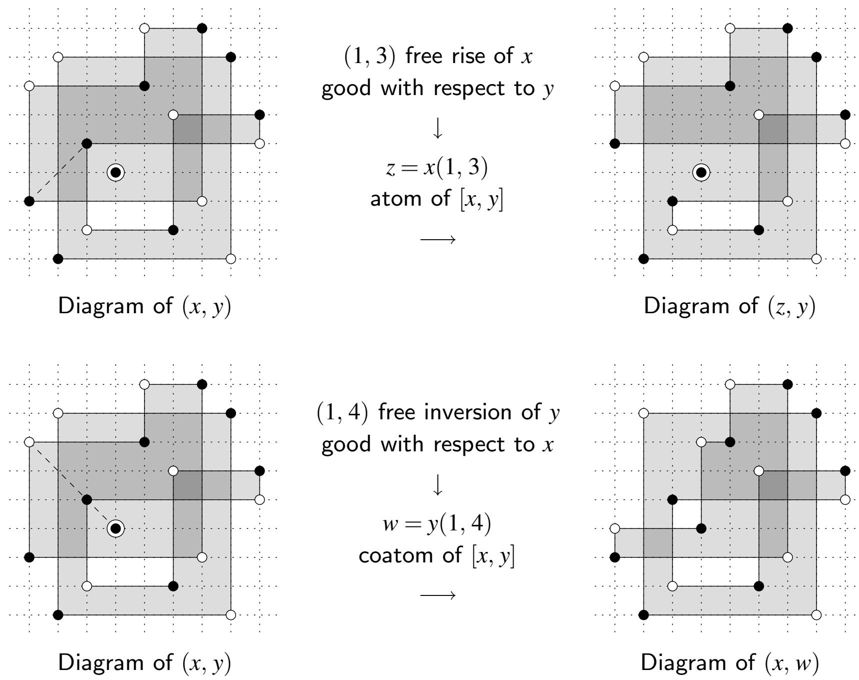


Figure 3: Atoms and coatoms.

4. The Stair Method

In this section, we derive an algorithm for generating all the increasing paths in BG between two given permutations x and y. As a corollary, by Theorem 2.4, we obtain a recursive formula for computing the $\widetilde{R}$-polynomial associated with (x, y).

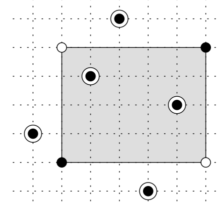
It is known that in the symmetric group S_n the reflections are the transpositions:

$$T = \{(i, j) \colon i, j \in [n]\}$$

and that a possible reflection ordering on them is the lexicographic order. From now on we assume this order fixed on the transpositions. For instance, in S_4:

$$(1, 2) \prec (1, 3) \prec (1, 4) \prec (2, 3) \prec (2, 4) \prec (3, 4).$$

Let $x, y \in S_n$ with $x < y$. We start with the case in which (x, y) is an edge of BG, that is $y = x(i, j)$ for some rise (i, j) of x. In this case we have $\Omega(x, y) = Rect(i, j, x(i), x(j))$. A possible diagram of (x, y) is the following:



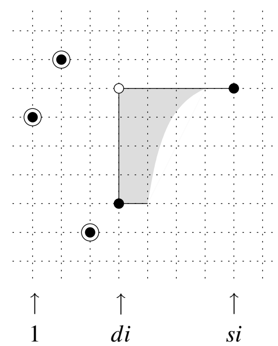
In this case we have

$$\ell(x, y) = 1 + 2 fix_\Omega(x, y).$$

Now, let (x, y) be any pair of permutations with $x < y$. Consider an increasing path in BG from x to y:

$$x = x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} x_k = y. \quad (4.1)$$

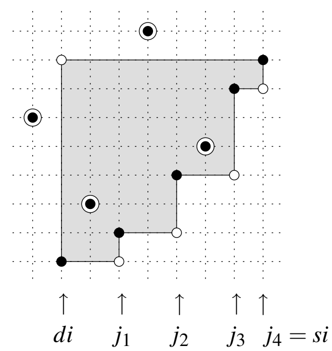
We recall that the *difference index* of (x, y), denoted by $di(x, y)$ (or simply di), is the minimal k such that $x(k) \neq y(k)$ (see [8]). We also call $x^{-1}y(di)$ the *stair index* of (x, y), and denote it by $si(x, y)$ (or simply si):



Note that, by definition, $di < si$. Also note that in (4.1) necessarily $t_1 = (di, j_1)$, for some j_1, otherwise the path would not be increasing. We now consider the case $t_h = (di, j_h)$, for every $h \in [k]$, with $j_k = si$:

$$x = x_0 \xrightarrow{(di, j_1)} x_1 \xrightarrow{(di, j_2)} \cdots \xrightarrow{(di, j_{k-1})} x_{k-1} \xrightarrow{(di, si)} x_k = y.$$

We call such a path a *stair path*, because of the shape of the support: it is a stair with k steps. Here is an example:



Note that in this case the length of $[x, y]$ is

$$\ell(x, y) = k + 2 fix_{\Omega}(x, y).$$

In the general case, the increasing path in (4.1) is a sequence of stair paths. It is useful to give the following definitions, which formalize the notion of stair.

Definition 4.1. *Let $x \in S_n$. A* stair *of x is an increasing sequence*

$$s = (j_0, j_1, \dots, j_k) \in [n]^{k+1}$$

such that $(x(j_0), x(j_1), \dots, x(j_k))$ is also increasing. The stair area *associated with s is the subset of $\mathbf{R}^2$ defined by*

$$Stair_x(s) = \bigcup_{i \in [k]} Rect(j_0, j_i, x(j_{i-1}), x(j_i)).$$

The length *of the stair s is $|s| = k$. We also say that the permutation*

$$x(j_0, j_1)(j_0, j_2) \cdots (j_0, j_k) = x(j_0, j_k, \dots, j_2, j_1)$$

is obtained *from x by* performing *the stair s, and denote it by $\varphi_s(x)$.*

Definition 4.2. *Let $x, y \in S_n$ with $x < y$. A stair s of x is said to be* good *with respect to y if*

$$Stair_x(s) \subseteq \Omega(x, y).$$

Then, we have the following (cfr. Proposition 3.4).

Proposition 4.3. *Let $x, y \in S_n$ with $x < y$ and s be a stair of x. Then $\varphi_s(x) \leq y$ if and only if s is good with respect to y.*

Proof. Set $z = \varphi_s(x)$. As it can be easily checked, for every $(h, k) \in \mathbf{R}^2$, we have

$$(z, y)[h, k] = \begin{cases} (x, y)[h, k] - 1, & \text{if } (h, k) \in Stair_x(s); \\ (x, y)[h, k], & \text{otherwise.} \end{cases}$$

Then, by Theorem 3.1, $z \leq y$ if and only if $(x, y)[h, k] \geq 1$ for every $(h, k) \in Stair_x(s)$, that is if and only if $Stair_x(s) \subseteq \Omega(x, y)$. ∎

Definition 4.4. *Let $x, y \in S_n$ with $x < y$. An* initial stair *of (x, y) is a stair s of x good with respect to y which starts with di and ends with si*:

$$s = (di, j_1, j_2, \dots, j_{k-1}, si).$$

Examples of initial stairs will be shown in Figure 4. Note that, if $x < y$, then an initial stair of (x, y) always exists. One can be obtained, for example, by choosing all its indices as small as possible.

We are now able to give the general algorithm which allows to generate all possible increasing paths in *BG* from x to y.

Algorithm (the stair method). Given $x, y \in S_n$ with $x \leq y$, do the following:

(1) if $x = y$, then return the empty path, otherwise do steps (2) to (5);
(2) choose an initial stair $s = (di, j_1, \dots, j_{k-1}, si)$ of (x, y);
(3) let x_1 be the permutation obtained from x by performing the stair s (note that $x_1 \leq y$, by Proposition 4.3) and consider the path

$$x \xrightarrow{(di,j_1)} \bullet \xrightarrow{(di,j_2)} \cdots \xrightarrow{(di,j_{k-1})} \bullet \xrightarrow{(di,si)} x_1;$$

(4) recursively apply the procedure on (x_1, y), obtaining a path $x_1 \xrightarrow{\Delta_1} y$;
(5) return the union of the two paths:

$$x \xrightarrow{(di,j_1)} \bullet \xrightarrow{(di,j_2)} \cdots \xrightarrow{(di,j_{k-1})} \bullet \xrightarrow{(di,si)} x_1 \xrightarrow{\Delta_1} y.$$

Theorem 4.5. *Let $x, y \in S_n$ with $x < y$. The stair method can be applied to produce all the increasing paths from x to y in BG.*

Proof. First of all, the paths generated with the stair method are increasing. We prove it by induction on the length of the path. The stair path obtained in step (3) is itself increasing. By induction, we may assume that Δ_1 is also increasing. Finally, the first label of Δ_1 is greater than (di, si). In fact $x_1(di) = x(si) = y(di)$, which implies $di(x_1, y) > di$.

Then, every increasing path from x to y can be generated by the algorithm. We prove it again by induction on the length. An increasing path from x to y has to start with a sequence of labels $(di, j_1), \dots, (di, j_{k-1}), (di, si)$, otherwise it would not be increasing. By induction, the remaining part of the path can be generated by the algorithm, so the whole path has this property. ∎

An example is shown in Figure 4. Here x and y are the same permutations as in Figure 1, and we apply the algorithm to (x, y). The corresponding increasing path in BG that comes out has length 9 and it is the following:

$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y.$$

We could have chosen, for example, the initial stairs $(1, 5)$ or $(1, 3, 5)$ of (x, y), instead of $(1, 4, 5)$, or the initial stair $(6, 7, 8, 9)$ of (x_5, y), instead of $(6, 8, 9)$. Considering all possible choices, all increasing paths from x to y can be obtained, and it turns out in this example that they are 10 and that:

$$\widetilde{R}_{x,y}(q) = q^{13} + 4q^{11} + 4q^9 + q^7.$$

Note that the only increasing path in BG from x to y of length $\ell(x, y)$, whose existence and uniqueness are guaranteed by the EL-shellability, is obtained by choosing in the algorithm always the lexicographically minimal stair.

As a consequence of Theorem 4.5, we obtain a recursive formula for computing $\widetilde{R}_{x,y}(q)$. Let $InitStairs(x, y)$ denote the set of all the initial stairs of (x, y).

Corollary 4.6. *Let $x, y \in S_n$ with $x < y$. Then*

$$\widetilde{R}_{x,y}(q) = \sum_{s \in InitStairs(x,y)} q^{|s|} \widetilde{R}_{\varphi_s(x),y}(q).$$

Proof. This is a consequence of Theorems 2.4 and 4.5. ∎

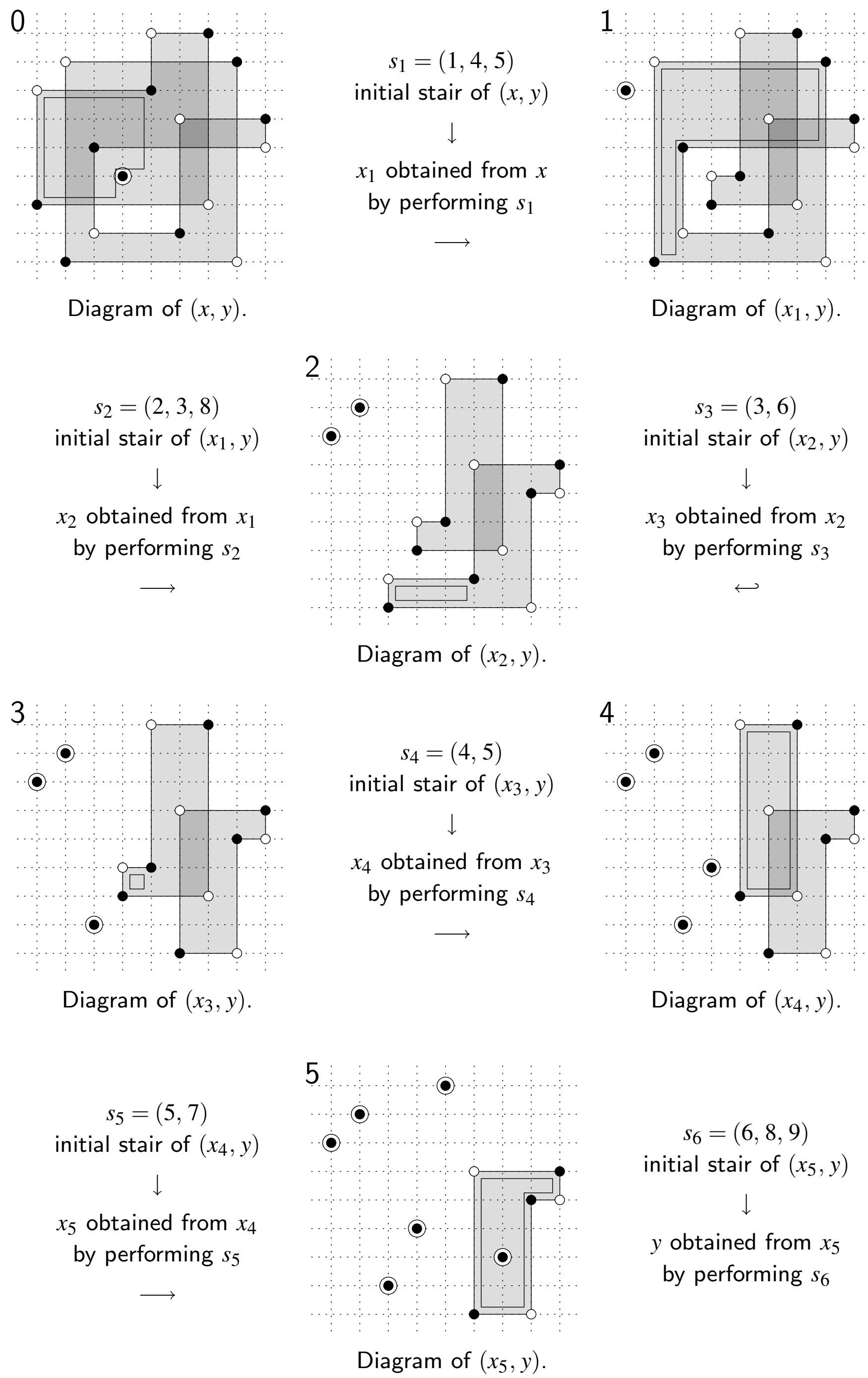


Figure 4: The stair method.

Note that the formula given in Corollary 4.6 is closely related to the one given in [3, Eq. 19]. However, our techniques are different from those used in [3], and in particular the notion of initial stair allows us to say precisely what are the stairs s such that $\varphi_s(x) \leq y$. This leads to a generalization of the results in [3], as we will point out in Section 6.

5. Consequences

In this section, we derive some consequences of the stair method, in particular showing how the computation of the R-polynomials is affected by some fixed points.

We start with introducing the following notation. Given $x \in S_n$ and $I \subseteq [n]$ with $|I| = m$, we denote by $x|_I$ the permutation of S_m whose diagram is obtained from that of x, by considering only the dots corresponding to the indices in I, removing the others, and renumbering the remaining indices and values from 1 to m, in the same order as they were in x. We call $x|_I$ the *subpermutation* of x induced by I.

A first simplification is provided by the fact that all the information about $\widetilde{R}_{x,y}(q)$ is contained in the support of (x, y).

Proposition 5.1. *Let $x, y \in S_n$ with $x < y$. Set $x_\Omega = x\big|_{I_\Omega(x,y)}$ and $y_\Omega = y\big|_{I_\Omega(x,y)}$. Then*

$$\widetilde{R}_{x,y}(q) = \widetilde{R}_{x_\Omega, y_\Omega}(q).$$

Proof. This is a consequence of the fact that in the stair method we always work only on the support, by "removing" at each step one stair. ∎

We also have the following result for the factorization of the polynomial $\widetilde{R}_{x,y}(q)$, in case the support is not connected.

Proposition 5.2. *Let $x, y \in S_n$ with $x < y$. Suppose that $\Omega(x, y)$ is not connected. Let $I_1, I_2, \ldots, I_p$ be the sets of indices associated with the connected components of $\Omega(x, y)$ and set $x_r = x|_{I_r}$ and $y_r = y|_{I_r}$, for $r \in [p]$. Then*

$$\widetilde{R}_{x,y}(q) = \widetilde{R}_{x_1,y_1}(q) \cdot \widetilde{R}_{x_2,y_2}(q) \cdots \widetilde{R}_{x_p,y_p}(q).$$

Proof. By induction, it is enough to prove the result for $p = 2$. By the stair method, it is possible to construct a bijection

$$\psi\colon Paths^{\prec}(x_1, y_1) \times Paths^{\prec}(x_2, y_2) \to Paths^{\prec}(x, y)$$

such that, for every $(\Gamma, \Lambda) \in Paths^{\prec}(x_1, y_1) \times Paths^{\prec}(x_2, y_2)$, we have

$$|\psi(\Gamma, \Lambda)| = |\Gamma| + |\Lambda|.$$

Given $(\Gamma, \Lambda) \in Paths^{\prec}(x_1, y_1) \times Paths^{\prec}(x_2, y_2)$, the path $\psi(\Gamma, \Lambda)$ is obtained by "merging" the two paths, in such a way to keep the labels increasing. The fact that Γ and Λ act on different connected components, ensures that they do not interfere with each other. By Theorem 2.4, the existence of such a bijection implies our result. In fact

$$\widetilde{R}_{x,y}(q) = \sum_{\Delta \in Paths^{\prec}(x,y)} q^{|\Delta|} = \sum_{\substack{\Gamma \in Paths^{\prec}(x_1,y_1) \\ \Lambda \in Paths^{\prec}(x_2,y_2)}} q^{|\Gamma|+|\Lambda|} = \widetilde{R}_{x_1,y_1}(q) \cdot \widetilde{R}_{x_2,y_2}(q). \qquad \blacksquare$$

According to Proposition 5.1, $\widetilde{R}_{x,y}(q)$ is not affected by the fixed points whose corresponding dots are out of the support of (x, y). In what follows we show how some other fixed points affect the $\widetilde{R}$-polynomial.

Definition 5.3. *Let $x, y \in S_n$ with $x < y$. A fixed point $i \in Fix(x, y)$ is said to be* simple *if $(i, x(i))$ belongs to a connected component of $\Omega(x, y)$, say Ω', such that*

$$(x, y)[h, k] = 1, \quad \textit{for every } (h, k) \in \Omega'.$$

Let $x, y \in S_n$ with $x < y$, and let $i \in Fix(x, y)$ be a simple fixed point of (x, y). Let Δ be an increasing path from x to y. If i does not belong to any of the labels of Δ, then we say that Δ *ignores* i, otherwise we say that Δ *moves* i.

Theorem 5.4. *Let $x, y \in S_n$ with $x < y$. Let $i \in Fix(x, y)$ be a simple fixed point. Set $x\backslash i = x\big|_{[n]\backslash\{i\}}$ and $y\backslash i = y\big|_{[n]\backslash\{i\}}$. Then*

$$\widetilde{R}_{x,y}(q) = (q^2+1)\widetilde{R}_{x\backslash i, y\backslash i}(q).$$

Proof. There is an obvious bijection between the increasing paths from $x\backslash i$ to $y\backslash i$ of a given length and the increasing paths from x to y which ignore i of the same length. On the other hand, we claim that the increasing paths from x to y of length k which ignore i are also in bijection with those from x to y of length $k+2$ which move i. Consider an increasing path Δ from x to y of length k which ignores i. Since i is simple, there is exactly one edge of the path, say

$$x_k \xrightarrow{(j_1, j_2)} x_{k+1},$$

such that $j_1 < i < j_2$ and $x_k(j_1) < x(i) < x_k(j_2)$. We call the associated rectangle $Rect(j_1, j_2, x_k(j_1), x_k(j_2))$ the *main rectangle*. Among the rectangles associated with the other edges of the path, we say that $Rect(a, b, c, d)$ is *covered* by the main rectangle if $a < i < b$ and $Rect(a, b, c, x(i)) \subseteq \Omega(x, y)$. It is easy to check that, if $Rect(a, b, c, d)$ is covered by the main rectangle, then either $y(a) = d$, or $x(b) = d$: we respectively say that $Rect(a, b, c, d)$ is *supported on the left* or *on the right*. Now consider the path Δ' obtained from Δ by doing the following:

1. substitute the label (j_1, j_2) corresponding to the main rectangle with the three labels (j_1, i), (j_1, j_2), (i, j_2);
2. for every rectangle $Rect(a, b, c, d)$ covered by the main rectangle, substitute the label (a, b) with (a, i) or (i, b), depending on whether $Rect(a, b, c, d)$ is supported on the left or on the right.
3. reorder the labels in an increasing way.

Then, Δ' is a path from x to y of length $k+2$ which moves i. The procedure described can be easily reversed, so that the original path Δ can be reconstructed starting from Δ'. Thus, it is a bijection. An example is shown in Figure 5. Here $(x, y) = (615384729, 965283714)$ and we consider the simple fixed point $i = 5$ and the increasing path

$$x \xrightarrow{(1,7)} \bullet \xrightarrow{(1,9)} \bullet \xrightarrow{(2,4)} \bullet \xrightarrow{(2,6)} \bullet \xrightarrow{(2,7)} \bullet \xrightarrow{(4,8)} \bullet \xrightarrow{(7,9)} y,$$

which has length 7 and ignores i (picture on the left). The main rectangle is $Rect(1, 9, 7, 9)$ and the rectangles covered by it are $Rect(1, 7, 6, 7)$, $Rect(2, 7, 4, 6)$ and $Rect(2, 6, 3, 4)$. According to the procedure described, we substitute the label $(1, 9)$ with the three labels $(1, 5)$, $(1, 9)$, $(5, 9)$, and $(1, 7)$ with $(5, 7)$, $(2, 7)$ with $(2, 5)$, and $(2, 6)$ with $(5, 6)$. Reordering the labels in increasing order, we obtain the path

$$x \xrightarrow{(1,5)} \bullet \xrightarrow{(1,9)} \bullet \xrightarrow{(2,4)} \bullet \xrightarrow{(2,5)} \bullet \xrightarrow{(4,8)} \bullet \xrightarrow{(5,6)} \bullet \xrightarrow{(5,7)} \bullet \xrightarrow{(5,9)} \bullet \xrightarrow{(7,9)} y,$$

which has length 9 and moves i (picture on the right).

Thus, we have bijections between any two of the following three sets:

$$A = \left\{\Delta \in Paths^{\prec}(x\backslash i, y\backslash i) \colon |\Delta| = k\right\},$$

$$B = \left\{\Delta \in Paths^{\prec}(x, y) \colon |\Delta| = k, \Delta \text{ ignores } i\right\},$$

$$C = \left\{\Delta \in Paths^{\prec}(x, y) \colon |\Delta| = k+2, \Delta \text{ moves } i\right\}.$$

By Theorem 2.4, this implies

$$\left[q^k\right] \widetilde{R}_{x,y}(q) = \left[q^{k-2}\right] \widetilde{R}_{x\backslash i, y\backslash i}(q) + \left[q^k\right] \widetilde{R}_{x\backslash i, y\backslash i}(q),$$

and the result follows. ∎

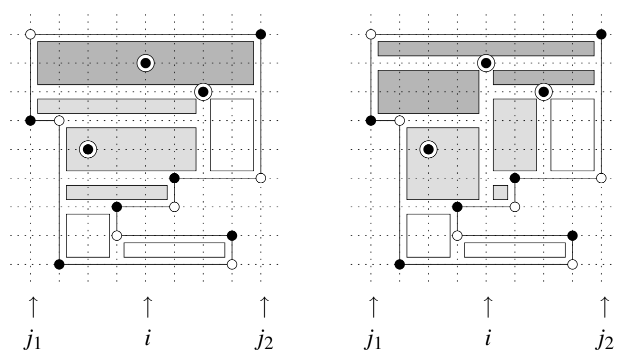


Figure 5: A bijection in the proof of Theorem 5.4.

6. Explicit Formulas

The stair method and the considerations in the previous section allow to compute the $\widetilde{R}$-polynomials for some general classes of pairs (x, y).

Definition 6.1. *Let $x, y \in S_n$ with $x < y$. We say that*

(1) (x, y) *has the* 01-multiplicity property *if*

$$(x, y)[h, k] \in \{0, 1\} \quad \textit{for every } (h, k) \in \mathbf{R}^2;$$

(2) (x, y) *is* simple *if it has the* 01*-multiplicity property and* $Fix(x, y) = \emptyset$;

(3) (x, y) *is a* permutomino *if it is simple and* $\Omega(x, y)$ *is connected and simply connected* (*that is, it has no holes*).

Examples are shown in Figure 6.

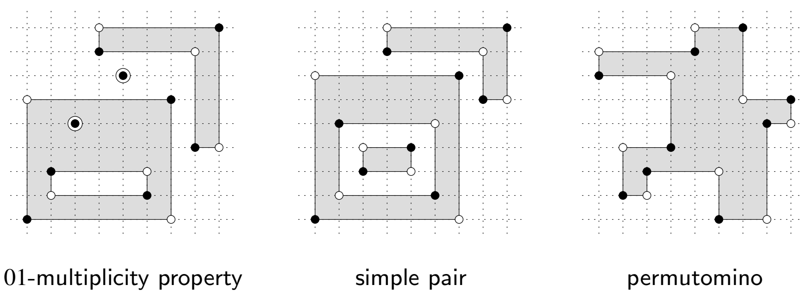


Figure 6: General classes of pairs of permutations.

In the following we give closed expressions of the $\widetilde{R}$-polynomials for these classes.

Theorem 6.2. *Let $x, y \in S_n$ with $x < y$.*

(1) *If (x, y) is simple, then*

$$\widetilde{R}_{x,y}(q) = q^{\ell(x,y)}.$$

(2) *If (x, y) has the* 01*-multiplicity property, then*

$$\widetilde{R}_{x,y}(q) = (q^2 + 1)^{fix_\Omega(x,y)} q^{\ell(x,y) - 2 fix_\Omega(x,y)}.$$

Proof. (1) Applying the stair method in this case, it turns out that at each step there is only one choice. Thus the increasing path from x to y of length $\ell(x, y)$ is actually the only increasing path from x to y.

(2) This follows from Proposition 5.1, Theorem 5.4 (applied to every simple fixed point), and (1). ∎

Note that Theorem 6.2 generalizes [3, Theorem 4.6] and [11, Theorem 5.4]. In fact, [3, Theorem 4.6] is concerned with the case of a stair path, which is a special kind of pair with the 01-multiplicity property. If $y = x(i, j)(p, p+1)$, with $i < p < j-1$ and $x(i) < x(p+1) < x(p) < x(j)$, then (x, y) has the 01-multiplicity property (since $(x, y)[h, k] = 1$ if $(h, k) \in Rect(i, j, x(i), x(j)) \setminus Rect(p, p+1, x(p+1), x(p))$ and $(x, y)[h, k] = 0$ otherwise) and Theorem 6.2 in this case is just [11, Theorem 5.4].

Corollary 6.3. *Let $x, y \in S_n$ with $x < y$.*

(1) *If (x, y) is simple, then*

$$R_{x,y}(q) = (q-1)^{\ell(x,y)}.$$

(2) *If (x, y) has the 01-multiplicity property, then*

$$R_{x,y}(q) = (q^2 - q + 1)^{fix_\Omega(x,y)} (q-1)^{\ell(x,y) - 2fix_\Omega(x,y)}.$$

Proof. This is a consequence of Theorem 6.2 and Proposition 2.3. ∎

It is worth mentioning that for the special classes of pairs introduced in this section, there are nice combinatorial expressions for $\ell(x, y) = inv(x, y)$ and $a\ell(x, y)$. Note that a pair (x, y) has the 01-multiplicity property if and only if the paths in the diagram of (x, y) are nonintersecting. In this case we call *positive* those paths whose leftmost vertical segment is positive (see Section 3), and *negative* the others. We denote by $cyc^+(x, y)$ (respectively, $cyc^-(x, y)$) the number of positive (respectively, negative) paths in the diagram of (x, y). Then we have the following.

Corollary 6.4. *Let $x, y \in S_n$ with $x < y$.*

(1) *If (x, y) is a permutomino (necessarily with $2n$ sides), then*

$$\ell(x, y) = a\ell(x, y) = n - 1.$$

(2) *If (x, y) is simple, then*

$$\ell(x, y) = a\ell(x, y) = n - cyc^+(x, y) + cyc^-(x, y).$$

(3) *If (x, y) has the 01-multiplicity property, then*

$$\ell(x, y) = |I_\Omega(x, y)| + fix_\Omega(x, y) - cyc^+(x, y) + cyc^-(x, y),$$

$$a\ell(x, y) = n - fix(x, y) - cyc^+(x, y) + cyc^-(x, y).$$

Proof. The expressions for the length can be easily proved by induction on $\ell(x, y)$. Those for the absolute length follow from Theorem 6.2 and Corollary 2.6. ∎

We finally mention that the class of permutominoes has been recently studied in detail in [6]; the interested reader can find several enumerative results therein.

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