Results Math (2024) 79:247
Online First

https://doi.org/10.1007/s00025-024-02276-3

Results in Mathematics

On Sums of Sums Involving the Von Mangoldt Function

Isao Kiuchi and Wataru Takeda

Abstract. Let Λ denote the von Mangoldt function, and (n, q) be the greatest common divisor of positive integers n and q. For any positive real numbers x and y, we shall consider several asymptotic formulas for sums of sums involving the von Mangoldt function; $S_k(x,y) := \sum_{n\le y}\left(\sum_{q\le x} \sum_{d|(n,q)} d\Lambda\left(\frac{q}{d}\right)\right)^k$ for $k = 1, 2$.

Mathematics Subject Classification. 11N37, 11A25, 11M06.

Keywords. Asymptotic results on arithmetical functions, von Mangoldt function, riemann zeta-function, exponential sums, anderson–apostol sums.

1. Introduction

For any integers $n \ge 1$, we define

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ someprime } p \text{ andsome } m \ge 1, \\ 0 & \text{otherwise,} \end{cases}$$

which is the von Mangoldt function. Let $s = \sigma + it$ be the complex variable, where σ and t are real, and let $\zeta(s)$ denote the Riemann zeta-function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

Published online: 15 September 2024

and $\zeta'(s)$ its first derivative. The Riemann zeta function can be analytically continued to the whole plane. We define the following sum over the von Mangoldt function

$$s_q(n) := \sum_{d|(n,q)} d\Lambda\left(\frac{q}{d}\right), \tag{1.1}$$

where (n,q) denotes the greatest common divisor of integers n and q. This sum is a special type of Anderson–Apostol sum defined by $\sum_{d|(n,q)} f(d)g\left(q/d\right)$ with any arithmetical functions f and g (see [1,2]). We use the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} \quad (\mathrm{Re}\ s > 1) \tag{1.2}$$

to deduce the Dirichlet series with the coefficients $s_q(n)$, namely

$$\sum_{q=1}^{\infty} \frac{s_q(n)}{q^s} = -\sigma_{1-s}(n)\frac{\zeta'(s)}{\zeta(s)} \tag{1.3}$$

for Re $s > 1$ with the divisor function $\sigma_{1-s}(n) = \sum_{d|n} d^{1-s}$. For any large positive real numbers x and y, we let the double sums

$$S_k(x,y) := \sum_{n \le y} \left(\sum_{q \le x} s_q(n)\right)^k \qquad (k = 1, 2). \tag{1.4}$$

The double sum of the type (1.4) was first considered by Chan and Kumchev [3], who proved several interesting asymptotic formulas concerning the Ramanujan sum $c_q(n)$, defined by $c_q(n) = \sum_{d|(n,q)} d\mu\left(q/d\right)$ with μ being the Möbius function, instead of $s_q(n)$. In 2015, Minamide, Tanigawa, and the first author [7] were inspired by their work, and considered square-free numbers instead of the Möbius function in the Ramanujan sum, and derived the precise asymptotic formulas. Robles and Roy [16] studied an analogue of the type (1.4) concerning the generalized Ramanujan sums, known as the Cohen–Ramanujan sums. Moreover, the first author considered some sums of the type (1.4) concerning square-full numbers [8], cube-full numbers [10], the Liouville function [11] and others (see [9,12,13,15]). This study aims to derive several asymptotic formulas for (1.4) with $k = 1$ and 2.

1.1. Evaluation of $S_1(x,y)$

Following the same procedure as in [3] (see also [7,8,10,13,15,16]), we obtain some interesting theorems for the double sum $S_k(x,y)$. First, the case $k = 1$ implies the following theorem, namely

Theorem 1.5. *Let the notation be as above. Let x and y be large real numbers such that $x\log x \ll y \ll \frac{x^2}{\log x}$. Then, we have*

$$S_1(x,y) = yx(\log x - 1) + \frac{\zeta'(2)}{4\zeta(2)}x^2 + O\left(xy^{\frac{1}{3}}\log x + y\log x + \frac{x^3}{y}\right). \quad (1.6)$$

Remark 1.7. Substituting $y = x^{\frac{3}{2}}$ and $y = x\log x$ into (1.6), we obtain

$$S_1(x,x^{\frac{3}{2}}) = x^{\frac{5}{2}}(\log x - 1) + \frac{\zeta'(2)}{4\zeta(2)}x^2 + O\left(x^{\frac{3}{2}}\log x\right),$$

and

$$S_1(x, x\log x) = x^2\log x(\log x - 1) + \frac{\zeta'(2)}{4\zeta(2)}x^2 + O\left(\frac{x^2}{\log x}\right), \quad (1.8)$$

respectively.

If we could use an alternative method to investigate an asymptotic behavior for $S_1(x,y)$ under the condition $y \ll x\log x$, then we may use some analytic method to study the asymptotic formulas for (1.4) for $k=1$. We use analytic properties between the Riemann zeta-function and the von Mangoldt function to investigate the asymptotic behavior of sharp approximate formulas for (1.4), and whose form yields an interesting formula. Before elucidating the statement, let $\kappa(u)$ denote the Fourier integral given by

$$\kappa(u) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\zeta(-\frac{1}{4}-it)\zeta'(\frac{9}{4}+it)}{\zeta(\frac{9}{4}+it)} \frac{e^{itu}}{(\frac{1}{4}+it)(\frac{9}{4}+it)}dt \quad (1.9)$$

with $u := \log\frac{x}{y}$. It follows from (3.15) below that $|\kappa(u)|$ is given by the inequality

$$|\kappa(u)| \le \frac{8}{(2\pi)^{\frac{7}{4}}}\frac{\zeta(\frac{5}{4})\zeta(\frac{9}{4})\zeta'(\frac{9}{4})}{\zeta(\frac{9}{2})}\left(\frac{\pi}{9}+1\right). \quad (1.10)$$

Here, the integral is a computable constant. We use a contour integral of the generating Dirichlet series (the method in [9]) and some properties of the Riemann zeta-function to obtain

Theorem 1.11. *Let the notation be as above. Let x and y be large real numbers such that $1 \ll y \ll \frac{x^{7/5}}{\log^2 x}$. Then, we have*

$$S_1(x,y) = yx(\log x - 1) + \frac{\zeta'(2)}{4\zeta(2)}x^2 + x^2\left(\frac{x}{y}\right)^{\frac{1}{4}}\kappa(u) + O\left(xy^{\frac{1}{3}}\log^4 x + yx^{\frac{1}{2}}\log^{\frac{5}{2}} x\right), \quad (1.12)$$

where $\kappa(u)$ denotes the Fourier integral given by (1.9).

For $y = x$, we have an interesting formula, namely

Remark 1.13. We substitute $y = x$ into (1.12), then we obtain

$$S_1(x, x) = x^2 \log x + \left(\frac{\zeta'(2)}{4\zeta(2)} + \kappa(0) - 1 \right) x^2 + O\left(x^{\frac{3}{2}} \log^{\frac{5}{2}} x \right),$$

where $\kappa(0)$ is a computable constant.

Remark 1.14. Furthermore, we substitute $y = x \log x$ into (1.12) to deduce

$$S_1(x, x \log x) = x^2 \log x (\log x - 1) + \frac{\zeta'(2)}{4\zeta(2)} x^2 + \frac{x^2}{\log^{\frac{1}{4}} x} \kappa\left(\log \frac{1}{\log x} \right) + O\left(x^{\frac{3}{2}} \log^{\frac{7}{2}} x \right).$$

It follows from (1.8) and the above that

$$\kappa\left(\log \frac{1}{\log x} \right) = O\left((\log x)^{-\frac{3}{4}} \right).$$

1.2. Evaluation of $S_2(x, y)$

For the case $k = 2$, two different methods to handle function $S_2(x, y)$ exist. We use an elementary lattice point counting argument to obtain the formula (1.16) below and use the generating Dirichlet series and the properties of the Riemann zeta-function to prove (1.18) below, which we state as

Theorem 1.15. *Let x and y denote large real numbers such that $y \gg \frac{x^2}{\log^3 x}$. We have*

$$S_2(x, y) = \frac{1}{3\zeta(2)} y x^2 \log^3 x + O\left(y x^2 \log^2 x + x^4 \right). \tag{1.16}$$

To establish the precise asymptotic formula of $S_2(x, y)$, we let γ denote the Euler–Mascheroni constant, and γ_1, γ_2 denote the Stieltjes constants defined by (5.5) below. Let $c_1, \ldots, c_6$ denote the constants given by

$$c_1 = \frac{1}{\zeta(2)} \left(\gamma - 1 - \frac{\zeta'(2)}{\zeta(2)} \right),$$

$$c_2 = \frac{1}{\zeta(2)} \left(1 - 2(\gamma + \gamma_1) + 2(1 - \gamma) \frac{\zeta'(2)}{\zeta(2)} + 2 \left(\frac{\zeta'(2)}{\zeta(2)} \right)^2 - \frac{\zeta''(2)}{\zeta(2)} \right),$$

$$c_3 = \frac{1}{\zeta(2)} \left(c_0 - \gamma + 2\gamma_1 + 2\gamma_2 + (2\gamma + 2\gamma_1 - 1) \frac{\zeta'(2)}{\zeta(2)} + 2(\gamma - 1) \left(\frac{\zeta'(2)}{\zeta(2)} \right)^2 \right)$$
$$+ \frac{1}{\zeta(2)} \left((1 - \gamma) \frac{\zeta''(2)}{\zeta(2)} - \frac{\zeta'''(2)}{3\zeta(2)} + 2 \frac{\zeta'(2)}{\zeta(2)} \frac{\zeta''(2)}{\zeta(2)} - 2 \left(\frac{\zeta'(2)}{\zeta(2)} \right)^3 \right),$$

$$c_4 = \frac{1}{\zeta(2)} \left(\frac{1}{2} + \gamma \right), \quad c_5 = \frac{2}{\zeta(2)} \left(\gamma^2 + \gamma - \gamma_1 + 1 \right),$$

and

$$c_6 = \frac{2}{\zeta(2)} \left(\gamma^3 + \gamma^2 + 2\gamma - \gamma_1 - 3\gamma\gamma_1 + 3\gamma_2 + 1\right),$$

where c_0 is given by

$$c_0 := \frac{1}{2\pi i} \int_{\frac{5}{4}-i\infty}^{\frac{5}{4}+i\infty} \frac{\zeta'(s)\zeta'(2-s)}{s(2-s)} ds$$

which is a computable constant. In the last section, the value of c_0 is evaluated by

$$c_0 \leq 0.425\cdots \left|\zeta'\left(\frac{5}{4}\right)\right| \int_0^\infty \frac{\left|\zeta'\left(\frac{3}{4}(1+iy)\right)\right|}{1+y^2} dy.$$

The integral on the right-hand side of the above is a computable constant. We obtain

Theorem 1.17. *Let the notation be as above. Let x and y be large real numbers such that $x \log^{16} x \ll y \ll \frac{x^2}{\log^{16} x}$. Then, we have*

$$\begin{aligned} S_2(x,y) = &\frac{1}{3\zeta(2)} yx^2 \log^3 x + c_1 yx^2 \log^2 x + c_2 yx^2 \log x + (c_3 - c_6) yx^2 \\ &+ \frac{1}{6\zeta(2)} yx^2 \log^3 \frac{x^2}{y} - c_4 yx^2 \log^2 \frac{x^2}{y} + c_5 yx^2 \log \frac{x^2}{y} + E(x,y), \end{aligned} \tag{1.18}$$

where the error term $E(x,y)$ is estimated by

$$E(x,y) = O\left(x^{5/3} y L^8 + x^2 y L^{10} \left(\left(\frac{x}{y}\right)^{1/2} + \left(\frac{y}{x^2}\right)^{1/2}\right)\right) \tag{1.19}$$

with $L = \log(xy)$.

1.3. Open Problems

Here we list two open problems concerning some functions discussed above.

1. Investigate asymptotic formulas of the type (1.4) for a fixed integers $k \geq 3$.
2. Investigate asymptotic formulas of

$$\sum_{n \leq y} \left(\sum_{q \leq x} \sum_{d|(n,q)} d \cdot f\left(\frac{q}{d}\right) \right)^k$$

for any arithmetic functions f with a fixed integers $k \geq 1$. For example, we may consider the divisor function $\tau (:= \mathbf{1} * \mathbf{1})$, the sum-of-sum divisors function $\sigma (:= \mathbf{1} * \mathrm{id})$, and the Euler totient function $\phi (:= \mathrm{id} * \mu)$ in place of f, respectively.

2. Proof of Theorem 1.5

2.1. Exponent Pair

To prove Theorem 1.5, we need the following Lemma. Let $\psi(x) = x - [x] - \frac{1}{2}$ denote the first periodic Bernoulli function. Then, we have

Lemma 2.1. *Let (κ, λ) be an exponent pair. If I is a subinterval in $(N, 2N]$, we have*

$$\sum_{n \in I} \psi\left(\frac{y}{n}\right) \ll y^{\frac{\kappa}{\kappa+1}} N^{\frac{\lambda-\kappa}{\kappa+1}} + N^2 y^{-1}.$$

In particular, if we take the exponent pair $(\kappa, \lambda) = \left(\frac{1}{2}, \frac{1}{2}\right)$, we obtain

$$\sum_{n \in I} \psi\left(\frac{y}{n}\right) \ll y^{\frac{1}{3}} + N^2 y^{-1}. \tag{2.2}$$

Proof. This lemma is given by Lemma 2.1 in [3] (see also [4]). □

2.2. Proof of (1.6).

Using (1.1) and (1.4) with $k = 1$, we have

$$S_1(x, y) = \sum_{n \le y} \sum_{q \le x} s_q(n) = \sum_{n \le y} \sum_{\substack{dk \le x \\ d \mid n}} d\Lambda(k).$$

Changing the order of summation, we find that

$$\begin{aligned} S_1(x, y) &= y \sum_{dk \le x} \Lambda(k) - \frac{1}{2} \sum_{dk \le x} d\Lambda(k) - \sum_{dk \le x} d\Lambda(k) \psi\left(\frac{y}{d}\right) \\ &=: S_{1,1}(x, y) - S_{1,2}(x, y) - S_{1,3}(x, y), \end{aligned} \tag{2.3}$$

Consider $S_{1,1}(x, y)$. We use the identity $\sum_{d \mid n} \Lambda(d) = \log n$ and the summation formula $\sum_{n \le x} \log n = x \log x - x + O(\log x)$ to obtain

$$\begin{aligned} S_{1,1}(x, y) &= y \sum_{n \le x} \sum_{k \mid n} \Lambda(k) = y \sum_{n \le x} \log n \\ &= yx \log x - yx + O(y \log x). \end{aligned} \tag{2.4}$$

We use (1.2) and the summation formula $\sum_{n \le x} \frac{\Lambda(n)}{n} = \log x + O(1)$ to deduce

$$\begin{aligned} S_{1,2}(x, y) &= \frac{1}{2} \sum_{k \le x} \Lambda(k) \sum_{d \le \frac{x}{k}} d \\ &= \frac{x^2}{4} \sum_{k \le x} \frac{\Lambda(k)}{k^2} + O\left(x \sum_{k \le x} \frac{\Lambda(k)}{k}\right) \\ &= -\frac{\zeta'(2)}{4\zeta(2)} x^2 + O(x \log x). \end{aligned} \tag{2.5}$$

Let $N_j = N_{j,k} = \left(\frac{x}{k}\right) 2^{-j}$. We use the theory of exponent pairs to obtain

$$\begin{aligned} S_{1,3}(x,y) &= \sum_{k\leq x} \Lambda(k) \sum_{d\leq \frac{x}{k}} d\psi\left(\frac{y}{d}\right) \\ &\ll \sum_{k\leq x} \Lambda(k) \sum_{j=0}^{\infty} N_j \sup_I \left| \sum_{d\in I} \psi\left(\frac{y}{d}\right)\right|, \end{aligned}$$

where sup is over all subintervals I in $(N_j, 2N_j]$. From (2.2) in Lemma 2.1, we have

$$\begin{aligned} S_{1,3}(x,y) &\ll \sum_{k\leq x} \Lambda(k) \sum_{j=0}^{\infty} \left\{ N_j y^{1/3} + N_j^3 y^{-1} \right\} \\ &\ll \sum_{k\leq x} \Lambda(k) \left\{ \left(\frac{x}{k}\right) y^{1/3} + \left(\frac{x}{k}\right)^3 y^{-1} \right\} \\ &\ll \sum_{k\leq x} \frac{\Lambda(k)}{k} \cdot xy^{1/3} + \sum_{k\leq x} \frac{\Lambda(k)}{k^3} \cdot x^3 y^{-1} \\ &\ll xy^{1/3} \log x + x^3 y^{-1}. \end{aligned} \tag{2.6}$$

Substituting (2.4), (2.5), and (2.6) into (2.3), we obtain the assertion of Theorem 1.5. □

3. Proof of Theorem 1.11

3.1. Lemmas

To prove theorem 1.11, we utilize the following Lemmas.

Lemma 3.1. *Suppose that the Dirichlet series* $\alpha(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *absolutely converges for* $\operatorname{Re} s > \sigma_a$. *If* $\sigma_0 > \max(0, \sigma_a)$ *and* $x > 0,\ T > 0$, *then*

$$\sum_{n\leq x}{}' a_n = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} ds + R,$$

where

$$R \ll \sum_{\substack{\frac{x}{2} < n < 2x \\ n \neq x}} |a_n| \min\left(1, \frac{x}{T|x-n|}\right) + \frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}.$$

and $\sum'$ *indicates that the last term is to be halved if* x *is an integer.*

Proof. This is the famous Perron's formula (see Theorem 5.2 and Corollary 5.3 in [14]). □

Lemma 3.2. *For $t \geq t_0 > 0$ uniformly in σ, we have*

$$\zeta(\sigma+it) \ll \begin{cases} t^{\frac{1}{6}(3-4\sigma)}\log t & \left(0 \leq \sigma \leq \frac{1}{2}\right), \\ t^{\frac{1}{3}(1-\sigma)}\log t & \left(\frac{1}{2} \leq \sigma \leq 1\right), \\ \log t & (1 \leq \sigma < 2), \\ 1 & (\sigma \geq 2), \end{cases} \tag{3.3}$$

and

$$\zeta'(\sigma+it) \ll \begin{cases} t^{\frac{1}{6}(3-4\sigma)}\log^2 t & \left(0 \leq \sigma \leq \frac{1}{2}\right), \\ t^{\frac{1}{3}(1-\sigma)}\log^2 t & \left(\frac{1}{2} \leq \sigma \leq 1\right), \\ \log^2 t & (1 \leq \sigma < 2), \\ 1 & (\sigma \geq 2). \end{cases} \tag{3.4}$$

Furthermore, we have

$$\frac{1}{\zeta(\sigma+it)} \ll \begin{cases} \log t & (1 \leq \sigma < 2), \\ 1 & (\sigma \geq 2). \end{cases} \tag{3.5}$$

Proof. The formula (3.3) follows from Theorem II.3.8 in Tenenbaum [17], and Ivić [6]. The formula (3.4) follows from Lemmas 2.3 and 2.4 in Tóth and Zhai [19]. The estimate (3.5) follows from Titchmarsh [18]. □

Lemma 3.6. *Let* Re $z \leq 0$, *and let $\sigma_{z,b}(n)$ denote the generalization of the divisor function defined by $\sigma_{z,b}(n) = \sum_{d^b|n} d^{bz}$. Then, we have*

$$\sideset{}{'}\sum_{n\leq x} \sigma_{z,b}(n) = D_{z,b}(x) + \Delta_{z,b}(x),$$

where $\sum'$ indicates that the last term is to be halved if x is an integer, and

$$\Delta_{z,b}(x) = O\left(x^{\frac{1}{3}}\log^2 x\right)$$

uniformly for $b \geq 1$ and $D_{z,b}(x)$ is given by the following:

(i) *If $b = 1, 2$ and $-\frac{2}{3b^2} <$ Re $z \leq 0$, then*

$$D_{z,b}(x) = \zeta(b(1-z))x + \frac{1}{1+bz}\zeta\left(z+\frac{1}{b}\right)x^{z+\frac{1}{b}}. \tag{3.7}$$

(ii) *If $b \geq 3$ and $-1 <$ Re $z \leq 0$, then*

$$D_{z,b}(x) = \zeta(b(1-z))x.$$

Proof. The proof of this result is found in Theorem 1.4 in [16]. □

3.2. Proof of (1.12).

We assume that $1 \le y \le x^M$ for some constant M. Without loss of generality, we can assume that $x,\ y \in \mathbb{Z} + \frac{1}{2}$. Suppose that $\alpha \ge 1 + \frac{1}{\log x}$ and T is a real parameter at our disposal. We apply Lemma 3.1 with (1.3) to deduce

$$\sum_{q\le x} s_q(n) = -\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} \sigma_{1-s}(n)\frac{\zeta'(s)}{\zeta(s)}\ \frac{x^s}{s}ds + E_1(x;n), \tag{3.8}$$

where $E_1(x;n)$ is the error term given by

$$\begin{aligned} E_1(x;n) &\ll \sigma_0(n) \sum_{\substack{\frac{x}{2}<q<2x\\ q\ne x}} \Lambda(q)\min\left(1, \frac{x}{T|x-q|}\right) + \sigma_0(n)\frac{(4x)^\alpha}{T}\sum_{q=1}^{\infty}\frac{\Lambda(q)}{q^\alpha} \\ &\ll \sigma_0(n)\log x\left(1+\frac{x}{T}\sum_{1\le k\le x}\frac{1}{k}\right) \ll \sigma_0(n)\left(1+\frac{x}{T}\log x\right)\log x. \end{aligned}$$

We substitute $b = 1$ and $z = 1 - s$ into Lemma 3.6 and use the well-known estimate $\sum_{n\le y}\sigma_0(n) \ll y\log y$ to deduce

$$\begin{aligned} S_1(x,y) &= -\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\sum_{n\le y}\sigma_{1-s}(n)\frac{\zeta'(s)}{\zeta(s)}\ \frac{x^s}{s}ds \\ &\quad + O\left(\left(\frac{x\log^2 x}{T}+\log x\right)\sum_{n\le y}\sigma_0(n)\right) \\ &= K_1 + K_2 + O\left(xy^{\frac{1}{3}}\log^2 y\log^2 T\right) + O\left(\frac{xy\log^3 x}{T}\right) + O\left(y\log^2 x\right), \end{aligned} \tag{3.9}$$

where

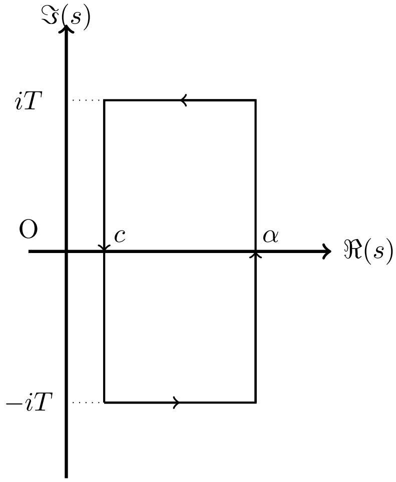
$$K_1 := -\frac{y}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\zeta'(s)\frac{x^s}{s}ds,$$

and

$$K_2 := \frac{y^2}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{\zeta(2-s)\zeta'(s)}{\zeta(s)}\ \frac{\left(\frac{x}{y}\right)^s}{(s-2)s}ds.$$

3.3. Calculation of K_1

Moving the line of integration to Re $s = c$ $(:= \frac{1}{2})$, we consider the following rectangular contour formed by the line segments joining the points $\alpha - iT$, $\alpha + iT$, $c + iT$, $c - iT$, and $\alpha - iT$ in the counter-clockwise sense.



We observe that $s = 1$ is a double pole of the integrand. Note that the Laurent expansion of the first derivative of the Riemann zeta-function at its pole s = 1 is given by

$$\zeta'(s) = -\frac{1}{(s-1)^2} + O(1).$$

Thus, we obtain the main term from the sum of the residue coming from the pole $s = 1$. Hence, using the Cauchy residue theorem, we have

$$K_1 = -\frac{y}{2\pi i}\left\{\int_{c+iT}^{\alpha+iT} + \int_{c-iT}^{c+iT} + \int_{\alpha-iT}^{c-iT}\right\}\zeta'(s)\frac{x^s}{s}ds + xy(\log x - 1). \quad (3.10)$$

The second term (the left vertical line segment) on the right-hand side of (3.10) contributes the quantity

$$\frac{y}{2\pi}\int_{-T}^{T}\zeta'\left(\frac{1}{2}+it\right)\frac{x^{\frac{1}{2}+it}}{\frac{1}{2}+it}dt$$
$$\ll yx^{\frac{1}{2}} + yx^{\frac{1}{2}}\left(\int_{2\pi}^{T}\frac{\left|\zeta'\left(\frac{1}{2}+it\right)\right|^2}{1+|t|}dt\right)^{\frac{1}{2}}\log^{\frac{1}{2}} T \ll yx^{\frac{1}{2}}\log^{\frac{5}{2}} T \quad (3.11)$$

using $\int_{2\pi}^{T}\left|\zeta'\left(\frac{1}{2}+iv\right)\right|^2\frac{dv}{v} \ll \log^4 T$ (see (172) in Hall [5]) and Cauchy–Schwarz's inequality. We can estimate the contributions coming from the upper horizontal line (the lower horizontal line is similar), noting that $T = x^{12}$. We define function $F(t)$ as

$$F(t) := \frac{1}{2\pi}\int_{\frac{1}{2}}^{1+\frac{1}{\log x}}\zeta'(\sigma+it)\frac{x^{\sigma+it}}{\sigma+it}d\sigma.$$

Then, we set

$$Q := \int_{\frac{T}{2}}^{T} \left| \int_{\frac{1}{2}}^{1+\frac{1}{\log x}} \zeta'(\sigma+it)\frac{x^{\sigma+it}}{\sigma+it}d\sigma \right| dt.$$

Using Lemma 3.2, we obtain

$$\begin{aligned} Q &\ll \int_{\frac{1}{2}}^{1+\frac{1}{\log x}} \int_{\frac{T}{2}}^{T} |\zeta'(\sigma+it)|\frac{x^{\sigma}}{t} dt d\sigma \\ &\ll T^{\frac{1}{3}}\log^2 T \int_{\frac{1}{2}}^{1+\frac{1}{\log x}} \left(\frac{x}{T^{\frac{1}{3}}}\right)^{\sigma} d\sigma \ll \left(x^{\frac{1}{2}}T^{\frac{1}{6}}+x\right)\log^2 T. \end{aligned}$$

Then, $T^* \in [\frac{T}{2}, T]$ exists such that $|F(T^*)|$ is minimum and

$$|F(T^*)| \ll \frac{1}{T}\cdot\left(x^{\frac{1}{2}}T^{\frac{1}{6}}+x\right)\log^2 T \ll x^{-8}$$

by setting $T = x^{12}$. Hence, using horizontal lines of height $\pm T^*$ to move the line of integration in (3.10), we find that the total contribution of the horizontal lines, in absolute value, is $\ll yx^{-8}$. Collecting the error estimates (3.11) and the above, we obtain the total contribution of all error terms, that is,

$$yx^{\frac{1}{2}}\log^{\frac{5}{2}} x + yx^{-8} \ll yx^{\frac{1}{2}}\log^{\frac{5}{2}} x.$$

Hence, we have

$$K_1 = xy(\log x - 1) + O\left(yx^{\frac{1}{2}}\log^{\frac{5}{2}} x\right). \tag{3.12}$$

3.4. Calculation of K_2

We consider the rectangular contour formed by the line segments joining the points $\alpha - iT$, $\alpha + iT$, $\frac{9}{4}+iT$, $\frac{9}{4}-iT$, and $\alpha - iT$ in the clockwise sense, and we observe that $s = 2$ is a simple pole of the integrand. We denote the integrals over the horizontal line segments by $K_{2,1}$ and $K_{2,3}$, and the integral over the vertical line segment by $K_{2,2}$, respectively. A simple pole exists at $s = 2$ of the integral K_2 with the residue $-\frac{\zeta'(2)}{4\zeta(2)}\left(\frac{x}{y}\right)^2$ using $\zeta(0) = -\frac{1}{2}$. For $K_{2.2}$, we use the functional equation of the Riemann zeta-function

$$\zeta(s) = \chi(s)\zeta(1-s) \qquad \text{with} \qquad \chi(s) \asymp \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma} \quad \text{for} \quad |t| \geq T_0 \tag{3.13}$$

and Lemma 3.2 to deduce

$$\begin{aligned} K_{2,2} &= \frac{y^2}{2\pi i}\int_{\frac{9}{4}-i\infty}^{\frac{9}{4}+i\infty} \frac{\zeta(2-s)\zeta'(s)}{\zeta(s)} \frac{\left(\frac{x}{y}\right)^s}{(s-2)s} ds \\ &\quad + O\left(x^2\left(\frac{x}{y}\right)^{\frac{1}{4}} \int_{T}^{\infty} \frac{\left|\zeta\left(-\frac{1}{4}-it\right)\right|}{(1+|t|)^2} dt\right) \end{aligned}$$

$$= x^2 \left(\frac{x}{y}\right)^{\frac{1}{4}} \kappa(u) + O\left(x^2 \left(\frac{x}{y}\right)^{\frac{1}{4}} \frac{\log T}{T^{\frac{1}{4}}}\right), \tag{3.14}$$

where the Fourier integral $\kappa(u)$ is given by

$$\kappa(u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\zeta(-\frac{1}{4} - it)\zeta'(\frac{9}{4} + it)}{\zeta(\frac{9}{4} + it)} \frac{\mathrm{e}^{itu}}{(\frac{1}{4} + it)(\frac{9}{4} + it)} dt$$

with $u := \log \frac{x}{y}$. Because the absolute value of $\kappa(u)$ is

$$\begin{aligned}
|\kappa(u)| &\le \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\zeta(\frac{5}{4} + it)\zeta'(\frac{9}{4} + it)}{\zeta(\frac{9}{4} + it)} \right| \left| \frac{\chi(-\frac{1}{4} - it)}{(\frac{1}{4} + it)(\frac{9}{4} + it)} \right| dt \\
&\le \frac{2}{(2\pi)^{\frac{7}{4}}} \frac{\zeta(\frac{5}{4})\zeta(\frac{9}{4})\zeta'(\frac{9}{4})}{\zeta(\frac{9}{2})} \int_0^{\infty} \frac{t^{\frac{3}{4}}}{t^2 + \left(\frac{3}{4}\right)^2} dt \\
&\le \frac{2}{(2\pi)^{\frac{7}{4}}} \frac{\zeta(\frac{5}{4})\zeta(\frac{9}{4})\zeta'(\frac{9}{4})}{\zeta(\frac{9}{2})} \left(\int_0^1 \frac{1}{t^2 + \left(\frac{3}{4}\right)^2} dt + \int_1^{\infty} t^{-\frac{5}{4}} dt \right) \\
&\le \frac{8}{(2\pi)^{\frac{7}{4}}} \frac{\zeta(\frac{5}{4})\zeta(\frac{9}{4})\zeta'(\frac{9}{4})}{\zeta(\frac{9}{2})} \left(\frac{\pi}{9} + 1 \right)
\end{aligned} \tag{3.15}$$

using (3.13), the inequalities $\sqrt{\left(t^2 + \left(\frac{1}{4}\right)^2\right)} \sqrt{\left(t^2 + \left(\frac{9}{4}\right)^2\right)} \ge t^2 + \left(\frac{3}{4}\right)^2$ for $t \ge 0$, $|\frac{1}{\zeta(s)}| \le \frac{\zeta(\sigma)}{\zeta(2\sigma)}$ for $\sigma > 1$, and $|\chi(-\frac{1}{4} - it)| \asymp \left(\frac{|t|}{2\pi}\right)^{\frac{3}{4}}$. Hence $|\kappa(u)|$ is a bound. We define the function $G(t)$ as

$$G(t) := \frac{1}{2\pi} \int_{\alpha}^{\frac{9}{4}} \frac{\zeta(2 - \sigma - it)\zeta'(\sigma + it)}{\zeta(\sigma + it)} \frac{\left(\frac{x}{y}\right)^{\sigma + it}}{(\sigma - 2 + it)(\sigma + it)} d\sigma.$$

Then, we set

$$R := \int_{\frac{T}{2}}^{T} \left| \int_{\alpha}^{\frac{9}{4}} \frac{\zeta(2 - \sigma - it)\zeta'(\sigma + it)}{\zeta(\sigma + it)} \frac{\left(\frac{x}{y}\right)^{\sigma + it}}{(\sigma - 2 + it)(\sigma + it)} d\sigma \right| dt.$$

We use Lemma 3.2 and (3.13) to obtain

$$\begin{aligned}
R &\ll \log^3 T \int_{\alpha}^{\frac{3}{2}} \left(\frac{x}{y}\right)^{\sigma} \int_{\frac{T}{2}}^{T} \frac{|\zeta(2 - \sigma - it)|}{t^2} dt d\sigma \\
&\quad + \log^3 T \int_{\frac{3}{2}}^{\frac{9}{4}} \left(\frac{x}{y}\right)^{\sigma} \int_{\frac{T}{2}}^{T} \frac{|\chi(2 - \sigma - it)\zeta(\sigma - 1 + it)|}{t^2} dt d\sigma \\
&\ll \frac{\log^3 T}{T^{1+\frac{1}{3}}} \int_{\alpha}^{\frac{3}{2}} \left(\frac{xT^{\frac{1}{3}}}{y}\right)^{\sigma} d\sigma + \frac{\log^3 T}{T^{2-\frac{1}{6}}} \int_{\frac{3}{2}}^{\frac{9}{4}} \left(\frac{xT^{\frac{2}{3}}}{y}\right)^{\sigma} d\sigma
\end{aligned}$$

$$\ll \frac{\log^3 T}{T} \cdot \frac{x}{y} \left(1 + T^{\frac{1}{6}} \left(\frac{x}{y} \right)^{\frac{1}{2}} + T^{\frac{2}{3}} \left(\frac{x}{y} \right)^{\frac{5}{4}} \right).$$

Hence, $T^* \in [\frac{T}{2}, T]$ exists such that $|G(T^*)|$ is minimum and

$$\begin{aligned} |G(T^*)| &\ll \frac{1}{T} \cdot \frac{\log^3 T}{T} \cdot \frac{x}{y} \left(1 + T^{\frac{1}{6}} \left(\frac{x}{y} \right)^{\frac{1}{2}} + T^{\frac{2}{3}} \left(\frac{x}{y} \right)^{\frac{5}{4}} \right) \\ &\ll \frac{1}{y^2} \left(\frac{x}{y} \right)^{\frac{1}{4}} \frac{\log^3 x}{x^{15}} \ll x^{-10} \end{aligned}$$

by setting $T = x^{12}$. For a similar manner as in K_1, we have the weak estimates, that is, $K_{2,1},\ K_{2,3} \ll x^{-10}$. Collecting the error estimates (3.14) and the above, we obtain the total contribution of all error terms, that is, $\ll x^{-\frac{3}{4}}$. Therefore, we obtain

$$K_2 = \frac{\zeta'(2)}{4\zeta(2)} x^2 + x^2 \left(\frac{x}{y} \right)^{\frac{1}{4}} \kappa(u) + O\left(x^{-\frac{3}{4}} \right) \tag{3.16}$$

with $T = x^{12}$.

3.5. Conclusion

Inserting (3.12) and (3.16) into (3.9), we obtain the formula (1.12), which proves Theorem 1.11.

4. Proof of Theorem 1.15

From (1.1) and the identity $(m, n)[m, n] = mn$ for any integers m and n, we have

$$\begin{aligned} S_2(x, y) &= \sum_{n \le y} \left(\sum_{\substack{dk \le x \\ d|n}} d\Lambda(k) \right)^2 = \sum_{d_1 k_1 \le x} d_1 \Lambda(k_1) \sum_{d_2 k_2 \le x} d_2 \Lambda(k_2) \sum_{\substack{n \le y \\ d_1|n,\ d_2|n}} 1 \\ &= \sum_{d_1 k_1 \le x} \sum_{d_2 k_2 \le x} d_1 d_2 \Lambda(k_1) \Lambda(k_2) \left[\frac{y}{[d_1, d_2]} \right] \\ &= y \sum_{d_1 k_1 \le x} \sum_{d_2 k_2 \le x} (d_1, d_2) \Lambda(k_1) \Lambda(k_2) + O(E), \end{aligned}$$

where

$$\begin{aligned} E &:= \sum_{d_1 k_1 \le x} \sum_{d_2 k_2 \le x} d_1 d_2 \log k_1 \log k_2 \\ &\ll x^2 \sum_{k_1 \le x} \frac{\log k_1}{k_1^2} \cdot x^2 \sum_{k_2 \le x} \frac{\log k_2}{k_2^2} \ll x^4. \end{aligned}$$

We use $\sum_{d|n}\phi(d)=n$, $\sum_{d|n}\Lambda(d)=\log n$, and $\sum_{d\le x}\log d=(\log x-1)x+O(\log x)$ to obtain

$$\begin{aligned}\sum_{d_1k_1\le x}\sum_{d_2k_2\le x}(d_1,d_2)\Lambda(k_1)\Lambda(k_2)&=\sum_{d\le x}\phi(d)\sum_{l_1k_1\le x/d}\sum_{l_2k_2\le x/d}\Lambda(k_1)\Lambda(k_2)\\&=\sum_{d\le x}\phi(d)\left(\sum_{mk\le x/d}\Lambda(k)\right)^2=\sum_{d\le x}\phi(d)\left(\sum_{n\le x/d}\sum_{k|n}\Lambda(k)\right)^2\\&=x^2(\log x-1)^2\sum_{d\le x}\frac{\phi(d)}{d^2}-2x^2(\log x-1)\sum_{d\le x}\frac{\phi(d)}{d^2}\log d\\&\quad+x^2\sum_{d\le x}\frac{\phi(d)}{d^2}\log^2 d+O\left(x\log^2 x\sum_{d\le x}\frac{\phi(d)}{d}+\log^2 x\sum_{d\le x}\phi(d)\right).\end{aligned}$$

Using well-known formulas $\sum_{n\le x}\frac{\phi(n)}{n^2}=\frac{1}{\zeta(2)}\log x+O(1)$, $\sum_{n\le x}\frac{\phi(n)}{n^2}\log n=\frac{1}{2\zeta(2)}\log^2 x+O(1)$, and $\sum_{n\le x}\frac{\phi(n)}{n^2}\log^2 n=\frac{1}{3\zeta(2)}\log^3 x+O(1)$ we have

$$\sum_{d_1k_1\le x}\sum_{d_2k_2\le x}(d_1,d_2)\Lambda(k_1)\Lambda(k_2)=\frac{1}{3\zeta(2)}x^2\log^3 x+O\left(x^2\log^2 x\right).$$

Hence, we have

$$S_2(x,y)=\frac{1}{3\zeta(2)}yx^2\log^3 x+O\left(x^4+yx^2\log^2 x\right),$$

which completes the proof of Theorem 1.15. □

5. Proof of Theorem 1.17

5.1. Lemmas

We need the following Lemmas to prove Theorem 1.17, namely

Lemma 5.1. *Let $G(s_1,s_2;y)$ be a sum function defined by*

$$G(s_1,s_2;y)=\sum_{n\le y}\sigma_{1-s_1}(n)\sigma_{1-s_2}(n) \tag{5.2}$$

and $L=\log y$. Then, we have

$$G(s_1,s_2;y)=\sum_{j=1}^{4}R_j(s_1,s_2;y)+O\left(yL^6\left(y^{-\frac12}+\frac{1}{T}\right)\right) \tag{5.3}$$

for $\mathrm{Re}\ s_j\ge 1/2$ *and* $|\mathrm{Im}\ s_j|\le T$ $(j=1,2)$, *where*

$$R_1(s_1,s_2;y)=y\frac{\zeta(s_1)\zeta(s_2)\zeta(s_1+s_2-1)}{\zeta(s_1+s_2)},$$

$$R_2(s_1,s_2;y)=y^{2-s_1}\frac{\zeta(2-s_1)\zeta(1-s_1+s_2)\zeta(s_2)}{(2-s_1)\zeta(2-s_1+s_2)},$$
$$R_3(s_1,s_2;y)=y^{2-s_2}\frac{\zeta(2-s_2)\zeta(1+s_1-s_2)\zeta(s_1)}{(2-s_2)\zeta(2+s_1-s_2)},$$
$$R_4(s_1,s_2;y)=y^{3-s_1-s_2}\frac{\zeta(3-s_1-s_2)\zeta(2-s_2)\zeta(2-s_1)}{(3-s_1-s_2)\zeta(4-s_1-s_2)}.$$

Proof. The proof of this lemma follows from (4.12) in [3]. □

To calculate $S_{2,1}(x,y)$ (See (5.15) with $j=1$ below), we use the Laurent expansions of the Riemann zeta-function at $s=1$, namely

$$\zeta(s)=\frac{1}{s-1}+\gamma+\gamma_1(s-1)+\gamma_2(s-1)^2+\gamma_3(s-1)^3+\cdots, \tag{5.4}$$

where γ is the Euler–Mascheroni constant, and

$$\gamma_k:=\frac{(-1)^k}{k!}\lim_{N\to\infty}\left(\sum_{m=1}^{N}\frac{\log^k m}{m}-\frac{\log^{k+1}N}{k+1}\right)\qquad(k=1,2,\dots) \tag{5.5}$$

are known as Stieltjes constants. Then, we have

$$\zeta'(s)=-\frac{1}{(s-1)^2}+\gamma_1+2\gamma_2(s-1)+3\gamma_3(s-1)^2+\cdots, \tag{5.6}$$
$$\frac{\zeta'(s)}{\zeta(s)}=-\frac{1}{s-1}+\gamma+(2\gamma_1-\gamma^2)(s-1)+(\gamma^3-3\gamma\gamma_1+3\gamma_2)(s-1)^2+\cdots \tag{5.7}$$

as $s\to 1$. We need the following residues, namely

Lemma 5.8. *Let the notation be as above. We have*

$$\begin{aligned}
&\operatorname*{Res}_{s=1}\frac{\zeta(s)\zeta'(s)}{\zeta(s+1)}\left(\log\frac{x}{e}-\frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s}\\
&=-\frac{1}{2\zeta(2)}x\log^3 x-\frac{1}{\zeta(2)}\left(\gamma-\frac{3}{2}-\frac{3}{2}\frac{\zeta'(2)}{\zeta(2)}\right)x\log^2 x\\
&\quad-\frac{1}{\zeta(2)}\left(2(1-\gamma)+(3-2\gamma)\frac{\zeta'(2)}{\zeta(2)}+3\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2-\frac{3}{2}\frac{\zeta''(2)}{\zeta(2)}\right)x\log x\\
&\quad-\frac{1}{\zeta(2)}\left(\gamma-1+2(\gamma-1)\frac{\zeta'(2)}{\zeta(2)}+(2\gamma-3)\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2-3\left(\frac{\zeta'(2)}{\zeta(2)}\right)^3\right)x\\
&\quad-\frac{1}{\zeta(2)}\left(\left(\frac{3}{2}-\gamma\right)\frac{\zeta''(2)}{\zeta(2)}-\frac{1}{2}\frac{\zeta'''(2)}{\zeta(2)}+3\frac{\zeta'(2)}{\zeta(2)}\frac{\zeta''(2)}{\zeta(2)}\right)x,
\end{aligned}\tag{5.9}$$

$$\begin{aligned}
&\operatorname*{Res}_{s=1}\frac{(\zeta'(s))^2}{\zeta(s+1)}\frac{x^s}{s}=\frac{1}{6\zeta(2)}x\log^3 x-\frac{1}{2\zeta(2)}\left(1+\frac{\zeta'(2)}{\zeta(2)}\right)x\log^2 x\\
&+\frac{1}{\zeta(2)}\left(\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2+\frac{\zeta'(2)}{\zeta(2)}-\frac{\zeta''(2)}{2\zeta(2)}+1-2\gamma_1\right)x\log x
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\zeta(2)}\left(\left(\frac{\zeta'(2)}{\zeta(2)}\right)^3+\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2+(1-2\gamma_1)\frac{\zeta'(2)}{\zeta(2)}-\frac{\zeta'(2)}{\zeta(2)}\frac{\zeta''(2)}{\zeta(2)}\right)x\\
&+\frac{1}{\zeta(2)}\left(\frac{\zeta''(2)}{2\zeta(2)}-\frac{\zeta'''(2)}{6\zeta(2)}+2(\gamma_1-2\gamma_2)-1\right)x, \qquad (5.10)
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname*{Res}_{s=1}\ \zeta(2-s)\zeta'(s)\frac{\zeta'(s)}{\zeta(s)}\frac{u^s}{(2-s)s^2}\\
&\quad=-\frac{1}{6}u\log^3 u+\left(\frac{1}{2}+\gamma\right)u\log^2 u-2\left(\gamma^2+\gamma-\gamma_1+1\right)u\log u\\
&\qquad+2\left(\gamma^3+\gamma^2+2\gamma-\gamma_1-3\gamma\gamma_1+3\gamma_2+1\right)u \qquad (5.11)
\end{aligned}
$$

with $u=x^2/y$.

Proof. Suppose that $g(s)$ is regular in the neighborhood at $s=1$, and $f(s)$ has only a triple pole at $s=1$, then the Laurent expansion of $f(s)$ implies

$$
f(s):=\frac{a}{(s-1)^3}+\frac{b}{(s-1)^2}+\frac{c}{s-1}+h(s),
$$

where $h(s)$ is regular in the neighborhood of its pole, and a,b,c are computable constants. We use the residue calculation to deduce

$$
\operatorname*{Res}_{s=1} f(s)g(s)=\frac{a}{2}g''(1)+bg'(1)+cg(1).
$$

To prove (5.9), we use (5.4) and (5.6) to obtain

$$
\zeta(s)\zeta'(s)=\frac{-1}{(s-1)^3}+\frac{-\gamma}{(s-1)^2}+\gamma_2+\gamma\gamma_1+O\left(|s-1|\right)
$$

as $s\to 1$. We set $g(s):=\frac{1}{\zeta(s+1)}\left(\log\frac{x}{e}-\frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s}$, then

$$
\begin{aligned}
g'(1)=&\frac{1}{\zeta(2)}x\log^2 x-\frac{2}{\zeta(2)}\left(1+\frac{\zeta'(2)}{\zeta(2)}\right)x\log x\\
&+\frac{1}{\zeta(2)}\left(1+2\frac{\zeta'(2)}{\zeta(2)}+2\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2-\frac{\zeta''(2)}{\zeta(2)}\right)x,
\end{aligned}
$$

and

$$
\begin{aligned}
g''(1)=&\frac{1}{\zeta(2)}x\log^3 x-\frac{3}{\zeta(2)}\left(1+\frac{\zeta'(2)}{\zeta(2)}\right)x\log x\\
&+\frac{2}{\zeta(2)}\left(2+\frac{\zeta'(2)}{\zeta(2)}+3\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2-\frac{3}{2}\frac{\zeta''(2)}{\zeta(2)}\right)x\log x\\
&-\frac{1}{\zeta(2)}\left(2+4\frac{\zeta'(2)}{\zeta(2)}+6\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2\right.
\end{aligned}
$$

$$+6\left(\frac{\zeta'(2)}{\zeta(2)}\right)^3 - 6\frac{\zeta'(2)}{\zeta(2)}\frac{\zeta''(2)}{\zeta(2)} - 3\frac{\zeta''(2)}{\zeta(2)} + \frac{\zeta'''(2)}{\zeta(2)}\Bigg) x.$$

Hence, we have

$$\operatorname*{Res}_{s=1} \frac{\zeta(s)\zeta'(s)}{\zeta(s+1)}\left(\log\frac{x}{e} - \frac{\zeta'(s+1)}{\zeta(s+1)}\right)\frac{x^s}{s} = -\frac{1}{2}g''(1) - \gamma g'(1).$$

We use the same method as above to prove (5.10) and (5.11). □

5.2. Expressions of $S_{2,j}(x,y)$ for $j = 1, 2, 3, 4$

We assume that $1 \le y \le x^M$ for some constant M. Without loss of generality, we can assume that $x,\ y \in \mathbb{Z} + \frac{1}{2}$. Suppose that T is a real parameter at our disposal. Let $\alpha_1 = 1 + \frac{2}{\log x}$ and $\alpha_2 = 1 + \frac{3}{\log x}$. Applying (3.8) with $\alpha = \alpha_j$ $(j = 1, 2)$ we have

$$\left(\sum_{q\le x} s_q(n)\right)^2 = \frac{1}{(2\pi i)^2}\int_{\alpha_1 - iT}^{\alpha_1+iT}\int_{\alpha_2-iT}^{\alpha_2+iT} F(s_1, s_2; n) ds_2 ds_1 + E_2(x; n), \tag{5.12}$$

where

$$F(s_1, s_2; n) := \sigma_{1-s_1}(n)\sigma_{1-s_2}(n)\frac{\zeta'(s_1)\zeta'(s_2)}{\zeta(s_1)\zeta(s_2)}\frac{x^{s_1+s_2}}{s_1 s_2}$$

and

$$\begin{aligned} E_2(x; n) &:= E_1(x; n)\Bigg(\frac{1}{2\pi i}\int_{\alpha_1-iT}^{\alpha_1+iT} \sigma_{1-s_1}(n)\frac{\zeta'(s_1)}{\zeta(s_1)}\frac{x^{s_1}}{s_1} ds_1 \\ &\quad + \frac{1}{2\pi i}\int_{\alpha_2-iT}^{\alpha_2+iT} \sigma_{1-s_2}(n)\frac{\zeta'(s_2)}{\zeta(s_2)}\frac{x^{s_2}}{s_2} ds_2 + E_1(x; n)\Bigg) \\ &\ll \frac{x^2}{T}\sigma_0(n)^2 \log^4 T. \end{aligned}$$

Summing (5.12) over n and using the inequality $\sum_{n\le y}\sigma_0(n)^2 \ll y\log^3 y$, we obtain

$$\begin{aligned} S_2(x, y) = \frac{1}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-iT}^{\alpha_2+iT} G(s_1, s_2; y)\frac{\zeta'(s_1)\zeta'(s_2)}{\zeta(s_1)\zeta(s_2)}\frac{x^{s_1+s_2}}{s_1 s_2} ds_2 ds_1 \\ + O\left(\frac{x^2 y L^7}{T}\right), \end{aligned} \tag{5.13}$$

where $G(s_1, s_2; y) := \sum_{n\le y}\sigma_{1-s_1}(n)\sigma_{1-s_2}(n)$ and $L = \log(Txy)$.

Now, we shall evaluate the integral of (5.13). Substituting (5.3) into (5.13), we obtain

$$S_2(x, y) = \sum_{j=1}^{4} S_{2,j}(x, y) + O\left(x^2 y L^{10}\left(\frac{1}{T} + y^{-1/2}\right)\right), \tag{5.14}$$

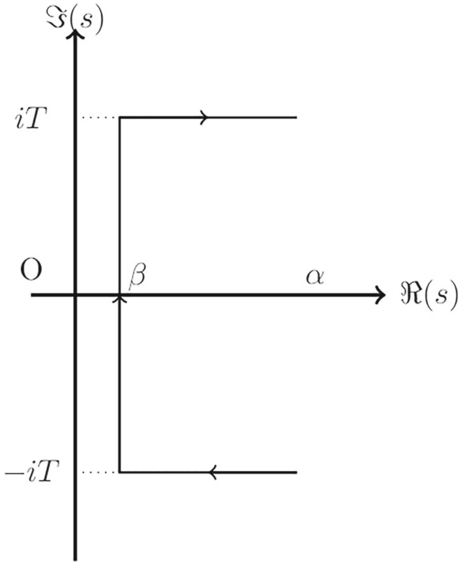


FIGURE 1. $\Gamma(\alpha,\beta,T)$

where

$$S_{2,j}(x,y)=\frac{1}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-iT}^{\alpha_2+iT}R_j(s_1,s_2;y)\frac{\zeta'(s_1)\zeta'(s_2)}{\zeta(s_1)\zeta(s_2)}\;\frac{x^{s_1+s_2}}{s_1s_2}ds_2ds_1. \tag{5.15}$$

Note that we substitute $T=x$ with a small positive constant c into the error term on the right-hand side of (5.14) to obtain

$$\ll xyL^{10}\left(1+\frac{x}{y^{1/2}}\right). \tag{5.16}$$

5.3. Evaluation of $S_{2,1}(x,y)$.

Let $\alpha_1=1+\frac{2}{\log x}$ and $\alpha_2=1+\frac{3}{\log x}$. From the definition of $R_1(s_1,s_2,y)$, we obtain

$$S_{2,1}(x,y)=\frac{y}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-iT}^{\alpha_2+iT}\frac{\zeta'(s_1)\zeta'(s_2)\zeta(s_1+s_2-1)}{\zeta(s_1+s_2)}\frac{x^{s_1+s_2}}{s_1s_2}ds_2ds_1. \tag{5.17}$$

Let $\Gamma(\alpha,\beta,T)$ denote the following contour comprising the line segments $[\alpha-iT,\beta-iT]$, $[\beta-iT,\beta+iT]$, and $[\beta+iT,\alpha+iT]$ (Fig. 1).

In (5.17), we move the integration with respect to s_2 to $\Gamma(\alpha_2,\frac{1}{2}+\frac{1}{\log x},T)$. We denote the integrals over the horizontal line segments by $J_{1,1}$ and $J_{1,3}$, and the integral over the vertical line segment by $J_{1,2}$, respectively. Then, using the weak estimate $\int_1^T |\zeta'(\alpha_1+it)|dt \ll T\log T$ and Lemma 3.2, we have

$$\begin{aligned} J_{1,1}, J_{1,3} &\ll \frac{xyL}{T}\int_{-T}^{T}\frac{|\zeta'(\alpha_1+it_1)|}{1+|t_1|}\int_{\frac{1}{2}+\frac{1}{\log x}}^{\alpha_2}|\zeta'(\sigma_2+iT)\zeta(\alpha_1+\sigma_2-1 \\ &\quad +i(t_1+T))|x^{\sigma_2}d\sigma_2 dt_1 \\ &\ll \frac{xyL^4}{T}\int_{-T}^{T}\frac{|\zeta'(\alpha_1+it_1)|}{1+|t_1|}\int_{\frac{1}{2}+\frac{1}{\log x}}^{\alpha_2}T^{\frac{2}{3}(1-\sigma_2)}x^{\sigma_2}d\sigma_2 dt_1 \\ &\ll \frac{x^2yL^6}{T^{2/3}}\left(x^{-1/2}+T^{-1/3}\right). \end{aligned}$$

For the integral along the vertical line, we have

$$\begin{aligned} J_{1,2} &\ll yx^{3/2}L \\ &\times \int_{-T}^{T}\int_{-T}^{T}\frac{|\zeta'(\alpha_1+it_1)\zeta'\left(\frac{1}{2}+\frac{1}{\log x}+it_2\right)\zeta(\alpha_1+\frac{1}{\log x}-\frac{1}{2}+i(t_1+t_2))|}{(1+|t_1|)(1+|t_2|)}dt_1dt_2 \\ &\ll yx^{3/2}L^3\int_{-2T}^{2T}\left|\zeta\left(\frac{1}{2}+\frac{1}{\log x}+iu\right)\right|\int_{-T}^{T}\frac{|\zeta'(\frac{1}{2}+\frac{1}{\log x}+it)|}{(1+|t|)(1+|t-u|)}dtdu. \end{aligned}$$

Now, we use Lemma 3.2 to obtain the estimate

$$\begin{aligned} \int_{-T}^{T}\frac{|\zeta'(\frac{1}{2}+\frac{1}{\log x}++it)|}{(1+|t|)(1+|t-u|)}dt &\ll T^{1/6}L^2\left(\int_{|t-u|>\frac{1}{2}|u|}+\int_{|t-u|\le\frac{1}{2}|u|}\right)\frac{1}{(1+|t|)(1+|t-u|)}dt \\ &\ll \frac{T^{1/6}L^3}{1+|u|}, \end{aligned}$$

and use the Cauchy–Schwarz inequality and the above to deduce

$$\begin{aligned} J_{1,2} &\ll yx^{3/2}T^{1/6}L^6\int_{-2T}^{2T}\frac{\left|\zeta\left(\frac{1}{2}+\frac{1}{\log x}+iu\right)\right|}{1+|u|}du \\ &\ll yx^{3/2}T^{1/6}L^8. \end{aligned} \tag{5.18}$$

It remains to evaluate the residues of the poles of the integrand when we move the line of integration to $\Gamma(\alpha_2, \frac{1}{2}+\frac{1}{\log x}, T)$. A simple pole exists at $s_2 = 2 - s_1$ with residue

$$\frac{\zeta'(s_1)\zeta'(2-s_1)}{\zeta(2)s_1(2-s_1)}\,x^2 =: H_1(s_1)x^2,$$

and a double pole at $s_2 = 1$ with residue

$$\begin{aligned} &-\frac{(\zeta'(s_1))^2}{\zeta(s_1+1)s_1}x^{s_1+1}-\frac{\zeta(s_1)\zeta'(s_1)}{\zeta(s_1+1)s_1}\left(\log\frac{x}{e}-\frac{\zeta'(s_1+1)}{\zeta(s_1+1)}\right)\,x^{s_1+1} \\ &\quad =: H_2(s_1)x^{s_1+1}+H_3(s_1)x^{s_1+1}. \end{aligned}$$

The contributions to $S_{2,1}(x,y)$ from these residues are

$$\frac{x^2y}{2\pi i}\int_{\alpha_1-iT}^{\alpha_1+iT} H_1(s_1)ds_1+\frac{xy}{2\pi i}\int_{\alpha_1-iT}^{\alpha_1+iT} H_2(s_1)x^{s_1}ds_1+\frac{xy}{2\pi i}\int_{\alpha_1-iT}^{\alpha_1+iT} H_3(s_1)x^{s_1}ds_1$$
$$=: I_1+I_2+I_3, \text{ say.}$$

For I_1, moving the line of integration to $\Gamma(\frac{5}{4},\alpha_1,T)$, we have

$$\begin{aligned} I_1 &= \frac{x^2y}{2\pi i}\int_{5/4-i\infty}^{5/4+i\infty} H_1(s_1)ds_1 + O\left(x^2y\int_T^\infty \left|H_1\left(\frac{5}{4}\pm it_1\right)\right|dt_1\right) \\ &= \frac{c_0}{\zeta(2)}x^2y + O\left(\frac{x^2yL^4}{T^{11/12}}\right), \end{aligned}$$

where the computable constant c_0 is given by

$$c_0 := \frac{1}{2\pi i}\int_{\frac{5}{4}-i\infty}^{\frac{5}{4}+i\infty} \frac{\zeta'(s_1)\zeta'(2-s_1)}{s_1(2-s_1)}ds_1. \tag{5.19}$$

For I_2, we move the line of integration to $\Gamma(\alpha_1,\frac{1}{2}+\frac{1}{\log x},T)$. Using Lemma 3.2, the integrals over the horizontal lines are

$$\ll \frac{xyL^5}{T}\int_{\frac{1}{2}+\frac{1}{\log x}}^{\alpha_1} T^{\frac{2}{3}(1-\sigma_1)}x^{\sigma_1}d\sigma_1 \ll \frac{x^{3/2}yL^5}{T}\left(x^{1/2}+T^{1/3}\right)$$

and that over the vertical line is

$$\ll xyL\int_{-T}^{T}\frac{|\zeta'(\frac{1}{2}+it_1)|^2}{1+|t_1|}x^{1/2}dt_1 \ll x^{3/2}yL^5$$

using $\int_{2\pi}^{T}\left|\zeta'\left(\frac{1}{2}+iv\right)\right|^2\frac{dv}{v} \ll \log^4 T$ (see (172) in Hall [5]). Moving the path of integration, a pole of order 4 exists at $s_1=1$. Hence, we use Cauchy's theorem and (5.10) to obtain

$$\begin{aligned} I_2 = &-\frac{1}{6\zeta(2)}x^2y\log^3 x + \frac{1}{2\zeta(2)}\left(1+\frac{\zeta'(2)}{\zeta(2)}\right)x^2y\log^2 x \\ &- \frac{1}{\zeta(2)}\left(\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2+\frac{\zeta'(2)}{\zeta(2)}-\frac{\zeta''(2)}{2\zeta(2)}+2\gamma_1+1\right)x^2y\log x \\ &+ \frac{1}{\zeta(2)}\left(\left(\frac{\zeta'(2)}{\zeta(2)}\right)^3+\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2+(2\gamma_1+1)\frac{\zeta'(2)}{\zeta(2)}-\frac{\zeta'(2)}{\zeta(2)}\frac{\zeta''(2)}{\zeta(2)}\right)x^2y \\ &- \frac{1}{\zeta(2)}\left(\frac{\zeta''(2)}{2\zeta(2)}-\frac{\zeta'''(2)}{6\zeta(2)}-2(\gamma_1+\gamma_2)-1\right)x^2y \\ &+ O\left(\frac{x^{3/2}yL^5}{T}\left(x^{1/2}+T^{1/3}\right)\right) \\ &+ O(x^{3/2}yL^5), \end{aligned}$$

where γ_1 and γ_2 are the Stieltjes constants.

Similarly to I_2, we move the line of integration to $\Gamma(\alpha_1, \frac{1}{2} + \frac{1}{\log x}, T)$ to calculate I_3. The integrals over the horizontal lines are

$$\ll \frac{xyL^6}{T} \int_{\frac{1}{2}+\frac{1}{\log x}}^{\alpha_1} T^{\frac{2}{3}(1-\sigma_1)} x^{\sigma_1} d\sigma_1 \ll \frac{x^{3/2}yL^6}{T} \left(x^{1/2} + T^{1/3} \right)$$

and the integral over the vertical line is

$$\ll xyL^4 \int_{-T}^{T} \frac{|\zeta(\frac{1}{2}+it_1)\zeta'(\frac{1}{2}+it_1)|}{1+|t_1|} x^{1/2} dt_1 \ll x^{3/2}yL^7$$

using $\int_{2\pi}^{T} \left|\zeta\left(\frac{1}{2}+iv\right)\zeta'\left(\frac{1}{2}+iv\right)\right| \frac{dv}{v} \ll \log^3 T$ (see (173) in Hall [5]). Furthermore, when moving the path of integration, a triple pole exists at $s_1 = 1$. Hence, using Cauchy's theorem and (5.9) we have

$$\begin{aligned}
I_3 = {} & \frac{1}{2\zeta(2)} x^2 y \log^3 x + \frac{1}{\zeta(2)} \left(\gamma - \frac{3}{2} - \frac{3}{2}\frac{\zeta'(2)}{\zeta(2)} \right) x^2 y \log^2 x \\
& + \frac{1}{\zeta(2)} \left(2(1-\gamma) + (3-2\gamma)\frac{\zeta'(2)}{\zeta(2)} + 3\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2 - \frac{3}{2}\frac{\zeta''(2)}{\zeta(2)} \right) x^2 y \log x \\
& + \frac{1}{\zeta(2)} \left(\gamma - 1 + 2(\gamma-1)\frac{\zeta'(2)}{\zeta(2)} + (2\gamma-3)\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2 - 3\left(\frac{\zeta'(2)}{\zeta(2)}\right)^3 \right) x^2 y \\
& + \frac{1}{\zeta(2)} \left(\left(\frac{3}{2} - \gamma\right) \frac{\zeta''(2)}{\zeta(2)} - \frac{1}{2}\frac{\zeta'''(2)}{\zeta(2)} + 3\frac{\zeta'(2)}{\zeta(2)}\frac{\zeta''(2)}{\zeta(2)} \right) x^2 y \\
& + O\left(\frac{x^{3/2}yL^6}{T} \left(x^{1/2} + T^{1/3} \right) \right) + O(x^{3/2}yL^7),
\end{aligned}$$

where γ is the Euler–Mascheroni constant. Combining these results, we have

$$\begin{aligned}
S_{2,1}(x,y) = {} & \frac{1}{3\zeta(2)} yx^2 \log^3 x + \frac{1}{\zeta(2)} \left(\gamma - 1 - \frac{\zeta'(2)}{\zeta(2)} \right) yx^2 \log^2 x \\
& + \frac{1}{\zeta(2)} \left(1 - 2(\gamma+\gamma_1) + 2(1-\gamma)\frac{\zeta'(2)}{\zeta(2)} + 2\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2 \right. \\
& \left. - \frac{\zeta''(2)}{\zeta(2)} \right) yx^2 \log x + \frac{1}{\zeta(2)} \left(c_0 + \gamma + 2(\gamma_1+\gamma_2) + (2\gamma + 2\gamma_1 - 1) \right. \\
& \left. \frac{\zeta'(2)}{\zeta(2)} + 2(\gamma-1)\left(\frac{\zeta'(2)}{\zeta(2)}\right)^2 \right) yx^2 \\
& + \frac{1}{\zeta(2)} \left((1-\gamma)\frac{\zeta''(2)}{\zeta(2)} - \frac{\zeta'''(2)}{3\zeta(2)} + 2\frac{\zeta'(2)}{\zeta(2)}\frac{\zeta''(2)}{\zeta(2)} - 2\left(\frac{\zeta'(2)}{\zeta(2)}\right)^3 \right) yx^2 \\
& + O(x^{5/3}yL^8). \qquad (5.20)
\end{aligned}$$

Here, we substitute $T = x$ into the error term of $S_{2,1}(x,y)$.

5.4. Estimation of $S_{2,4}(x,y)$.

This is determined explicitly by

$$S_{2,4}(x,y)=\frac{y^3}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-iT}^{\alpha_2+iT} \frac{\zeta(3-s_1-s_2)\zeta(2-s_1)\zeta(2-s_2)}{\zeta(4-s_1-s_2)(3-s_1-s_2)s_1s_2}\frac{\zeta'(s_1)\zeta'(s_2)}{\zeta(s_1)\zeta(s_2)}\left(\frac{x}{y}\right)^{s_1+s_2} ds_2ds_1.$$

For this purpose, we move the line of integral with respect to s_2 to contour $\Gamma(\beta,\alpha_2,T)$, where $\beta=\frac{5}{2}-\alpha_1=\frac{3}{2}-\frac{2}{\log x}$. No poles are present when we deform the path of the integral over s_2. The contribution from the horizontal lines is

$$J_{4,1},J_{4,3}\ll xy^2\left(\frac{x}{y}\right)^{\frac{1}{\log x}}\int_{-T}^{T}\frac{\left|\zeta\left(1-\frac{2}{\log x}-it_1\right)\zeta'\left(1+\frac{2}{\log x}+it_1\right)\right|}{\left|\zeta\left(1+\frac{2}{\log x}+it_1\right)\right|(1+|t_1|)}dt_1$$
$$\times\int_{\alpha_2}^{\beta}\frac{\left|\zeta\left(2-\frac{2}{\log x}-\sigma_2-i(t_1+T)\right)\zeta\left(2-\sigma_2-iT\right)\zeta'\left(\sigma_2+iT\right)\right|}{\left|\zeta\left(3-\frac{2}{\log x}-\sigma_2-i(t_1+T)\right)\zeta\left(\sigma_2+iT\right)\right|(1+|t_1+T|)T}\left(\frac{x}{y}\right)^{\sigma_2}d\sigma_2.$$

The inner integral is estimated as

$$\ll\frac{L^5}{T(1+|t_1+T|)}\left(\frac{x}{y}\right)\left(1+T^{1/6}\left(\frac{x}{y}\right)^{\frac{1}{2}}\right),$$

where we have used Lemma 3.2 and assumption $y\ll x^M$. Hence, we have

$$J_{4,1},J_{4,3}\ll\frac{x^2yL^8}{T}\left(1+T^{\frac{1}{6}}\left(\frac{x}{y}\right)^{\frac{1}{2}}\right)\int_{-T}^{T}\frac{\left|\zeta\left(1-\frac{2}{\log x}-it_1\right)\right|}{(1+|t_1|)(1+|t_1|+T)}dt_1$$
$$\ll\frac{x^2yL^{10}}{T^2}\left(1+T^{\frac{1}{6}}\left(\frac{x}{y}\right)^{\frac{1}{2}}\right).$$

For the integral on the vertical line, we find that

$$J_{4,2}\ll y^3\int_{-T}^{T}\int_{-T}^{T}\frac{\left|\zeta(\frac{1}{2}-i(t_1+t_2))\zeta(1-\frac{2}{\log x}-it_1)\zeta(\frac{1}{2}+\frac{2}{\log x}-it_2)\right|}{(1+|t_1+t_2|)(1+|t_1|)(1+|t_2|)}$$
$$\times\frac{\left|\zeta'\left(1+\frac{2}{\log x}+it_1\right)\zeta'\left(\frac{3}{2}-\frac{2}{\log x}-it_2\right)\right|}{\left|\zeta\left(1+\frac{2}{\log x}+it_1\right)\zeta\left(\frac{3}{2}-\frac{2}{\log x}-it_2\right)\right|}\left(\frac{x}{y}\right)^{5/2}dt_1dt_2$$
$$\ll y^3\left(\frac{x}{y}\right)^{5/2}L^6\int_{-2T}^{2T}\frac{\left|\zeta\left(\frac{1}{2}-iu\right)\right|}{1+|u|}\int_{-T}^{T}\frac{\left|\zeta\left(\frac{1}{2}+\frac{2}{\log x}-it_2\right)\right|}{(1+|t_2|)(1+|u-t_2|)}dt_2du$$
$$\ll x^2yL^{10}\left(\frac{x}{y}\right)^{1/2}$$

using a well-known estimate $\int_1^T |\zeta(\frac{1}{2}+it)|^2 \frac{dt}{t} \ll \log^2 T$. Hence, we take $T = x$ to obtain

$$S_{2,4}(x,y) \ll x^2 y L^{10} \left(\frac{x}{y}\right)^{1/2}. \tag{5.21}$$

5.5. Estimation of $S_{2,3}(x,y)$.

This is determined explicitly by

$$S_{2,3}(x,y) = \frac{y^2}{(2\pi i)^2} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \int_{\alpha_2 - iT}^{\alpha_2 + iT} \frac{\zeta(2-s_2)\zeta(1+s_1-s_2)\zeta'(s_1)\zeta'(s_2)}{\zeta(2+s_1-s_2)\zeta(s_2)(2-s_2)} \frac{x^{s_1+s_2} y^{-s_2}}{s_1 s_2} ds_2 ds_1.$$

We move the path of integration with respect to s_2 to $\Gamma(\frac{3}{2}, \alpha_2, T)$. No poles with this deformation exist. The contribution from the horizontal lines is

$$\begin{aligned} J_{3,1}, J_{3,3} &\ll \frac{y^2 x L}{T^2} \int_{-T}^{T} \frac{|\zeta'(\alpha_1 + it_1)|}{1+|t_1|} \\ &\quad \int_{\alpha_2}^{3/2} \frac{|\zeta(2-\sigma_2 - iT)\zeta(1+\alpha_1-\sigma_2+i(t_1-T))\zeta'(\sigma_2+iT)|}{|\zeta(\sigma_2+iT)|} \\ &\quad \times \left(\frac{x}{y}\right)^{\sigma_2} d\sigma_2 dt_1 \\ &\ll \frac{y^2 x L^6}{T^2} \int_{-T}^{T} \frac{|\zeta'(\alpha_1+it_1)|}{1+|t_1|} \\ &\quad \int_{\alpha_2}^{3/2} T^{\frac{1}{3}(-1+\sigma_2)} (1+|t_1-T|)^{\frac{1}{3}(-1+\sigma_2)} \left(\frac{x}{y}\right)^{\sigma_2} d\sigma_2 dt_1 \\ &\ll y x^2 L^8 \left(T^{-2} + T^{-5/3}\left(\frac{x}{y}\right)^{1/2}\right) \end{aligned}$$

using Lemma 3.2. In contrast, the contribution from the vertical lines is

$$\begin{aligned} J_{3,2} &\ll y^2 x \int_{-T}^{T} \frac{|\zeta'(\alpha_1+it_1)|}{1+|t_1|} \\ &\quad \int_{-T}^{T} \frac{|\zeta(\frac{1}{2}-it_2)\zeta(\frac{1}{2}+\frac{2}{\log x}+i(t_1-t_2))\zeta'(\frac{3}{2}+it_2)|}{|\zeta(\frac{3}{2}+\frac{2}{\log x}+i(t_1-t_2))|(1+|t_2|)^2|\zeta(\frac{3}{2}+it_2)|} \left(\frac{x}{y}\right)^{3/2} dt_2 dt_1 \\ &\ll y^2 x \left(\frac{x}{y}\right)^{3/2} L^6 \int_{-T}^{T} \int_{-2T}^{2T} \frac{|\zeta(\frac{1}{2}-it_2)|}{(1+|t_2|)^2} \frac{|\zeta(\frac{1}{2}+\frac{1}{\log x}+iu)|}{1+|u+t_2|} du dt_2 \\ &\ll y^2 x \left(\frac{x}{y}\right)^{3/2} L^8. \end{aligned}$$

Hence, we substitute $T = x$ into the above to obtain

$$S_{2,3}(x,y) \ll x^2 y L^8 \left(\frac{x}{y}\right)^{1/2}. \tag{5.22}$$

5.6. Evaluation of $S_{2,2}(x, y)$.

The explicit form of $S_{2,2}(x, y)$ is given by

$$S_{2,2}(x,y) = \frac{y^2}{(2\pi i)^2} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \int_{\alpha_2 - iT}^{\alpha_2 + iT} \frac{\zeta(2-s_1)\zeta(1-s_1+s_2)\zeta'(s_1)\zeta'(s_2)}{\zeta(2-s_1+s_2)\zeta(s_1)(2-s_1)} \frac{x^{s_1+s_2} y^{-s_1}}{s_1 s_2} ds_2 ds_1. \tag{5.23}$$

First, we move the integral line from s_1 to $\Gamma(\frac{3}{2}, \alpha_1, T)$. The estimates over the horizontal and vertical lines are the same as that of $S_{2,3}(x, y)$, but a simple pole exists at $s_1 = s_2$ inside this contour. The residue of the integrand of (5.23) at this pole is

$$-\frac{\zeta(2-s_2)(\zeta'(s_2))^2}{\zeta(2)\zeta(s_2)(2-s_2)s_2^2} x^{2s_2} y^{-s_2}.$$

The contribution from the horizontal lines is

$$\begin{aligned} J_{2,1}, J_{2,3} &\ll \frac{y^2 x L}{T^2} \int_{-T}^{T} \frac{|\zeta'(\alpha_2 + it_2)|}{1+|t_2|} \\ &\quad \int_{\alpha_1}^{3/2} \frac{|\zeta(2-\sigma_1 - iT)\zeta(1+\alpha_2-\sigma_1+i(t_2-T))\zeta'(\sigma_1+iT)|}{|\zeta(\sigma_1+iT)|} \times \\ &\quad \times \left(\frac{x}{y}\right)^{\sigma_1} d\sigma_1 dt_2 \\ &\ll \frac{y^2 x L^6}{T^2} \int_{-T}^{T} \frac{|\zeta'(\alpha_2 + it_2)|}{1+|t_2|} \\ &\quad \int_{\alpha_1}^{3/2} T^{\frac{1}{3}(-1+\sigma_1)} (1+|t_2 - T|)^{\frac{1}{3}(-1+\sigma_1)} \left(\frac{x}{y}\right)^{\sigma_1} d\sigma_2 dt_1 \\ &\ll y x^2 L^8 \left(T^{-2} + T^{-5/3} \left(\frac{x}{y}\right)^{1/2}\right) \end{aligned}$$

using Lemma 3.2. In contrast, the contribution from the vertical lines is

$$\begin{aligned} J_{2,2} &\ll y^2 x \int_{-T}^{T} \frac{|\zeta'(\alpha_2 + it_2)|}{1+|t_2|} \\ &\quad \int_{-T}^{T} \frac{|\zeta(\frac{1}{2} - it_1)\zeta(\frac{1}{2} + \frac{3}{\log x} + i(t_2 - t_1))\zeta'(\frac{3}{2} + it_1)|}{|\zeta(\frac{3}{2} + \frac{3}{\log x} + i(t_2-t_1))|(1+|t_1|)^2 |\zeta(\frac{3}{2} + it_1)|} \left(\frac{x}{y}\right)^{3/2} dt_1 dt_2 \\ &\ll y^2 x \left(\frac{x}{y}\right)^{3/2} L^6 \int_{-T}^{T} \int_{-2T}^{2T} \frac{|\zeta(\frac{1}{2} - it_1)|}{(1+|t_1|)^2} \frac{|\zeta(\frac{1}{2} + \frac{3}{\log x} + iu)|}{1+|u+t_1|} du dt_1 \end{aligned}$$

$$\ll y^2 x \left(\frac{x}{y}\right)^{3/2} L^8.$$

Hence, we substitute $T = x$ into the above to obtain

$$J_{2,1}, J_{2,2}, J_{2,3} \ll x^2 y L^8 \left(\frac{x}{y}\right)^{1/2}.$$

Hence, we have

$$S_{2,2}(x,y) = \frac{y^2}{\zeta(2)} Q(x,y) + O\left(x^2 y L^8 \left(\frac{x}{y}\right)^{1/2}\right),$$

where

$$Q(x,y) := -\frac{1}{2\pi i}\int_{\alpha_2 - iT}^{\alpha_2 + iT} \zeta(2-s_2)\zeta'(s_2)\frac{\zeta'(s_2)}{\zeta(s_2)} \cdot \frac{1}{(2-s_2)s_2^2}\left(\frac{x^2}{y}\right)^{s_2} ds_2.$$

It remains to evaluate the integral $Q(x,y)$. We move the integration with respect to s_2 to $\Gamma(\alpha_2, \alpha_0, T)$ with $\alpha_0 = 1 - \frac{c}{\log T}$, where c is a small positive constant, and denote the integrals over the horizontal line segments by $Q_1(x,y)$ and $Q_3(x,y)$, and the integral over the vertical line segment by $Q_2(x,y)$, respectively. Using Lemma 3.2 and the estimate $\left|-\frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)}\right| \ll \log T$ for $\sigma \geq \alpha_0$, we have

$$\begin{aligned} Q_1(x,y) &\ll \int_{\alpha_0}^{\alpha_2} |\zeta(2-\sigma-iT)\zeta'(\sigma+iT)| \left|-\frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)}\right| \cdot \frac{1}{T^3}\left(\frac{x^2}{y}\right)^{\sigma} d\sigma \\ &\ll \left(\frac{x^2}{y}\right)^{\alpha_2} L^4(\alpha_2 - \alpha_0)T^{-2} \ll \frac{x^2}{y}\frac{L^3}{T^2}, \end{aligned}$$

and similarly, $Q_3(x,y) \ll \frac{x^2}{y}\frac{L^3}{T^2}$, and

$$\begin{aligned} Q_2(x,y) \ll \int_{-T}^{T} &\left|\zeta\left(1 + \frac{c}{\log T} - it_2\right)\zeta'\left(1 - \frac{c}{\log T} + it_2\right)\right| \\ &\left|-\frac{\zeta'\left(1 - \frac{c}{\log T} + it_2\right)}{\zeta\left(1 - \frac{c}{\log T} + it_2\right)}\right| \frac{1}{1+|t_2|^2}\left(\frac{x^2}{y}\right)^{\alpha_0} dt_2 \ll \left(\frac{x^2}{y}\right)^{\alpha_0} L^4. \end{aligned}$$

Therefore, using Cauchy's theorem, (5.11) with $u = x^2/y$ in Lemma 5.8 and taking $T = x$ in the above we have

$$\begin{aligned} S_{2,2}(x,y) = &\frac{1}{6\zeta(2)} y x^2 \log^3 \frac{x^2}{y} - \frac{2\gamma+1}{2\zeta(2)} y x^2 \log^2 \frac{x^2}{y} \\ &+ \frac{2\left(\gamma^2 + \gamma - \gamma_1 + 1\right)}{\zeta(2)} y x^2 \log \frac{x^2}{y} \\ &- \frac{2\left(\gamma^3 + \gamma^2 + 2\gamma - \gamma_1 - 3\gamma\gamma_1 + 3\gamma_2 + 1\right)}{\zeta(2)} y x^2 \end{aligned}$$

$$+ O\left(x^2 y L^8 \left(\frac{x}{y}\right)^{1/2}\right). \tag{5.24}$$

5.7. Asymptotic Formula of (1.18).

Now, we substitute (5.16), (5.20), (5.21), (5.22), and (5.24) into (5.14) to obtain the assertion of theorem 1.15. □

6. Evaluation of c_0

We use (5.19) and Lemma 3.2 to obtain

$$c_0 = \frac{1}{2\pi i}\int_{5/4-iT}^{5/4+iT} \frac{\zeta'(s)\zeta'(2-s)}{s(2-s)} ds + O\left(T^{-1/2}\right),$$

then

$$|c_0| \le \frac{\left|\zeta'(\frac{5}{4})\right|}{2\pi}\int_{-T}^{T} \frac{\left|\zeta'\left(\frac{3}{4}-it\right)\right|}{\left|\frac{3}{4}-it\right|\left|\frac{5}{4}+it\right|} dt + O\left(T^{-1/2}\right).$$

As $T \to \infty$, then we have

$$\begin{aligned} |c_0| &\le \frac{\left|\zeta'(\frac{5}{4})\right|}{\pi}\int_{0}^{\infty} \frac{\left|\zeta'\left(\frac{3}{4}+it\right)\right|}{\sqrt{((\frac{3}{4})^2+t^2)((\frac{5}{4})^2+t^2)}} dt \\ &\le \frac{4\left|\zeta'(\frac{5}{4})\right|}{3\pi}\int_{0}^{\infty} \frac{\left|\zeta'\left(\frac{3}{4}(1+iy)\right)\right|}{1+y^2} dy \\ &\le 0.425\cdots \left|\zeta'\left(\frac{5}{4}\right)\right| \int_{0}^{\infty} \frac{\left|\zeta'\left(\frac{3}{4}(1+iy)\right)\right|}{1+y^2} dy. \end{aligned}$$

Here, the integral on the right-hand side of the above is absolutely convergent, and it is a computable constant.

Funding The first author was supported by JSPS Grant-in-Aid for Scientific Research(C) (JP21K03205). The second author was supported by JSPS Grant-in-Aid for Early-Career Scientists (JP22K13900).

Data availibility Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

References

[1] Anderson, D.R., Apostol, T.M.: The evaluation of Ramanujan's sum and generalizations. Duke Math. J. **20**, 211–216 (1952)

[2] Apostol, T.M.: Introduction to Analytic Number Theory. Springer, Cham (1976)

[3] Chan, T.H., Kumchev, A.V.: On sums of Ramanujan sums. Acta Arith. **152**, 1–10 (2012)

[4] Graham, S.W., Kolesnik, G.: Van der Corput's Method of Exponential Sums. London Mathematical Society Lecture Note Series, vol. 126. Cambridge University Press, Cambridge (1991)

[5] Hall, R.R.: The behaviour of the Riemann zeta-function on the critical line. Mathematika **46**, 281–313 (1999)

[6] Ivić, A.: The Riemann Zeta-Function. Dover Publications, Garden City (2003)

[7] Kiuchi, I., Minamide, M., Tanigawa, Y.: On a sum involving the Möbius function. Acta Arith. **169**, 149–168 (2015)

[8] Kiuchi, I.: On sums of sums involving squarefull numbers. Acta Arith. **200**, 197–211 (2021)

[9] Kiuchi, I.: On a sum involving squarefull numbers. Rocky Mountain. J. Math. **52**, 1713–1718 (2022)

[10] Kiuchi, I.: On sums of sums involving cube-full numbers. Ramanujan J. **59**, 279–296 (2022)

[11] Kiuchi, I.: On sums of sums involving the Liouville function. Funct. Approx. Comment. Math. **70**, 245–262 (2024)

[12] Kiuchi, I.: Sums of logarithmic weights involving r-full numbers. Ramanujan J. **64**, 1045–1059 (2024)

[13] Kühn, P., Robles, N.: Explicit formulas of a generalized Ramanujan sum, Inter. J. Number Theory. **12**, 383–408 (2016)

[14] Montgomery, H.L., Vaughan, R.C.: Multiplicative Number Theory I. Classical Theory. Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (2007)

[15] Robles, N.: Twisted second moments and explicit formulae of the Riemann zeta-function, Ph.D. Thesis, Universität Zürich, 25–52 (2015)

[16] Robles, N., Roy, A.: Moments of averages of generalized Ramanujan sums. Monatsh. Math. **182**, 433–461 (2017)

[17] Tenenbaum, G.: Introduction to Analytic and Probabilistic Number Theory, Graduate Studies, vol. 163. American Mathematical Society, Providence (2008)

[18] Titchmarsh, E.C.: The Theory of the Riemann Zeta-Function, 2nd edn. Oxford University Press, Oxford (1986)

[19] Tóth, L., Zhai, W.: On the error term concerning the number of subgroups of the groups $Z_m \times Z_n$ with $m, n \le x$. Acta Arith. **183**, 285–299 (2018)

Isao Kiuchi
Department of Mathematical Sciences
Yamaguchi University
1677-1 Yoshida
Yamaguchi, Yamaguchi 753-8512
Japan

Wataru Takeda
Department of Mathematics
Toho University
2-2-1 Miyama
Funabashi, Chiba 274-8510
Japan
e-mail: wataru.takeda@sci.toho-u.ac.jp

Received: October 6, 2023.
Accepted: August 21, 2024.